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Using the homogenization method we shall discuss several clases of double-porosity models. We introduce the model of partially fissured medium in which there is some fluid flow directly through the cell structure. The intensity of the direct diffusion will give at the macroscale different equations.

1. INTRODUCTION

Every attempt to model laminar flow through highly inhomogeneous media leads to singular problems of partial differential equations with rapidly oscillating coefficients.

A partially fissured medium is a fissured medium in which there are a substantial paths directly joining the cells in addition to the predominant connection with the surrounding fissure system. That means that the cells are not completely isolated from one another by the fissure system, and the matrix is somewhat connected.

The classical example of such flow is the parabolic system

$$
(1.1)\qquad \begin{aligned} a\frac{\partial u_1}{\partial t} - \nabla \cdot (A \nabla u_1) + H(u_1 - u_2) = f \\ b\frac{\partial u_2}{\partial t} - \nabla \cdot (B \nabla u_2) - H(u_1 - u_2) = f \end{aligned}
$$

introduced by G.I. Barenblatt, I.P Zheltov and I.N. Kochina [1]. Here u_1 represents the density of fluid in one material and u_2 the density in the second. The coefficients $a(x)$ and $A(x)$ are the porosity and permeability of the first material, respectively, whereas $b(x)$ and $B(x)$ are the corresponding property of the second material. Both of these equations are to be understood macroscopically. The third term in each equation is an attempt to quantify the exchange of fluid between the two components.

In order to obtain a mathematical proof for such a model introduced by heuristic justification, we refer to [2] [3] in which the derivation by homogenization of the distributed microstructute model of a totally fissured medium was made. For a general approach, by homogenization, of the problems of flow through porous media we indicate the book [4]. An extensive disscusion of different microstructure models of porous media was given in [5].

The aim of the present paper is to determine the diffusion from the system of fissures to the matrix of porous cells. The order of magnitude of such a transfer by diffusion gives rise to very different models at the macroscale. One of them is precisely the model described by the system (1.1). Note also there exists four other different models.

In section 2 we introduce the diffusion model and we discuss the boundary conditions. The homogenization technique will be applied in section 3. Concluding remarks are given in section 4.

Remark 1.1. For historical reasons the distributed microstructure model will be presented in the framework of a diffusion problem. Clearly it has a meaningful analog for the analogous problems of heat conduction or absorption of a dissolved chemical in a fluid flawing through a porous medium. Note also that the problem of convection-diffusion will be treated in the same manner, and we refer to the book [6].

2. THE DIFFUSION MODEL

We must consider a porous medium composed of two interwoven and connected components. The first one is the system of fissures and the second is the matrix of porous cells. The two components were periodically distributed in space. Note that such a construction is impossible in $\mathbb{R}^2$.

Let Ω be an open, bounded set of $\mathbb{R}^3$, composed by the two parts described before. We denote by

$$\Omega_{\varepsilon\alpha} = \{x \in \Omega; x \in \varepsilon Y_\alpha\} \qquad \alpha = 1, 2$$

such that $\Omega = \Omega_{\varepsilon_1} \cup \Omega_{\varepsilon 2}$.

At the microscale we search for $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2})$ solution to:

$$a^\varepsilon \frac{\partial u_{\varepsilon 1}}{\partial t} - \nabla \cdot (A^\varepsilon \nabla u_{\varepsilon 1}) = f \qquad \text{in } \Omega_{\varepsilon 1} \tag{2.1}$$

(2.2) $$b^{\varepsilon}\frac{\partial u_{\varepsilon 2}}{\partial t} - \nabla \cdot (B^{\varepsilon} \nabla u_{\varepsilon 2}) = f \qquad \text{in } \Omega_{\varepsilon 2}$$

The boundary and initial conditions are as follows:

(2.3) $$A^{\varepsilon} \nabla u_{\varepsilon 1} \cdot \vec{\nu} = B^{\varepsilon} \nabla u_{\varepsilon 2} \cdot \vec{\nu} \qquad \text{on } \Gamma_{\varepsilon}$$

(2.4) $$-A^{\varepsilon} \nabla u_{\varepsilon 1} \cdot \vec{\nu} = \varepsilon^{p} h^{\varepsilon}(u_{\varepsilon 1} - u_{\varepsilon 2}) \qquad \text{on } \Gamma_{\varepsilon}$$

(2.5) $$u_{\varepsilon 1} = 0 \qquad u_{\varepsilon 2} = 0 \ \text{ for } t = 0$$

The notation $\varphi^{\varepsilon}(x) = \varphi(\frac{x}{\varepsilon})$ means that the microscopic coefficients are rapidly oscillating. If we denote by Y the cell of periodicity and introduce the microscopic variable $y = \frac{x}{\varepsilon}$, we must understand that a function $\varphi(\frac{x}{\varepsilon}) = \varphi(y)$ is Y periodic.

Remark 2.1. The boundary conditions (2.3) (2.4) means that u_{ε} is not a continuous function. If we rewrite (2.3) and (2.4) as

$$\nabla u_1 \cdot \vec{\nu} = \nabla u_2 \, \vec{\nu}$$

$$- \nabla u_1 \cdot \vec{\nu} = h(u_1 - u_2)$$

it is clear that for $h \to \infty$ we obtain $u_1 = u_2$. Note also that the case $h = 0$ gives rise to a completly issolated medium, with no diffusion flux between the two components.

In order to describe such situations we rescale the boundary condition (2.4) by ε^{p}. Following the different values of p we obtain very different equations at the macroscale.

We must suppose that $h^{\varepsilon}(x)) = h(\frac{x}{\varepsilon}$ is a Y-periodic function in the variable $y = \frac{x}{\varepsilon}$, and that its mean value over the bundary Γ between the two components is

(2.6) $$\int_{\Gamma} h(y) \mathrm{d}s \neq 0$$

3. HOMOGENIZATION

We assume that our medium is strictly periodic. Than we are in the classical framework of the homogenization method [4][5][6].

We search for an asymptotic expansion of u_ε of the form:

$$u_{\varepsilon\alpha}(t,x) = u_\alpha^\circ(t,x,y) + \varepsilon u_\alpha^1(t,x,y) + \ldots \quad y = \frac{x}{\varepsilon}, \alpha = 1,2 \tag{3.1}$$

where $u_\alpha^k(t,x,y)$ are Y-periodic functions in y.

The method consists in incorparating expansions (3.1) into the equations (2.1)(2.2), indentifying the similar powers of ε and solving a set of boundary value problems in a characteristic cell Y. In the computation we must take into account the fact that x and y should be considered as independent variables and that the derivation operator is now expressed by

$$\nabla u_\varepsilon = \nabla_x u_\varepsilon + \frac{1}{\varepsilon} \nabla_y u_\varepsilon \tag{3.2}$$

We also introduce the mean value of a function $\varphi(y)$ by

$$\tilde{\varphi} = \frac{1}{|Y|} \int_Y \varphi(y) dy \tag{3.3}$$

The homogenization process, $\varepsilon \to 0$, produces a set of equations satisfied by u°, which in fact represents the macroscopic behaviour of our porous medium.

There are two critical values of $p : p = -1$ and $p = 1$.

3.1 The case $p = -1$

In this case we obtain $u_1^\circ(t,x) = u_2^\circ(t,x) = u(t,x)$. That mean at the macroscale we have only one diffusion equation:

$$\tilde{a}\frac{\partial u^\circ}{\partial t} - \nabla \cdot (\wedge \nabla u^\circ) = f \qquad \text{in } \Omega \tag{3.4}$$

where:

$$\tilde{a} = \frac{1}{|Y|}\left(\int_{Y_1} a(y)dy + \int_{Y_2} b(y)dy\right) \tag{3.5}$$

and the macroscopic permeability $\wedge$ is given by:

$$\wedge_{ij} = \frac{1}{|Y|}\left(\int_{Y_1} (a_{ij} + A_{ik}\frac{\partial w_{1k}}{\partial y_j})dy + \int_{Y_2} (B_{ij} + B_{ik}\frac{\partial w_{2k}}{\partial y_j})dy\right) \tag{3.6}$$

with the auxiliary functions w_{ik} and w_{qk}, the Y-periodic solutions of the cell problems:

$$(3.7)\qquad \begin{array}{ll} -\nabla_y \cdot (A(y) \nabla_y w_{1k}) = \nabla_y A(y) \vec{e}_k & y \in Y_1 \\ -\nabla_y \cdot (B(y) \nabla_y w_{2k}) = \nabla_y B(y) \vec{e}_k & y \in Y_2 \\ A(y) \nabla_y w_{1k} \cdot \vec{\nu} = B(y) \nabla_y w_{2k} \cdot \vec{\nu} & y \in \Gamma \\ A(y) \nabla_y w_{1k} \cdot \vec{\nu} + h(y)(w_{1k} - w_{2k}) = -A(y) \vec{e_k} \cdot \vec{\nu} & y \in \Gamma \end{array}$$

Remark 3.1 The effective macroscopic permeability depends on h. That means, at the macroscopic scale, we must consider not only the values of the microscopic permeabilities of the fissure and the porous block, but also the diffusion accross the boundary Γ*. Nevertheless, the structure of this model remains classical, i.e. only one equation at the macroscale* (3.4).

3.2 The case $p = 1$

At the first order we obtain $u_1^\circ = u_1^\circ(t,x)$ and $u_2^\circ = u_2^\circ(t,x)$, with $u_1^\circ(t,x) \neq u_2^\circ(t,x)$. It will be seen that the leading terms for the density in the fractures and in the matrix blockes, will be a pair of functions $u_1^\circ(t,x)$ and $u_2^\circ(t,x), x \in \Omega, t > 0$, which satisfy the system of equations:

$$(3.8)\qquad \tilde{a}\frac{\partial u_1^\circ}{\partial t} - \nabla \cdot (\tilde{A} \nabla u_1^\circ) + H(u_1^\circ - u_2^\circ) = \theta f$$

$$(3.9)\qquad \tilde{b}\frac{\partial u_2^\circ}{\partial t} - \nabla \cdot (\tilde{B} \nabla u_2^\circ) - H(u_1^\circ - u_2^\circ) = (1-\theta) f$$

with

$$(3.10)\qquad \tilde{A}_{ij} = \frac{1}{|Y|}\int_{Y_1} (A_{ij} + A_{ik}\frac{\partial \chi_{1k}}{\partial y_j})\mathrm{d}y$$

$$(3.11)\qquad \tilde{B}_{ij} = \frac{1}{|Y|}\int_{Y_2} (B_{ij} + B_{ik}\frac{\partial \chi_{2k}}{\partial y_j})\mathrm{d}y$$

$$(3.12)\qquad H = \frac{1}{|Y|}\int_{\Gamma} h(y)\mathrm{d}s$$

(3.13)
$$\theta = \frac{|Y_1|}{|Y_2|}$$

Now the auxiliary Y-periodic functions χ_{1k} and χ_{2k} are the solutions to:

(3.14)
$$\begin{array}{ll} -\nabla_y \cdot (A(y) \nabla_y \chi_{1k}) = \nabla_y A(y)\, \vec{e_k} & y \in Y_1 \\ A(y) \nabla_y \chi_{1k}\, \vec{\nu} = -A(y)\, \vec{e_k} \cdot \vec{\nu} & y \in \Gamma \end{array}$$

and

(3.15)
$$\begin{array}{ll} -\nabla_y (B(y) \nabla_y \chi_{2k}) = \nabla_y B(y)\, \vec{e_k} & y \in Y_2 \\ B(y) \nabla_y \chi_{2k}\, \vec{\nu} = -B(y)\, \vec{e_k} \vec{\nu} & y \in \Gamma \end{array}$$

Remark 3.2. The macroscopic behaviour is exactly the model described by the system (1.1) of G.I. Barenblat, I.P. Zheltov and I.N. Kochina [1]. Of course the hypothesis (2.6) is fundamental, and in such a way we can account for the flux across the boundary Γ.

3.3 Totally Fissured Media

A totally fissured medium consists of a matrix of porous and permeable material cells through which is intertwined a highly developed system of fissures. The flow occurs in the highly permeable system, and most of the storage of fluid is in the matrix of cells which accounts for almost all of the total volume. These fissured media characteristics are modeled by choosing very small values for the permeability in the blockes. Consequently in (2.2) one sets $B^\varepsilon = 0$, because there is no direct flow through the matrix of cells. Practically each cell of the matrix is isolated from adjacent cells by the fissure system. The resulting system at the macroscale is a parabolic-ordinary differential equations.

(3.16)
$$\begin{array}{l} \tilde{a}\dfrac{\partial u_1^\circ}{\partial t} - \nabla \cdot (\tilde{A} \nabla u_1^\circ) + H(u_1^\circ - u_2^\circ) = f \\ \tilde{b}\dfrac{\partial u_2^\circ}{\partial t} - H(u_1^\circ - u_2^\circ) = 0 \end{array}$$

and is called the first-order kinetic model [2][3].

3.4 The case $p = 0$

Of course this case is an intermediate one between those presented in sections 3.1 and 3.2. We have $u_1^\circ(t,x) = u_2^\circ(t,x) = u^\circ(t,x)$ and at the macroscale an equation of the form (3.4):

$$\tilde{a}\frac{\partial u^\circ}{\partial t} - \nabla \cdot (\widetilde{\Lambda} \nabla u^3) = f \qquad \text{in}\Omega \tag{3.17}$$

where the macroscopic permeability $\tilde{\Lambda}$ is given by (3.6) wityh χ_{1k} and χ_{2k} instead of w_{1k} and w_{2k}. Of course χ_{1k} and χ_{2k} are given by (3.14) and (3.15).

3.5 Other cases

It is clear that we have also two extremely different cases: $p < -1$ and $p > 1$.

In the case $p < -1$ we have practically the diffusion problem with classical transmission boundary conditions: continuity of normal fluxes and continuity of u_ε. The macroscopic result is only one diffusion equation with classical homogenized coefficients [7].

The case $p > 1$ corresponds to the macroscopic equations (3.8) (3.9) with $H = 0$. The porous blocks are completely isolated and no interaction occurs between them and the system of fissures. Such a model coresponds to the case described in the section 3.3 with the suplementary hypothesis of small values of b^ε in (2.1).

3.6 Estimates

In order to prove the convergence of the homogenization process we need to extend each $u_{\varepsilon\alpha}(\alpha = 1,2)$, defined in $\Omega_{\varepsilon\alpha}$, in the oposite part $\Omega_{\varepsilon\beta}(\beta \neq \alpha,\ \alpha,\beta = 1,2)$. To due that, if we use the extension operator introduced by D. Ciorănescu and J. Saint Jean Paulin [8], denoting by $\widehat{u}_{\varepsilon 1}$ and $\widehat{u}_{\varepsilon 2}$ the extension of each $u_{\varepsilon\alpha}$, we obtain for $p \geq 0$ the estimates:

$$||\widehat{u}_{\varepsilon 1} - \widehat{u}_{\varepsilon 2}||^2_{L^2(r)} \leq \varepsilon^{1-p} C \tag{3.19}$$

Now it is clear that $p = 1$ is the critical value.

In the case $p < -1$ we have the same estimates (3.19) with $\varepsilon^{\frac{p+1}{2}}$, and we can see that $p = -1$ is also a critical value.

CONCLUDING REMARKS

The basic distributed microstructure model is obtained as the limit by homogenization of a corresponding exact but highly singular partial differential equation with rapidly oscillating coefficients.

Experience suggests that the distributed microstructure models are easy to work with, they provide accurate models which include the fine scales and geometry, and their theory can be developed using conventional techniques. Note also that the formula for the macroscopic coefficients is exact.

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