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ON THE INDEX FORMULA OF MELROSE AND NISTOR

ROBERT LAUTER AND SERGIU MOROIANU

ABSTRACT. We improve two results of a paper by Melrose and Nistor [2]. We propose a simpler expression for the index of fully elliptic cusp pseudo-differential operators. We show that the local term in the index formula is given by a convergent indefinite integral.

The main goal of this note is to improve Proposition 16 from the paper [2] by Melrose and Nistor. Namely, let A, B be cusp operators. We claim that we obtain the index functional $IF(A, B)$ defined in [2, (82)] as a sum of only two terms instead of the eight in [2, (90)], provided that $A \otimes B$ is a Hochschild 1-cycle in $\mathcal{A}$. Moreover, we prove that the Atiyah-Singer-type term in the index formula from the Index Theorem of [2] is given by a convergent integral. This term is defined *a priori* as the regularization of a diverging integral. Our second result improves the Reduced Index Theorem from [2], where the integral is shown to be convergent under some extra assumptions.

To fix notations, recall the following definition of the "trace" functionals, equivalent to [2, (44)]:

$$z\tau Tr(Ax^zQ^{-\tau}) = Tr_{\partial,\sigma}(A) + \tau\widehat{Tr}_\partial(A) + z\widehat{Tr}_\sigma(A) + \mathrm{O}(z^2, z\tau, \tau^2). \tag{1}$$

Proposition 1. *Let A, B be cusp operators. Assume that $A \otimes B$ defines a Hochschild cycle in $\mathcal{A}$ (i.e. $[A, B]$ belongs to the residual ideal $\mathcal{I}$). Then*

$$IF(A, B) = \widehat{\mathrm{Tr}}_\partial(A[B, \log x]) + \widehat{\mathrm{Tr}}_\sigma([\log Q, B]A).$$

Proof: The cycle condition means that $Tr([A, B])$ is well-defined, hence the function $Tr([A, B]x^zQ^{-\tau})$ is regular at $z = 0, \tau = 0$. So

$$\begin{aligned}
IF(A, B) &= Tr([A, B]x^zQ^{-\tau})_{z=0,\tau=0} \\
&= Tr\left[(A(B - x^zBx^{-z})x^zQ^{-\tau}) + (Q^\tau BQ^{-\tau} - B)Ax^zQ^{-\tau})\right]_{z=0,\tau=0} \\
&= Tr\left[zA([B, \log x] + zH_1(z))x^zQ^{-\tau}\right]_{z=0,\tau=0} \\
&\quad + Tr\left[\tau([\log Q, B] + \tau H_2(\tau))Ax^zQ^{-\tau}\right]_{z=0,\tau=0},
\end{aligned}$$

where $H_1(z), H_2(\tau)$ are entire families of operators of fixed order.

Recall [2, Lemma 4] that for any holomorphic family $C(z, \tau)$ of cusp operators of fixed integral order, the function $Tr(Cx^zQ^{-\tau})$ has at most simple poles in each complex variable at real integers. It follows that the terms containing $H_1(z), H_2(\tau)$ do not contribute any constant term. We are left with the regularized values at $z = 0, \tau = 0$ of $Tr(zA[B, \log x]x^zQ^{-\tau})$ and $Tr(\tau[\log Q, B]Ax^zQ^{-\tau})$, which by (1) are just $\widehat{\mathrm{Tr}}_\partial(A([B, \log x])$, resp. $\widehat{\mathrm{Tr}}_\sigma([\log Q, B]A)$. $\square$

Let now A be a fully elliptic (or equivalently Fredholm) cusp operator. Then A has a parametrix B which inverts it up to remainders in the residual ideal $\mathcal{I}$. Note that $A \otimes B$ is a Hochschild cycle in $\mathcal{A}$. Let us make the following definitions: $\widehat{\mathrm{Tr}}_\partial(A([B, \log x]) := -\frac{1}{2}\eta(A)$ and $\widehat{\mathrm{Tr}}_\sigma([\log Q, B]A) := \overline{AS}(A)$ (the second definition differs from ([2], (87)) by at least a minus sign). Of course, both the definitions of $\eta(A)$ and $\overline{AS}(A)$ are independent of the choice of the parametrix B. In fact, it suffices that B inverts A up to trace class remainders.

Theorem 2 (Variant of [2, Index Theorem]). *Let A be a fully elliptic cusp operator. Then*

$$Ind(A) = \overline{AS}(A) - \frac{1}{2}\eta(A).$$

Proof: Apply Proposition 1 to the cycle $A \otimes B$, where B is a pseudo-inverse of A such that

$$BA = I - P_{\ker A}, AB = I - P_{\mathrm{coker} A}. \tag{2}$$

Notice now that $IF(A, B) = Ind(A)$. □

We claim now that the conclusion of [2, Lemma 15] holds provided only that A is fully elliptic, without any extra assumption. In other words:

Proposition 3. *The integral for $\overline{AS}(A)$ converges without regularization in x.*

Proof: Consider the function $f(\epsilon) = \int_{^cS^*X\cap\{x=\epsilon\}}([\log Q, B]A)_{-n} \imath_{\partial_x} \imath_{\mathcal{R}} \omega_c^n$. By [2, (55)],

$$\overline{AS}(A) = \int_{^cS^*X\cap\{x\geq x_0\}}([\log Q, B]A)_{-n} \imath_{\mathcal{R}} \omega_c^n + \mathrm{LIM}_{\epsilon\to 0} \int_\epsilon^{x_0} f(x)dx. \tag{3}$$

Here LIM is the Hadamard regularization of the limit, defined as in [1]. We remark that the volume form $\imath_{\partial_x} \imath_{\mathcal{R}} \omega_c^n$, and hence also $f(x)$, is a smooth multiple of x^{-2}. Replace f by its asymptotic expansion at $x = 0$. As far as convergence is concerned, only the coefficients of x^{-2} and x^{-1} matter. The product on the boundary algebra $\mathcal{A}_\partial$ has as leading term the suspended product, and the other terms are multiples of x. So let A_0, B_0, Q_0 be the indicial operators of A, B, Q in the suspended algebra. The term in (3) diverging like ϵ^{-1} is given by the leading component of f, i.e. $f_{-2} = \int_{^cS^*X\cap\{x=0\}}([\log Q_0, B_0]A_0)_{-n} \imath_{\partial_x} \imath_{\mathcal{R}} \omega_c^n$.

Lemma 4. $f_{-2} = 0$.

Proof: We can express f_{-2} as the residue of $\overline{Tr}(Q_0^\tau[\log Q_0, B_0]A_0)$ at $\tau = 0$. The rest of the lemma is formally identical to the index formula [2, (3)]. We claim that f_{-2} is equal to $\overline{Tr}([B_0, A_0])$, which equals zero (because $\overline{Tr}$ is a trace, and also because $[B_0, A_0] = 0$). Going backwards,

$$\begin{aligned}\overline{Tr}([B_0, A_0]) &= \overline{Tr}(Q_0^\tau[B_0, A_0])_{\tau=0} = \overline{Tr}([Q_0^\tau, B_0]A_0)_{\tau=0} \\ &= \overline{Tr}\left(\tau Q_0^\tau \frac{B_0 - Q_0^{-\tau} B_0 Q_0^\tau}{\tau} A_0\right)_{\tau=0}.\end{aligned}$$

Let $F(\tau) = (B_0 - Q_0^{-\tau} B_0 Q_0^{\tau})/\tau$. Then $F(\tau)$ is entire, of fixed order, and $F(0) = [\log Q_0, B_0]$. Note that $\overline{Tr}(Q_0^{\tau} F(\tau) A_0)$ has at most simple poles at real integer τ, which implies the claim. $\square$

Let us now examine the other diverging term. It is given by the coefficient of x^{-1} in $f(x)$, which by [2, (55)] is just $\mathrm{Tr}_{\partial,\sigma}([\log Q, B]A)$. By [2, Lemma 12], we have $\mathrm{Tr}_{\partial,\sigma}([\log Q, B]A) = \widehat{\mathrm{Tr}}_{\partial}([A, B])$. By assumption, $[A, B]$ belongs to $\mathcal{I}$ and so $\widehat{\mathrm{Tr}}_{\partial}([A, B]) = 0$.

Therefore, only integrable terms are left in the asymptotic expansion of the integrand in (3). This finishes the proof of the proposition. $\square$

As for the η term, this is not quite the η invariant from [1], even with the assumptions on Q at the boundary from [2, Proposition 17]. The difference arises from the fact that the regular value at $z = 0$ of $\overline{Tr}(Q^z A)$ and $\overline{Tr}(A)$ do not always agree. As shown by Nistor [3], they do agree for differential operators A, in particular for Dirac operators.

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References

[1] R. B. Melrose. The eta invariant and families of pseudodifferential operators. *Math. Res. Letters*, 2:541–561, 1995.

[2] R. B. Melrose and V. Nistor. Homology of pseudodifferential operators I. Manifolds with boundary. Version 3.0A, to appear in Amer. J. Math.

[3] V. Nistor. An index theorem for families of elliptic operators invariant with respect to a bundle of Lie groups. in preparation.

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