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The Intertwining Lifting Theorem for a Class of Reproducing Kernel Spaces

Călin-Grigore Ambrozie and Dan Timotin

0. Introduction.

Originating in the work of Schur, Carathéodory, Fejer and others, interpolation problems for bounded analytic functions have been studied for more than a century. The simplest (and more easily generalized to different contexts) is probably the Nevanlinna-Pick problem, which prescribes the value of the function in a finite number of points and asks for the minimum value of the uniform norm. Starting with the paper of Sarason ([S]), it has been realized that there exists a natural operatorial frame which unifies all function theoretic problems. This line of research culminated in the intertwining lifting theorem of Sz.-Nagy and Foiaş ([SNF]), which has found subsequently many applications, including applied areas as systems theory.

In recent years, there has been an increasing trend towards the investigation of several variables variants of the interpolation problems. As expected, complications arise, and a large variety of techniques has been used in order to cope with them (see, for instance, [Ag2], [Ag3], [AgM], [BB], [CS]) .

The aim of this paper is to discuss a certain class of reproducing kernel Hilbert spaces that seems to be the most suitable for the direct extension of the classical results. Indeed, the considered class, while defined in an extremely general manner, allows one to recapture most of the interpolation theorems, including their operatorial generalizations. It seems to include all known examples in which these generalizations are true.

The framework is the following: we consider an abstract set of indices Λ and a Hilbert space $\mathcal{H}$ of complex functions defined on Λ, such that the point evaluations $f \mapsto f(\lambda)$ are continous. A standard procedure is then to introduce the reproducing vectors for these evaluations, which we will denote with C_λ; that is, $f(\lambda) = \langle f, C_\lambda \rangle$ for any $f \in \mathcal{H}$. The reproducing kernel for the space is then the function $C : \Lambda \times \Lambda \to \mathbf{C}$, defined by $C(\lambda, \mu) = C_\mu(\lambda)$, which is a function of positive type.

Now, the particular class of reproducing kernels spaces that we will consider, apart from some natural conditions that are fulfilled by basically all the spaces that appear in applications, satisfies a special property which is essential to interpolation. Namely, we will assume that the function $1 - 1/C$ is also of positive type on $\Lambda \times \Lambda$. This condition has been pointed out by Quiggin ([Q]) in connection with the Nevanlinna-Pick problem; see also [McC], where it is shown it actually characterizes kernels which satisfy a (matrix-valued) Nevanlinna-Pick theorem. We will develop here more of its consequences; it is in fact surprising that the condition contains all the function theory that is usually involved in the proof of the interpolation results. There is no need of further complex function theory of one or of several variables, nor of any Lebesgue spaces and measures.

Another remarkable fact is that, although the techniques we use, which originate in the work of Agler ([Ag2]), are closely connected with concepts of systems theory (for instance, the main representation theorem below can be interpreted as a realization formula, and there exists even a "controlability" operator that can be proved to be contractive—see Proposition 1), there is actually no corresponding "time domain" interpretation.

To state the main results, we have to introduce also the multiplier algebra on $\mathcal{H}$. This is the algebra $\mathcal{M}$ of all bounded operators of the form $T(f)(\lambda) = \phi(\lambda)f(\lambda)$. The function ϕ is called the *symbol* of T, and we will denote $T_\phi = T$. A closed subspace $\mathcal{X} \subset \mathcal{H}$ will be called *invariant* if $\mathcal{M}^*\mathcal{X} \subset \mathcal{X}$ (that is, if it is invariant to the adjoints of multipliers).

The main result of the paper is the following intertwining lifting theorem.

Theorem A. *Suppose $\mathcal{X} \subset \mathcal{H}$ is an invariant subspace, and denote $\mathcal{M}_\mathcal{X} = P_\mathcal{X}\mathcal{M}|\mathcal{X}$. If $X \in \mathcal{L}(\mathcal{X})$ and $X \in \mathcal{M}'_\mathcal{X}$ (the commutant of $\mathcal{M}_\mathcal{X}$), then there exists ϕ multiplier, $\|\phi\|_\mathcal{M} = \|X\|$, such that $X = P_\mathcal{X}T_\phi|\mathcal{X}$.*

We will give several applications of Theorem A, including Nevannlina-Pick or Carathéodory-Fejer type results, as well as applications to model theory; other consequences will be pursued elsewhere.

A central role in the proof of Theorem A is played by a fractional transform representation for contractive multipliers, which can be of independent interest. To state it, note that, since $1 - 1/C$ is of positive type, there exists an (essentially unique) Hilbert space $\mathcal{E}$, and a function $F : \Lambda \to \mathcal{E}$, such that

$$(1 - \frac{1}{C})(\lambda, \mu) = \langle F(\mu), F(\lambda) \rangle_\mathcal{E}.$$

Since $||F(\lambda)|| < 1$ for any $\lambda \in \Lambda$, we may define, for any Hilbert space K, a strict contraction $Z(\lambda) : \mathcal{E} \otimes K \to K$ by its adjoint

$$Z(\lambda)^* k = F(\lambda) \otimes k$$

and state the representation theorem.

Theorem B. *A function $\phi : \Lambda \to \mathbf{C}$ satisfies $\phi \in \mathcal{M}$ and $||\phi||_{\mathcal{M}} \leq 1$ if and only if there exists a Hilbert space K and a unitary operator $U : K \oplus \mathbf{C} \to (\mathcal{E} \otimes K) \oplus \mathbf{C}$, $U = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$, such that*

$$\phi(\lambda) = \mathbf{d} + \mathbf{c}(I - Z(\lambda)\mathbf{a})^{-1} Z(\lambda)\mathbf{b}.$$

Theorem A is actually Theorem 2 in section 4. Theorem B follows by combining Theorem 1 with Corollary 1. The ideas of the proof originate in the work of Agler; the actual development is inspired by [BLTT].

1. Preliminaries.

Let $\mathcal{H}$ be a Hilbert space of complex functions defined on the set Λ, such that the point evaluations $f \mapsto f(\lambda)$ are continous. Define $C_\lambda \in \mathcal{H}$ by $f(\lambda) = \langle f, C_\lambda \rangle$ for any $f \in \mathcal{H}$. The reproducing kernel for the space is the function $C : \Lambda \times \Lambda \to \mathbf{C}$, defined by $C(\lambda, \mu) = C_\mu(\lambda)$; it is a function of positive type: $\sum_1^N C(\lambda_i, \lambda_j) z_i \bar{z}_j \geq 0$ for any $\lambda_1, \dots, \lambda_N \in \Lambda$ and $z_1, \dots, z_N \in \mathbf{C}$.

There is actually a one-to-one relationship between functions of positive type and reproducing kernel Hilbert spaces: if $k : \Lambda \times \Lambda \to \mathbf{C}$ is of positive type, then there exists an essentially unique Hilbert space of functions $\mathcal{E}$ on Λ which has k as reproducing kernel. In particular, there exists a function $F : \Lambda \to \mathcal{E}$, such that

$$k(\lambda, \mu) = \langle F(\mu), F(\lambda) \rangle.$$

A multiplier on $\mathcal{H}$ is a function ϕ defined on Λ, such that $\phi f \in \mathcal{H}$ for any $f \in \mathcal{H}$. The closed graph theorem implies then that T_ϕ defined by $T_\phi(f) = \phi f$ is a bounded operator on $\mathcal{H}$. The algebra of such multiplication operators is denoted by $\mathcal{M}$. We will denote (a slight abuse of notation) $||\phi||_{\mathcal{M}} = ||T_\phi||$. Note also that $T_\phi^* C_\lambda = \overline{\phi(\lambda)} C_\lambda$ for any $\lambda \in \Lambda$.

Functions of positive type play an important role in our proofs; we collect now a series of results that will be needed. First, the following characterizations are well known ([Aro], [Sa]).

Lemma 1. *Suppose $f, \phi : \Lambda \to \mathbf{C}$. Then:*

(i) $f \in \mathcal{H}$ and $\|f\| \leq 1$ if and only if the function $C(\lambda, \mu) - f(\lambda)\overline{f(\mu)}$ is of positive type.

(ii) ϕ is a multiplier and $\|\phi\|_{\mathcal{M}} \leq 1$ if and only if the function $C(\lambda, \mu)(1 - \phi(\lambda)\overline{\phi(\mu)})$ is of positive type.

A related result that is easy to prove (and that implies (ii) in Lemma 1) is

Lemma 2. *Suppose $\mathcal{G}_1, \mathcal{G}_2$ are Hilbert spaces, and $g_i : \Lambda \to \mathcal{G}_i$. There exists a contraction $T : \mathcal{G}_1 \to \mathcal{G}_2$ such that $T(g_1(\lambda)) = g_2(\lambda)$ for any $\lambda \in \Lambda$ if and only if the function $\langle g_1(\lambda), g_1(\mu)\rangle - \langle g_2(\lambda), g_2(\mu)\rangle$ is of positive type.*

Finally, a classical theorem of Schur states that the (pointwise) product of two functions of positive type is also of positive type.

We will now specialize to a more restricted class of reproducing kernel Hilbert spaces. From now on we assume that the following conditions are satisfied:

(i) The constant functions belong to $\mathcal{H}$, and $\|\mathbf{1}\| = 1$;

(ii) $\mathcal{M}\mathbf{1}$ is dense in $\mathcal{H}$;

(iii) $C(\lambda, \mu) \neq 0$ for any $\lambda, \mu \in \Lambda$;

(iv) $1 - 1/C : \Lambda \times \Lambda \to \mathbf{C}$ is a function of positive type.

A few comments concerning these definitions are in order. Conditions (i) and (ii) are quite natural, and all the reproducing kernel spaces usually met satisfy them. We will denote by P the orthogonal projection onto the constant functions; from the equality $\langle C_\lambda - \mathbf{1}, \mathbf{1}\rangle = \langle C_\lambda, \mathbf{1}\rangle - \langle \mathbf{1}, \mathbf{1}\rangle = 1 - 1 = 0$ it follows that $PC_\lambda = \mathbf{1}$ for any $\lambda \in \Lambda$.

(iii) is also true in most applications; moreover, it plays an important role in model theory (see [Ag1], [AMV]).

The main restriction is given by condition (iv); it is the one that singles out a class of spaces which are extremely convenient for interpolation. It has already appeared in [Q], where it was shown that it implies a Nevanlinna-Pick property (see Corollary 2), and in [McC]. In fact, we will use the consequence of condition (iv) that states that there exists an (essentially unique) Hilbert space $\mathcal{E}$ and a function $F : \Lambda \to \mathcal{E}$, such that

$$(1 - \frac{1}{C})(\lambda, \mu) = \langle F(\mu), F(\lambda)\rangle_{\mathcal{E}}; \tag{1}$$

$\mathcal{E}$ and F will be fixed in the sequel. For any $\lambda \in \Lambda$ we have

$$\|F(\lambda)\|^2 = 1 - \frac{1}{C(\lambda, \bar{\lambda})} < 1.$$

Finally, in view of the lifting theorem, we will consider later a subspace $\mathcal{X} \subset \mathcal{H}$, such that $\mathcal{H} \ominus \mathcal{X}$ is invariant to $\mathcal{M}$; we will call such an $\mathcal{X}$ an *invariant subspace* (it is of course invariant to the adjoint of multipliers in $\mathcal{M}$).

2. The fractional transform.

We will introduce a fractional transform that produces contractive multipliers on $\mathcal{H}$; later it will be shown that it yields a representation formula for all multipliers. If K is an arbitrary Hilbert space and $\lambda \in \Lambda$ is fixed, define $Z(\lambda) : \mathcal{E} \otimes K \to K$ by its adjoint

$$Z(\lambda)^* k = F(\lambda) \otimes k.$$

It is easy to check that the action of $Z(\lambda)$ on simple tensors is given by $Z(\lambda)(\xi \otimes k) = \langle \xi, F(\lambda) \rangle k$. Also, $Z(\lambda)$ is a strict contraction, since $\|F(\lambda)\| < 1$.

Theorem 1. *Suppose K a Hilbert space and $U : K \oplus \mathbf{C} \to (\mathcal{E} \otimes K) \oplus \mathbf{C}$ is a unitary operator such that with respect to the above decompositions we have $U = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$. If*

$$\phi(\lambda) = \mathbf{d} + \mathbf{c}(I - Z(\lambda)\mathbf{a})^{-1} Z(\lambda)\mathbf{b} \tag{2}$$

(with $Z(\lambda)$ defined above), then ϕ is a multiplier and $\|\phi\|_{\mathcal{M}} \leq 1$.

Proof. By Lemma 1, we have to prove that the function $C(\lambda, \mu)(1 - \phi(\lambda)\overline{\phi(\mu)})$ is of positive type. We have

$$\overline{\phi(\lambda)} = \mathbf{d}^* 1 + \mathbf{b}^* Z(\lambda)^* (I - \mathbf{a}^* Z(\lambda)^*)^{-1} \mathbf{c}^* 1.$$

Let us denote $g(\lambda) = (I - \mathbf{a}^* Z(\lambda)^*)^{-1} \mathbf{c}^* 1$. The system of relations

$$\begin{aligned} g(\lambda) &= \mathbf{a}^* Z(\lambda)^* g(\lambda) + \mathbf{c}^* 1, \\ \overline{\phi(\lambda)} &= \mathbf{b}^* Z(\lambda)^* g(\lambda) + \mathbf{d}^* 1 \end{aligned} \tag{3}$$

can be written (using the definition of $Z(\lambda)^*$)

$$\begin{pmatrix} g(\lambda) \\ \overline{\phi(\lambda)} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^* & \mathbf{c}^* \\ \mathbf{b}^* & \mathbf{d}^* \end{pmatrix} \begin{pmatrix} F(\lambda) \otimes g(\lambda) \\ 1 \end{pmatrix}$$

Since $\begin{pmatrix} \mathbf{a}^* & \mathbf{c}^* \\ \mathbf{b}^* & \mathbf{d}^* \end{pmatrix}$ is unitary (in particular contractive), Lemma 2 implies that the function

$$\langle F(\lambda), F(\mu)\rangle\langle g(\lambda), g(\mu)\rangle + 1 - \langle g(\lambda), g(\mu)\rangle - \overline{\phi(\lambda)}\phi(\mu)$$

is of positive type. Using the definition of F (formula (1)), it follows that

$$\left(1 - \frac{1}{C(\mu,\lambda)}\right)\langle g(\lambda), g(\mu)\rangle + 1 - \langle g(\lambda), g(\mu)\rangle - \overline{\phi(\lambda)}\phi(\mu) = -\frac{1}{C(\mu,\lambda)}\langle g(\lambda), g(\mu)\rangle + 1 - \overline{\phi(\lambda)}\phi(\mu)$$

is of positive type. But the product of this function with the positive type function $C(\mu,\lambda)$ is also of positive type (by Schur's theorem). Thus $C(\mu,\lambda)(1-\overline{\phi(\lambda)}\phi(\mu)) - \langle g(\lambda), g(\mu)\rangle$ is of positive type, and therefore also $C(\mu,\lambda)(1-\overline{\phi(\lambda)}\phi(\mu))$. The theorem is proved. ∎

In Section 4 we will obtain a converse to theorem 1, showing that contractive multipliers are actually characterized by the representation (2). Meanwhile, keeping the same notations, we prove a related result that will be used in the sequel.

Proposition 1. *The map $\Psi(C_\lambda) = g(\lambda)$ can be extended to a contraction $\Psi : \mathcal{H} \to K$.*

Proof. By Lemma 2, we have to prove that $\langle C_\lambda, C_\mu\rangle - \langle g(\lambda), g(\mu)\rangle$ is of positive type. The first equation in (3) can be written

$$g(\lambda) = \begin{pmatrix} \mathbf{a}^* & \mathbf{c}^* \end{pmatrix} \begin{pmatrix} F(\lambda)\otimes g(\lambda) \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} \mathbf{a}^* & \mathbf{c}^* \end{pmatrix}$ is contractive, applying again Lemma 2 implies that

$$\begin{aligned}\langle F(\lambda), F(\mu)\rangle\langle g(\lambda), g(\mu)\rangle + 1 - \langle g(\lambda), g(\mu)\rangle &= \left(1 - \frac{1}{C(\mu,\lambda)}\right)\langle g(\lambda), g(\mu)\rangle + 1 - \langle g(\lambda), g(\mu)\rangle \\ &= \frac{1}{C(\mu,\lambda)}(C(\mu,\lambda) - \langle g(\lambda), g(\mu)\rangle)\end{aligned}$$

is of positive type. Appying Schur's theorem ends the proof. ∎

3. An auxiliary operator.

We introduce in this section a kind of "multiplication operator" coresponding to $Z(\lambda)$. It will be denoted by $S : \mathcal{E}\otimes\mathcal{H} \to \mathcal{H}$; as in the case of $Z(\lambda)$, we actually start with its adjoint $S^* : \mathcal{H} \to \mathcal{E}\otimes\mathcal{H}$. This is defined by the requirement that

$$S^* C_\lambda = F(\lambda)\otimes C_\lambda.$$

We apply Lemma 2, in order to show that S^* (and hence S) can be extended to a contraction; indeed, the function

$$\langle C_\lambda, C_\mu\rangle - \langle C_\lambda, C_\mu\rangle\langle F(\lambda), F(\mu)\rangle = C(\mu,\lambda)(1-(1-1/C(\mu,\lambda))) = 1,$$

is obviously of positive type.

We can also compute the action of S on a simple tensor; we have

$$\begin{aligned} S(\eta\otimes f)(\lambda) &= \langle S(\eta\otimes f), C_\lambda\rangle = \langle \eta\otimes f, S^*C_\lambda\rangle \\ &= \langle \eta\otimes f, F(\lambda)\otimes C_\lambda\rangle = \langle \eta, F(\lambda)\rangle f(\lambda). \end{aligned}$$

For a fixed $\eta\in\mathcal{E}$, denote $F_\eta(\lambda) = \langle\eta, F(\lambda)\rangle$; we have thus $S(\eta\otimes f)(\lambda) = F_\eta(\lambda)f(\lambda)$. Also, since S is a contraction, it follows that F_η is a multiplier and $\|F_\eta\|_{\mathcal{M}} \le \|\eta\|$.

Lemma 3. *(i) $SS^* = I - P$.*

(ii) With the notations in Proposition 1, we have

$$\begin{aligned} \Psi &= \mathbf{a}^*(I_{\mathcal{E}}\otimes\Psi)S^* + \mathbf{c}^*P \\ PT_\phi^* &= \mathbf{b}^*(I_{\mathcal{E}}\otimes\Psi)S^* + \mathbf{d}^*P \end{aligned} \tag{4}$$

Proof. All statements are equalities between bounded operators defined on $\mathcal{H}$, and it is therefore enough to check them on the total family C_λ, $\lambda\in\Lambda$. We have

$$\begin{aligned} (SS^*C_\lambda)(\mu) &= S(F(\lambda)\otimes C_\lambda)(\mu) = \langle F(\lambda), F(\mu)\rangle C_\lambda(\mu) \\ &= (1-1/C(\mu,\lambda))C(\mu,\lambda) = C_\lambda(\mu) - 1 = (I-P)(C_\lambda)(\mu), \end{aligned}$$

which proves (i). As for (ii), the system of equations (4) applied to a function C_λ becomes (3). ∎

Consider now an invariant subspace $\mathcal{X}\subset\mathcal{H}$; we can then "restrict" the action of S^* to $\mathcal{X}$.

Lemma 4. *With the above notations, $S^*(\mathcal{X})\subset\mathcal{E}\otimes\mathcal{X}$.*

Proof. Suppose $f\in\mathcal{X}$, $g\in\mathcal{X}^\perp$, $\eta\in\mathcal{E}$. Then

$$\langle S^*f, \eta\otimes g\rangle = \langle f, S(\eta\otimes g)\rangle = \langle f, F_\eta g\rangle = 0,$$

where the last equality is true since F_η is a multiplier. Since $(\mathcal{E}\otimes\mathcal{X})^\perp = \mathcal{E}\otimes\mathcal{X}^\perp$, it follows that $S^*f\in\mathcal{E}\otimes\mathcal{X}$. ∎

4. The intertwining lifting theorem.

Theorem 2. *Suppose $\mathcal{X} \subset \mathcal{H}$ is an invariant subspace, and denote $\mathcal{M}_{\mathcal{X}} = P_{\mathcal{X}}\mathcal{M}|\mathcal{X}$. If $X \in \mathcal{L}(\mathcal{X})$, $\|X\| \leq 1$, and $X \in \mathcal{M}'_{\mathcal{X}}$ (the commutant of $\mathcal{M}_{\mathcal{X}}$), then there exists a multiplier ϕ with $\|\phi\|_{\mathcal{M}} \leq 1$, such that $X = P_{\mathcal{X}}T_{\phi}|\mathcal{X}$. Moreover, ϕ admits a representation of type (2) for some Hilbert space K and unitary operator U.*

Proof. We will use some supplementary notation. Define $Y : \mathcal{E} \otimes \mathcal{X} \to \mathcal{X}$ to be $Y = P_{\mathcal{X}}S|\mathcal{E} \otimes \mathcal{X}$. By Lemma 4, we have $Y^* = S^*|\mathcal{X}$. Also, since $SS^* = I - P$, $YY^* = P_{\mathcal{X}}(I - P)|\mathcal{X}$.

Using elementary tensors $\eta \otimes f$, we have

$$XY(\eta \otimes f) = XP_{\mathcal{X}}S(\eta \otimes f) = XP_{\mathcal{X}}F_{\eta}f.$$

Since $X \in \mathcal{M}'_{\mathcal{X}}$, the last term is

$$P_{\mathcal{X}}F_{\eta}Xf = Y(I_{\mathcal{E}} \otimes X)(\eta \otimes f).$$

Therefore

$$XY = Y(I_{\mathcal{E}} \otimes X), \quad Y^*X^* = (I_{\mathcal{E}} \otimes X^*)Y^* \tag{5}$$

By hypothesis, X is a contraction, and so $I - XX^* = D_{X^*}^2$, where we have denoted by D_{X^*} the defect operator of X^*. Therefore,

$$I_{\mathcal{E}\otimes\mathcal{X}} - (I_{\mathcal{E}} \otimes X)(I_{\mathcal{E}} \otimes X^*) = I_{\mathcal{E}} \otimes D_{X^*}^2.$$

Multiplying to the left with Y and to the right with Y^* and using formulas (5), it follows that

$$YY^* - XYY^*X^* = Y(I \otimes D_{X^*}^2)Y^*$$

whence

$$P_{\mathcal{X}}(I - P) - XP_{\mathcal{X}}(I - P)X^* = Y(I \otimes D_{X^*}^2)Y^*.$$

After rearranging the terms, applying the equality to an $h \in \mathcal{X}$ and taking the scalar product with h, we obtain

$$\|Ph\|^2 + \|(I \otimes D_{X^*})Y^*h\|^2 = \|PX^*h\|^2 + \|D_{X^*}h\|^2$$

Consequently, the map

$$\begin{pmatrix} (I_{\mathcal{E}} \otimes D_{X^*})Y^*h \\ Ph \end{pmatrix} \mapsto \begin{pmatrix} D_{X^*}h \\ PX^*h \end{pmatrix}$$

is an isometry on the respective domains (defined by taking all $h \in \mathcal{X}$). We can then embed $\mathcal{H}$ into a larger Hilbert space K, such that $K \ominus \mathcal{H}$ has infinite codimension; denote by $\Phi : \mathcal{X} \to K$ the composition between D_{X^*} and this embedding. Then the map

$$\begin{pmatrix} (I_{\mathcal{E}} \otimes \Phi)Y^*h \\ Ph \end{pmatrix} \mapsto \begin{pmatrix} \Phi h \\ PX^*h \end{pmatrix}$$

is an isometry between its domain and range, which are subspaces of infinite codimension in $(\mathcal{E} \otimes K) \oplus \mathbf{C}$ and $K \oplus \mathbf{C}$ respectively. It can be extended to an everywhere defined unitary, that we will denote $U^* = \begin{pmatrix} \mathbf{a}^* & \mathbf{c}^* \\ \mathbf{b}^* & \mathbf{d}^* \end{pmatrix}$; therefore

$$\begin{aligned} \Phi h &= \mathbf{a}^*(I \otimes \Phi)Y^*h + \mathbf{c}^*Ph, \\ PX^*h &= \mathbf{b}^*(I \otimes \Phi)Y^*h + \mathbf{d}^*Ph. \end{aligned} \tag{6}$$

Define then $\phi(\lambda) = \mathbf{d} + \mathbf{c}(I - Z(\lambda)\mathbf{a})^{-1}Z(\lambda)\mathbf{b}$, which, by the first half of the proof, is a contractive multiplier in $\mathcal{H}$. Consider the map Ψ associated to ϕ as in Proposition 1; it satisfies relations (4). By substracting the first relations of (4) and (6) we obtain, if $\Phi_0 = \Phi - \Psi|\mathcal{X}$,

$$\Phi_0 h = \mathbf{a}^*(I \otimes \Phi_0)Y^*h \qquad \text{for any } h \in \mathcal{X}.$$

Since $\mathbf{a}^*$ is contractive,

$$\Phi_0^*\Phi_0 = Y(I \otimes \Phi_0^*)\mathbf{a}\mathbf{a}^*(I \otimes \Phi_0)Y^* \le Y(I \otimes \Phi_0^*\Phi_0)Y^*.$$

Denote then $Q_0 = \Phi_0^*\Phi_0$; then $Q_0 \ge 0$, and

$$Q_0 \le Y(I \otimes Q_0)Y^*. \tag{7}$$

We will now show that any positive contraction Q_0 on $\mathcal{X}$ which satisfies relation (7) has to be 0. It is enough to show that any positive contraction $Q \in \mathcal{L}(\mathcal{H})$ that satisfies $Q \le S(I \otimes Q)S^*$ has to be 0, since we may then apply the result to $P_{\mathcal{X}}Q_0P_{\mathcal{X}}$ (and take into account Lemma 4). If we denote $M : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $M(T) = S(I \otimes T)S^*$, then M is monotone, and we have $(I - M)(Q) \le 0$. Applying to this inequality the monotone operator $I + M + \cdots + M^{p-1}$ (for some integer p), we obtain $Q - M^p(Q) \le 0$, or $\langle Qh, h\rangle - \langle M^p(Q)h, h\rangle \le 0$.

But it can be shown by induction that $M^p(Q) \to 0$ strongly. Indeed, since $\|M^p(Q)\| \le 1$, it is enough to show that $M^p(Q)(C_\lambda) \to 0$ for any $\lambda \in \Lambda$. Since $M^p(Q) = S(I \otimes M^{p-1}(Q))S^*$, we have

$$M^p(Q)(C_\lambda) = S(I \otimes M^{p-1}(Q))S^*(C_\lambda) = S(F(\lambda) \otimes M^{p-1}(Q)(C_\lambda)).$$

But S is a contraction, and therefore

$$\|M^p(Q)(C_\lambda)\| \leq \|F(\lambda)\| \cdot \|M^{p-1}(Q)(C_\lambda)\|.$$

As $\|F(\lambda)\| < 1$, this proves that $M^p(Q)(C_\lambda) \to 0$.

Thus $Q_0 = 0$ in equation (7) and $\Phi = \Psi$. The second formulas in (4) and (6) say then that $PX^*h = PT_\phi^*h$ for any $h \in \mathcal{X}$; or, equivalently, $\langle X^*h, \mathbf{1}\rangle = \langle T_\phi^*h, \mathbf{1}\rangle$. Remember now that the space $\mathcal{X}$ is invariant to T_f^* for any $f \in \mathcal{M}$; writing the relation for T_f^*h, it follows that

$$\langle X^*h, T_f\mathbf{1}\rangle = \langle T_f^*X^*h, \mathbf{1}\rangle = \langle X^*T_f^*h, \mathbf{1}\rangle = \langle T_\phi^*T_f^*h, \mathbf{1}\rangle = \langle T_f^*T_\phi^*h, \mathbf{1}\rangle = \langle T_\phi^*h, T_f\mathbf{1}\rangle.$$

Since we have assumed that vectors of type $T_f\mathbf{1}$ are dense in $\mathcal{H}$, it follows that $X^* = T_\phi^*|\mathcal{X}$ or $X = P_\mathcal{X}T_\phi|\mathcal{X}$. The theorem is proved. ∎

Important consequences of Theorem 2 are obtained in two extreme cases: either for $\mathcal{X}$ equal to the whole space or for $\mathcal{X}$ finite dimensional. First, by taking $\mathcal{X} = \mathcal{H}$ we obtain a converse of Theorem 1.

Corollary 1. *If $\phi \in \mathcal{M}$ and $\|\phi\|_\mathcal{M} \leq 1$, then there exists a Hilbert space K and a unitary operator $U : K \oplus \mathbf{C} \to (\mathcal{E} \otimes K) \oplus \mathbf{C}$, $U = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$, such that*

$$\phi(\lambda) = \mathbf{d} + \mathbf{c}(I - Z(\lambda)\mathbf{a})^{-1}Z(\lambda)\mathbf{b}.$$

Secondly, suppose $z_1, \ldots, z_n \in \Lambda$, and $w_1, \ldots, w_n \in \mathbf{C}$. Take as $\mathcal{X}$ the linear space generated by the functions C_{z_j} with $j = 1, \ldots, n$. It is invariant to $\mathcal{M}^*$, since any C_λ is an eigenvector for any T_ϕ^*, with eigenvalue $\overline{\phi(\lambda)}$; we may then apply Lemma 2 to obtain the Nevanlinna-Pick property for the space $\mathcal{H}$ (which has been proved by different methods in [Q]).

Corollary 2. *If $z^1, \ldots, z^n \in \Lambda$ and $w_1, \ldots, w_n \in \mathbf{C}$, then there exists $\phi \in \mathcal{M}$ with $\|\phi\|_\mathcal{M} \leq 1$ and $\phi(z^j) = w_j$ for any j if and only if the Pick matrix*

$$\mathbf{P} = \left((1 - w_i\bar{w}_j)C(z^i, z^j)\right)_{i,j=1}^n \tag{8}$$

is positive definite.

Corollary 2 can also be extended by replacing $z^1, \ldots, z^n$ with an arbitrary subset $E \subset \Lambda$.

5. Spaces of analytic functions.

Up to this stage it was not necessary to mention analiticity. However, most of the reproducing spaces of interest are indeed spaces of analytic functions on domains in $D \subset \mathbf{C}^d$ (that is, $\Lambda = D$). In this case the function C is analytic in the first variable and antianalytic in the second, and F defined by formula (1) is antianalytic. Also, the multipliers on $\mathcal{H}$ are analytic functions.

There are some other possible invariant subspaces of interest in this case. For instance, evaluation of partial derivatives in points of D are also bounded functionals, and we have

$$\partial^\alpha f(w) = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_d^{\alpha_d}}(w) = \langle f, C_w^{(\alpha)} \rangle,$$

where

$$C_w^{(\alpha)} = \frac{\partial^{|\alpha|} C_w}{\partial \bar{z}_1^{\alpha_1} \cdots \partial \bar{z}_d^{\alpha_d}}.$$

Thus we may consider analogues of the Carathéodory-Fejer problem: supposing, for instance, that we are given $w \in D$ and $a_\alpha \in \mathbf{C}$, $\alpha \in \mathbf{N}^d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq N$, find necessary and sufficient conditions for the existence of a multiplier ϕ with $\|\phi\|_{\mathcal{M}} \leq 1$ and

$$\partial^\alpha f(w) = a_\alpha$$

for $|\alpha| \leq N$. In this case, we have to apply Theorem 2 to the case when $\mathcal{M}$ is the finite dimensional space spanned by $C_w^{(\alpha)}$, for $|\alpha| \leq N$. This space is invariant, since its orthogonal is $\{f \in \mathcal{H} \mid \partial^\alpha f(w) = 0 \text{ for } |\alpha| \leq N\}$.

Proposition 2. *The above Carathéodory-Fejer problem is solvable if and only if* $\|X\| \leq 1$, *where the operator* $X \in \mathcal{L}(\mathcal{X})$ *is defined by* $X^* C_w^{(\alpha)} = \sum_{\beta+\gamma=\alpha} \bar{a}_\beta C_w^{(\gamma)}$.

Proof. If ϕ is a multiplier, then for any $f \in \mathcal{H}$ we have

$$\begin{aligned} \langle f, T_\phi^* C_w^{(\alpha)} \rangle &= \langle \phi f, C_w^{(\alpha)} \rangle = \partial^\alpha (\phi f)(w) = \sum_{\beta+\gamma=\alpha} \partial^\beta \phi(w) \partial^\gamma f(w) \\ &= \sum_{\beta+\gamma=\alpha} \partial^\beta \phi(w) \langle f, C_w^{(\gamma)} \rangle = \sum_{\beta+\gamma=\alpha} \langle f, \overline{\partial^\beta \phi(w)} C_w^{(\gamma)} \rangle \end{aligned}$$

The proof now follows from Theorem 2. ∎

Naturally, the condition $\|X\| \leq 1$ can be written as the positivity of a certain matrix depending on the data a_α, but its precise form is more complicated than the Pick matrix (8). We may imagine also other sets of given coefficients in the Taylor development of f in w. The corresponding invariant subspace $\mathcal{X}$ is then spanned by a finite number of $C_w^{(\alpha)}$'s (for different values of α); of course, they have to satisfy the restriction arising from the invariance of $\mathcal{X}$. Also, generalizations of the Fejer-Hermite problem can be stated, fixing the value of the function and certain derivatives in a finite number of points.

Another interesting application in this area is connected with model theory for d-tuples of operators which satisfy a certain positivity condition. This is an extensive area of research (see, for instance, [At], [CV], [MV]); we are rather interested by the general results appearing in [Ag], [AgM] or [AEM]. To obtain a neat statement, we will make below the most convenient assumptions; they can actually be slightly relaxed. Concerning the space $\mathcal{H}$, apart from conditions (i)–(iv) in section 1, we suppose that the multiplications T_{χ_j} with the d coordinate functions $\chi_j(z) = z_j$ are continous, and that

$$\dim \bigcap_{j=1}^{d} \ker(T_{\chi_j}^* - \mu_j) = 1$$

for any $\mu \in D$. The last condition, which has appeared in [AS] and [AgM], insures that the commutant of $\{T_{\chi_1}, \ldots, T_{\chi_d}\}$ coincides with $\mathcal{M}'$.

Suppose now that $T = (T_1, \ldots, T_d)$ is a commuting multioperator on some Hilbert space $\mathcal{K}$, having the Taylor spectrum in D and satisfying the condition

$$\frac{1}{C}(T) \geq 0$$

where C is, as above, the reproducing kernel of $\mathcal{H}$, while $\frac{1}{C}(T)$ is defined via Agler's "hereditary" functional calculus. It follows then ([AEM]) that T is the restriction to an invariant subspace of the adjoint of the multishift $T_{\chi_1}, \cdots, T_{\chi_d}$, where $\chi_j(z) = z_j$ (the j-th coordinate), acting actually on the Hilbert space of the vector valued functions $\mathcal{H} \otimes H$, where H is a separable Hilbert space.

In order to obtain an intertwining dilation for this class of operators, we have to extend Theorem 2 to the case of vector valued functions. This extension does not present any difficulty, and thus we may state its consequence for the model space.

Proposition 3. *Suppose $T = (T_1, \ldots, T_d)$ is a commuting multioperator which satisfies the positivity condition*

$$\frac{1}{C}(T) \geq 0$$

and is thus unitary equivalent to the restriction to an invariant subspace $\mathcal{X}$ of $T_{\chi_1}^* \otimes I_H, \cdots, T_{\chi_d}^* \otimes I_H$, acting on $\mathcal{H} \otimes H$ (H some separable Hilbert space). If $X \in \mathcal{L}(\mathcal{X})$ satisfies $XT_i^* = T_i^* X$, then there exists $\tilde{X} \in \mathcal{M} \otimes \mathcal{L}(H)$, with $\|\tilde{X}\| = \|X\|$, such that $X = P_{\mathcal{X}} \tilde{X} | \mathcal{X}$.

6. Examples.

There is a large class of examples of spaces of analytic functions that satisfy the four conditions in the preliminaries.

6.1. The Hardy space on the unit disc $D \subset \mathbf{C}$. This is the basic example. We have

$$C(\lambda, \mu) = \frac{1}{1 - \lambda\bar{\mu}}, \qquad 1 - 1/C(\lambda, \mu) = \lambda\bar{\mu}.$$

Theorem A becomes Sarason's interpolation theorem([S]), while Theorem B is a well known result about the realization of analytic functions bounded in the unit disc.

6.2. Consider, in the unit ball $\mathbf{B} \subset \mathbf{C}^d$, $\mathbf{B} = \{\lambda \in \mathbf{C}^d \mid \|\lambda\|^2 = |\lambda_1|^2 + \cdots + \|\lambda_d\|^2 < 1\}$, the reproducing kernel $C(\lambda, \mu) = (1 - \langle \lambda, \mu \rangle)^{-1}$. We obtain then the space denoted by Arveson with H^2 and extensively studied in [Ar]. Interpolation results for this space (including a commutant lifting theorem) have been obtained by Popescu ([P1], [P2]) as a consequence of his study of noncommutative d-contractions.

In this case we have $\mathcal{E} = \mathbf{C}^d$, $F(\lambda) = \lambda$, while

$$Z(\lambda) : \overbrace{K \otimes \cdots \otimes K}^{d \text{ copies}} \to K$$

is the row operator $(\lambda_1 I_K \quad \cdots \quad \lambda_d I_K)$. Introducing these values in (2) yields a concrete representation formula for contractive multipliers on this space.

6.3. The above example can be generalized to a class of spaces studied in [Po]. Let $p = p(x) = \sum_{\gamma \in \mathbf{Z}_+^n} a_\gamma \lambda^\gamma$ be a polynomial with $p(0) = 0$, all $a_\gamma \geq 0$ and the coefficients of the linear terms λ_j are nonzero. Define

$$D = \{\lambda \in \mathbf{C}^n \mid p(|\lambda_1|^2, \ldots, |\lambda_n|^2) < 1\}.$$

Let ρ_α be the Taylor coefficients in $\lambda = 0$ of the function $(1-p)^{-1}$, and define $\mathcal{H}_p$ to be the Hilbert space of all formal series $f(\lambda) = \sum_\alpha c_\alpha \lambda^\alpha$ such that $\|f\|_{\mathcal{H}_p} = \sum_\alpha |c_\alpha|^2/\rho_\alpha < \infty$, endowed with the corresponding inner product. Then $\mathcal{H}_p$ is a Hilbert space of analytic functions over D with reproducing kernel

$$C(\lambda, \mu) = \frac{1}{1 - p(\lambda_1\bar{\mu}_1, \ldots, \lambda_n\bar{\mu}_n)}, \qquad \lambda, \mu \in D,$$

which satisfies conditions (i)–(iv).

6.4. Dirichlet spaces. The Dirichlet space $\mathcal{D}$ on the unit disc is originally defined as the space of analytic functions in the unit disc $\mathbf{D}$ with

$$\int\int |f'(z)|^2 dA \leq \infty,$$

where dA denotes the area measure. To avoid factorizing through the constants, one should add a supplementary term; in [Ag3] it is shown that, if the norm on $\mathcal{D}$ is defined by adding the usual Hardy space norm

$$\|f\|_{\mathcal{D}}^2 = \int\int |f'(z)|^2 dA + \int_{\mathbf{T}} |f|^2,$$

then we obtain the reproducing kernel

$$C(\lambda, \mu) = \frac{1}{\lambda\bar{\mu}} \log\left(\frac{1}{1 - \lambda\bar{\mu}}\right),$$

which satisfies all conditions (i)–(iv). The argument has been generalized by [Q] to a larger class of spaces on the unit disc; the same proof can be extended to the following more general setting.

Let $w_k > 0$ ($k \geq 0$) be a sequence with $w_0 = 1$, w_k/w_{k+1} increasing and $\underline{\lim}_{k\to\infty} w_k^{1/k} \geq 1$. Define $\mathcal{H}$ to be the space of all series $f = \sum_{\alpha \in \mathbf{Z}_+^n} a_\alpha z^\alpha$ such that $\sum_\alpha \alpha!(|\alpha|!)^{-1} w_{|\alpha|} |a_\alpha|^2 < \infty$. We have then a corresponding scalar product, given, for $g \in \mathcal{H}$ with coefficients b_α, by $\langle f, g\rangle_{\mathcal{H}} = \sum_\alpha \alpha!(|\alpha|!)^{-1} w_{|\alpha|} a_\alpha \bar{b}_\alpha$. Then $\mathcal{H}$ becomes a Hilbert space of analytic functions over the unit ball of $\mathbf{C}^n$. Its reproducing kernel

$$C(\lambda, \mu) = \sum_\alpha |\alpha|!(\alpha! w_{|\alpha|})^{-1} \lambda^\alpha \bar{\mu}^\alpha = \sum_{k\geq 0} w_k^{-1} \langle \lambda, \mu\rangle^k$$

satisfies condition (i)–(iv). The only nontrivial part is again (iv); this is proved, as in [Q], by showing reccurently that

$$1 - \frac{1}{C(\lambda, \mu)} = \sum_{k\geq 0} a_k \langle \lambda, \mu\rangle^k,$$

with $a_k \geq 0$.

A particular case is the Sobolev space $W^1H(\mathbf{B}_d)$ $(d \geq 2)$ for the unit ball considered in [AS]. The weights are originally given in that case by

$$w_k = \frac{(d+k)!}{k!(k^2+dk+1)};$$

it is easy to show that one can change them to obtain an equivalent norm and have the required conditions satisfied.

On the other hand, it is rather well known that many classical spaces of analytic functions do not satisfy the positivity condition (iv). This is the case for the Hardy or the Bergmann spaces for the unit ball or the unit polydisc in more than one dimension, for instance. In some of these cases (see [Ag2] or [BLTT]), it can be shown that a satisfactory statement is obtained by replacing the positivity of the Pick matrix (8) with other positivity conditions.

References

[Ag1] J. Agler, The Arveson extension theorem and coanalytic models, *Integral Equations Operator Theory* **5** (1982), 608-631.

[Ag2] J. Agler, On the representation of certain holomorphic functions defined on a polydisc, in *Topics in Operator Theory: Ernst D. Hellinger Memorial Volume*, Operator Theory: Advances and Applications Vol. 48, Birkhäuser-Verlag, Basel, 1990, pp. 47-66.

[Ag3] J. Agler, Interpolation, preprint.

[AgM] J. Agler and J.E. McCarthy, Nevanlinna-Pick interpolation on the bidisk, preprint.

[AS] O.P Agrawal and N. Salinas, Sharp kernels and canonical subspaces, *Amer. Jour. of Math.* **110** (1988), 23-48.

[AEM] C.G. Ambrozie, M. Engliš and V. Müller, Analytic models over general domains in $\mathbf{C}^n$, *J. Operator Theory*, to appear.

[Aro] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337-404.

[Arv] W.B. Arveson, Subalgebras of C*-algebras III, Multivariable operator theory, *Acta Mathematica* **181** (1998), 159-228.

[At] A. Athavale, Holomorphic kernels and commuting operators, *Trans. Amer. Math. Soc.* **304** (1987), 101-110.

[BB] F. Beatrous and J. Burbea, Positive-definiteness and its applications to interpolation problems for holomorphic functions, *Trans. Amer. Math. Soc.*, **284** (1984), 247-270.

[BLTT] J.A. Ball, W.S. Li, D. Timotin and T.T. Trent, A commutant lifting theorem on the polydisc, *Indiana Univ. Math. Journal* **48** (1999), 653-675.

[CS] M. Cotlar and C. Sadosky, Nehari and Nevanlinna-Pick problems and holomorphic extensions in the polydisc in terms of restricted BMO, *J. Functional Analysis* **124** (1994), 205-210.

[CV1] R. Curto, F.-H. Vasilescu: Standard operator models in the polydisc, *Indiana Univ. Math. J.* **42** (1993), 791-810.

[McC] S. McCullough, The local De Branges-Rovnyak construction and complete Nevanlinna-Pick kernels, in *Algebraic Methods in Operator Theory*, ed. by Raul Curto and Palle E.T. Jorgensen, Birkhauser Boston, 1994, 15-24.

[MV] V. Müller and F.-H. Vasilescu: Standard models for some commuting multioperators, *Proc. Amer. Math. Soc.*, **117** (1993), 979-989.

[P1] G. Popescu, Interpolation problems in several variables, *J. Math. Anal. Appl.* **227** (1998), 227-250.

[P2] G. Popescu, Commutant lifting, tensor algebras, and functional calculus, preprint.

[Po] S. Pott, Standard models under polynomial positivity conditions, *J. Operator Theory* **41** (1999), 365-389.

[Q] P. Quiggin, For which reproducing kernel Hilbert spaces is Pick's theorem true?, *Integral Equations Operator Theory* **16** (1993), 244-266.

[Sa] S. Saitoh, *Theory of Reproducing Kernels and its Applications*, Pitman Research Notes in math. no 189, Longman, 1988.

[S] D. Sarason, Generalized interpolation in H^∞, *Trans. Am. Math. Soc.* **127** (1967), 2, 179-203.

[SNF] B. Sz.-Nagy, C. Foias, *Harmonic Analysis of Operators on Hilbert Space*,

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