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June, 2000

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Optimization and numerical approximation for micropolar fluids

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Summary. In the theory of micropolar fluids, a special case appears when the microrotation is equal to the vorticity of the fluid. The aim of this paper is to determine an external field which realises this case. An existence result for the proposed control problem is obtained and the necessary conditions of optimality are derived. For solving the optimality system, an iterative algorithm is proposed and its convergence is obtained. The discretization of the approximation is studied; stability and convergence theorems are proved.

1. Introduction

The classical Navier-Stokes theory is incapable of describing some physical phenomena; for a class of fluids which exhibit certain microscopic effects arising from the local structure and micro-motions of the fluid elements a new theory was introduced by Eringen in [1]. A subclass of these fluids is the micropolar fluids, which exhibit micro-rotational effects and micro-rotational inertia. Animal blood, liquid crystals, and certain polymeric fluids are a few examples of fluids which may be represented by the mathematical model of micropolar fluids. This model was introduced by Eringen in [2]. From the physical point of view, micropolar fluids are characterized by the following property: fluid points contained in a small volume element, in addition to its usual rigid motion, can rotate about the centroid of the volume element in an average sense, described by the gyration tensor, ω, which is skew-symmetric.

Due to its importance in industrial and engineering applications, micropolar fluids were studied in several papers such as: [3], [4].

Both from the mathematical and from the physical points of view, it is interesting to consider optimal control problems associated with the micropolar fluids motion. Such a problem was studied in [5]. In [5], a viscosity coefficient was considered as the control variable, in order to obtain a desired field of the microrotation velocity.

The present paper deals with another optimal control problem associated with micropolar fluids. Let us denote by $\vec{v}$ the velocity of the micropolar fluid and by ω the scalar function (microrotation velocity), which replaces the skew-symmetric gyration tensor in the 2 D case. Since in the theory of micropolar fluids a special case (see [2]) appears when the microrotation is constrained by:

$$\omega = \mathrm{rot}\vec{v}, \tag{1.1}$$

we considered in this paper the following control problem: *Find the external field,* g, *which minimizes the functional:*

$$J(g) = \frac{1}{2}\int_{\Omega_T} (\omega_g - \mathrm{rot}\vec{v}_g)^2 dx dt, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is the bounded region of motion, $\Omega_T = \Omega \times]0, T[$ and $\vec{v}_g$, ω_g are the velocity and the microrotation of the fluid respectively, corresponding to g.

The paper is organized as follows: in Section 2 we introduce the system of equations which describes the non stationnary flow of a micropolar fluid, and its variational formulation. We also define the notation used throughout the paper and we state a well known existence and uniqueness result. Section 3 deals with the optimal control problem; an existence result is proved and the necessary conditions of optimality are obtained. Section 4 is devoted to the approximation of the optimality system. In the first part of this section we introduce an algorithm for solving this system and we discuss its convergence. The discretization of this approximation is studied in the second part. We propose fully implicit schemes as Scheme 5.1 of [6], p. 334, and we obtain stability and convergence theorems. The most difficult to prove is the stability of the scheme corresponding to the adjoint system. While the scheme for the initial system is unconditionally stable, the one corresponding to the adjoint system is only conditionally stable.

2. The physical problem. Existence and uniqueness results.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set, with $\partial\Omega$ of class C^2 and T a given positive constant. Taking into account the constitutive equations for micropolar fluids given in [2], the non stationnary flow of such a fluid is described by the following coupled system

$$(2.1)\qquad \begin{cases} \vec{v}' + (\vec{v}\cdot\nabla)\vec{v} - (\mu+\chi)\,\triangle\,\vec{v} + \nabla p - \chi \mathrm{rot}\ \omega = \vec{f} \ \text{ in } \Omega_T, \\ j\omega' + j\vec{v}\cdot\nabla\omega - \gamma\,\triangle\,\omega + 2\chi\omega - \chi\mathrm{rot}\ \vec{v} = g \ \text{ in } \Omega_T, \\ \mathrm{div}\ \vec{v} = 0 \ \text{ in } \Omega_T, \\ \vec{v} = \vec{0},\ \omega = 0 \ \text{ on } \partial\Omega_T, \\ \vec{v}(x,0) = \vec{0},\ \omega(x,0) = 0 \ \text{ in } \Omega, \end{cases}$$

where χ, μ, j, γ are positive given constants associated to the properties of the material, $\vec{f}$, g are the given external fields, $\vec{v}$, ω, p are the unknown of the problem: the velocity, the microrotation and the pressure of the micropolar fluid, respectively, and $\mathrm{rot}\ \vec{v} = \dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2}$, $\mathrm{rot}\ \omega = (\dfrac{\partial \omega}{\partial x_2}, -\dfrac{\partial \omega}{\partial x_1})$.

For obtaining the variational formulation of the system (2.1) we use the following spaces (for their properties see, e. g. [6]):

$$\begin{cases} V = \{\vec{u} \in (H_0^1(\Omega))^2 \,/\, \mathrm{div}\, \vec{u} = 0\}, \\ H = \{\vec{u} \in (L^2(\Omega))^2 \,/\, \mathrm{div}\, \vec{u} = 0,\ \vec{u}\cdot\vec{n}/_{\partial\Omega} = 0\}, \\ W(0,T;X,X') = \{u \in L^2(0,T;X)/u' \in L^2(0,T;X')\},\ X\text{-Hilbert space}, \\ H^{2,1}(\Omega_T) = \{u \in L^2(\Omega_T)\,/\, \dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x_i}, \dfrac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega_T),\ i,j = 1,2\}, \\ L_0^2(\Omega) = \{p \in L^2(\Omega)\,/\, \int_\Omega p dx = 0\} \end{cases}$$

and notation:

$(\cdot,\cdot)$ the scalar product , $|\cdot|$ the norm in $L^2(\Omega)$ or $(L^2(\Omega))^2$,

$((\cdot,\cdot))_0$ the scalar product , $\|\cdot\|_0$ the norm in $H_0^1(\Omega)$ or $(H_0^1(\Omega))^2$,

$\langle\cdot,\cdot\rangle_{X',X}$ the duality pairing between a space X and its dual X',

$B_1(\vec{u},\vec{v}) = (\vec{u}\cdot\nabla)\vec{v},\ \ B_2(\vec{u},\varphi) = \vec{u}\cdot\nabla\varphi\ \ \forall \vec{u},\vec{v} \in (H_0^1(\Omega))^2,\ \varphi \in H_0^1(\Omega)$.

In the sequel we shall use the following estimates for B_1 and B_2 (see [6], p. 292, for B_1):

$$(2.2)\begin{cases} |(B_1(\vec{u},\vec{v}),\vec{w})| \le \sqrt{2}\,|\vec{u}|^{1/2}\|\vec{u}\|_0^{1/2}|\vec{v}|^{1/2}\|\vec{v}\|_0^{1/2}\|\vec{w}\|_0^{1/2},\ \forall \vec{u} \in V,\ \vec{v},\vec{w} \in (H_0^1(\Omega))^2, \\ |(B_2(\vec{u},\omega),\rho)| \le \sqrt{2}\,|\vec{u}|^{1/2}\|\vec{u}\|_0^{1/2}|\omega|^{1/2}\|\omega\|_0^{1/2}\|\rho\|_0^{1/2}, \forall \vec{u} \in V,\ \omega,\rho \in H_0^1(\Omega). \end{cases}$$

For $\vec{f} \in L^2(0,T;V')$, $g \in L^2(0,T;H^{-1}(\Omega))$, the variational formulation of the problem (2.1) is given by

$$(2.3)\quad \begin{cases} \vec{v} \in W(0,T;V,V'),\ \omega \in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)), \\ \langle \vec{v}'(t),\vec{z}\rangle_{V',V} + (\mu+\chi)((\vec{v}(t),\vec{z}))_0 + \langle B_1(\vec{v}(t),\vec{v}(t)),\vec{z}\rangle_{V',V} \\ \qquad -\chi(\mathrm{rot}\ \omega(t),\vec{z}) = \langle \vec{f}(t),\vec{z}\rangle_{V',V}\ \forall \vec{z} \in V, \\ j\langle \omega'(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \gamma((\omega(t),\zeta))_0 + j\langle B_2(\vec{v}(t),\omega(t)),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ +2\chi(\omega(t),\zeta) - \chi(\mathrm{rot}\ \vec{v}(t),\zeta) = \langle g(t),\zeta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \forall \zeta \in H_0^1(\Omega), \\ \vec{v}(0) = \vec{0},\ \omega(0) = 0. \end{cases}$$

The next theorem gives the existence and the uniqueness of the solution of the problem (2.3). Since the main steps in obtaining these results are similar to those well known for the Navier-Stokes equations (see e.g. [6]), we shall skip the proof.

Theorem 2.1 *The problem (2.3) has a unique solution $(\vec{v}, \omega)$. Moreover, there exists $p \in \mathcal{D}'(\Omega_T)$, unique up to the addition of a distribution in $(0,T)$, which satisfies, together with $(\vec{v}, \omega)$, the system (2.1).*

3. The optimal control problem. The optimality system.

We shall consider in the sequel further regularity for the external given functions, i. e. $\vec{f} \in (L^2(\Omega_T))^2$, $g \in L^2(\Omega_T)$. Taking into account the regularity of Ω, the initial conditions and the regularity of the data, it can be proved, as in [6], that all the duality pairings $\langle \cdot, \cdot \rangle_{V',V}$ and $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ can be replaced by $(\cdot, \cdot)$.

We define the functional: $J : L^2(\Omega_T) \mapsto \mathbb{R}$, given by (1.2), with $(\vec{v}_g, \omega_g)$ the unique solution of (2.3) corresponding to g. We introduce:

$$B_r = \{g \in L^2(\Omega_T) \,/\, \|g\|_{L^2(\Omega_T)} \leq r\}, \ \forall r > 0, \tag{3.1}$$

and we consider the control problem:

$$(CP) \qquad \begin{cases} \text{Find } g^* \in B_r \text{ such that} \\ J(g^*) = \min\{J(g) \,/\, g \in B_r\}, \end{cases}$$

with r an arbitrarily large constant. The physical interpretation of this problem is the following: we determine an external field g, which gives rise to the special case in the theory of micropolar fluids, $\omega = \text{rot } \vec{v}$.

The existence of at least a solution for (CP) is given by the next proposition:

Proposition 3.1 *The control problem (CP) has at least a solution.*

Proof. If we prove that the functional J is weakly lower semicontinuous, the assertion of the proposition will follow, by using a Weierstrass theorem. Let $\{g_n\}_{n \in N}$ be a weakly convergent sequence to an element $g \in L^2(\Omega_T)$. We denote by $(\vec{v}_n, \omega_n)$ the unique solution of (2.3) corresponding to g_n. The boundedness of the sequence $\{(\vec{v}_n, \omega_n)\}_n$ in $(H^{2,1}(\Omega_T))^2 \times H^{2,1}(\Omega_T)$ (with a constant depending on Ω, T, $\vec{f}$, the viscosity coefficients, and on r) follows with the same steps as those for Navier-Stokes equations (see e. g. [6]), by using the estimates (2.2). Since the embedding $H^{2,1}(\Omega_T) \subset L^2(\Omega_T)$ is compact, we get the following convergences (on subsequences):

$$(\vec{v}_n, \omega_n) \rightharpoonup (\vec{v}, \omega) \text{ weakly in } (H^{2,1}(\Omega_T))^3, \text{ when } n \to \infty,$$

$$(\vec{v}_n, \omega_n) \to (\vec{v}, \omega) \text{ strongly in } (L^2(\Omega_T))^3, \text{ when } n \to \infty.$$

Using the above convergences, we can now pass to the limit in (2.3) corresponding to g_n and we obtain that $(\vec{v}, \omega)$ is the unique solution of (2.3) corresponding to g. From the uniqueness of the solution of (2.3), it follows that the whole sequence $\{(\vec{v}_n, \omega_n)\}_n$ is convergent to $(\vec{v}, \omega)$ and, therefore, the property of J, stated at the beginning of the proof, is immediate.

The next property of J, given by Proposition 3.2., will lead us to the necessary conditions of optimality.

Proposition 3.2 *The functional J is G-differentiable on $L^2(\Omega_T)$ and $\forall g, g^* \in L^2(\Omega_T)$*

$$(3.2)\quad (J'(g^*), g - g^*)_{L^2(\Omega_T)} = \int_{\Omega_T} (\text{rot } (\vec{v}_0 - \vec{v}^*) - (\omega_0 - \omega^*))(\text{rot } \vec{v}^* - \omega^*)dxdt,$$

where $(\vec{v}^*, \omega^*)$ is the unique solution of (2.3) corresponding to g^* and $(\vec{v}_0, \omega_0)$ is the unique solution of the system:

$$(3.3)\quad \begin{cases} \vec{v}_0 \in W(0,T;V,V') \cap (H^{2,1}(\Omega_T))^2,\ \omega_0 \in W(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \cap H^{2,1}(\Omega_T)), \\ (\vec{v}_0'(t), \vec{z}) + (\mu+\chi)((\vec{v}_0(t), \vec{z}))_0 + (B_1(\vec{v}^*(t), \vec{v}_0(t) - \vec{v}^*(t)), \vec{z}) \\ +(B_1(\vec{v}_0(t), \vec{v}^*(t)), \vec{z}) - \chi(\text{rot } \omega_0(t), \vec{z}) = (\vec{f}(t), \vec{z}) \ \forall \vec{z} \in V, \\ j(\omega_0'(t), \zeta) + \gamma((\omega_0(t), \zeta))_0 + j(B_2(\vec{v}_0(t), \omega^*(t)), \zeta) \\ +jB_2(\vec{v}^*(t), \omega_0(t) - \omega^*(t)), \zeta) + 2\chi(\omega_0(t), \zeta) - \chi(\text{rot } \vec{v}_0(t), \zeta) \\ = (g(t), \zeta) \ \forall \zeta \in H_0^1(\Omega), \\ \vec{v}_0(0) = \vec{0},\ \omega_0(0) = 0, \end{cases}$$

Proof. The existence and uniqueness of $(\vec{v}_0, \omega_0)$ follow with similar techniques as those of Theorem 2.1. The regularity of the solution of (3.3) is a consequence of the remark made at the beggining of this section.

The differentiability of J and the relation (3.2) are obtained with standard techniques. We denote by $(\vec{v}_{\alpha g}, \omega_{\alpha g})$ the unique solution of (2.3) corresponding to $g^* + \alpha(g - g^*)$, for any $\alpha \in (0,1)$; we define the functions $\vec{v}_\alpha = \frac{(\vec{v}_{\alpha g} - \vec{v}^*)}{\alpha} + \vec{v}^*$; $\omega_\alpha = \frac{(\omega_{\alpha g} - \omega^*)}{\alpha} + \omega^*$. With the same remarks as those of the previous proposition, we establish boundedness results for $(\vec{v}_\alpha, \omega_\alpha)$ in $(H^{2,1}(\Omega_T))^3$, with a constant independent of α and, then, the convergences which give (3.2).

If we take in (3.2) g^* a solution of (CP), it follows:

$$\int_{\Omega_T} (\text{rot } (\vec{v}_0 - \vec{v}^*) - (\omega_0 - \omega^*))(\text{rot } \vec{v}^* - \omega^*)dxdt \geq 0. \tag{3.4}$$

In the sequel, we shall replace the constrained inequality (3.4) by an unconstrained one, given by the optimality system.

Let g^* be a minimum point for the functional J. We consider the following adjoint system:

$$\text{(3.5)} \quad \begin{cases} \vec{u}^* \in W(0,T;V,V') \cap (H^{2,1}(\Omega_T))^2, \ \rho^* \in W(0,T;H_0^1(\Omega), H^{-1}(\Omega)) \cap H^{2,1}(\Omega_T), \\ -(\vec{u}^{*\prime}(t), \vec{z}) + (\mu + \chi)((\vec{u}^*(t), \vec{z}))_0 + (B_1(\vec{z}, \vec{v}^*(t)), \vec{u}^*(t)) \\ -(B_1(\vec{v}^*(t), \vec{u}^*(t)), \vec{z}) + j(B_2(\vec{z}, \omega^*(t)), \rho^*(t)) - \chi(\text{rot } \rho^*(t), \vec{z}) \\ = (\text{rot } \vec{v}^*(t) - \omega^*(t), \text{ rot } \vec{z}) \ \forall \vec{z} \in V, \\ -j(\rho^{*\prime}(t), \zeta) + \gamma((\rho^*(t), \zeta))_0 - j(B_2(\vec{v}^*(t), \rho^*(t)), \zeta) \\ +2\chi(\rho^*(t), \zeta) - \chi(\text{rot } \vec{u}^*(t), \zeta) = -(\text{rot } \vec{v}^*(t) - \omega^*(t), \zeta) \ \forall \zeta \in H_0^1(\Omega), \\ \vec{u}^*(T) = \vec{0}, \ \rho^*(T) = 0, \end{cases}$$

where $(\vec{v}^*, \omega^*)$ is the unique solution of (2.3) corresponding to g^*.

Computing (3.3)-$(2.3)_{g=g^*}$ for $(\vec{z}, \zeta) = (\vec{u}^*(t), \rho^*(t))$ and (3.5) for $(\vec{z}, \zeta) = (\vec{v}^*(t) - \vec{v}_0(t), \omega^*(t) - \omega_0(t))$ and adding all the obtained equalities, the inequality (3.4) becomes:

$$\int_{\Omega_T} \rho^*(g - g^*)dxdt \geq 0 \ \forall g \in B_r. \tag{3.6}$$

Therefore, we can state the following theorem:

Theorem 3.3 *Let g^* be an optimal control. Then, there exists the unique elements:*

$(\vec{v}^*, \omega^*)$, *the solution of (2.3) corresponding to* g^*, $(\vec{u}^*, \rho^*)$, *the solution of the adjoint system (3.5) which satisfy the following optimality system:*

$$(OS)_1 \begin{cases} (\vec{v}^{*\prime}(t), \vec{z}) + (\mu + \chi)((\vec{v}^*(t), \vec{z}))_0 + (B_1(\vec{v}^*(t), \vec{v}^*(t)), \vec{z}) \\ -\chi(\text{rot } \omega^*(t), \vec{z}) = (\vec{f}(t), \vec{z}) \ \forall \vec{z} \in V, \\ j(\omega^{*\prime}(t), \zeta) + \gamma((\omega^*(t), \zeta))_0 + j(B_2(\vec{v}^*(t), \omega^*(t)), \zeta) \\ +2\chi(\omega^*(t), \zeta) - \chi(\text{rot } \vec{v}^*(t), \zeta) = (g^*(t), \zeta) \ \forall \zeta \in H_0^1(\Omega), \\ \vec{v}^*(0) = \vec{0}, \ \omega^*(0) = 0, \end{cases}$$

$$(OS)_2 \begin{cases} -(\vec{u}^{*\prime}(t), \vec{z}) + (\mu + \chi)((\vec{u}^*(t), \vec{z}))_0 + (B_1(\vec{z}, \vec{v}^*(t)), \vec{u}^*(t)) \\ -(B_1(\vec{v}^*(t), \vec{u}^*(t)), \vec{z}) + j(B_2(\vec{z}, \omega^*(t)), \rho^*(t)) \\ -\chi(\text{rot } \rho^*(t), \vec{z}) = (\text{rot } \vec{v}^*(t) - \omega^*(t), \ \text{rot } \vec{z}) \ \forall \vec{z} \in V, \\ -j(\rho^{*\prime}(t), \zeta) + \gamma((\rho^*(t), \zeta))_0 - j(B_2(\vec{v}^*(t), \rho^*(t)), \zeta) \\ +2\chi(\rho^*(t), \zeta) - \chi(\text{rot } \vec{u}^*(t), \zeta) = -(\text{rot } \vec{v}^*(t) - \omega^*(t), \zeta) \ \forall \zeta \in H_0^1(\Omega), \\ \vec{u}^*(T) = \vec{0}, \ \rho^*(T) = 0, \end{cases}$$

$$(OS)_3 \qquad \int_{\Omega_T} \rho^*(g - g^*) dx dt \geq 0 \ \forall g \in B_r.$$

Proof. The regularity, the existence and the uniqueness of $(\vec{u}^*, \rho^*)$ can be proved with similar techniques as those mentioned in Theorem 2.1. The inequality $(OS)_3$ was previously obtained, hence the proof is complete.

The next section deals with the approximation of the optimality system.

4. The approximation of the optimality system.

In order to solve the optimality system, we propose, in the first part of this section, an iterative algorithm and we study its convergence.

The second part of this section deals with the discretization of this scheme; stability and convergence theorems are established.

4.1 The approximation scheme.

We propose the following iterative scheme: given an initial guess $g_0 \in B_r$, find, for any $m = 0, 1, 2, ...$, the elements $\vec{v}_m$, ω_m, $\vec{u}_m$, ρ_m, g_{m+1} satisfying:

$$(OS)_1^m \begin{cases} (\vec{v}_m'(t), \vec{z}) + (\mu+\chi)((\vec{v}_m(t), \vec{z}))_0 + (B_1(\vec{v}_m(t), \vec{v}_m(t)), \vec{z}) \\ -\chi(\text{rot } \omega_m(t), \vec{z}) = (\vec{f}(t), \vec{z}) \ \forall \vec{z} \in V, \\ j(\omega_m'(t), \zeta) + \gamma((\omega_m(t), \zeta))_0 + j(B_2(\vec{v}_m(t), \omega_m(t)), \zeta) \\ +2\chi(\omega_m(t), \zeta) - \chi(\text{rot } \vec{v}_m(t), \zeta) = (g_m(t), \zeta) \ \forall \zeta \in H_0^1(\Omega), \\ \vec{v}_m(0) = \vec{0}, \ \omega_m(0) = 0, \end{cases}$$

$$(OS)_2^m \begin{cases} -(\vec{u}_m'(t), \vec{z}) + (\mu+\chi)((\vec{u}_m(t), \vec{z}))_0 + (B_1(\vec{z}, \vec{v}_m(t)), \vec{u}_m(t)) \\ -(B_1(\vec{v}_m(t), \vec{u}_m(t)), \vec{z}) + j(B_2(\vec{z}, \omega_m(t)), \rho_m(t)) - \chi(\text{rot } \rho_m(t), \vec{z}) = \\ (\text{rot } \vec{v}_m(t) - \omega_m(t), \text{rot } \vec{z}) \ \forall \vec{z} \in V, \\ -j(\rho_m'(t), \zeta) + \gamma((\rho_m(t), \zeta))_0 - j(B_2(\vec{v}_m(t), \rho_m(t)), \zeta) \\ +2\chi(\rho_m(t), \zeta) - \chi(\text{rot } \vec{u}_m(t), \zeta) = -(\text{rot } \vec{v}_m(t) - \omega_m(t), \zeta) \ \forall \zeta \in H_0^1(\Omega), \\ \vec{u}_m(T) = \vec{0}, \ \rho_m(T) = 0, \end{cases}$$

$$(OS)_3^{m+1} \qquad g_{m+1} = \begin{cases} P_{B_r}(g_m - \delta_m \dfrac{\rho_m}{\|\rho_m\|_{L^2(\Omega_T)}}), \text{ if } \|\rho_m\|_{L^2(\Omega_T)} \neq 0, \\ g_m, \text{ if } \|\rho_m\|_{L^2(\Omega_T)} = 0, \end{cases}$$

where δ_m is a positive suitable constant and P_{B_r} denotes the projection map of $L^2(\Omega_T)$ on B_r.

We prove next a convergence theorem for this iterative algorithm.

Theorem 4.1 Let $\{g_{m_k}\}_{k\in\mathbb{N}} \subset \{g_m\}_{m\in\mathbb{N}}$ be a weakly convergent subsequence to an element $g^* \in L^2(\Omega_T)$. Then the sequence $\{\vec{v}_{m_k}, \omega_{m_k}, \vec{u}_{m_k}, \rho_{m_k}\}_{k\in\mathbb{N}}$, with the elements given by $(OS)_{1-2}^{m_k}$, is weakly convergent in $(H^{2,1}(\Omega_T))^6$ to the unique solution $(\vec{v}^*, \omega^*, \vec{u}^*, \rho^*)$ of $(OS)_{1-2}$, corresponding to the above weak limit, g^*.

Moreover, for any $m \in \mathbb{N}$, there exists $\delta_m > 0$ such that:

$$(4.1) \qquad \begin{cases} J(g_{m+1}) \leq J(g_m) \text{ and} \\ \lim_{m\to\infty} [J(g_m) - J(g_{m+1})] = 0 \Rightarrow \lim_{m\to\infty} \|J'(g_m)\|_{L^2(\Omega_T)} = 0. \end{cases}$$

Proof. The sequence $\{g_m\}_{m\in N}$ being bounded in $L^2(\Omega_T)$, it contains at least a weakly convergent subsequence. Let us denote by $\{g_{m_k}\}_{k\in N}$ such a subsequence, and by $g^* \in L^2(\Omega_T)$ its weak limit. We obtain the boundedness of $\{\vec{v}_{m_k}, \omega_{m_k}, \vec{u}_{m_k}, \rho_{m_k}\}_{k\in N}$, in $(H^{2,1}(\Omega_T))^6$ with the same remarks as those of Proposition 3.1. The first assertion of the theorem follows, passing to the limit in $(OS)_{1-2}^{m_k}$ (on a subsequence) and using the uniqueness of the solution of $(OS)_{1-2}$.

For obtaining the second assertion of the theorem, we shall prove first the Lipschitz continuity of J' on B_r.

Let g_1, g_2 be two elements of B_r. We denote by $(\vec{v}_i, \omega_i, \vec{u}_i, \rho_i)$ the unique solution of $(OS)_{1-2}$ corresponding to g_i, $i = 1,2$ and $(\vec{v}, \omega, \vec{u}, \rho) = (\vec{v}_1, \omega_1, \vec{u}_1, \rho_1) - (\vec{v}_2, \omega_2, \vec{u}_2, \rho_2)$.

Computing $(OS)_{1(g=g_1)}$-$(OS)_{1(g=g_2)}$ for $(\vec{z}, \zeta) = (\vec{v}(t), \omega(t))$ and adding the equalities we get:

$$\begin{aligned}(4.2)\qquad &\frac{1}{2}\frac{d}{dt}(|\vec{v}(t)|^2 + j|\omega(t)|^2) + (\mu+\chi)\|\vec{v}(t)\|_0^2 + \gamma\|\omega(t)\|_0^2 + 2\chi|\omega(t)|^2 = \\ &(g(t), \omega(t)) - (B_1(\vec{v}(t), \vec{v}_2(t)), \vec{v}(t)) - j(B_2(\vec{v}(t), \omega_2(t)), \omega(t)) + \\ &\chi(\text{rot } \omega(t), \vec{v}(t)) + \chi(\text{rot } \vec{v}(t), \omega(t)),\end{aligned}$$

since $(B_1(\vec{u}, \vec{v}), \vec{v}) = 0\ \forall \vec{u} \in V, \vec{v} \in (H_0^1(\Omega))^2$ and $(B_2(\vec{u}, \omega), \omega) = 0\ \forall \vec{u} \in V, \omega \in H_0^1(\Omega)$.

Using the estimates (2.2) and standard computations, it follows:

$$(4.3)\qquad \|\vec{v}\|_{L^2(0,T;V)} + \|\omega\|_{L^2(0,T;H_0^1(\Omega))} \leq M(r)\|g\|_{L^2(\Omega_T)}.$$

Computing now $(OS)_{2(g=g_1)}$-$(OS)_{2(g=g_2)}$ for $(\vec{z}, \zeta) = (\vec{u}(t), \rho(t))$, adding the equalities and using (2.2) together with (4.3), we obtain:

$$(4.4)\qquad \|\vec{u}\|_{L^2(0,T;V)} + \|\rho\|_{L^2(0,T;H_0^1(\Omega))} \leq L(r)\|g\|_{L^2(\Omega_T)},$$

which yields:

$$(4.5)\qquad \|J'(g_1) - J'(g_2)\|_{L^2(\Omega_T)} \leq L(r)\|g_1 - g_2\|_{L^2(\Omega_T)}, \forall g_1, g_2 \in B_r.$$

We note that the Lipschitz constant depends on r. For $r \to \infty$, $L(r)$ also converges to ∞.

This property of the functional J allows us to make a convergent choice of δ_m, i. e. $\delta_m \in [\frac{ac}{L(r)}||\rho_m||_{L^2(\Omega_T)}, \frac{c}{L(r)}||\rho_m||_{L^2(\Omega_T)}]$, with a, c arbitrarily chosen in $]0,1[$. For this choice, the properties (4.1) are fulfilled and, hence, the proof is achieved.

4.2 The discretization of the approximating system.

This subsection is concerned with the discretization of $(\mathrm{OS})^m_{1-2}$, $(\mathrm{OS})^{m+1}_3$, both in the space and in the time variables. For the discretization in the space variables we use an internal approximation and for the discretization in the time variable, a backward Euler scheme. After the description of the scheme, we prove its stability and convergence.

Let h be a parameter converging to 0; we denote by W_h, V_h, S_h and M_h internal approximations for V, $H^1_0(\Omega)$, $L^2_0(\Omega)$ and $L^2(\Omega)$, respectively, and we define:

$$K_h = M_h \cap \{g \in L^2(\Omega) \, / \, |g| \leq \frac{r}{\sqrt{T}}\}. \tag{4.6}$$

We divide the interval $[0, T]$ into n intervals of length $\Delta t = \frac{T}{n}$, $n \in \mathbb{N}^*$.

For any $f \in L^2(\Omega_T)$ we define the elements $f^1_{hn}, f^2_{hn}, ..., f^n_{hn} \in M_h$ given by

$$f^k_{hn} = \frac{1}{\Delta t}\int_{(k-1)\Delta t}^{k\Delta t} f_h dt, \; k = 1, ..., n, \tag{4.7}$$

where $f_h \in L^2(0, T; M_h)$ is the space discretization of f.

a) The discretization of $(\mathrm{OS})^m_1$.

For $(\mathrm{OS})^m_1$ we propose the following scheme:

Scheme$^m_1$. When $(\vec{v}_m)^0_{hn}, ..., (\vec{v}_m)^{k-1}_{hn}$ and $(\omega_m)^0_{hn}, ..., (\omega_m)^{k-1}_{hn}$ are known, let us find $((\vec{v}_m)^k_{hn}, (\omega_m)^k_{hn}) \in W_h \times V_h$ as the solution of the problem:

$$\tag{4.8} \begin{cases} \frac{1}{\Delta t}((\vec{v}_m)^k_{hn}, \vec{z}_h) + (\mu+\chi)(((\vec{v}_m)^k_{hn}, \vec{z}_h))_0 + (B_1((\vec{v}_m)^{k-1}_{hn}, (\vec{v}_m)^k_{hn}), \vec{z}_h) \\ -\chi(\mathrm{rot}\,(\omega_m)^k_{hn}, \vec{z}_h) = \frac{1}{\Delta t}((\vec{v}_m)^{k-1}_{hn}, \vec{z}_h) + (\vec{f}^k_{hn}, \vec{z}_h) \; \forall \vec{z}_h \in W_h, \\ \frac{j}{\Delta t}((\omega_m)^k_{hn}, \zeta_h) + \gamma(((\omega_m)^k_{hn}, \zeta_h))_0 + j(B_2((\vec{v}_m)^{k-1}_{hn}, (\omega_m)^k_{hn}), \zeta_h) \\ +2\chi((\omega_m)^k_{hn}, \zeta_h) - \chi(\mathrm{rot}(\vec{v}_m)^k_{hn}, \zeta_h) = \frac{j}{\Delta t}((\omega_m)^{k-1}_{hn}, \zeta_h) + ((g_m)^k_{hn}, \zeta_h) \; \forall \zeta_h \in V_h, \end{cases}$$

with $((\vec{v}_m)^0_{hn}, (\omega_m)^0_{hn}) = P_{W_h \times V_h}(\vec{v}_m(0), \omega_m(0)) = (\vec{0}, 0)$.

We define $a_h : (W_h \times V_h)^2 \mapsto \mathbb{R}$ and $L_h : W_h \times V_h \mapsto \mathbb{R}$ by:

$$a_h((\vec{v},\omega),(\vec{z},\zeta)) = \frac{1}{\Delta t}(\vec{v},\vec{z}) + (\mu+\chi)((\vec{v},\vec{z}))_0 + (B_1((\vec{v}_m)^{k-1}_{hn},\vec{v}),\vec{z}) - \chi(\text{rot }\omega,\vec{z})$$

$$+\frac{j}{\Delta t}(\omega,\zeta) + \gamma((\omega,\zeta))_0 + j(B_2((\vec{v}_m)^{k-1}_{hn},\omega),\zeta) + 2\chi(\omega,\zeta) - \chi(\text{rot }\vec{v},\zeta),$$

$$L_h(\vec{z},\zeta) = \frac{1}{\Delta t}((\vec{v}_m)^{k-1}_{hn},\vec{z}) + (\vec{f}^k_{hn},\vec{z}) + \frac{j}{\Delta t}((\omega_m)^{k-1}_{hn},\zeta) + ((g_m)^k_{hn},\zeta).$$

The system (4.8) is equivalent to the following linear equation:

$$a_h(((\vec{v}_m)^k_{hn},(\omega_m)^k_{hn}),(\vec{z}_h,\zeta_h)) = L_h(\vec{z}_h,\zeta_h)\ \forall(\vec{z}_h,\zeta_h)\in W_h\times V_h \tag{4.9}$$

and therefore the existence and the uniqueness of the solution of (4.8) is a consequence of the properties of a_h, L_h and of the classical Lax-Milgram theorem.

We shall prove some a priori estimates, which will give the stability of $Scheme^m_1$.

Theorem 4.2 *The solution $((\vec{v}_m)^k_{hn},(\omega_m)^k_{hn})$ of (4.8) satisfies:*

$$|(\vec{v}_m)^k_{hn}|^2 \le C(r),\ k=1,\dots,n, \tag{4.10}$$

$$\sum_{k=1}^{n} |(\vec{v}_m)^k_{hn} - (\vec{v}_m)^{k-1}_{hn}|^2 \le C(r), \tag{4.11}$$

$$\Delta t\sum_{k=1}^{n} \|(\vec{v}_m)^k_{hn}\|_0^2 \le C(r), \tag{4.12}$$

$$|(\omega_m)^k_{hn}|^2 \le C(r),\ k=1,\dots,n, \tag{4.13}$$

$$\sum_{k=1}^{n} |(\omega_m)^k_{hn} - (\omega_m)^{k-1}_{hn}|^2 \le C(r), \tag{4.14}$$

$$\Delta t\sum_{k=1}^{n} \|(\omega_m)^k_{hn}\|_0^2 \le C(r). \tag{4.15}$$

Proof. Taking $\vec{z}_h = (\vec{v}_m)^k_{hn}$ in $(4.8)_1$, $\zeta_h = (\omega_m)^k_{hn}$ in $(4.8)_2$, adding the two equalities and using the identity

$$2(a-b,a) = |a|^2 - |b|^2 + |a-b|^2\ \ \forall a,b\in L^2(\Omega),$$

we obtain:

$$
(4.16)\quad
\begin{aligned}
&\frac{1}{2\Delta t}(|(\vec{v}_m)^k_{hn}|^2 - |(\vec{v}_m)^{k-1}_{hn}|^2 + |(\vec{v}_m)^k_{hn} - (\vec{v}_m)^{k-1}_{hn}|^2)\\
&+\frac{j}{2\Delta t}(|(\omega_m)^k_{hn}|^2 - |(\omega_m)^{k-1}_{hn}|^2 + |(\omega_m)^k_{hn} - (\omega_m)^{k-1}_{hn}|^2)\\
&+(\mu+\chi)\|(\vec{v}_m)^k_{hn}\|_0^2 + \gamma\|(\omega_m)^k_{hn}\|_0^2 + 2\chi|(\omega_m)^k_{hn}|^2\\
&-\chi(\text{rot }(\omega_m)^k_{hn}, (\vec{v}_m)^k_{hn}) - \chi(\text{rot }(\vec{v}_m)^k_{hn}, (\omega_m)^k_{hn})\\
&= (\vec{f}^k_{hn}, (\vec{v}_m)^k_{hn}) + ((g_m)^k_{hn}, (\omega_m)^k_{hn}),\ k = 1, \ldots n.
\end{aligned}
$$

Next, using the equality

$$(\omega, \text{rot }\vec{v}) = (\vec{v}, \text{rot }\omega)\ \forall\vec{v} \in (H_0^1(\Omega))^2,\ \omega \in H_0^1(\Omega),$$

and the inequality

$$(4.17)\qquad (\omega, \text{rot }\vec{v}) \leq \sqrt{2}|\omega|\|\vec{v}\|_0,\ \forall\vec{v} \in (H_0^1(\Omega))^2,\ \omega \in H_0^1(\Omega),$$

(4.16) yields, after the addition of the obtained inequalities, for $k = 1, \ldots, n$:

$$
(4.18)\quad
\begin{aligned}
&|(\vec{v}_m)^n_{hn}|^2 + \sum_{k=1}^{n}|(\vec{v}_m)^k_{hn} - (\vec{v}_m)^{k-1}_{hn}|^2 + \mu\Delta t\sum_{k=1}^{n}\|(\vec{v}_m)^k_{hn}\|_0^2\\
&+j|(\omega_m)^n_{hn}|^2 + j\sum_{k=1}^{n}|(\omega_m)^k_{hn} - (\omega_m)^{k-1}_{hn}|^2 + \gamma\Delta t\sum_{k=1}^{n}\|(\omega_m)^k_{hn}\|_0^2\\
&\leq C_\Omega\Delta t(\frac{1}{\mu}\sum_{k=1}^{n}|\vec{f}^k_{hn}|^2 + \frac{1}{\gamma}\sum_{k=1}^{n}|(g_m)^k_{hn}|^2)
\end{aligned}
$$

and, hence (4.11), (4.12), (4.14) and (4.15) are obtained. If we add now the inequalities for $k = 1, \ldots q$, with $q \leq n$, we get (4.10) and (4.13) and the proof is achieved.

We define the functions:

$$
(4.19)\quad
\begin{cases}
(\vec{v}_m)_{hn}(t) = (\vec{v}_m)^k_{hn},\ \forall t \in [(k-1)\Delta t, k\Delta t[,\\
(\omega_m)_{hn}(t) = (\omega_m)^k_{hn},\ \forall t \in [(k-1)\Delta t, k\Delta t[,\ k = 1, \ldots, n.
\end{cases}
$$

An immediate consequence of the above theorem is the following stability result:

Theorem 4.3 *The functions $(\vec{v}_m)_{hn}$ $((\omega_m)_{hn})$ defined by (4.19) are unconditionally $L^\infty(0,T;(L^2(\Omega))^2)$ and $L^2(0,T;(H_0^1(\Omega))^2)$ $(L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;H_0^1(\Omega))$ stable.*

The next theorem gives the convergence of the $Scheme_1^m$.

Theorem 4.4 For $h \to 0$ and $n \to \infty$ the following convergences can be proved:

$$(\vec{v}_m)_{hn} \to \vec{v}_m \text{ strongly in } L^2(0,T;(L^2(\Omega))^2), \text{ weakly star in } L^\infty(0,T;(L^2(\Omega))^2),$$

$$\text{weakly in } L^2(0,T;V),$$

$$(\omega_m)_{hn} \to \omega_m \text{ strongly in} L^2(\Omega_T), \text{ weakly star in } L^\infty(0,T;L^2(\Omega)),$$

$$\text{weakly in } L^2(0,T;(H_0^1(\Omega)).$$

Proof. For obtaining the above convergences, we follow the same steps as in [6], p. 357-363, for Navier-Stokes equations, so we shall skip the proof.

The computation of the solution of (4.8) is not easy because of the constraint $\mathrm{div}(\vec{v}_m)_{hn}^k = 0$. To overcome this difficulty we introduce, as for Navier-Stokes equations, the following equivalent system:

$$(4.20)\quad \begin{cases} \frac{1}{\Delta t}((\vec{v}_m)_{hn}^k, \vec{z}_h) + (\mu+\chi)(((\vec{v}_m)_{hn}^k, \vec{z}_h))_0 + (B_1((\vec{v}_m)_{hn}^{k-1}, (\vec{v}_m)_{hn}^k), \vec{z}_h) \\ -\chi(\mathrm{rot}\ (\omega_m)_{hn}^k, \vec{z}_h) - ((p_m)_{hn}^k, \mathrm{div}\ \vec{z}_h) = \\ \frac{1}{\Delta t}((\vec{v}_m)_{hn}^{k-1}, \vec{z}_h) + (\vec{f}_{hn}^k, \vec{z}_h)\ \ \forall \vec{z}_h \in (V_h)^2, \\ \frac{j}{\Delta t}((\omega_m)_{hn}^k, \zeta_h) + \gamma(((\omega_m)_{hn}^k, \zeta_h))_0 + j(B_2((\vec{v}_m)_{hn}^{k-1}, (\omega_m)_{hn}^k), \zeta_h) \\ +2\chi((\omega_m)_{hn}^k, \zeta_h) - \chi(\mathrm{rot}\ (\vec{v}_m)_{hn}^k, \zeta_h) = \\ \frac{j}{\Delta t}((\omega_m)_{hn}^{k-1}, \zeta_h) + ((g_m)_{hn}^k, \zeta_h)\ \ \forall \zeta_h \in V_h, \\ (\mathrm{div}\ (\vec{v}_m)_{hn}^k, s_h) = 0\ \forall s_h \in S_h,\ \ k = 1, ..., n. \end{cases}$$

This system is solved by using Uzawa algorithm (see e. g. [6], p. 389).

We pass next to:

b) The discretization of $(\mathrm{OS})_2^m$.

First, we define the new functions:

$$(4.21)\quad \begin{cases} \vec{w}_m(x,t') = \vec{u}_m(x, T-t'), \\ \varphi_m(x,t') = \rho_m(x, T-t') \end{cases}$$

and we obtain the following equivalent form of $(\mathrm{OS})_2^m$:

$$(4.22)\begin{cases} (\vec{w}'_m(t),\vec{z})+(\mu+\chi)((\vec{w}_m(t),\vec{z}))_0+(B_1(\vec{z},\vec{v}_m(T-t)),\vec{w}_m(t))- \\ (B_1(\vec{v}_m(T-t),\vec{w}_m(t)),\vec{z})+j(B_2(\vec{z},\omega_m(T-t)),\varphi_m(t))-\chi(\mathrm{rot}\ \varphi_m(t),\vec{z}) \\ =(\mathrm{rot}\ \vec{v}_m(T-t)-\omega_m(T-t),\mathrm{rot}\ \vec{z})\ \forall\vec{z}\in V, \\ j(\varphi'_m(t),\zeta)+\gamma((\varphi_m(t),\zeta))_0-j(B_2(\vec{v}_m(T-t),\varphi_m(t)),\zeta)+2\chi(\varphi_m(t),\zeta) \\ -\chi(\mathrm{rot}\ \vec{w}_m(t),\zeta)=-(\mathrm{rot}\ \vec{v}_m(T-t)-\omega_m(T-t),\zeta)\ \forall\zeta\in H_0^1(\Omega), \\ \vec{w}_m(0)=\vec{0},\ \varphi_m(0)=0. \end{cases}$$

For solving (4.22) we introduce the iterative scheme:

Scheme$_2^m$. *When* $(\vec{w}_m)_{hn}^0,\dots,(\vec{w}_m)_{hn}^{k-1}$ *and* $(\varphi_m)_{hn}^0,\dots,(\varphi_m)_{hn}^{k-1}$ *are known, let us find* $((\vec{w}_m)_{hn}^k,(\varphi_m)_{hn}^k)\in W_h\times V_h$ *as the solution of the problem:*

$$(4.23)\begin{cases} \frac{1}{\Delta t}((\vec{w}_m)_{hn}^k,\vec{z}_h)+(\mu+\chi)(((\vec{w}_m)_{hn}^k,\vec{z}_h))_0+(B_1(\vec{z}_h,(\vec{v}_m)_{hn}^{n-k+1}),(\vec{w}_m)_{hn}^{k-1}) \\ -(B_1((\vec{v}_m)_{hn}^{n-k+1},(\vec{w}_m)_{hn}^k),\vec{z}_h)+j(B_2(\vec{z}_h,(\omega_m)_{hn}^{n-k+1}),(\varphi_m)_{hn}^{k-1}) \\ -\chi(\mathrm{rot}\ (\varphi_m)_{hn}^k,\vec{z}_h)= \\ \frac{1}{\Delta t}((\vec{w}_m)_{hn}^{k-1},\vec{z}_h)+(\mathrm{rot}\ (\vec{v}_m)_{hn}^{n-k+1}-(\omega_m)_{hn}^{n-k+1},\mathrm{rot}\ \vec{z}_h)\ \forall\vec{z}_h\in W_h, \\ \frac{j}{\Delta t}((\varphi_m)_{hn}^k,\zeta_h)+\gamma(((\varphi_m)_{hn}^k,\zeta_h))_0-j(B_2((\vec{v}_m)_{hn}^{n-k+1},(\varphi_m)_{hn}^k),\zeta_h) \\ +2\chi((\varphi_m)_{hn}^k,\zeta_h)-\chi(\mathrm{rot}\ (\vec{w}_m)_{hn}^k,\zeta_h)= \\ \frac{j}{\Delta t}((\varphi_m)_{hn}^{k-1},\zeta_h)-(\mathrm{rot}\ (\vec{v}_m)_{hn}^{n-k+1}-(\omega_m)_{hn}^{n-k+1},\zeta_h)\ \forall\zeta_h\in V_h, \end{cases}$$

with $((\vec{w}_m)_{hn}^0,(\varphi_m)_{hn}^0)=P_{W_h\times V_h}(\vec{w}_m(0),\varphi_m(0))=(\vec{0},0)$.

We define now $a_h:(W_h\times V_h)^2\mapsto\mathbb{R}$ and $L_h:W_h\times V_h\mapsto\mathbb{R}$ by:

$$a_h((\vec{w},\varphi),(\vec{z},\zeta))=\frac{1}{\Delta t}(\vec{w},\vec{z})+(\mu+\chi)((\vec{w},\vec{z}))_0-(B_1((\vec{v}_m)_{hn}^{n-k+1},\vec{w}),\vec{z})-\chi(\mathrm{rot}\ \varphi,\vec{z})$$

$$+\frac{j}{\Delta t}(\varphi,\zeta)+\gamma((\varphi,\zeta))_0-j(B_2((\vec{v}_m)_{hn}^{n-k+1},\varphi),\zeta)+2\chi(\varphi,\zeta)-\chi(\mathrm{rot}\ \vec{w},\zeta),$$

$$L_h(\vec{z},\zeta)=\frac{1}{\Delta t}((\vec{w}_m)_{hn}^{k-1},\vec{z})-(B_1(\vec{z}_h,(\vec{v}_m)_{hn}^{n-k+1}),(\vec{w}_m)_{hn}^{k-1})-j(B_2(\vec{z}_h,(\omega_m)_{hn}^{n-k+1}),(\varphi_m)_{hn}^{k-1})$$

$$+\frac{j}{\Delta t}((\omega_m)_{hn}^{k-1},\zeta)+(\mathrm{rot}(\vec{v}_m)_{hn}^{n-k+1}-(\omega_m)_{hn}^{n-k+1},\mathrm{rot}\ \vec{z})-(\mathrm{rot}(\vec{v}_m)_{hn}^{n-k+1}-(\omega_m)_{hn}^{n-k+1},\zeta).$$

The system (4.23) is equivalent to the linear equation (4.9), with $((\vec{v}_m)_{hn}^k,(\omega_m)_{hn}^k)$ replaced by $((\vec{w}_m)_{hn}^k,(\varphi_m)_{hn}^k)$.

The properties of a_h and L_h allow us to apply Lax-Milgram theorem which yields the existence and the uniqueness of the solution of (4.23).

The most difficult part of this section is to obtain the a priori estimates of the type (4.10)-(4.15), for the functions $((\vec{w}_m)^k_{hn}, (\varphi_m)^k_{hn})$, $k = 1, ..., n$.

In the sequel we shall need the following inequalities (see [6], p. 333):

$$(4.24) \qquad \begin{cases} |u_h| \leq d_0 \|u_h\|_0 \ \forall u_h \in V_h, \\ \|u_h\|_0 \leq S(h)|u_h| \ \forall u_h \in V_h, \end{cases}$$

where $S(h) \to \infty$ as $h \to 0$.

We shall prove:

Theorem 4.5 If h and n satisfy the inequality

$$(4.25) \qquad \Delta t S^2(h) < \max\left(\frac{\mu}{32C(r)}, \frac{\gamma}{32j^2C(r)}\right),$$

then:

$$(4.26) \qquad |(\vec{w}_m)^k_{hn}|^2 \leq D(r),\ k = 1, ..., n,$$

$$(4.27) \qquad \sum_{k=1}^{n} |(\vec{w}_m)^k_{hn} - (\vec{w}_m)^{k-1}_{hn}|^2 \leq D(r),$$

$$(4.28) \qquad \Delta t \sum_{k=1}^{n} \|(\vec{w}_m)^k_{hn}\|_0^2 \leq D(r),$$

$$(4.29) \qquad |(\varphi_m)^k_{hn}|^2 \leq D(r),\ k = 1, ..., n,$$

$$(4.30) \qquad \sum_{k=1}^{n} |(\varphi_m)^k_{hn} - (\varphi_m)^{k-1}_{hn}|^2 \leq D(r),$$

$$(4.31) \qquad \Delta t \sum_{k=1}^{n} \|(\varphi_m)^k_{hn}\|_0^2 \leq D(r).$$

Proof. For simplicity, any function $(f_m)^k_{hn}$ will be denoted in the sequel f^k. Taking $\vec{z}_h = \vec{w}^k$ in $(4.23)_1$, $\zeta_h = \varphi^k$ in $(4.23)_2$, adding the obtained equalities and using again the identity of Theorem 4.2, we obtain:

$$(4.32) \qquad \begin{aligned} &\frac{1}{2\Delta t}(|\vec{w}^k|^2 - |\vec{w}^{k-1}|^2 + |\vec{w}^k - \vec{w}^{k-1}|^2) + (\mu + \chi)\|\vec{w}^k\|_0^2 \\ &+\frac{j}{2\Delta t}(|\varphi^k|^2 - |\varphi^{k-1}|^2 + |\varphi^k - \varphi^{k-1}|^2) + \gamma\|\varphi^k\|_0^2 + 2\chi|\varphi^k|^2 \\ &= (\mathrm{rot}\ \vec{v}^{n-k+1} - \omega^{n-k+1}, \mathrm{rot}\ \vec{w}^k) - (\mathrm{rot}\ \vec{v}^{n-k+1} - \omega^{n-k+1}, \varphi^k) \\ &+\chi(\mathrm{rot}\ \varphi^k, \vec{w}^k) + \chi(\mathrm{rot}\ \vec{w}^k, \varphi^k) \\ &-(B_1(\vec{w}^k, \vec{v}^{n-k+1}), \vec{w}^{k-1}) - j(B_2(\vec{w}^k, \omega^{n-k+1}), \varphi^{k-1}). \end{aligned}$$

We shall introduce in (4.32) the following computations:

$$-(B_1(\vec{w}^k, \vec{v}^{n-k+1}), \vec{w}^{k-1}) = -(B_1(\vec{w}^k - \vec{w}^{k-1}, \vec{v}^{n-k+1}), \vec{w}^{k-1})$$
$$-(B_1(\vec{w}^{k-1}, \vec{v}^{n-k+1}), \vec{w}^{k-1}) \leq \text{(we use } (2.2)_1)$$
$$\sqrt{2}|\vec{w}^k - \vec{w}^{k-1}|^{1/2}||\vec{w}^k - \vec{w}^{k-1}||_0^{1/2}|\vec{v}^{n-k+1}|^{1/2}||\vec{v}^{n-k+1}||_0^{1/2}||\vec{w}^{k-1}||_0$$
$$+\sqrt{2}|\vec{w}^{k-1}|||\vec{w}^{k-1}||_0||\vec{v}^{n-k+1}||_0 \leq \text{(we use } (4.24)_2 \text{ for the first term)}$$
$$\sqrt{2}S(h)|\vec{w}^k - \vec{w}^{k-1}||\vec{v}^{n-k+1}|||\vec{w}^{k-1}||_0 + \sqrt{2}|\vec{w}^{k-1}|||\vec{w}^{k-1}||_0||\vec{v}^{n-k+1}||_0,$$

$$-j(B_2(\vec{w}^k, \omega^{n-k+1}), \varphi^{k-1}) = -j(B_2(\vec{w}^k - \vec{w}^{k-1}, \omega^{n-k+1}), \varphi^{k-1})$$
$$-j(B_2(\vec{w}^{k-1}, \omega^{n-k+1}), \varphi^{k-1}) \leq \text{(we use } (2.2)_2)$$
$$j\sqrt{2}|\vec{w}^k - \vec{w}^{k-1}|^{1/2}||\vec{w}^k - \vec{w}^{k-1}||_0^{1/2}|\omega^{n-k+1}|^{1/2}||\omega^{n-k+1}||_0^{1/2}||\varphi^{k-1}||_0$$
$$+j\sqrt{2}|\vec{w}^{k-1}|^{1/2}||\vec{w}^{k-1}||_0^{1/2}||\omega^{n-k+1}||_0|\varphi^{k-1}|^{1/2}||\varphi^{k-1}||_0^{1/2} \leq$$
$$\text{(we use } (4.24)_2 \text{ for the first term and } 2ab \leq a^2 + b^2 \text{ for the second)}$$
$$j\sqrt{2}S(h)|\vec{w}^k - \vec{w}^{k-1}||\omega^{n-k+1}|||\varphi^{k-1}||_0 + j\frac{\sqrt{2}}{2}|\vec{w}^{k-1}|||\vec{w}^{k-1}||_0||\omega^{n-k+1}||_0$$
$$+j\frac{\sqrt{2}}{2}|\varphi^{k-1}|||\varphi^{k-1}||_0||\omega^{n-k+1}||_0,$$

$$\chi(\text{rot } \varphi^k, \vec{w}^k) + \chi(\text{rot } \vec{w}^k, \varphi^k) = 2\chi(\text{rot } \vec{w}^k, \varphi^k) \leq \text{(we use (4.17))}$$
$$2\sqrt{2}\chi|\varphi^k|||\vec{w}^k||_0 \leq \chi(||\vec{w}^k||_0^2 + 2|\varphi^k|^2),$$

$$(\text{rot } \vec{v}^{n-k+1} - \omega^{n-k+1}, \text{rot } \vec{w}^k) \leq \text{(we use (4.17))}$$
$$\sqrt{2}|\text{rot } \vec{v}^{n-k+1} - \omega^{n-k+1}|||\vec{w}^k||_0 \leq \sqrt{2}(\sqrt{2}||\vec{v}^{n-k+1}||_0 + |\omega^{n-k+1}|)||\vec{w}^k||_0 \leq$$
$$\leq \text{(we use } (4.24)_1)\sqrt{2}(\sqrt{2}||\vec{v}^{n-k+1}||_0 + d_0||\omega^{n-k+1}||_0)||\vec{w}^k||_0$$
$$\leq \frac{\mu}{2}||\vec{w}^k||_0^2 + \frac{4}{\mu}||\vec{v}^{n-k+1}||_0^2 + \frac{2d_0^2}{\mu}||\omega^{n-k+1}||_0^2$$
$$\leq \frac{\mu}{2}||\vec{w}^k||_0^2 + a(||\vec{v}^{n-k+1}||_0^2 + ||\omega^{n-k+1}||_0^2),$$

where $a = \max(\frac{4}{\mu}, \frac{2d_0^2}{\mu})$,

$$-(\text{rot } \vec{v}^{n-k+1} - \omega^{n-k+1}, \varphi^k) \leq \text{(we use } (4.24)_1)$$

$$d_0 |\text{rot } \vec{v}^{n-k+1} - \omega^{n-k+1}| \|\varphi^k\|_0 \leq \text{(we use the same inequalities as above)}$$

$$\frac{\gamma}{2}\|\varphi^k\|_0^2 + b(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2),$$

where $b = \max(\frac{2d_0^2}{\gamma}, \frac{d_0^4}{\gamma})$.

Introducing all these inequalities in (4.32), it follows:

$$\begin{aligned}
&|\vec{w}^k|^2 - |\vec{w}^{k-1}|^2 + |\vec{w}^k - \vec{w}^{k-1}|^2 + j(|\varphi^k|^2 - |\varphi^{k-1}|^2 + |\varphi^k - \varphi^{k-1}|^2) \\
&+\Delta t\mu\|\vec{w}^k\|_0^2 + \Delta t\gamma\|\varphi^k\|_0^2 \leq 2(a+b)\Delta t(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2) + \\
(4.33)\quad & 2\sqrt{2}\Delta t S(h)|\vec{w}^k - \vec{w}^{k-1}||\vec{v}^{n-k+1}|\|\vec{w}^{k-1}\|_0 + 2\sqrt{2}\Delta t|\vec{w}^{k-1}|\|\vec{w}^{k-1}\|_0\|\vec{v}^{n-k+1}\|_0 + \\
& 2j\sqrt{2}\Delta t S(h)|\vec{w}^k - \vec{w}^{k-1}||\omega^{n-k+1}|\|\varphi^{k-1}\|_0 + j\sqrt{2}\Delta t|\vec{w}^{k-1}|\|\vec{w}^{k-1}\|_0\|\omega^{n-k+1}\|_0 \\
& + j\sqrt{2}\Delta t|\varphi^{k-1}|\|\varphi^{k-1}\|_0\|\omega^{n-k+1}\|_0.
\end{aligned}$$

Majorating the right-hand side of the inequality (4.33) we obtain:

$$\begin{aligned}
&|\vec{w}^k|^2 - |\vec{w}^{k-1}|^2 + \frac{1}{2}|\vec{w}^k - \vec{w}^{k-1}|^2 + j(|\varphi^k|^2 - |\varphi^{k-1}|^2 + |\varphi^k - \varphi^{k-1}|^2) \\
&+\Delta t\mu\|\vec{w}^k\|_0^2 + \Delta t\gamma\|\varphi^k\|_0^2 \leq 2(a+b)\Delta t(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2) \\
(4.34)\quad &+\Delta t(8\Delta t S^2(h)|\vec{v}^{n-k+1}|^2 + \frac{\mu}{4})\|\vec{w}^{k-1}\|_0^2 \\
&+\Delta t(8j^2\Delta t S^2(h)|\omega^{n-k+1}|^2 + \frac{\gamma}{4})\|\varphi^{k-1}\|_0^2 + \frac{16}{\mu}\Delta t|\vec{w}^{k-1}|^2\|\vec{v}^{n-k+1}\|_0^2 \\
&+\frac{4j^2}{\mu}\Delta t|\vec{w}^{k-1}|^2\|\omega^{n-k+1}\|_0^2 + \frac{2j^2}{\gamma}\Delta t|\varphi^{k-1}|^2\|\omega^{n-k+1}\|_0^2.
\end{aligned}$$

Using in (4.34) the inequalities (4.10), (4.13) and the hypothesis of the theorem, it follows, for $k = 1, ..., n$:

$$\begin{aligned}
&|\vec{w}^k|^2 - |\vec{w}^{k-1}|^2 + \frac{1}{2}|\vec{w}^k - \vec{w}^{k-1}|^2 + j(|\varphi^k|^2 - |\varphi^{k-1}|^2 + |\varphi^k - \varphi^{k-1}|^2) \\
(4.35)\quad &+\Delta t\mu\|\vec{w}^k\|_0^2 - \Delta t\frac{\mu}{2}\|\vec{w}^{k-1}\|_0^2 + \Delta t\gamma\|\varphi^k\|_0^2 - \Delta t\frac{\gamma}{2}\|\varphi^{k-1}\|_0^2 \\
&\leq 2(a+b)\Delta t(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2) \\
&+\Delta t\alpha(|\vec{w}^{k-1}|^2 + j|\varphi^{k-1}|^2)(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2),
\end{aligned}$$

where $\alpha = \max(\frac{16}{\mu}, \frac{4j^2}{\mu}, \frac{2j}{\gamma})$. Adding the inequalities (4.35) for $k = 1, ..., q$, with

$q \leq n$, we get:

$$
(4.36)\quad
\begin{aligned}
&|\vec{w}^q|^2 + \frac{1}{2}\sum_{k=1}^{q}|\vec{w}^k - \vec{w}^{k-1}|^2 + j(|\varphi^k|^2 + \sum_{k=1}^{q}|\varphi^k - \varphi^{k-1}|^2)\\
&+\Delta t\frac{\mu}{2}\sum_{k=1}^{q}\|\vec{w}^k\|_0^2 + \Delta t\frac{\gamma}{2}\sum_{k=1}^{q}\|\varphi^k\|_0^2\\
&\leq 2(a+b)\Delta t\sum_{k=1}^{q}(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2)\\
&+\Delta t\alpha\sum_{k=1}^{q}(|\vec{w}^{k-1}|^2 + j|\varphi^{k-1}|^2)(\|\vec{v}^{n-k+1}\|_0^2 + \|\omega^{n-k+1}\|_0^2),\ q = 1,\dots n.
\end{aligned}
$$

We introduce the notations:

$$
(4.37)\quad
\begin{cases}
x_i = \Delta t\alpha(\|\vec{v}^i\|_0^2 + \|\omega^i\|_0^2),\ i = 1,\dots n,\\
\lambda_i = \dfrac{2(a+b)}{\alpha}[(x_n + \dots + x_{n-i+1}) + \dots + x_n\cdot\dots\cdot x_{n-i+1}],\ i = 1,\dots n.
\end{cases}
$$

We shall now prove recursively that:

$$
(4.38)\qquad |\vec{w}^i|^2 + j|\varphi^i|^2 \leq \lambda_i\ \ i = 1,\dots,n.
$$

Using the notation $(4.37)_1$, it follows from (4.36), for $q = 1,\dots,n$:

$$
(4.39)\quad |\vec{w}^q|^2 + j|\varphi^q|^2 \leq \frac{2(a+b)}{\alpha}\sum_{k=1}^{q}x_{n-k+1} + \sum_{k=1}^{q}(|\vec{w}^{k-1}|^2 + j|\varphi^{k-1}|^2)x_{n-k+1}.
$$

Taking $q = 1$ in (4.39) and using $\vec{w}^0 = \vec{0},\ \varphi^0 = 0$ we obtain:

$$
(4.40)\qquad |\vec{w}^1|^2 + j|\varphi^1|^2 \leq \frac{2(a+b)}{\alpha}x_n,
$$

i.e. (4.38) for $i = 1$. We suppose that the inequality (4.38) holds for $i = 1,\dots q-1$ and we introduce it in (4.39). This yields:

$$
(4.41)\quad
\begin{aligned}
&|\vec{w}^q|^2 + j|\varphi^q|^2 \leq \frac{2(a+b)}{\alpha}\{\sum_{k=1}^{q}x_{n-q+1} + x_nx_{n-1} + [(x_n + x_{n-1}) + x_nx_{n-1}]x_{n-2}\\
&+\dots + [(x_n + \dots + x_{n-q+2}) + \dots + x_n\cdot\dots\cdot x_{n-q+2}]x_{n-q+1}\},
\end{aligned}
$$

which is (4.38) for $i = q$.

Now, from (4.37) it is obvious that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$.

The next computation will give, together with (4.36), the assertion of the theorem.

$$\lambda_n = \frac{2(a+b)}{\alpha}[(x_n + ... + x_1) + ... + x_n \cdot ... \cdot x_1] =$$
$$\frac{2(a+b)}{\alpha}[(1+x_1) \cdot ... \cdot (1+x_n) - 1] =$$
$$\frac{2(a+b)}{\alpha}(\exp((\ln(1+x_1) \cdot ... \cdot (1+x_n)) - 1) \leq \text{(we use } \ln(1+x) \leq x,\ \forall x \geq 0)$$
$$\frac{2(a+b)}{\alpha}(\exp(x_1 + ... + x_n) - 1) = \frac{2(a+b)}{\alpha}(\exp(\alpha \Delta t \sum_{k=1}^{n}(\|\vec{v}^k\|_0^2 + \|\omega^k\|_0^2)) - 1) \leq$$
$$\text{(we use (4.10) and (4.13))}\frac{2(a+b)}{\alpha}(\exp(2\alpha C(r)) - 1).$$

We define the functions:

$$(4.42)\quad \begin{cases} (\vec{w}_m)_{hn}(t) = (\vec{w}_m)_{hn}^k,\ \forall t \in [(k-1)\Delta t, k\Delta t[, \\ (\varphi_m)_{hn}(t) = (\varphi_m)_{hn}^k,\ \forall t \in [(k-1)\Delta t, k\Delta t[,\ k = 1, ..., n, \\ (\vec{u}_m)_{hn}(t) = (\vec{w}_m)_{hn}(T-t), \\ (\rho_m)_{hn}(t) = (\varphi_m)_{hn}(T-t). \end{cases}$$

The stability and convergence resutlts follow, with the same techniques as for $Scheme_1^m$.

c) The discretization of $(\mathbf{OS})_3^{m+1}$.

We define

$$(4.43)\qquad g_{hn}(t) = g_{hn}^k\ \forall t \in [(k-1)\Delta t, k\Delta t[,$$

where g_{hn}^k is given by the definition (4.7).

We first prove that if $g_h \in K_h$, then $g_{hn} \in B_r$. Indeed:

$$\|g_{hn}\|_{L^2(\Omega_T)}^2 = \sum_{k=1}^{n} \int_{(k-1)\Delta t}^{k\Delta t} |g_{hn}^k|^2 dt =$$
$$\sum_{k=1}^{n} \int_{(k-1)\Delta t}^{k\Delta t} (\int_\Omega (\frac{1}{\Delta t} \int_{(k-1)\Delta t}^{k\Delta t} g_h dt)^2 dx) ds \leq$$
$$\sum_{k=1}^{n} \int_{(k-1)\Delta t}^{k\Delta t} (\frac{1}{\Delta t} (\int_\Omega \int_{(k-1)\Delta t}^{k\Delta t} g_h^2 dt dx) ds =$$
$$\sum_{k=1}^{n} \int_{(k-1)\Delta t}^{k\Delta t} (\frac{1}{\Delta t} \int_{(k-1)\Delta t}^{k\Delta t} |g_h|^2 dt) ds \leq$$
$$\sum_{k=1}^{n} \int_{(k-1)\Delta t}^{k\Delta t} \frac{1}{\Delta t} \frac{r^2}{T} \Delta t ds = r^2.$$

We consider the following scheme:

$Scheme_3^{m+1}$. When $(g_0)_{hn}, ..., (g_m)_{hn}$ are known, let us find $(g_{m+1})_{hn}$ given by:

$$(4.44)\qquad (g_{m+1})_{hn}(t) = P_{B_r}((g_m)_{hn}(t) - (\delta_m)_{hn} \frac{(\rho_m)_{hn}(t)}{\|(\rho_m)_{hn}\|_{L^2(\Omega_T)}}),$$

with $(\rho_m)_{hn}$ given by (4.42), $(\delta_m)_{hn}$ defined in the sequel and $(g_0)_{hn}$ the approximation of a given $g_0 \in B_r$. We define

(4.45) $$(\delta_m)_{hn} = \begin{cases} \delta_m \text{ if } \|\rho_m\|_{L^2(\Omega_T)} > 0, \\ 0 \text{ if } \|\rho_m\|_{L^2(\Omega_T)} = 0, \ \|(\rho_m)_{hn}\|_{L^2(\Omega_T)} = 0, \\ \delta_m \Delta t S^2(h) \text{ if } \|\rho_m\|_{L^2(\Omega_T)} = 0, \ \|(\rho_m)_{hn}\|_{L^2(\Omega_T)} > 0. \end{cases}$$

The last result of this section is a convergence theorem for $Scheme_3^{m+1}$.

Theorem 4.6 When $h \to 0$ and $n \to \infty$, satisfying the hypothesis of Theorem 4.5, $(g_{m+1})_{hn} \to g_{m+1}$ strongly in $L^2(\Omega_T)$.

Proof. We shall prove this convergence recursively. The convergence $(g_0)_{hn} \to g_0$ strongly in $L^2(\Omega_T)$ when $h \to 0$ and $n \to \infty$ is given by known results of the literature. We suppose that $(g_m)_{hn} \to g_m$ strongly in $L^2(\Omega_T)$.

i) If $\|\rho_m\|_{L^2(\Omega_T)} > 0$ it follows, from the convergence of $Scheme_2^m$: there exists $h_0 > 0$, $n_0 \in \mathbb{N}^*$ so that

$$\|(g_{m+1})_{hn} - g_{m+1}\|_{L^2(\Omega_T)} \leq \|(g_m)_{hn} - g_m\|_{L^2(\Omega_T)}$$
$$+\delta_m \left\| \frac{(\rho_m)_{hn}}{\|(\rho_m)_{hn}\|_{L^2(\Omega_T)}} - \frac{\rho_m}{\|\rho_m\|_{L^2(\Omega_T)}} \right\|_{L^2(\Omega_T)} \forall h < h_0, \ n > n_0.$$

Using again the convergence theorem for $Scheme_2^m$, the proof is complete in this case.

ii) If $\|\rho_m\|_{L^2(\Omega_T)} = 0$ we get:

$$\|(g_{m+1})_{hn} - g_{m+1}\|_{L^2(\Omega_T)} \leq \|(g_m)_{hn} - g_m\|_{L^2(\Omega_T)} + (\delta_m)_{hn},$$

and, hence, the assertion of the theorem follows from the definition of $(\delta_m)_{hn}$.

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