INSTITUTUL DE MATEMATICA
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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I. THE NON–LACUNAR DOMAIN

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Preprint nr. 7/2000

BUCURESTI

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July, 2000

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HALPHEN-CASTELNUOVO THEORY FOR SMOOTH CURVES IN P^n I. THE NON-LACUNAR DOMAIN

Ovidiu Păsărescu

Contents

1 Introduction

In this article we work over an algebraically closed field of characteristic zero (e.g. $\mathbf{C}$ = the complex field) (see §4 also). We use the standard notations from [Ha 1].

By a *curve* (resp. *surface*) we mean a $\mathbf{C}$-algebraic integral scheme of dimension 1 (resp. 2).The curves and surfaces used in this paper will be non-singular.

We begin with some comments. Let C be a (smooth, irreducible) curve. Because any such curve can be embedded in a projective space using a morphism associated to some linear system, we can consider the curve C, simultaneously, in three hypostases: abstract curve, polarised curve and embedded curve. In the classification of **abstract curves** it is considered the numerical invariant named genus $g = h^0(\omega_C) = h^1(\mathcal{O}_C)$ (comming from the classification of compact Riemannian Surfaces) and, for fixed g, it is considered the space of continuous invariants $\mathcal{M}_g$ (called the moduli space) containing the isomorphism classes of curves of genus g. In this theory there are considered various problems concerning $\mathcal{M}_g$: dimension, quasiprojectivity, unirationality, singular locus etc. In the theory of **polarised curves** there are considered pairs (C, g^n_d), where g^n_d is a linear system (producing rational applications $C \to \mathbf{P}^n$) of degree d and dimension n. In the Brill-Noether theory it is considered, for instance, the object

$$W^n_d = W^n_d(C) = \{g^s_d | g^s_d \text{ complete on } C, \quad s \geq n\}.$$

There are studied problems concerning the structure of W^n_d (a determinantal variety): irreducibility, dimension, singular locus etc. But the first problem is: when $W^n_d \neq \emptyset$? An answer is given using the Brill-Noether number $\rho = \rho(g, n, d) = g - (n+1)(g - d + n)$: namely, $\rho \geq 0 \Rightarrow W^n_d \neq \emptyset$. An **embedded curve** is a curve $C \subset \mathbf{P}^n$. The theory of *non-degenerate curves* from $\mathbf{P}^n$ (i.e. not contained in any hyperplane) is related to the theory of polarised curves (the study of W^n_d) and to the theory of embedded curves, both of them being two faces of the same problem. If $C \subset \mathbf{P}^n$ we associate to it, as usual, the genus g (comming from Riemannian Surfaces), the degree d (which is a projective invariant) and the embedding dimension n (both of them comming from the theory of polarised curves). The Hilbert schemes $H^n_{d,g}$ of *smooth*, *irreducible* and *non-degenerate* curves from $\mathbf{P}^n$ (the closure in the general Hilbert scheme of the open set corresponding to smooth, irreducible, non-degenerate curves from $\mathbf{P}^n$) are usually studied in the theory of embedded curves. There are studied topics about $H^n_{d,g}$: reducibility, projectivity, singular locus, tangent spaces, "good" or "bad" components etc. But, first of all, similar to the Brill-Noether theory, what is *important to know is in which conditions we have* $H^n_{d,g} \neq \emptyset$.

*We call **Halphen-Castelnuovo theory** the study of existence, for a fixed triplet of integers* (n, d, g), $n \geq 2$, $d \geq n$, $g \geq 0$ *and a given property* $\mathcal{P}$, *of non-degenerate curves* $C \subset \mathbf{P}^n$ *of degree* d, *genus* g *and having the property* $\mathcal{P}$. Then it is natural to study the (nonempty) families $\mathcal{F}^{n,\mathcal{P}}_{d,g}$ of curves as before.

The property $\mathcal{P}$ may be, for instance, irreducibility, smoothness, linear normality, projective normality, maximal rank etc. If $\mathcal{P}$ = smoothness, then $\mathcal{F}^{n,\mathcal{P}}_{d,g} = H^n_{d,g}$, for instance.

The numbers d = degree, g = genus, n = the embedding dimension appear naturally from the necessity for the theories of abstract, polarised and embedded curves to be compatible. This allows us to study the connections between the *extrinsic geometry* (represented by properties of projective enbeddings and of the Hilbert scheme) and the *intrinsic geometry* (represented by abstract properties and the moduli space) of families of algebraic curves. This comparison is often best represented by the natural maps $\pi : H^n_{d,g} \to \mathcal{M}_g$. So, *it's important again to know when* $H^n_{d,g} \neq \emptyset$, so "when the map π is not the empty map ?". In this context some new properties $\mathcal{P}$ appear: general or particular moduli, expected number of moduli etc.

We recall now the *Castelnuovo bound* (found in 1893) (for a modern proof, see [H], ch. 3):

Theorem A (Castelnuovo [C]): *If $n \in Z$, $n \geq 3$, and $C \subset P^n$ is a reduced and irreducible (possibly singular) non-degenerate curve of degree d and geometrical genus g, then $d \geq n$ and $0 \leq g \leq \pi_0(d,n)$. The curves for which the bound is attained lie on a rational normal surface of degree $n-1$ in P^n (i.e. either rational scrolls or the Veronese surface in P^5) and can be completely described (these curves are called extremal curves).*

This result is a generalization of a previous similar result obtained by Halphen and Noether in 1882 for $n = 3$ ([Hl], [N]).

Here, $\pi_0(d,n)$ is the first one from the Harris-Eisenbud numbers $\pi_p(d,n)$ (see [H], ch. 3: Castelnuovo theory), given (for $0 \leq p \leq n-2$) by:

$$\pi_p = \pi_p(d,n) = \frac{m_p(m_p-1)}{2}(n+p-1) + m_p(\varepsilon_p + p) + \mu_p, \tag{1.1}$$

where

$$m_p = m_p(d,n) = [(d-1)/(n+p-1)]_* \tag{1.2}$$

(we denote, during this article by $[x]_*$ the integer part of $x \in \mathbf{R}$)

$$\varepsilon_p = \varepsilon_p(d,n) = d - 1 - m_p(n+p-1) \tag{1.3}$$

$$\mu_p = \mu_p(d,n) = \max(0, [(p-n+2+\varepsilon_p)/2]_*). \tag{1.4}$$

We remark that $\mu_0 = 0$ and $\pi_p = d^2/(2(n+p-1)) + O(d)$.

We recall that, if the property $\mathcal{P}$ is irreducibility (or nodality), the complete answer in Halphen-Castelnuovo theory is given by the following

Theorem B (Tannenbaum [T1], [T2]): *For $n \geq 2$ and any $d, g \in \mathbf{Z}$, $d \geq n$, $0 \leq g \leq \pi_0(d,n)$ there is a non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and geometric genus g with only nodes.*

This theorem generalize a similar result of Severi from 1915 for $n = 2$ ([Sv]).

If $\mathcal{P}$ = smoothness we arrive to the **Halphen-Castelnuovo Problem** (related with the intrinsic and extrinsic geometry of curves), denoted $HC(n)$ for $n \geq 2$, $n \in \mathbf{Z}$:

$HC(n)$: For which pairs of integers(d,g), $d \geq n$, $0 \leq g \leq \pi_0(d,n)$ do we have $H^n_{d,g} \neq \emptyset$?

This is the Problem which we'll consider in this article. We recall now what is known on $HC(n)$.

$HC(2)$ is simple: $H^2_{d,g} \neq \emptyset \Leftrightarrow d \geq 2$ and $g = (d-1)(d-2)/2$. But, for $n \geq 3$, $HC(n)$ becomes highly non-trivial. A (correct) solution for $HC(3)$ has been proposed by Halphen in 1882 ([Hl]), but his proof was, partially, incorrect. A complete proof was given by Gruson and Peskine 100 years later ([GP1], [GP2]). The answer is contained in the following

Theorem C (Halphen-Gruson-Peskine): *Let there be $d, g \in \mathbf{Z}$, $d \geq 3$ and*

$$0 \leq g \leq \pi_0(d,3) = \begin{cases} \frac{1}{4}d^2 - d + 1, & d \text{ even} \\ \frac{1}{4}(d^2-1) - d + 1, & d \text{ odd.} \end{cases}$$

a) If $\left[\frac{1}{6}d(d-3)\right]_* + 1 = \pi_1(d,3) < g \leq \pi_0(d,3)$, then any non-degenerate (smooth, irreducible) curve $C \subset \mathbf{P}^3$ of degree d and genus g is contained in a quadric surface; so, in this case $H^3_{d,g} \neq \emptyset \Leftrightarrow (\exists) a, b \in \mathbf{Z}$, $a, b \geq 0$ such that $d = a + b$ and $g = (a-1)(b-1)$.

b) If $0 \leq g \leq \pi_1(d,3)$ for any pair (d,g) there is a (smooth, irreducible) non-degenerate curve $C \subset \mathbf{P}^3$ of degree d and genus g; then the necessary curves can be found on surfaces of degree 4 (singular: [GP2] or not: [Mo]) if $0 \leq g \leq \frac{1}{8}(d-1)^2$ and on cubic surfaces if $\frac{1}{8}(d-1)^2 < g \leq \pi_1(d,3)$.

Definition 1.1: *A pair of integers (d,g), $d \geq n$, $0 \leq g \leq \pi_0(d,n)$ is called a gap for $HC(n)$ if there is no non-degenerate (smooth, irreducible) curve $C \subset \mathbf{P}^n$ of degree d and genus g.*

So, for $HC(3)$ there are two domains in the (d, g)-plane:

$D_1^3 : 0 \leq g \leq \pi_1(d,3),\ d \geq 3$ where there is no gap.

$D_2^3 : \pi_1(d,3) < g \leq \pi_0(d,3),\ d \geq 3$ where there are gaps.

$HC(n)$ has been solved for $n = 4, 5$ by Rathmann ([Ra]), $n = 6$ by Ciliberto ([Ci]) and "almost" solved by Ciliberto if $n = 7$ ([Ci]). The situation is similar to the case $n = 3$ in the sense that there are two domains: D_1^n, where there is no gap and D_2^n, where there are gaps.

Let's review that is known for $n \geq 3$ general. The construction of Gruson-Peskine has two steps: *step 1* consists in the construction of curves on a quartic surface and *step 2* consists in the construction of curves on a smooth cubic surface (see [GP 2] or [Ha 2]).

Step 1 has been generalised to $\mathbf{P}^n$ $(n \geq 3)$ as follows:

Theorem D (Păsărescu [P 1], Ciliberto-Sernesi [CS]): *If $d, g \in Z$, $d \geq n$, $\delta \in \{2,3,4\}$, $n \geq 2\delta - 1$, $0 \leq g \leq (d-n)^2/2(2n-\delta)$, there is a (smooth, irreducible) curve $C \subset P^n$ of degree d and genus g which is non-degenerate in P^n, on a surface of degree $2n - \delta$ in P^n (so $H^n_{d,g} \neq \emptyset$ for such pairs (d, g)).*

For $\delta = 2$ see [P 1], for $\delta \in \{3,4\}$ see [CS]. The case $\delta = 2$, $n = 3$ is exactly the Gruson-Peskine construction.

A generalisation of Step 2 was initiated by Păsărescu and Rathmann ([P1], [Ra]) who applied the Gruson-Peskine construction on Del Pezzo surfaces from $\mathbf{P}^4$ and $\mathbf{P}^5$. Ciliberto ([Ci]) studied a generalisation of these 3 initial cases (the Del Pezzo surfaces from $\mathbf{P}^3, \mathbf{P}^4, \mathbf{P}^5$). He constructed curves on some rational surfaces with hyperelliptic hyperplane sections $X_k^n \subset \mathbf{P}^n$ of degree $n + k - 1$, $k = [n/3]_*$ (see §3 section 2.2 for X_k^n).

Precisely let's denote by

$$(1.5)\quad A(d,n) := \begin{cases} \pi_k(d,n) & ,\text{if } n \equiv 0,1 \pmod 3 \\ \pi_k(d,n) - \mu_k + \max(0, \varepsilon_k - 3k - 1) & ,\text{if } n \equiv 2 \pmod 3). \end{cases}$$

(see (1.1) - (1.4)).

Ciliberto proved the following:

Theorem E ([Ci]): *Let there be $n \in Z$, $n \geq 6$ and $k = [n/3]_*$. Then there are two functions $d_0(n)$ (of degree 3/2 in n) and $\varphi(d,n)$ (of degree 3/2 in d), given explicitly, so that for any $d \geq d_0(n)$, $d \in Z$ and $\varphi(d,n) <$*

$g \leq A(d,n)$, $g \in Z$ there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g (hence, in this range $H_{d,g}^n \neq \emptyset$).

If $n = 3$ and $k = 1$ one obtains exactly the Gruson-Peskine construction.

Combining Theorems D ($\delta = 4$) and E, Ciliberto defined 4 domains for $HC(n) : A, B, C, D$ proving that there is no gap in the domain C, the situation from A, B, D being unknown ([Ci]). In this article we obtain results in the domains A and B (and C, incidentally). Precisely, we replace the 4 domains of Ciliberto with two domains: D_1^n (without gaps) and D_2^n (containing gaps), as in the cases $n \in \{3,4,5,6,7\}$. The domains D_1^n and D_2^n will be defined in §2, where we will also state the Main Theorem. In §3 we give the (long) proof of Main Theorem. Our contribution is for "small" degrees, less than $D(n) =$ *a* quadratic function. The *method* used in the proofs is a new one and it isn't o generalization of Gruson-Peskine methods. It will be briefly explained in section 3.1 of §3. Finally, further developments may concern the construction of curves with some special properties $\mathcal{P}$ (topics belonging to Halphen-Castelnuovo theory) or the construction of "good" components of $H_{d,g}^n$. These aspects will be briefly discussed in §4, where something concerning the domain D_2^n will be said.

We remark that the Mori construction for smooth cuartic surface ([Mo]) has been generalised to $\mathbf{P}^n$ by Rathmann, in his thesis, obtaining curves in the range of Theorem D.

We end this § by remarking that the case $n = 3$ of $HC(n)$ has some intrinsic importance, because any (smooth, irreducible) curves in some $\mathbf{P}^n$, $n \geq 4$, can be projected (so, conserving the degree) isomorphically (so, concerving the genus) on $\mathbf{P}^3$. But, from the more sophisticated reasons, concerning in the comparison between the extrinsic and intrinsic properties, the natural problem is the general $HC(n)$, $n \geq 3$.

Convension: *Formula (a,b) means that it is formula b from §a and (a,b,c) means that it is formula c from section b, §a; similarly for the statements of Lemmas, Propositions and Theorems.*

Aknowledgements:

1) During the preparation of this article, I have worked to the Institute of Mathematics of Romanian Academy and I have visited the University of Ferrara, the University of Stockholm, the University of Nice, and the University of Angers (in cronological order). I have had interesting discussion on the subject with J. Alexander, J. Coandă, Ph. Ellia, A. Hirschowitz, J. Kleppe, D. Laksov, A. Lascu, C. Peskine, R. Strano. Many of these discus-

sions were facilitated by the existence of the European Network in Projective Geometry, EUROPROJ.

2) I want to thank also Ş. Basarab, A. Buium, A. Dimca, D. Popescu for general stimulating discussions.

3) A preliminary version of parts of this article circulated under the title "Curves on rational surfaces with hyperelliptic hyperplane sections".

2 The functions $\alpha_p(d,n)$; the domains D_1^n, D_2^n and A_p^n; the Main Theorem

At the beginning of this § we *define* for any $p, n \in \mathbf{Z}$, $p \geq 0$, $n \geq 3$ the following numerical functions $\alpha_p(d,n) = \dfrac{d^2}{2(n+p-1)} + O(d)$ given by:

$$(2.1) \qquad \alpha_p = \alpha_p(d,n) := \frac{x_p(x_p-1)}{2}(n+p-1) + x_p(t_p+p) + \mu_p,$$

where

$$(2.2) \qquad x_p = x_p(d,n) := [(d-a_p^n)/(n+p-1)].$$

$$(2.3) \qquad a_p = a_p^n := [(n-p)/2]_* + 1$$

$$(2.4) \qquad t_p = t_p(d,n) := d - 1 - x_p(n+p-1)$$

$$(2.5) \qquad \mu_p = \mu_p(d,n) := [(p-n+1+t_p(d,n))/2]_*.$$

We also consider the functions

$$(2.6) \quad d_1(n) := \begin{cases} \max\left(2n+1, \frac{1}{6}(3+(4k-1)\sqrt{24k-33})\right) & , n \equiv 0 \pmod 3 \\ \max\left(2n+1, \frac{4k}{4k+1}(-4k+\sqrt{32k^2+16k^2-2k-1})\right) & , n \equiv 1 \pmod 3 \\ \max(2n+1, 5k+3+(2k+1)\sqrt{48k+6}) & , n \equiv 2 \pmod 3 \end{cases}$$

where $k = [n/3]_*$ and $n \in \mathbf{Z}$, $n \geq 3$.

We'll need

$$(2.7) \quad B(d,n) = \begin{cases} \left.\begin{array}{ll} \alpha_{k+1}(d,n), & \text{if } n \equiv 1,2 \pmod 3 \\ \alpha_k(d,n), & \text{if } n \equiv 0 \pmod 3 \end{array}\right\} & , \text{if } 2n+1 \leq d < d_1(n) \\ A(d,n) & , \text{if } d \geq d_1(n) \end{cases}$$

with k from before and $A(d,n)$ from (1.5).

We consider the following graphs (contained in the (d,g)-plane), of equations:

$$B_0^n : g = \pi_0(d,n)$$

$$B_k^n : g = \begin{cases} \pi_0(d,n), & n \le d \le 2n \\ B(d,n), & d \ge 2n+1 \end{cases}, k = [n/3]_*$$

$$C_p^n : g = \alpha_p(d,n),\ p \ge n/3,\ p,n \in \mathbf{Z},\ n \ge 3.$$

Now we are ready to *define* the **non-lacunar domain** D_1^n (bounded by $g=0$ and B_k^n) and the **lacunar domain** D_2^n (bounded by B_k^n and B_0^n), contained in the (d,g)-plane:

$$\text{(2.8)} \qquad D_1^n : 0 \le g \le \begin{cases} \pi_0(d,n), & n \le d \le 2n \\ B(d,n), & d \ge 2n+1 \end{cases}$$

$$\text{(2.9)} \qquad D_2^n : B(d,n) < g \le \pi_0(d,n),\ d \ge 2n+1.$$

The Main Theorem belonging to Halphen-Castelnuovo theory which we'll prove in this article is:

MAIN THEOREM: *In the domain D_1^n there is no gap for the problem $HC(n)$, for any $n \ge 3$, $n \in \mathbf{Z}$.*

Remark 2.1: *We recalled in §1 that the Main Theorem is already proved for $3 \le n \le 7$ due to the contributions of Rathmann ([Ra]), Ciliberto, Sernesi ([Ci], [CS]) and of the author ([P1]). Moreover, the Main Theorem is true for $d > D(n)$ (= a quadratic functions in n) (see [Ci], [CS], [P1]). So, our main contribution here is for "small" degrees, $2n+1 \le d < D(n)$. We will suppose, that $n \ge 8$ (but our arguments work, in principle, for $5 \le n \le 7$ also).*

During the proof of Main Theorem, the domain D_1^n will be divided in subdomains A_p^n, namely:

$$\text{(2.10)} \qquad A'^n_{n-3} : 0 \le g \le \alpha_{n-3}(d,n),\ d \ge 2n+1 \text{ (between } g=0 \text{ and } C^n_{n-3})$$

$$\text{(2.11)} \qquad A_p^n : \alpha_{p+1}(d-1,n) \le g \le \alpha_p(d,n),\ d \ge 2n+1,\ p \ge n/3$$

(containing the domain between C_{p+1}^n and C_p^n for $p \le n-1$; see section 3.3 from §3).

$$\text{(2.12)} \qquad \tilde{A}_k^n : \alpha_{k+1}(d-1,n) < g \le B(d,n),\ d \ge 2n+1,\ k = [n/3]_*$$

(containing the domain between C^n_{k+1} and B^n_k).

(2.13) $$A^n_0 : 0 \leq g \leq \pi_0(d,n), \quad n \leq d \leq 2n.$$

Lemma 2.2: *In the domain A^n_0 then is no gap for $HC(n)$, $n \geq 8$.*
Proof: We use projections of nonspecial curves. For $d = 2n$ and $g = \pi_0(d,n) = \pi_0(2n,n)$ we consider the extremal curve.

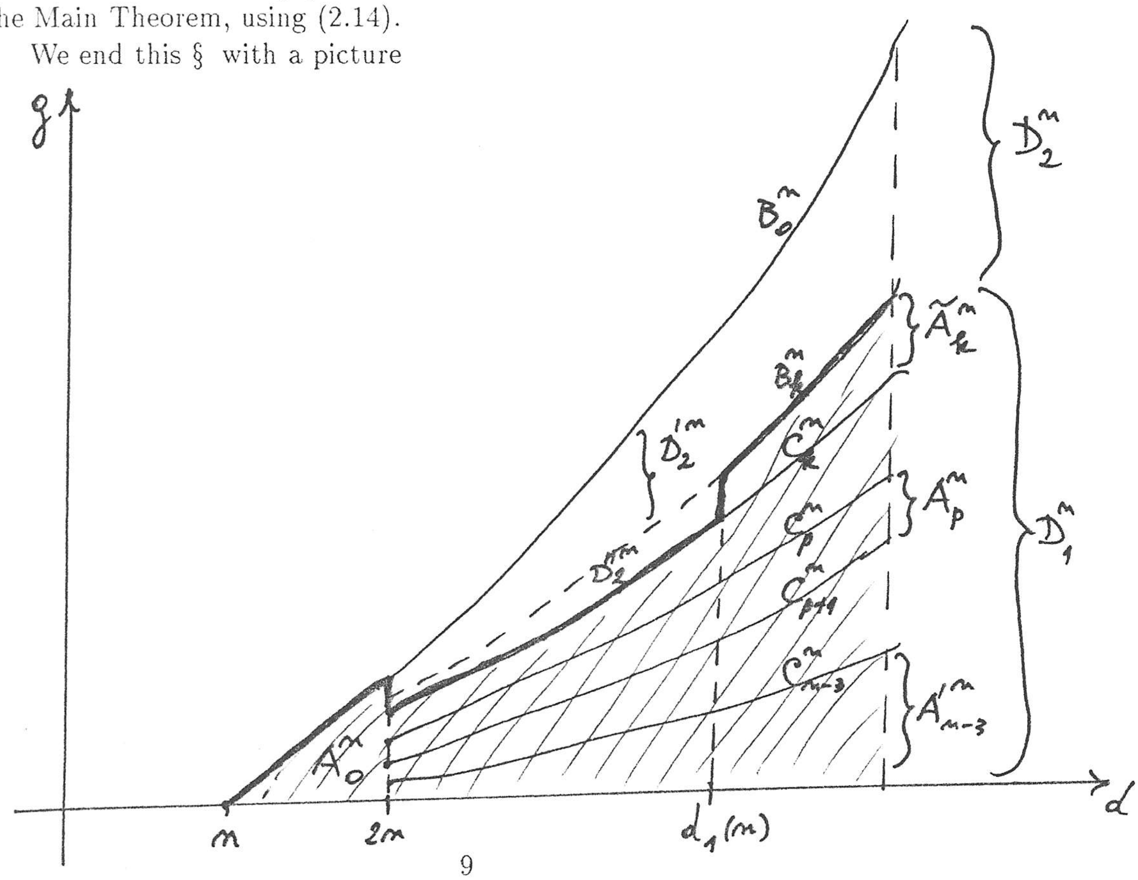
* * *

The following equality holds (see Lemma 3.3.1 b)):

(2.14) $$D^n_1 = A^n_0 \cup A'^n_{n-3} \cup \Big(\bigcup_{\frac{n}{3} \leq p \leq n-4} A^n_p \Big) \cup \tilde{A}^n_k$$

We just solved the elementary step (curves in A^n_0) of $HC(n)$, $n \geq 8$ in the previous lemma. The curves from $\bigcup_{\frac{n}{3} \leq p \leq n-4} A^n_p$ will appear in section 3.5 (§3)-Theorem 3.5.11, the curves from $\tilde{A}^n_k$, in section 3.6 (§3)-Theorem 3.6.1 and the curves from A'^n_{n-3}, in section 3.6 (§3)-Theorem 3.6.2, thus proving the Main Theorem, using (2.14).

We end this § with a picture



The hashureted domain is D_1^n (the non-lacunar one).

3 The proof of Main Theorem

3.1 Methods

Let's fix some notations. Let there be $\Sigma \subset \mathbf{P}^2$ a finite set of (distinct) points $\Sigma = \{P_0, P_1, \ldots, P_s\}$. We denote by $S = BL_\Sigma(\mathbf{P}^2) \to \mathbf{P}^2$ the blow up of $\mathbf{P}^2$ in Σ. Then Pic $(S) \cong \mathbf{Z} \oplus \mathbf{Z}^{s+1}$ with $(l; -e_0, -e_1, \ldots, -e_s)$ a $\mathbf{Z}$-basis (here l is the class of the inverse image in S of a line $L \subset \mathbf{P}^2$ and e_i are the classes of the exceptional divisors $E_i \subset S$ corresponding to the points P_i, $0 \le i \le s$). We recall that the intersection form on S is given by:

$$(l \cdot e_i) = 0,\ (e_i^2) = -1,\ i = \overline{0,s},\ (e_i \cdot e_j) = 0,\ (\forall) i \neq j,\ (l^2) = 1.$$

If $\mathcal{L} = al - \sum_{i=0}^{s} b_i e_i \in Pic(S)$, we write $\mathcal{L} = (a; b_0, b_1, \ldots, b_s)$.

If $\mathcal{L} = \mathcal{O}_S(D)$, $D \in Div(S)$, we denote by $[\mathcal{L}] = |D|$ the complete linear system associated to $\mathcal{L}$ (resp. D). We write $[(a; b_0, b_1, \ldots, b_s)] := [a; b_0, b_1, \ldots, b_s]$.

Let's now analyse the Gruson-Peskine type construction given by Ciliberto ([Ci]). The surfaces X_k^n used by Ciliberto, containing the necessary curves ($k = [n/3]_*$) are obtained blowing up $3k - n + 6$ points from $\mathbf{P}^2$ in *general position* (i.e. any 3 noncollinear and not all on a smooth conic) and embedding the abstract surface obtained using the very ample invertible sheaf $(k+2; k, 1^{3k-n+5})$ (we denoted $(a; b, \underbrace{c \ldots, c}_{t \text{ times}})$ by $(a; b, c^t)$). Because $3k - n + 6 \in \{4, 5, 6\}$ there is a well-known criterion giving sufficient conditions for a linear system for containing (smooth, irreducible) curves. Precisely, if $\mathcal{L} = (a;, b_0, b_1, \ldots, b_{s_k^n}) \in Pic(X_k^n)$, where $s_k^n = 3k - n + 5$ and

(3.1.1) $\qquad a \ge b_0 + b_1 + b_2,\ b_0 \ge b_1 \ge \ldots \ge b_{s_k^n} \ge 0,\ a > 0$

then $[\mathcal{L}] = [a; b_0, b_1, \ldots, b_s]$ contains a curve.

Due to this criterion (depending only on coefficients), the construction of curves on the surfaces X_k^n has two parts: an *arithmetical part*, where the necessary degrees and arithmetical genera are realised by using the well-known formulae of genus and degree for $\mathcal{L} \in Pic(X_k^n)$ and a *geometric part*, where the necessary curves are obtained by applying (3.1.1) to the sheaves $\mathcal{L}$ used in order to solve the arithmetical part.

The main contribution of this section to $HC(n)$ is the construction of invertible sheaves (and then curves) having the degree and genus in the domains A_p^n, on some surfaces X_p^n. Precisely, let's denote by $S_p = S_p^n := Bl_{\Sigma_p}(\mathbf{P}^2)$, $\Sigma_p = \Sigma_p^n = \{P_0, P_1, \ldots, P_{s_p^n}\}$, $s_p^n = 3p - n + 5$, Σ_p^n containing general points. Then $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n}) \in Pic(S_p^n)$ is very ample (see section 3.2, Proposition 3.2.2) and then $X_p^n := \text{Im } \varphi_{[\mathcal{H}_p^n]} \subset \mathbf{P}^n$, $\deg X_p^n = n + p - 1$. If $p = k$ one obtains the surfaces X_k^n from [Ci].

If we try to use an argument similar to the argument from [GP 2] for X_p^n, so to divide the proof in two parts, an arithmetical one and a geometrical one (the smoothing) we need a criterion similar to (2.1.1) for an *arbitrary* number of points in Σ_p. But this is, anyway, very complicated by itself (see, for instance, the Hirschowitz conjecture [Hi]). On the other hand, such a general criterion must include the conditions (2.1.1) (*normalisation*, using quadratic transformations); but these conditions allow us to construct curves only for $d >$ *a* function of degree 3/2 in n as it can be seen if we try to apply an argument as in [Ci]. So, in order to produce the necessary curves *for small degrees*, we need a new technique, which we shall explain here.

The main idea is to consider the arithmetical and the geometrical parts of the construction of curves on X_p^n *entirely linked*, i.e. to construct linear systems containing the necessary curves directly and explicitely enough using combinations between some simple sheaves (so, somehow easy to understand them).

Explicitely, we'll proceed as follows (in order to construct the necessary curves from the domains A_p^n on X_p^n): we start with a simple *initial family* $\mathcal{D}_0$ of sheaves

$$(3.1.2) \qquad \mathcal{D}_0 = (a+2; a, 1^t, 0^{s_p^n - t}) \in Pic(X_p^n), \quad 0 \le t \le s_p^n, \ a \in \mathbf{Z}$$

of arithmetical genus a and some degrees, realising the necessary sheaves in some initial intervals in d (like $x_p(d, x) = 0$, for instance - see (2.2) - but not only). The families (3.1.2) are the only ones which are good for the initial intervals (we'll explain this later). After the initial construction we continue using a number of inductive arguments (after $x_p(d, n)$, for instance, but not only), adding repeatingly to $\mathcal{D}_0$ (by tensorisation) some simple (and well understood) invertible sheaves, similar to the class of hyperplane section (see Lemma 3.2.4). Let's explain, shortly, why the inductive processes works. Let's suppose that we need to construct curves on the surfaces X_p^n, of degree d and genus $g = \pi_p(d, n)$ (see (1.1)). Let's suppose that we have constructed the necessary curves for $m_p(d,n)=0$ (see (1.2)). Because

(3.1.3) $\pi_p(d{+}(n{+}p{-}1)) = \pi_p(d){+}(d{+}p{-}1);\ m_p(d{+}(n{+}p{-}1)) = m_p(d){+}1$

in order to construct curves with $(\deg(C), g(C)) = (d, g)$, we proceed as follows: if the curves C have degrees d and genera $g = \pi_p(d,n)$ in the domain $m_p(d,n) = m$, then the curves $C' := C + H_p^n$, $H_p^n \in [p+2; p, 1^{s_p^n}]$ have degrees $d' = d + (n+p-1)$ and genera $g' = \pi_p(d',n)$ in the next domain $m_p(d',n) = m+1$ (use (3.1.3)). Hence, we obtain invertible sheaves for all pairs (d,g), $g = p_a = \pi_p(d,n)$ as far as we succeed to construct these invertible sheaves in the domain $m_p(d,n) = 0$. Moreover, we can test when the associated linear system appearing contain curves, because the appearing linear systems $|D_0 + lH_p^n|$, $l \geq 0$ are simple (specialise the points from Σ_p^n on a rational plane curve of degree $p+1$, having a singular point P_0 with multiplicity p, see Proposition 3.2.2 and Lemma 3.2.4).

In our case, in order to construct the necessary curves from the domains A_p^n it is necessary to change in some way the functions π_p so that the initial verifications (corresponding to $m_p(d,n) = 0$) to be made using sheaves as in (3.1.2) and a property as (3.1.3) to still hold (and some others are appearing, see section 3.3, Lemma 3.3.1). We get in such a way the (unique) functions $\alpha_p(d,n)$ from section 2 (see (2.1)).

Moreover, the method shortly explained here works only if $\#\ \Sigma_p^n = 3p-n+6 \geq 12$ ($\#A$ means the cardinal of the finite set A) so, if $p \geq \dfrac{n}{3}+2$. So, for $\#\Sigma_p^n \leq 11$ we need *another method* in order to construct curves on X_p^n. Because we need such curves for *small degrees* also, the inductive argument is used again, as possible. But in this second method of construction we will be, partially inspired from [GP 2], because in this case a smoothing criterion of (3.1.1) type will be good enough (it is obtained specialising the points from Σ_p^n on a smooth plane cubic curve, see Proposition 3.2.1 and Proposition 3.2.3). In this case, a *suplimentar property* of the function α_p will be necessary, namely that these functions must be related in a "good" way with the genus formula (see section 3.3, Lemma 3.3.2); and this is possible exactly because one uses initial families of type (3.1.2) (so these families are obligatory).

The smoothing criteria used in the methods are collected in section 3.2, in Proposition 3.2.1 and Lemma 3.2.4.

In section 3.4 we'll construct invertible sheaves from $Pic(X_p^n)$ in A_p^n using both methods. In section 3.5 we'll apply the two smoothing criteria to sheaves from section 3.4 in order to get curves. We remark that the domain of applicability of the only inductive method is (R1) and for the other method (using some of Gruson-Peskine ideas) is (R2) where:

$$(3.1.4) \qquad (R1): n \geq 9, \quad \frac{n}{3} + 2 \leq n - 4 \text{ (so } \#\Sigma_p^n \geq 12)$$

$$(3.1.5) \qquad (R2): n \geq 8, \quad \frac{n}{3} \leq p < \frac{n}{3} + 2 \text{ (so } 6 \leq \#\Sigma_p^n \leq 11).$$

In section 3.6 we finish the proof of Main Theorem by briefly comparing our results from section 3.5 with two previous theorems of Ciliberto and Ciliberto-Sernesi.

We end this section remarking that it is possible to see that the constructions from sections 3.4 and 3.5 have *no degree of freedom*, but they cover all the necessary ranges. Let's remark that the method used in (R2) cannot be used in general because the condition (c5) from Proposition 3.2.3 gives curves only for $d >$ a quadratic function in n.

And now, let's systematically analyse the domain D_1^n.

3.2 Two smoothing criteria; the surfaces X_p^n

Let there be $\Sigma \subset \mathbf{P}^2$, $\Sigma = \{P_0, P_1, \ldots, P_s\}$ a set of general points and $S := Bl_\Sigma(\mathbf{P}^2) \xrightarrow{\pi} \mathbf{P}^2$ the blow up of $\mathbf{P}^2$ in Σ.

The first smoothing crierion is ii) from the next Proposition (reformulated in Proposition 3.2.3).

Proposition 3.2.1: *If $\mathcal{D} = (a; b_0, b_1, \ldots, b_s) \in Pic(S)$ is so that $a \geq b_0 \geq b_1 \geq \ldots \geq b_s > 0$, then:*

i) each one of conditions (i.1), (i.2), (i.2)', (i.3), (i.3)' implies $h^1(\mathcal{D}) = 0$, where (i.1): $a - \sum_{l=0}^{s} b_l \geq -1$; (i.2): $2 \leq s \leq 7$, $a \geq \sum_{l=0}^{2} b_l$; $(i.2)' : 2 \leq s \leq 7$, $a - \sum_{l=0}^{2} b_l \geq -1$, $b_0 > b_1$; $(i.3) : s \geq 8$, $a \geq \sum_{l=0}^{2} b_l$, $3a - \sum_{l=0}^{s} b_l \geq 1$; $(i.3)' : s \geq 8$, $a - \sum_{l=0}^{2} b_l \geq -1$, $3a - \sum_{l=0}^{s} b_l \geq 1$, $b_0 > b_1$;

ii) each one of the conditions (ii.1), (ii.2), (ii.2)', (ii.3), (ii.3)' implies $h^0(\mathcal{D}) \neq 0$, $[\mathcal{D}]$ has no base point and contains a (smooth, irreducible) curve, where (ii.1): $a \geq \sum_{l=0}^{s} b_l$; (ii.2): $2 \leq s \leq 6$, $a \geq \sum_{l=0}^{2} b_l$; $(ii.2)' : 2 \leq s \leq 7$,

$a - \sum_{l=0}^{2} b_l \geq -1$, $b_0 \geq b_1 + 2$; *(ii.3)*: $s \geq 7$, $a \geq \sum_{l=0}^{2} b_l$, $3a - \sum_{l=0}^{s} b_l \geq 2$; $(ii.3)'$: $s \geq 8$, $a - \sum_{l=0}^{2} b_l \geq -1$, $3a - \sum_{l=0}^{s} b_l \geq 2$, $b_0 \geq b_1 + 2$.

Remark: This is a Harbourne type result ([Hb 1]).

Proof: A direct proof can be obtained specialising the points from Σ on a smooth plane cubic $\Gamma_0 \subset \mathbf{P}^2$ (general on Γ_0), using induction on s and standard exact sequences. Details are left to the reader (or see [P 2]).

* * *

Proposition 3.2.2: *Let there be $\mathcal{D} = (p+2; p, 1^s) \in Pic(S)$, $p \in Z$, $p \geq 1$. Then: i) if $s \leq 3p+3$, then $h^0(\mathcal{D}) \neq 0$, $[\mathcal{D}]$ has no base point and contains a (smooth, irreducible) curve; ii) if $s \leq 3p$, then $\mathcal{D}$ is very ample.*

Remark: A proof of ii) can be found in [Gi].

Proof: A direct proof can be obtained specialising the points from Σ (except the last one) on a rational irreducible plane curve $\Delta_0 \subset \mathbf{P}^2$ of degree $p+1$ having only one ordinary singularity of multiplicity p : P_0 will be the singular point, $P_1, \ldots, P_{s-1} \in \Delta_0$, $P_s \notin \Delta_0$. Then use standard exact sequences and standard techniques in order to separate points and tangent vectors. Details are left to the reader (or see [P 2]).

* * *

Now, we are ready to *define* the surfaces X_p^n. Precisely, let there be $p \in \mathbf{Z}$, $k \leq p \leq n-4$, $k = [n/3]_*$, $n \in \mathbf{Z}$, $n \geq 5$ and $\Sigma_p = \Sigma_p^n := \{P_0, P_1, \ldots, P_{s_p^n}\} \subset \mathbf{P}^2$ ($s_p^n := 3p - n + 5$) be a set of general points. From Proposition 3.2.2 ii) follows that $\mathcal{H}_p^n := (p+2; p, 1^{s_p^n}) \in Pic(S_p^n)$ ($S_p^n := Bl_{\Sigma_p^n}(\mathbf{P}^2)$) is very ample on S_p^n. Then:

$$X_p^n := \mathrm{Im}\,(\varphi_{[\mathcal{H}_p^n]})$$

X_p^n are rational surfaces with hyperelliptic hyperplane sections and we can easily check (doing computations) that $X_p^n \subset \mathbf{P}^n$ and $\deg(X_p^n) = n + p - 1$. If $p = k$ one obtains the surfaces used in [Ci].

Now we want to reformulate Proposition 3.2.3 (ii) in the Gruson-Peskine coordinates $(d, r; \theta_1, \ldots \theta_{s_p^n})$ ([GP 2], step 2), (this *is possible*). So, for $\mathcal{D} \in Pic(X_p^n) \cong \mathbf{Z} \oplus \mathbf{Z}^{s_p^n+1}$,

$$\mathcal{D} = \mathcal{O}_{X_p^n}(D) = (a; b_0, b_1, \ldots, b_{s_p^n}),\ a \geq b_0 \geq b_1, \geq \ldots \geq b_{s_p^n} \geq 0,$$

for some $D \in Div(X_p^n)$, we consider the change of coordinates (in $Pic(X_p^n) \otimes$

$\mathbb{Q}$):

(3.2.1) $$r := a - b_0, \ \theta_i := \frac{1}{2}r - b_i, \quad i = \overline{1, s_p^n}.$$

If $d = \deg D := (D \cdot H_p^n)$ $(H_p^n \in [\mathcal{H}_p^n])$ and $g = p_a(D)$, using the genus formula one obtains:

(3.2.2) $$d = \frac{n+p-5}{2}r + 2a + \sum_{i=1}^{s_p^n} \theta_i; \quad g = F_d(r) - \frac{1}{2}\sum_{i=1}^{s_p^n} \theta_i^2$$

(3.2.3) $$F_d(r) = F_d^{p,n}(r) := \frac{1}{2}\left[d(r-1) + (p-1)r - \frac{n+p-1}{4}r^2\right] + 1$$

Proposition 3.2.1 ii) can be reformulated now as:

Proposition 3.2.3: *Let there be*

$$\mathcal{D} = \mathcal{O}_{X_p^n}(D) \in Pic(X_p^n), \quad \mathcal{D} = (a; b_0, b_1, \ldots, b_{s_p^n}).$$

If the conditions (c1), (c2), (c3), (c4), (c5) are simultaneously satisfied by $\mathcal{D}$ *in the coordinates* $(d, r; \theta_1, \ldots, \theta_{s_p^n})$, *then* $[\mathcal{D}] = |D| \neq \emptyset$, *has no base point and contains a (smooth, irreducible) curve, where:*

(c1) $$\theta_i \equiv \frac{1}{2}r \ (\mathrm{mod}\ 1), \quad i = \overline{1, s_p^n};$$

(c2) $$d + \frac{1}{2}(p - n + 5) - \sum_{i=1}^{s_p^n} \theta_i \equiv 0 \ (\mathrm{mod}\ 2);$$

(c3) $$|\theta_1| \leq \theta_2 \leq \ldots \leq \theta_{s_p^n} \leq \frac{r}{2};$$

(c4) $$-\theta_1 + \sum_{i=2}^{s_p^n} \theta_i \leq d - \frac{1}{2}(n - p + 1)r;$$

(c5) $$d \geq (p-1)r + 2.$$

Proof: Left to the reader (similar to the case $n = 3$, see [Ha 2], [GP 2]).

* * *

We end this section with a technical lemma (which represents the second smoothing criterion), essential in section 3.5.

Lemma 3.2.4: *If $p, n \in Z$, $n \geq 3$ and $p \geq \frac{n}{3} + 2$, let's consider the following invertible sheaves from $Pic(X_p^n)$:*

$$\mathcal{H}_1 = \mathcal{H}_p^n := (p+2; p, 1^{s_p^n}),\ \mathcal{H}_2 = \tilde{\mathcal{H}}_p^n := (p+2; p, 1^{s_p^n-1}, 0),\ \mathcal{H}_3' = (\tilde{H}_{p-1,1}^{n+1})' :=$$

$$= (p+1; p-1, 1^{s_p^n-5}, 0^2, 1^2, 0),\ \mathcal{H}_4' = (\tilde{H}_{p-1,2}^{n+1})' := (p+1; p-1, 1^{s_p^n-7}, 0^2, 1^4, 0),$$

$$\mathcal{H}_5' = (\tilde{H}_{p-1,3}^{n+1})' := (p+1; p-1, 1^{s_p^n-9}, 0^2, 1^6, 0),$$

$$\mathcal{H}_6' = (\tilde{H}_{p-1,4}^{n+1})' := (p+1; p-1, 1^{s_p^n-11}, 0^2, 1^8, 0).$$

If $t_1, t_2, \ldots, t_6 \in Z$, $t_1, \ldots, t_6 \geq 0$, let there be

$$(3.2.4)\quad \mathcal{D} := \mathcal{D}_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}_3' + t_4\mathcal{H}_4' + t_5\mathcal{H}_5' + t_6\mathcal{H}_6' \in Pic(X_p^n)$$

where $\mathcal{D}_0 := (a+2; a, \nu_1, \nu_2, \ldots, \nu_{s_p^n})$, $\nu_j \in \{0,1\}$, $j = \overline{1, s_p^n}$ and $\nu_j = 1$ for u values of the index $j (0 \leq u \leq s_p^n)$. Let's denote by $t := t_1 + t_2 + \ldots + t_6$. Then, the condition

$$(3.2.5)\qquad 3a \geq u + n - 7 - (t+1)(n-4)$$

implies $[\mathcal{D}] \neq \emptyset$, $[\mathcal{D}]$ without base points and contains a (smooth, irreducible) curve.

Proof: Write $\mathcal{D} = t_1\mathcal{D}_{x_1} + t_2\mathcal{D}_{x_2} + t_3\mathcal{D}_{x_3} + t_4\mathcal{D}_{x_4} + t_5\mathcal{D}_{x_5} + t_6\mathcal{D}_{x_6} + \mathcal{R}$ for any $x_1, x_2, \ldots, x_6 \in \mathbf{Z}$, where $\mathcal{D}_{x_1} := (x_1+2; x_1, 1^{s_p^n})$, $\mathcal{D}_{x_2} := (x_2+2; x_2, 1^{s_p^n-1}, 0)$, $\mathcal{D}_{x_3} := (x_3+2; x_3, 1^{s_p^n-5}, 0^2, 1^2, 0)$, $\mathcal{D}_{x_4} := (x_4+2; x_4, 1^{s_p^n-7}, 0^2, 1^4, 0)$, $\mathcal{D}_{x_5} := (x_5+2; x_5, 1^{s_p^n-9}, 0^2, 1^6, 0)$, $\mathcal{D}_{x_6} := (x_6+2; x_6, 1^{s_p^n-11}, 0^2, 1^8, 0)$, $\mathcal{R} := (b+2; b, \nu_1, \nu_2, \ldots, \nu_{s_p^n})$ where $b := a + t(p-1) + (t_1+t_2) - \sum_{i=1}^{6} t_i x_i$.

Take then $x_1, \ldots, x_6 \geq 0$ *minimal* so that Proposition 3.2.2 i) applies for $\mathcal{D}_{x_1}, \ldots, \mathcal{D}_{x_6}$. We deduce that

$$(3.2.6)\qquad 3p - n + 2 \leq 3x_1 \leq 3p - n + 4$$

$$(3.2.7)\qquad 3p - n + 1 \leq 3x_2 \leq 3p - n + 3$$

$$(3.2.8)\qquad 3p - n - 1 \leq 3x_i \leq 3p - n + 1,\ i = \overline{3, 6}.$$

In order to obtain the conclusion from the lemma for $[\mathcal{D}]$, it follows that it's enough to have the same for $[\mathcal{R}]$ so, by Proposition 3.2.2 i) we need

$$(3.2.9) \qquad u \leq 3b+3,\ b = a + t(p-1) + (t_1+t_2) - \sum_{i=1}^{6} t_i x_i$$

Replacing b, (3.2.9) becomes $u \leq 3a+3t(p-1)+3(t_1+t_2)-\sum_{i=1}^{6} t_i(3x_i)+3$, or $\sum_{i=1}^{6} t_i(3x_i) \leq 3a+3t(p-1)+3(t_1+t_2)-u+3$. Using (3.2.6), (3.2.7), (3.2.8), we can see that this last inequality is implied by $t(3p-n+1)+3t_1+2t_2 \leq 3a+3t(p-1)+3(t_1+t_2)-u+3 \Leftrightarrow 3a \geq u+n-7-(t+1)(n-4)-t_2$. Because $t_2 \geq 0$, this last inequality follows from (3.2.5).

* * *

Corollary 3.2.5: *In the same hypotheses as in Lemma 3.2.4, the same conclusion holds for* $[\mathcal{D}]$ *if*

$$(3.2.10) \qquad 3a \geq 3p-(t+1)(n-4)-2.$$

Proof: Give to u the biggest value $u = s_p^n = 3p-n+5$ in (3.2.5)

3.3 Numerical properties of the functions $\alpha_p(d,n)$

We recall that the functions $\alpha_p = \alpha_p(d,n)$ were defined in section 2 ((2.1)-(2.5)). Let it be now

$$(3.3.1) \qquad \alpha'_p(d,n) := \alpha_{p-1}(d+1,n+1).$$

We prove now the following *key lemma:*

Lemma 3.3.1:

a) *Let's denote by* $d' := d+(n+p-1)$. *Then:*

$$x_p(d',n) = x_p(d,n)+1;\ t_p(d',n) = t_p(d,n);\ u_p(d',n) = u_p(d,n);$$

$$\alpha_p(d',n) = \alpha_p(d,n)+(d+p-1)(\Leftrightarrow \alpha_p(d-(n+p-1),n) = \alpha_p(d,n)-d+n);$$

the same for α'_p.

b)
$$\alpha_{p+1}(d-1,n) \leq \alpha_p(d,n),\quad (\forall)d \geq a_p^n+1,\ d \in \mathbb{Z};$$
$$\alpha_{p+1}(d,n) \leq \alpha_p(d,n),\quad (\forall)d \geq a_p^n+n+p,\ d \in \mathbb{Z}.$$

c) $x_{p+1}(d,n) = x_p(d+1,n+1)$; $t_{p+1}(d,n) = t_p(d+1,n+1) - 1$;

$$\alpha_p(d+1,n+1) = \alpha'_{p+1}(d,n) \in \{\alpha_{p+1}(d,n) - 1,\ \alpha_{p+1}(d,n)\}.$$

Proof: b) We will show first that

$$(3.3.2) \qquad \alpha_{p+1}(d-1,n) \le \alpha_{p+1}(d,n), \quad (\forall) d \ge a^n_{p+1} + 1,\ d \in \mathbf{Z}.$$

We'll use *induction* on $x_{p+1}(d-1,n)$, using a). Let's remark that

$$(3.3.3) \quad \begin{cases} x_{p+1}(d-1,n) = 0 \Rightarrow \alpha_{p+1}(d-1,n) = [(d-n+p)/2]_* \\ x_{p+1}(d-1,n) = 0 \Rightarrow \alpha_{p+1}(d,n) = \begin{cases} [(d-n+p+1)/2]_*\,, & d \ne a^n_{p+1} + (n+p) \\ [(3d-4n+1)/2]_*\,, & d = a^n_{p+1} + (n+p) \end{cases} \end{cases}$$

So, $x_{p+1}(d-1,n) = 0 \Rightarrow \alpha_{p+1}(d-1,n) \le \alpha_p(d,n)$ (doing some computations for $d = a^n_{p+1} + (n+p)$).

Suppose now that we have proved (3.3.2) for $x_{p+1}(d-1,n) = x$. Let d' be so that $x_{p+1}(d'-1,n) = x+1$. Put $d' := d' - (n+p)$. Using a) we deduce that $x_{p+1}(d-1,n) = x$. Then $\alpha_{p+1}(d',n) - \alpha_{p+1}(d'-1,n) = \alpha_{p+1}(d+(n+p),n) - \alpha_{p+1}(d-1+(n+p),n) =$ (use a) again) $(\alpha_{p+1}(d,n) - \alpha_{p+1}(d-1,n)) + 1 \ge 1 > 0$ (we used the induction hypothesis). Now (3.3.2) is proved.

We'll now prove the first inequality from b) using induction on $x_p(d-1,n)$. We have

$$(3.3.4) \quad x_p(d-1,n) = 0 \Rightarrow \alpha_p(d,n) = \begin{cases} [(d-n+p)/2]_*, & d \ne a^n_p + (n+p-1) \\ [(3d-4n+1)/2]_*, & d = a^n_p + (n+p-1). \end{cases}$$

It can be seen that

$$(3.3.5) \quad x_p(d-1,n) = 0 \Rightarrow [(d-n+p)/2]_* \le \alpha_p(d,n) \le [(d-n+p+1)/2]_*.$$

Because $x_p(d-1,n) = 0 \Rightarrow x_{p+1}(d-1,n) = 0$, from (3.3.3) we now deduce the first inequality from b) if $x_p(d-1,n) = 0$.

Suppose now that the inequality holds for $x_p(d-1,n) = x$ and let d' be so that $x_p(d'-1,n) = x+1$. Put $d := d' - (n+p-1)$. Then $x_p(d-1,n) = x$ and we have: $\alpha_p(d',n) - \alpha_{p+1}(d',n) = \alpha_p(d+(n+p-1),n) - \alpha_{p+1}(d-1+(n+p),n) =$ (use a)) $\alpha_p(d,n) - \alpha_{p+1}(d-1,n) \ge 0$ (induction hypotheses). So $\alpha_p(d',n) \ge \alpha_{p+1}(d',n)$. Using now (3.3.2) we get the first inequality b).

The second equality b) comes from the first one, as before.

* * *

Lemma 3.3.2 (for $F_d^{p,n}(r)$ see (3.2.3):
a) $F_{d+(n+p-1)}^{p,n}(r+2) = F_d^{p,n}(r) + (d+p-1)$;

b) i) $\alpha_p(d,n) = \left[F_d^{p,n}(2(x_p(d,n)+1)) - \dfrac{1}{2}\right]_*$;

ii) $\alpha_{p+1}(d-1,n) = [F_d^{p,n+1}(2(x_{p+1}(d-1,n)+1))]_*$.

Proof: b) Use a), Lemma 3.3.1 a) and induction on $x_p(d,n)$ (and $x_{p+1}(d-1,n)$, respectively).

* * *

Remark 3.3.3: The previous lemma shows us that the functions $\alpha_p(d,n)$ are related in a "good" way to the genus formula (see (3.2.2)) and this will be very important in sections 3.4 and 3.6.

3.4 Invertible sheaves from $Pic(X_p^n)$ in the domains $A_p^n(n \geq 5,\ p \geq \frac{n}{3})$

We recall that the domains A_p^n, $p \geq n/3$, $p \in \mathbf{Z}$ from (d,g)-plane were defined in §2, (2.11). Here we consider, for $p \geq n/3$, $p \in \mathbf{Z}$ the domains $\bar{A}_p^n$:

(3.4.1) $$\bar{A}_p^n : \alpha_{p+1}(d-1,n) \leq g \leq \alpha_p(d,n),\ d \geq a_p^n + 1,\ d,g \in \mathbf{Z}$$

(for the definition of a_p^n see (2.3)). Obviously

(3.4.2) $$\bar{A}_p^n \supset A_p^n, \quad (\forall)\ p \geq n/3,\ (\forall)\ n \geq 3,\ n \in \mathbf{Z}.$$

In this section we'll prove the following

Proposition 3.4.1: *Let there be $n \geq 5$, $n \in \mathbf{Z}$ and $(d,g) \in \bar{A}_p^n$, $p \geq n/3$. Then there is $\mathcal{D} \in Pic(X_p^n)$ such that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d,g)$.*

We recall that $\deg \mathcal{D} := (\mathcal{D} \cdot \mathcal{H}_p^n)$, where $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n}) \in Pic(X_p^n)$.
Proof: This is long and will be divided in 4 steps:
Step 1: Let there be $d,g \in \mathbf{Z}$, $d \geq a_{p+1}^n + 1$ so that

$$\alpha'_{p+1}(d-1,n) \leq g \leq \beta_{p+1}(d,n) - x_{p+1}(d,n),$$

where

$$\beta_p(d,n) = \begin{cases} \alpha'_p(d,n), & d \not\equiv a_p^n (\mathrm{mod}\ (n+p-1)) \\ \alpha_p(d,n), & d \equiv a_p^n (\mathrm{mod}\ (n+p-1)) \end{cases} \quad \text{(for } \alpha'_p \text{ see (3.3.1))}.$$

Then, there is $\mathcal{D} \in Pic(X_p^n)$ so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g)$.

Step 2: Let there be $d, g \in \mathbb{Z}$, $d \geq a_p^{n+1}$ so that $\alpha_p'(d, n+1) - x_p(d, n+1) \leq g \leq \alpha_p(d, n+1)$. Then there is $\mathcal{D} \in Pic(X_p^n)$ so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g)$.

Step 3: Let there be $d, g \in \mathbb{Z}$, $d \geq a_p^{n+1} - 1 = a_{p+1}^n$ so that $\alpha_{p+1}'(d, n) - x_{p+1}(d, n) \leq g \leq \alpha_{p+1}'(d, n)$. Then there is $\mathcal{D} \in Pic(X_p^n)$ so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g)$.

Step 4: The statement of Proposition 3.4.1.

Proof of Step 1: We use induction on $x_{p+1}(d-1, n)$. If $x_{p+1}(d-1, n) = 0$, we have:

$$[(d-n+p-1)/2]_* = \alpha_{p+1}'(d-1, n) \leq \beta_{p+1}(d, n) - x_{p+1}(d, n) =$$

$$(3.4.3) \qquad = \begin{cases} [(d-n+p)/2]_*, & d \neq a_{p+1}^n + (n+p) \\ [(3d-4n-1)/2]_*, & d = a_{p+1}^n + (n+p). \end{cases}$$

We consider the following invertible sheaves $\mathcal{D}_0 \in Pic(X_p^n)$:

$$(3.4.4) \qquad \mathcal{D}_0 = (g+2; g, 1^u, 0^{s_p^n - u}),\ 0 \leq u \leq s_p^n = 3p - n + 5.$$

Then $g = p_a(\mathcal{D}_0)$ and, if $d := \deg \mathcal{D}_0 = (\mathcal{D}_0 \cdot \mathcal{H}_p^n)$ it result that $g = (d - 2p + u - 4)/2$. Taking now

$$(3.4.5) \qquad u = 3p - n + 4,\ 3p - n + 3,\ 3p - n + 2 \geq 0$$

and using (3.4.3), one obtains invertible sheaves $\mathcal{D}_0 \in Pic(X_p^n)$ *having the last component zero* and degree and arithmetical genus in the range from Step 1, for $x_{p+1}(d-1, n) = 0$. Because

$$(3.4.6) \qquad \begin{cases} \alpha_{p+1}'(d-1+(n+p), n) = \alpha_{p+1}'(d-1, n) + (d+p-1) \\ (\beta_{p+1} - x_{p+1})(d+(n+p), n) = (\beta_{p+1} - x_{p+1})(d, n) + (d+p-1) \end{cases}$$

(see Lemma 3.3.1 a)) we obtain from (3.4.3) by induction on $x_{p+1}(d-1, n)$ the inequality $\alpha_{p+1}'(d-1, n) \leq \beta_{p+1}(d, n) - x_{p+1}(d, n)$, $(\forall) d \geq a_{p+1}^n + 1$.

Let there be $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n - 1}, 0) \in Pic(X_p^n)$. It is easy to check that, if $\mathcal{D} \in Pic(X_p^n)$ *has the last component zero*, then

$$(3.4.7) \qquad \begin{cases} \deg(\mathcal{D} + \bar{\mathcal{H}}_p^n) = d + (n+p), & d = \deg \mathcal{D} \\ p_a(\mathcal{D} + \bar{\mathcal{H}}_p^n) = g + (d+p-1), & g = p_a(\mathcal{D}) \end{cases}$$

Now, using (3.4.6) and (3.4.7) it follows that we cover the range from Step 1, adding to the sheaves $\mathcal{D}_0$, used for $x_{p+1}(d-1,n)=0$, succesively, the sheaf $\bar{\mathcal{H}}_p^n$ (at each addition, $x_{p+1}(d-1,n)$ increase by 1, see Lemma 3.3.1 a)).

Remark 3.4.2: *The invertible sheaves used in order to cover the domain from Step 1 are of the form:* $\mathcal{D}=\mathcal{D}_0'+t_2\mathcal{H}_2$ *(see Lemma 3.2.4), where* $\mathcal{H}_2=\bar{\mathcal{H}}_p^n=(p+2;p,1^{s_p^n-1},0), t_2\in Z,\ t_2\geq 0,\ \mathcal{D}_0'=(a+2;a,1^u,0^{s_p^n-u})$, $u\in\{3p-n+2,\ 3p-n+3,\ 3p-n+4\}$, $a\in\{(d_0'-n+p-2)/2,\ (d_0'-n+p-1)/2,\ (d_0'-n+p)/2\}\cap Z,\ d_0'=(\mathcal{D}_0'\cdot\mathcal{H}_p^n)\geq a_{p+1}^n+1,\ d_0'\leq a_{p+1}^n+(n+p)$ *(see (3.4.4), (3.4.5) and the construction.)*

Proof of Step 2: This step is the most difficult. We'll give two constructions, necessary in a complementary way in section 3.5.

Construction A (works for $\#\Sigma_p^n\geq 12$, hence $p\geq\frac{n}{3}+2$):

1) We'll prove here the existence of $\mathcal{D}\in Pic(X_p^{n+1})$ of degree d and arithmetical genus g for $g=\alpha_p(d,n+1)$ and any $d\geq a_p^{n+1}$. We recall that $X_p^{n+1}\subset \mathbf{P}^{n+1}$.

Let there be $\mathcal{H}_p^{n+1}=(p+2;p,1^{s_p^{n+1}})\in Pic(X_p^{n+1})$. Because $\deg(X_p^{n+1})=n+p$, it can be seen that, for any $\mathcal{D}\in Pic(X_p^{n+1})$

$$(3.4.8)\qquad \begin{cases}\deg(\mathcal{D}+\mathcal{H}_p^{n+1})=d+(n+p), & d=\deg(\mathcal{D})\\ p_a(\mathcal{D}+\mathcal{H}_p^{n+1})=g+(d+p-1), & g=p_a(\mathcal{D}).\end{cases}$$

Using (3.4.8) and Lemma 3.3.1 a), it follows that we can use the same argument as in the proof of Step 1, using induction on $x_p(d,n+1)$. So, we check the existence for $x_p(d,n+1)=0$ and we add succesively the class of hyperplane section of X_p^{n+1}, namely $\mathcal{H}_p^{n+1}$. If $x_p(d,n+1)=0$, then $\alpha_p(d,n+1)=[(d-n+p-1)/2]_*$. So, we cover this initial range with sheaves $\mathcal{D}_0\in Pic(X_p^{n+1})$ of the form

$$(3.4.9)\qquad \mathcal{D}_0=(g+2;g,1^u,0^{s_p^{n+1}-u}),\quad g=[(d-n+p-1)/2]_*,$$

with

$$(3.4.10)\qquad u=3p-n+2=s_p^{n+1}-2\quad\text{or}\quad u=3p-n+3=s_p^{n+1}-1.$$

Let's *remark* that the sheaves used here are of the form

$$(3.4.11)\quad (t+2l;t,l^{s_p^{n+1}-2},l-1\text{ or }l,l-1),\ \text{where } l=x_p(d,n+1)+1\in \mathbf{Z},\ l\geq 1.$$

2) Now we consider $\mathcal{H}_3 = \tilde{H}^{n+1}_{p-1,1} = (p+1; p-1, 1^{3p-n}, 0^2, 1^2) \in Pic(X^{n+1}_p)$. Then

$$(3.4.12) \qquad p_a(\tilde{H}^{n+1}_{p-1,1}) = p-1, \quad \deg(\tilde{H}^{n+1}_{p-1,1}) = n+p.$$

In the beginning we'll prove

$$(3.4.13) \qquad \begin{cases} \text{If } g = \alpha_p(d, n+1) - (x_p(d,n+1) - x)^2,\ x_p(d,n+1) \geq x,\ x \in \mathbf{Z},\ x \geq 0 \\ \text{then } (\exists)\mathcal{D} \in Pic(X^{n+1}_p) \text{ so that } (\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g). \end{cases}$$

We'll use induction on $x_p(d, n+1) \geq x$. In the induction process we'll need the following $\varepsilon(\mathcal{D})$:

$$(3.4.14) \qquad \begin{cases} \text{If } \mathcal{D} = (a; b_0, b_1, \ldots, b_{3p-n+4}) \in Pic(X^{n+1}_p), \\ \text{then } \varepsilon(\mathcal{D}) := a - b_0 - b_{3p-n+1} - b_{3p-n+2}. \end{cases}$$

(the indices $3p - n + 1$ and $3p - n + 2$ correspond to the two consecutive zeros in $\mathcal{H}_3$, so that $\varepsilon(\mathcal{H}_3) = 2$).

If $x_p(d, n+1) = x$, (3.4.13) is just 1). If $x_p(d, n+1) = x$, let's denote by $\mathcal{D}_x$ an invertible sheaf satisfying (3.4.13). Then $\varepsilon(\mathcal{D}_x) = 0$ $(= 2(x_p(d, n+1) - x))$, by (3.4.11). Actually, we'll prove by induction on $x_p(d, n+1) \geq x$ the following statement (stronger than (3.4.13)):

$$(3.4.15) \qquad \begin{cases} \text{If } g = \alpha_p(d, n+1) - (x_p(d,n) - x)^2,\ x_p(d, n+1) \geq x, \\ x \in \mathbf{Z},\ x \geq 0, \text{ then } (\exists)\mathcal{D} \in Pic(X^{n+1}_p) \text{ so that} \\ (\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g) \text{ and } \varepsilon(\mathcal{D}) = 2(x_p(d, n+1) - x). \end{cases}$$

We have just verified (3.4.15) for $x_p(d, n+1) = x$. In order to finish the inductive process, let's suppose that we have constructed invertible sheaves $\mathcal{D}_{x+t}$ having $(\deg(\mathcal{D}_{x+t}),\ p_a(\mathcal{D}_{x+t})) = (d, g)$ in the situation $(t \in \mathbf{Z},\ t \geq 0)$:

$$(3.4.16) \qquad \begin{cases} x_p(d, n+1) = x + t \\ g = \alpha_p(d, n+1) - t^2 = \alpha_p(d, n+1) - (x_p(d, n+1) - x)^2 \\ \text{so that } \varepsilon(\mathcal{D}_{x+t}) = 2t = 2(x_p(d, n+1) - x). \end{cases}$$

We need $\mathcal{D}_{x+t+1} \in Pic(X^{n+1}_p)$ to do the same job for the next range in $d(x_p(d', n+1) = x + t + 1)$. Or, using the induction hypothesis and Lemma 3.3.1 a) we can see that the sheaves $\mathcal{D}_{x+t+1} := \mathcal{D}_{x+t} + \tilde{H}^{n+1}_{p-1,1}$ are the good ones. Indeed:

$$\deg(\mathcal{D}_{x+t+1}) = d + (n+p) := d'; \ x_p(d', n+1) = x + t + 1.$$

$$p_a(\mathcal{D}_{x+t+1}) = p_a(\mathcal{D}_{x+t}) + (p-1) + (\mathcal{D}_{x+t}\cdot \tilde{H}^{n+1}_{p-1,1}) - 1 = p_a(\mathcal{D}_{x+t}) + (p-1) +$$
$$+\deg(\mathcal{D}_{x+t}) - \varepsilon(\mathcal{D}_{x+t}) - 1 = \text{(use the induction hypothesis)}$$
$$\alpha_p(d,n+1) - (x_p(d,n+1)-x)^2 + (p-1) + d - 2(x_p(d,n+1)-x) - 1 =$$
$$= \alpha_p(d,n+1) - (x_p(d,n+1)-x)^2 - 2(x_p(d,n+1)-x) + d + p - 2.$$

We replace now d with $d' - (n+p)$ and we use Lemma 3.3.1 a). We obtain

$$p_a(\mathcal{D}_{x+t+1}) = \alpha_p(d'-(n+p), n+1) - (x_p(d'-(n+p), n+1) - x)^2 -$$
$$-2(x_p(d'-(n+p), n+1) - x) + d' - (n+p) + p - 2 = \alpha_p(d', n+1) -$$
$$-d' + n + 1 - (x_p(d', n+1) - 1 - x)^2 - 2(x_p(d', n+1) - 1 - x) + d' - n - 2 =$$
$$= \alpha_p(d', n+1) - (x_p(d', n+1) - x)^2 + 2(x_p(d', n+1) - x) - 1 -$$
$$-2(x_p(d', n+1) - x) + 2 - 1 = \alpha_p(d', n+1) - (x_p(d', n+1) - x)^2.$$

Moreover, $\varepsilon(\mathcal{D}_{x+t+1}) = \varepsilon(\mathcal{D}_{x+t}) + 2 = 2(x_p(d', n+1) - 1)$. Now, the inductive process is finished. So (3.4.15) (hence (3.4.13) also) is proved.

Now, let a be *arbitrary*, $a \in Z$, $a \geq 0$ and put $x := a^2 - a \geq 0$. We obtain by (3.4.13) elements from $Pic(X^{n+1}_p)$ of degree d and arithmetical genus $p_a = \alpha_p(d,n+1) - (x_p(d,n+1) - a^2 + a)^2$ for *any* d so that $x_p(d,n+1) \geq a^2 - a$. Take now d so that $x_p(d,n+1) = a^2 (\geq x = a^2 - a)$. One obtains invertible sheaves of degree d and $p_a = \alpha_p(d,n+1) - a^2$ for $x_p(d,n+1) = a^2$. Now, adding succesively $\mathcal{H}^{n+1}_p$ and using Lemma 3.3.1 a), we obtain invertible sheaves of degree d and arithmetical genus g in the domain (d,g) defined by

$$(a) \qquad g = \alpha_p(d,n+1) - a^2, \quad x_p(x,n+1) \geq a^2.$$

3) Now, start again with the invertible sheaves covering the domain (a), from 2). Using the invertible sheaf $\mathcal{H}_4 = \tilde{H}^{n+1}_{p-1,2} = (p+1; p-1, 1^{3p-n-2}, 0^2, 1^4) \in$ $\in Pic(X^{n+1}_p)$ (instead of $\tilde{H}^{n+1}_{p-1,1}$) and using (3.4.11), we obtain, as before, invertible sheaves covering by degrees and arithmetical genera the domain from the (d,g)-plane defined by

$$g = \alpha_p(d,n+1) - a^2 - (x_p(d,n+1) - y)^2, \; x_p(d,n+1) \geq y$$

for any $y \geq a^2$, $y \in \mathbf{Z}$.

Take now $b \in \mathbf{Z}$, $b \geq 0$ *arbitrary* and put $y = a^2 + b^2 - b$ ($\geq a^2$). It follows that we obtained invertible sheaves from $Pic(X_p^{n+1})$ covering by degrees and arithmetical genera the domain

$$\begin{cases} g = \alpha_p(d, n+1) - a^2 - (x_p(d, n+1) - a^2 - b^2 + b)^2 \\ x_p(d, n+1) \geq a^2 + b^2 - b \end{cases}$$

Take now d such that $x_p(d, n+1) = a^2 + b^2$ ($\geq y = a^2 + b^2 - b$). We obtain invertible sheaves of degree d and arithmetical genus g for $g = \alpha_p(d, n+1) - (a^2 + b^2)$. Adding succesively $\mathcal{H}_p^{n+1}$ we get invertible sheaves covering by degrees and arithmetical genera the domain

$$(b) \qquad g = \alpha_p(d, n+1) - (a^2 + b^2), \quad x_p(d, n+1) \geq a^2 + b^2.$$

4), 5) Continuing for another two times as for (a) and (b), using (3.4.11) (and $s_p^{n+1} \geq 10$, because $p \geq \frac{n}{3}+2$) and the invertible sheaves $\mathcal{H}_5 = \tilde{H}_{p-1,3}^{n+1} = (p+1; p-1, 1^{3p-n-4}, 0^2, 1^6)$ and $\mathcal{H}_6 = \tilde{H}_{p-1,4}^{n+1} = (p+1; p-1, 1^{3p-n-6}, 0^2, 1^8)$, $\mathcal{H}_5$, $\mathcal{H}_6 \in Pic(X_p^{n+1})$, one obtains sheaves from $Pic(X_p^{n+1})$ covering by degrees and arithmetical genera the domain (d, g) given by

$$(e) \quad g = \alpha_p(d, n+1) - (a^2 + b^2 + c^2 + e^2), \quad x_p(d, n+1) \geq a^2 + b^2 + c^2 + e^2$$

for $a, b, c, e \in \mathbf{Z}$, $a, b, c, e \geq 0$ *arbitrary*.

6) The domain (e) is exactly the domain from Step 2, because any positive integer can be written as a sum of 4 squares of positive integers. Indeed, let there be $g \in \mathbf{Z}$ so that $\alpha_p(d, n+1) - x_p(d, n+1) \leq g \leq \alpha_p(d, n+1)$. Write $f = a^2 + b^2 + c^2 + e^2$, $a, b, c, e \in \mathbf{Z}$, $a, b, c, e \geq 0$ for $f := \alpha_p(d, n+1) - g$, $0 \leq f \leq x_p(d, n+1)$. Hence

$$\begin{cases} g = \alpha_p(d, n+1) - (a^2 + b^2 + c^2 + e^2) \; (= \alpha_p(d, n+1) - f) \\ x_p(d, n+1) \geq a^2 + b^2 + c^2 + e^2 \end{cases}$$

Because the domain (e) is covered by degrees and arithmetical genera corresponding to invertible sheaves from $Pic(X_p^{n+1})$ for *any* $a, b, c, e \in \mathbf{Z}$, $a, b, c, e \geq 0$, we deduce the existence of $\mathcal{D} \in Pic(X_p^{n+1})$ so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g)$.

Construction B (works for $\#\Sigma_p^n \geq 6$, hence $p \geq \frac{n}{3}$): because

$$(3.4.17) \quad \alpha_p(d, n+1) = \left[F_d^{p,n+1}(2(x_p(d, n+1) + 1) - \frac{1}{2}\right]_* \quad \text{(Lemma 3.3.2 b)i)}$$

we can use the Gruson-Peskine coordinates $(d, r; \theta_1, \theta_2, \dots, \theta_{s_p^{n+1}})$ where $r = a - b_0$, $\theta_i = \frac{1}{2}r - b_i$, $i = \overline{1, s_p^{n+1}}$ for $\mathcal{D} = (a; b_1, \dots, b_{s_p^{n+1}}) \in Pic(X_p^{n+1})$ (see (3.2.1)). Step 2 will be proved as far as for any (d, g) from the domain of Step 2 we'll prove the existence of a sheaf $\mathcal{D} \in Pic(X_p^{n+1})$ with $\deg \mathcal{D} = d$, such that, in the coordinates $(d, r; \theta_1, \dots, \theta_{s_p^{n+1}})$, $\mathcal{D}$ satisfies the conditions (c1) and (c2) from Proposition 3.2.3 and

$$F_d^{p,n+1}(r) - \frac{1}{2}\sum_{i=1}^{s_p^{n+1}} \theta_i^2 = g. \tag{3.4.18}$$

Indeed, the left member of (3.4.18) is just $p_a(\mathcal{D})$ (see (3.2.2)), hence $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d, g)$. Moreover, it is necessary, performing the transformation of coordinates of $\mathcal{D}$ from $(d, r; \theta_1, \dots, \theta_{s_p^{n+1}})$ to $(a; b_1, \dots, b_{s_p^{n+1}})$ that these last ones be integers. But (c1) means $b_i \in \mathbf{Z}$ (see (3.2.2)); then $b_0 \in \mathbf{Z}$, because $r \in \mathbf{Z}$.

Now, we'll prove the existence of such $\mathcal{D}$ covering the domain from Step 2 by degree and arithmetical genus. Using (3.4.17), take

$$r := 2(x_p(d, n+1) + 1). \tag{3.4.19}$$

Let there be $b \in \mathbf{Z}$ so that

$$1 \le b \le r = 2(x_p(d, n+1) + 1). \tag{3.4.20}$$

$$\begin{cases} \text{Write } b = \sum_{i=1}^{4} c_i^2,\ c_i \in \mathbf{Z},\ c_1 \ge c_2 \ge c_3 \ge c_4 \ge 0. \text{ Then take} \\ \theta_{s_p^{n+1}-j+1} := c_j,\ j = \overline{1,4},\ \theta_i = 0,\ i = \overline{1, s_p^{n+1} - 4}. \end{cases} \tag{3.4.21}$$

(recall that $s_p^{n+1} \ge 4$, because $p \ge n/3$).

Given r as in (3.4.19) and θ_i as in (3.4.21), $\mathcal{D}$ is determined by these numbers and d, performing the inverse transformation (3.2.1), with $d = \deg \mathcal{D}$.

We have $p_a(\mathcal{D}) = F_d^{p,n+1}(r) - \frac{1}{2}\sum_{i=1}^{s_p^{n+1}} \theta_i^2 = F_d^{p,n+1}(r) - \frac{1}{2}b$ (from (3.2.2) and (3.4.21)). Because b moves in the range (3.4.20), it follows from (3.4.17) that we get the necessary invertible sheaves for Step 2 as far as we check (c1) and (c2). Now, (c1) is clear, because $r \in 2\mathbf{Z}$ and $\theta_i \in \mathbf{Z}$. As for (c2),

$g = p_a(\mathcal{D}) \in \mathbf{Z}$ and $p_a(\mathcal{D}) = F_d^{p,n+1}(r) - \frac{1}{2}\sum_{i=1}^{s_p^{n+1}} \theta_i^2$; so $2F_d^{p,n+1}(r) - \sum_{i=1}^{s_p^{n+1}} \theta_i^2 \equiv 0$ (mod 2). Because $f^2 \equiv f$ (mod 2) if $f \in \mathbf{Z}$ and because $r \in 2\mathbf{Z}$, we can see that the left member of the above congruence is congruent (mod 2) with the left member of (c2).

Remark 3.4.3: *The Construction B is, obviously, shorter than Construction A, but in section 2.5 we'll need both of them.*

Proof of Step 3: The existence of invertible sheaves in the domain of Step 3 is a consequence of the construction from Step 2, using Lemma 3.3.1 c).

Recall that $X_p^n = \varphi_{[\mathcal{H}_p^n]}(S_p^n) \subset \mathbf{P}^n$, where $\varphi_{[\mathcal{H}_p^n]}$ is the embedding defined by the very ample sheaf $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n}) \in Pic(S_p^n)$ and $S_p^n = Bl_{\Sigma_p^n}(\mathbf{P}^2)$ ($\Sigma_p^n \subset \mathbf{P}^2$ a set of $s_p^n + 1$ general points). The classes from Step 2 were constructed by using the surfaces $X_p^{n+1} \subset \mathbf{P}^{n+1}$. If we consider them on the abstract surface S_p^{n+1}, these invertible sheaves can be considered also on S_p^n, obtained from S_p^{n+1} blowing up a new general point $P_{s_p^n}$ (considered in $\mathbf{P}^2$). Put $\Sigma_p^n = \Sigma_p^{n+1} \cup \{P_{s_p^n}\}$. The sheaves from $Pic(S_p^{n+1})$ used in Step 2, considered now in $Pic(S_p^n)$ have the last component equal with zero. Using Lemma 3.3.1. c), we obtain invertible sheaves $\mathcal{D} \in Pic(X_p^n)$ of degree d and $p_a = g$ in the domain

$$\alpha'_{p+1}(d-1,n) - x_{p+1}(d-1,n) \le g \le \alpha'_{p+1}(d-1,n),\ d \ge a_p^{n+1} = a_{p+1}^n + 1.$$

Now, putting 1 (instead of 0) on the last component of the previous sheaves, the arithmetical genus doens't change and the degree is translated by 1. We get exactly the necessary domain for the proof of Step 3.

Remark 3.4.4: *The invertible sheaves $\mathcal{D} \in Pic(X_p^n)$ used in order to cover the domain from Step 3, comming from Construction A in Step 2, are of the following form: $\mathcal{D} = \mathcal{D}_0'' + t_2\mathcal{H}_2 + t_3\mathcal{H}_3' + t_4\mathcal{H}_4' + t_5\mathcal{H}_5' + t_6\mathcal{H}_6'$, where $\mathcal{H}_2, \mathcal{H}_3', \mathcal{H}_4', \mathcal{H}_5', \mathcal{H}_6'$ are as in the Lemma 3.2.4, $t_2 \dots, t_6 \in \mathbf{Z}$, $t_2, \dots, t_6 \ge 0$, $\mathcal{D}_0'' = (a+2; a, 1^u, 0^{s_p^{n+1}-u}, 1)$, $u \in \{3p-n+2,\ 3p-n+3\}$, $a \in \{(d_0''-n+p-1)/2,\ (d_0''-n+p)/2\} \cap \mathbf{Z}$, $d_0'' = (\mathcal{D}_0'' \cdot \mathcal{H}_p^n) \ge a_{p+1}^n = [(n-p-1)/2]_*$, $d_0'' \le a_{p+1}^n + n + p - 1$.*

This follows from (3.4.9), (3.4.10), (3.4.11), the inductive processes, used in proving $(a), (b), (e)$ and the transformation from Step 3 (1 instead of a 0 on the last component, giving 1 on the last component of $\mathcal{D}_0''$ and translating d_0 by 1, giving a as in Remark).

Proof of Step 4: Let's remark for the beginning that, by putting together the Steps 1 and 3 and using Lemma 3.3.1 c), it follows that we covered with invertible sheaves from $Pic(X_p^n)$, by degrees and arithmetical genera the domain

$$\alpha'_{p+1}(d-1,n) \leq g \leq \alpha'_{p+1}(d,n), \quad g \geq a_{p+1}^n + 1. \tag{3.4.22}$$

We need sheaves from $Pic(X_p^n)$ that cover by degrees and arithmetical genera the domain

$$\alpha_{p+1}(d-1,n) \leq g \leq \alpha_p(d,n), \quad g \geq a_p^n + 1.$$

We'll do this by induction on $x_p(d-1,n)$ using (3.4.22). If $x_p(d-1,n) = 0$, then $x_{p+1}(d-1,n) = 0$. Hence $\alpha_{p+1}(d-1,n) = [(d-n+p)/2]_*$. Using (3.3.5), it follows that we need invertible sheaves which cover by degrees and arithmetical genera the domain $[(d-n+p)/2]_* \leq g \leq [(d-n+p+1)/2]_*$, $a_p^n + 1 \leq d \leq a_p^n + (n+p-1)$. But then the necessary degrees and genera are realised by

$$\mathcal{D}_0 = (g+2; g, 1^u, 0^{s_p^n - u}) \in Pic(X_p^n) \tag{3.4.23}$$

with

$$u = 3p-n+3,\ 3p-n+4 \text{ or } 3p-n+5. \tag{3.4.24}$$

Then, if we succeeded to construct invertible sheaves in the domain $\alpha_{p+1}(d-1,n) \leq g \leq \alpha_p(d,n)$, $x_p(d-1,n) = x \geq 0$, adding to them the hyperplane class $\mathcal{H}_p^n = (p+2; p, 1^{s_p^n})$ we obtain, as many times before, using Lemma 3.3.1 a), invertible sheaves from $Pic(X_p^n)$ covering by degrees and arithmetical genera the domain $\alpha_{p+1}(d,n) \leq g \leq \alpha_p(d,n)$ for $x_p(d-1,n) = x+1$. Filling on the left from (3.4.22) (using Lemma 3.3.1 c), last part) we finish the inductive process. Now Proposition 3.4.1 is proved.

* * *

Before ending this section, some remarks. The next Remark 3.4.5 is a consequence of Remarks 3.4.2, 3.4.4 and of the construction given in Step 4.

Remark 3.4.5: *The invertible sheaves $\mathcal{D} \in Pic(X_p^n)$ used in order to cover by degrees and arithmetical genera the domain from Proposition 3.4.1, using Construction A in Step 2 are of one of the following forms:*

a) $\mathcal{D} = \mathcal{D}'_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2$, *where $\mathcal{H}_1, \mathcal{H}_2$ as in Lemma 3.2.4, $t_1, t_2 \in Z$, $t_1, t_2 \geq 0$, $\mathcal{D}'_0 = (a+2; a, 1^u, 0^{s_p^n - u})$, $s_p^n = 3p-n+5$, $u \in \{3p-n+2,\ 3p-n+3,\ 3p-n+4,\ 3p-n+5\}$, $a \in \{(d'_0-n+p-2)/2,\ (d'_0-n+p-$*

$1)/2,\ (d'_0 - n + p)/2,\ (d'_0 - n + p + 1)/2\} \cap Z$, $d'_0 = (\mathcal{D}'_0 \cdot \mathcal{H}^n_p) \geq a^n_{p+1} + 1$, $d_0 \leq a^n_p + n + p$;

b) $\mathcal{D} = \mathcal{D}''_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}'_3 + t_4\mathcal{H}'_4 + t_5\mathcal{H}'_5 + t_6\mathcal{H}'_6$, *with* $\mathcal{H}_1, \ldots, \mathcal{H}'_6$ *as in Lemma 3.2.4*, $t_1, \ldots, t_6 \in Z$, $t_1, \ldots, t_6 \geq 0$, $\mathcal{D}'_0 = (a+2; a, 1^u, 0^{s^n_p - u}, 1)$, $u \in \{3p - n + 2,\ 3p - n + 3\}$, $a \in \{(d''_0 - n + p - 1)/2,\ (d''_0 - n + p)/2\} \cap Z$, $d''_0 = (\mathcal{D}''_0 \cdot \mathcal{H}^n_p) \geq a^n_{p+1}$ *and* $d''_0 \leq a^n_{p+1} + n + p - 1$.

From Remark 3.4.5 we deduce the following remark, which we'll use in the next section.

Remark 3.4.6: *If* $n \geq 9$ *and* $\frac{n}{3} \leq p \leq n-4$, $n, p \in Z$, *then the invertible sheaves* $\mathcal{D} \in Pic(X^n_p)$ *used in order to cover by degrees and arithmetical genera the domain from Proposition 3.4.1 (using Construction A in Step 2) are of the following form:* $\mathcal{D} = \mathcal{D}_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}'_3 + t_4\mathcal{H}'_4 + t_5\mathcal{H}'_5 + t_6\mathcal{H}'_6$ *with* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_3, \mathcal{H}'_4, \mathcal{H}'_5, \mathcal{H}'_6$ *as in Lemma 3.2.4 and* $\mathcal{D}_0 = (a+2; a, 1^u, 0^{s^n_p - u - 1}, \varepsilon)$, $\varepsilon \in \{0, 1\}$, $0 \leq u \leq s^n_p - 1 = 3p - n + 4$, $a \geq (d_0 - n + p - 1)/2$, $a \in Z$, $d_0 \in [a^n_{p+1}, a^n_p + n + p] \cap Z$ $(a^n_p = [(n-p)/2]_* + 1)$.

Put $d_0 = d'_0 - 1$ if $\mathcal{D}$ is of category a) in Remark 3.4.5 and $d_0 = d''_0$ if $\mathcal{D}$ is category b).

Remark 3.4.7: *Let there be* $\mathcal{D} = (a; b_1, b_2, \ldots, b_{s^n_p}) \in Pic(X^n_p)$ *one of the classes used in the proof of Proposition 3.4.1. Let there be* $d = \deg \mathcal{D} = (\mathcal{D} \cdot \mathcal{H}^n_p)$ *and* $r := a - b_0$. *Then* $r \leq 2(x_p(d, n) + 1)$.

Indeed, let's denote by $r(d, n) := 2(x_p(d, n) + 1)$. If $\mathcal{D}$ has been used in Step 1, then $r = 2(x_{p+1}(d-1, n) + 1) \leq r(d, n)$. If $\mathcal{D}$ has been used in Step 3 (with any construction in Step 2), then $r = 2(x_{p+1}(d, n) + 1) \leq r(d, n)$. If $\mathcal{D}$ has been used in Step 4, then $r = 2(x_p(d-1, n) + 1) \leq r(d, n)$ or $\mathcal{D}$ is obtained from some $\mathcal{D}_1 = (a^1; b^1_0, \ldots, b^1_{s^n_p}) \in Pic(X^n_p)$ adding a number, let's say t, of $\mathcal{H}^n_p = (p+2; p, 1^{s^n_p}) \in Pic(X^n_p)$; then $r_1 := a^1 - b^1_0 \leq r(d_1, n)$, where $d_1 = \deg \mathcal{D}_1$; now, adding t times $\mathcal{H}^n_p$, r_1 increases with at most $2t$ in order to become r and $r(d_1, n)$ increases with exactly $2t$ in order to become $r(d, n)$ (due to the form of the three previous r); so the inequality $r_1 \leq r(d_1, n)$ is transfered to $r \leq r(d, n)$.

3.5 Curves on the surfaces X^n_p in the domains A^n_p ($n \geq 8$, $\frac{n}{3} \leq p \leq n-4$)

We recall that the domains A^n_p were defined in §2 (2.11). In this section we'll prove *Theorem 3.5.11* (stated to the end of section). This theorem follows from:

Proposition 3.5.1: *Let there be $(d,g) \in A_p^n$, where $n, p \in Z$, $n \geq 8$, $n/3 \leq p \leq n-4$ and $k = [n/3]_*$. Then:*

i) *if $d \geq \frac{2}{3}(3p+n+9)$ there is a (smooth, irreducible) curve $C \subset X_p^n$, non-degenerate in P^n, so that $(\deg(C), g(C)) = (d,g)$;*

ii) *if $d < \frac{2}{3}(3p+n+9)$ there is a (smooth, irreducible) curve $C \subset P^n$, non-degenerate in P^n, so that $(\deg(C), g(C)) = (d,g)$; if $n \equiv 0$ (mod 3) then C can be found on X_k^n and if $n \equiv 1, 2$ (mod 3) then C can be found on X_{k+1}^n.*

Proof: The proof is a consequence of the analysis which we'll do on the linear systems associated to the invertible sheaves appearing in the proof of Proposition 3.4.1. Precisely, we'll test when these linear systems are nonempty and contain (smooth, irreducible) curves, using the smoothing criteria from section 3.2. Namely, for the situation (R1) (see (3.1.4)) we'll apply essentially the criterion given by Corollary 3.2.5 (considering *Construction A* in Step 2 of the proof of Proposition 3.4.1) and for the situation (R2) (see (3.1.5)) we'll apply the criterion given by Proposition 3.2.3 considering *Construction B* in Step 2 of the proof of Proposition (3.4.1). The situations (R1) and (R2) are complementary and their union cover the hypothesis.

The proof of Proposition 3.5.1 will be the consequence of a sequence of lemmas.

1) *For the situation (R1):*

Lemma 3.5.2: *If (p,n) is in the situation (R1), $(d,g) \in A_p^n$ and $\mathcal{D} \in Pic(X_p^n)$ is one of the invertible sheaves used in the proof of Proposition 3.4.1 (considering Construction A in Step 2) so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d,g)$, then we know that $\mathcal{D} = \mathcal{D}_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2 + t_3\mathcal{H}_3' + t_4\mathcal{H}_4' + t_5\mathcal{H}_5' + t_6\mathcal{H}_6'$ as in Remark 3.4.6; let's denote by $t := \sum_{i=1}^{6} t_i$ and suppose that $t \geq 3$. Then $[\mathcal{D}] \neq \emptyset$ and contain a (smooth, irreducible) curve C, non-degenerate in P^n (here $\deg \mathcal{D} = (\mathcal{D} \cdot \mathcal{H}_p^n)$).*

Proof: We use Corollary 3.2.5, i.e. we check that (3.2.10) is verified for $t \geq 3$. Indeed this becomes

$$3a \geq 3p - 4n + 14. \tag{3.5.1}$$

Because $a \geq (d_0 - n + p - 1)/2$, $d_0 \geq (n-p)/2$ (see Remark 3.4.6), minorating a (and d_0) as before, it follows that (3.5.1) holds if

$$13n \geq 9p + 62. \tag{3.5.2}$$

However, this is true in (R1) ($p \leq n-4$, $n \geq 9$).

Because $\deg \mathcal{H} \geq n+p-1$, $(\forall)\mathcal{H} \in \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_3, \mathcal{H}'_4, \mathcal{H}'_5, \mathcal{H}'_6\}$ and $\deg \mathcal{D}_0 \geq 0$ (with degree as in the statement of lemma), the curve C which we just obtained has $\deg C \geq 3(n+p-1) > n+p-1 = \deg X_p^n$. So C is non-degenerate in $\mathbf{P}^n$.

* * *

Lemma 3.5.3: *If $(p,n),(d,g),\mathcal{D},t$ are as in the previous lemma and $t=2$, then the same conclusion holds for $[\mathcal{D}]$.*

Proof: We apply again Corollary 3.2.5 for $t=2$ and we use that $a \in \mathbf{Z}$. Doing computations, it follows that the only case when (3.2.10) doesn't apply is for $p=n-4$ and $a=-1$. We'll consider separately this case.

However, it's easy to see that the sheaves $\mathcal{D}$ from the lemma are of the form $\mathcal{D} = (b+6; b, \eta_1, \eta_2, \ldots, \eta_{s_p^n})$, $\eta_i \in \{0,1,2,3\}$, $b \in \{a+2p-2,\ a+2p-1,\ a+2p\}$, $a \in \mathbf{Z}$ as in Remark 3.4.6. Let E be one of the exceptional divisors lying on X_p^n and corresponding to a $P_i \in \Sigma_p^n$. From the exact sequence

$$O \to \mathcal{D} \to \mathcal{D}(E) \to \mathcal{D}(E)|_E \to O$$

we deduce that the conclusion of lemma holds for $[\mathcal{D}]$ iff it holds for $[\mathcal{D}(E)]$, as far as $b_i = (\mathcal{D} \cdot \mathcal{O}_{X_p^n}(E)) \geq 1$. Moreover, it's easy to see that all $\eta_i = 3$ only if $b = a+2p$. So, to conclude the lemma it's enough to prove the conclusion for $\mathcal{D} \in \{\mathcal{D}', \mathcal{D}''\}$ where $\mathcal{D}' = (a+2p+4;\ a+2p-2,\ 3^{s_p^n-1}, 2)$, $\mathcal{D}'' = (a+2p+6;\ a+2p, 3^{s_p^n})$, of course for $p=n-4$ and $a=-1$. We'll study the case $\mathcal{D} = \mathcal{D}'$, the other one being similar.

So, $\mathcal{D}' = (2p+3; 2p-3, 3^{2p}, 2)$, $p = n-4 \geq 5$.

We specialize the points from Σ_p^n on a smooth cubic curve $\Gamma_0 \subset \mathbf{P}^2$; we denote this specialization by $\tilde{\Sigma}_p^n = \{\tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_{2p+1}\}$; we suppose, moreover that the points from $\tilde{\Sigma}_p^n$ are general on Γ_0 (see the sketch of proof of Proposition 3.2.1), in particular, every 3 of them are noncollinear. Let there be $\tilde{S}_p^n = Bl_{\tilde{\Sigma}_p^n}(\mathbf{P}^2)$. Then $(p+2; p, 1^{2p+1})$ is very ample on $\tilde{S}_p^n$ (actually, if $s \leq 2q+3$, $q \geq 1$, $s \geq 0$, $s,q \in \mathbf{Z}$ and $\Sigma = \{R_0, R_1, \ldots, R_s\} \subset \Gamma_0$ general on Γ_0, $S := Bl_\Sigma(\mathbf{P}^2)$, then $(q+2; q, 1^s)$ is very ample on S-see [Hb2]; for a direct proof see [P2]). Let's denote by $\tilde{X}_p^n := \varphi_{[p+2;p,1^{2p+1}]}(\tilde{S}_p^n) \subset \mathbf{P}^n$. Using Proposition 3.2.1 (i3)′ and (ii3)′ it results that, if $\tilde{\mathcal{D}}_1 := (2p; 2p-3, 2^{2p}, 1)$, then

(3.5.3) $\begin{cases} h^1(\tilde{\mathcal{D}}_1) = 0 \text{ and } [\tilde{\mathcal{D}}_1] \neq \emptyset, \text{ without base points and} \\ \text{containing a (smooth, irreducible) curve.} \end{cases}$

By semicontinuity (see [CS], Remark 1, p. 324) we deduce that (3.5.3) holds again if we replace $\tilde{P}_0$ with another point (denoted $\tilde{P}'_0$) from a small neighbourhood of $\tilde{P}_0$, non-belonging to Γ_0 and non-collinear with any other $\tilde{P}_i$ and $\tilde{P}_j$. Then, the proper transformation of Γ_0 in $\tilde{X}^n_p$ is $\Gamma \in [3; 0, 1^{2p+1}]$.

Now, we denote by δ_{ij} the quadratic transformation based on $\{\tilde{P}'_0, \tilde{P}_i, \tilde{P}_j\}$. Performing the succesive quadratic transformations $\delta_{12}, \delta_{34}, \ldots, \delta_{2p-1,2p}$, the curve Γ_0 becomes a curve $\Delta_0 \subset \mathbf{P}^2$ of degree $p+3$ having $2p+1$ singular points (with distinct tangents) with multiplicities $p, 2, \ldots, 2$ respectively; $\tilde{X}^n_p$ becomes $\bar{X}^n_p = \varphi_{[p+2;p,1^{2p-1}]}(\bar{S}^n_p) \subset \mathbf{P}^n$, where $\bar{S}^n_p = Bl_{\bar{\Sigma}^n_p}(\mathbf{P}^2)$, $\bar{\Sigma}^n_p = \{\bar{P}_0, \bar{P}_1, \ldots, \bar{P}_{2p}, \bar{P}_{2p+1}\}$, $\bar{P}_0, \bar{P}_1, \ldots, \bar{P}_{2p}$ being the singularities of Δ_0 with multiplicities $p, 2, \ldots, 2$ respectively and $\bar{P}_{2p+1} \in \Delta_0$ a smooth point so that every 3 points in $\bar{\Sigma}^n_p$ are non-collinear; of course $\bar{X}^n_p \cong \tilde{X}^n_p$ $(\cong \bar{S}^n_p = \tilde{S}^n_p)$; moreover, $\tilde{\mathcal{D}}_1$ becomes, in the new coordinates, $\bar{\mathcal{D}}_1 = (p; p-3, 1^{2p+3}) \in Pic(\bar{X}^n_p)$. If we consider $\bar{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_1$ on $\bar{S}^n_p = \tilde{S}^n_p$, then $\bar{\mathcal{D}}_1 = \tilde{\mathcal{D}}_1$. Considering (3.5.3) it results that

$$(3.5.4) \qquad \begin{cases} h^1(\bar{\mathcal{D}}_1) = 0 \text{ and } [\bar{\mathcal{D}}_1] \neq \emptyset, \text{ without base points and} \\ \text{containing a (smooth, irreducible) curve.} \end{cases}$$

Let's denote by $\Delta \in [p+3; p, 2^{2p}, 1]$ the proper transformation of Δ_0 in $\bar{X}^n_p$. We can see that $\bar{\mathcal{D}}_1 = \mathcal{D}'(-\Delta)$, where $\mathcal{D}'$ is our initial invertible sheaf (considered on $\bar{X}^n_p$). We obtain then the exact sequence of sheaves on X^n_p

$$0 \to \bar{\mathcal{D}}_1 \to \mathcal{D}' \to \mathcal{D}'/_\Delta \to 0.$$

Since $(\mathcal{D}' \cdot O_{\bar{X}^n_p}(\Delta)) = 7 \geq 2g(\Delta) = 2(= 2g(\Gamma))$, using (3.5.4) it results that $[\mathcal{D}'] \neq \emptyset$, without base points and containing a (smooth, irreducible) curve, if the points of Σ^n_p are specialized in $\bar{\Sigma}^n_p$. So, the same fact remains true on X^n_p, by semicontinuity.

* * *

Lemma 3.5.4: *If $(p,n), (d,g), \mathcal{D}$ are as in the Lemma 3.5.2 and $d = \deg \mathcal{D} \geq \max\left(2n+1, \frac{2}{3}(3p+n+9)\right)$, then $[\mathcal{D}] \neq \emptyset$, without base points and contains a (smooth, irreducible) curve C, non-degenerate in $\mathbf{P}^n$.*

Proof: Apply Corolary 3.2.5 for $t = 1$.

Preliminary Conclusion 3.5.5: *If $(d,g) \in A^n_p$ with (p,n) in situation (R1) and $d \geq \max\left(2n+1, \frac{2}{3}(3p+n+9)\right)$, then there is a (smooth, irreducible) curve $C \subset X^n_p$, non-degenerate in $\mathbf{P}^n$, with* $(\deg(C), g(C) = (d,g)$.

* * *

2) For the situation (R2):

Lemma 3.5.6: *If (p,n) is in the situation (R2), $(d,g) \in A_p^n$ and $\mathcal{D} \in Pic(X_p^n)$ is one of the invertible sheaves used in the proof of Proposition 3.4.1 (considering Construction B in Step 2) so that $(\deg \mathcal{D}, p_a(\mathcal{D})) = (d,g)$, then $\mathcal{D}$ satisfies condition (c3) from Proposition 3.2.3 in the coordinates $(d, r; \theta_1, \theta_2, \ldots, \theta_{s_p^n})$ (here $\deg \mathcal{D} := (\mathcal{D} \cdot \mathcal{H}_p^n)$).*

Proof: The sheaves used in Step 1 satisfy (c3). The invertible sheaves used in Step 3 come from the sheaves used in Step 2, putting 1 (instead 0) in the last component. So, it's enough to verify that the sheaves $\mathcal{D}$ used in Step 2 apply to

$$|\theta_1| \leq \theta_2 \leq \ldots \leq \theta_{s_p^n+1} < \frac{r}{2} \text{ for } d = \deg \mathcal{D} \geq 2n+2. \tag{3.5.5}$$

The inequalities $|\theta_1| \leq \theta_2 \leq \ldots \leq \theta_{s_p^n+1}$ come from (3.4.21). From $d \geq 2n+2$ we deduce that $r = 2(x_p(d,n+1)+1) \geq 4$ (cf. (3.4.19)). Then, $c_i > \frac{r}{2} \Rightarrow b \geq c_i^2 > \frac{r^2}{4}$ (see (3.4.21)); but $b \leq r$ (see (3.4.20)) and $r \leq \frac{r^2}{4}$, that is a contradiction ! So, $c_i \leq \frac{r}{2}$, hence $\theta_i \leq \frac{r}{2}$, $i = \overline{1, s_p^{n+1}}$. Moreover, $c_1 = \frac{r}{2} \Leftrightarrow r = 4$. Then we replace (if necessary) $(c_1, c_2, c_3, c_4) = (2,0,0,0)$ with $(c_1, c_2, c_3, c_4) = (1,1,1,1)$ and we get $\theta_{s_p^n+1} < \frac{r}{2}$. Now, the general invertible sheaves (from Step 4) satisfy (c3), because $\mathcal{H}_p^n$ satisfies this condition.

* * *

Lemma 3.5.7: *If $(p,n), (d,g)$ and $\mathcal{D}$ are as in the previous lemma, then $\mathcal{D}$ satisfies the condition (c4) from Proposition 3.2.3 in the coordinates $(d, r; \theta_1, \theta_2, \ldots, \theta_{s_p^n})$.*

Proof: (c4) means $b_0 \geq b_1$ if $\mathcal{D} = (a; b_1, b_1, \ldots, b_{s_p^n})$, in usual coordinates on $Pic(X_p^n)$. We will check this (to i)) for the sheaves used in Step 1 from the proof of Proposition 3.4.1 obtained by adding a (finite) numbers of $\mathcal{H}_p^n$ to the initial $\mathcal{D}_0$, for the sheaves used in Step 4 obtained by adding a (finite) number of $\mathcal{H}_p^n$ to the corresponding initial $\mathcal{D}_0$ and for sheaves used in Step 4 obtained by adding a (finite) number of $\mathcal{H}_p^n$ to the sheaves used in Step 1 (the invertible sheaves $\mathcal{D}$ from Remark 3.4.5 a) contains all these three categories of sheaves). We will check also (c4) for the sheaves used in Step 2, Construction B (to ii)). It's clear that from these two verifications, i) and ii), we obtain the conclusion of the lemma.

i) We consider sheaves $\mathcal{D} = \mathcal{D}_0 + t_1\mathcal{H}_1 + t_2\mathcal{H}_2$, $\mathcal{H}_1 = \mathcal{H}_p^n$, $\mathcal{H}_2 = \bar{\mathcal{H}}_p^n$, as in Remark 3.4.5 a). Put $t := t_1 + t_2$. If $t = 1$, because $d = \deg \mathcal{D} \geq 2n + 1$, it follows that $d_0 := \deg \mathcal{D}_0 \geq n - p + 1$. But, then $\mathcal{D} = (a + p + 4; a + p, 2^u, 1^{s_p^n - u - 1}, \varepsilon)$, $\varepsilon \in \{0, 1\}$. We need $a + p \geq 2$. Minorating a as in Remark 3.4.5 a) and d_0 from before, it results that we need $p \geq 3$, which is true in (R2). If $t \geq 2$, then $b_0 \geq b_1$ (i.e. (c4)) becomes $a + tp \geq t + 1$. Because the function (in t) $a + tp - t - 1$ is increasing, we can suppose that $t = 2$. Then we need $a + 2p \geq 3$. Minorating a and d_0 from Remark 3.4.5 a), it results that we need $p \geq (n + 14)/9$, which is satisfied in (R2).

ii) We are going to study now the sheaves used in the proof of Step 2 (Construction B). Their degree is $d \geq 2n + 2$, so $x_p(d, n + 1) \geq 1$.

1) *If* $x_p(d, n + 1) \geq 3$. We have: $-\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq \sum_{i=1}^{s_p^{n+1}} |\theta_i| = \sum_{i=1}^{4} c_i$ (see (3.4.21)) $\leq 2\sqrt{\sum_{i=1}^{4} c_i^2} \leq 2\sqrt{b} \leq 2\sqrt{2(x_p(d, n + 1) + 1)}$ (see 3.4.20)). Now, if we prove that

$$(3.5.6) \quad x_p(d, n+1) \geq 3 \Rightarrow d - (n-p+2)(x_p(d, n+1)+1) \geq 2(x_p(d, n+1)+1)$$

we obtain (because $\sqrt{2(x_p(d, n + 1) + 1)} \leq x_p(d, n + 1) + 1)$ $-\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq d - \frac{1}{2}(n - p + 2) \cdot 2(x_p(d, n + 1) + 1)$ which is exactly (c4), because $r = 2(x_p(d, n + 1) + 1)$ (see (3.4.19)). So, it remains to prove (3.5.6). But

$$(3.5.7) \qquad x_p(d, n + 1) \geq 3 \Rightarrow d \geq (7n + 5p + 2)/2.$$

a) *If* $p \geq 4$, we have $d - (n - p + 2)(x_p(d, n + 1) + 1) - 2(x_p(d, n + 1) + 1) \geq d - n(x_p(d, n + 1) + 1) \geq d - \frac{n}{2(n + p)}(2d + n + 3p - 2)$. This last number is ≥ 0 iff

$$(3.5.8) \qquad d \geq n(n + 3p - 2)/(2p).$$

But in (R2), (3.5.8) follows from (3.5.7). We get then (3.5.6).

b) *If* $p = 3$, then $n \in \{8, 9\}$. If $n = 8$, (3.5.6) becomes $d \geq 9(x_3(d, 9) + 1)$, true for $d \geq 27$. But from (3.5.7) it results that $d \geq 37$, so we are ready. If $n = 9$, (3.5.6) becomes $d \geq 10(x_3(d, 10) + 1)$, true for $d \geq 30$. But from (3.5.7) it results that $d \geq 40$ so we are ready again.

2) *If* $x_p(d, n+1) = 2$. Then

$$d \geq (5n + 3p + 2)/2. \tag{3.5.9}$$

In this case (c4) becomes

$$-\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq d - 3(n - p + 2). \tag{3.5.10}$$

Using (3.5.9), we obtain $d - 3(n-p+2) \geq (9p - n - 10)/2 \geq 4$ (because $p \geq \frac{n}{3}$, $n \geq 8$ and $p \geq \frac{n+1}{3} = 3$ if $n = 8$. Then, in order to obtain (3.5.10) it's enough to choose $\theta_1, \ldots, \theta_{s_p^{n+1}}$ such that $-\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq 4$. From (3.4.20) and (3.4.21) it results that it's enough to express each $b \in \{1, 2, \ldots, 6\}$ as $b = \sum_{i=1}^{4} c_i^2$ so that $\sum_{i=1}^{4} c_i \leq 4$, which is, obviously, possible.

3) *If* $x_p(d, n+1) = 1$, (c4) becomes

$$-\theta_1 + \sum_{i=2}^{s_p^{n+1}} \theta_i \leq d - 2(n - p + 2). \tag{3.5.11}$$

But $p \geq 3$ and $d \geq 2n + 2$, so $d - 2(n - p + 2) \geq 4$. Using now (3.4.20) and (3.4.21) it results that it's enough to express each $b \in \{1, 2, 3, 4\}$ as $b = \sum_{i=1}^{4} c_i^2$ so that $c_i \in \{0, 1\}$ for all i (hence $\sum_{i=1}^{4} c_i \leq 4$), which is possible.

* * *

Lemma 3.5.8: *If* $(p, n), (d, g), \mathcal{D}$ *are as in Lemma 3.5.6 and* $d = \deg \mathcal{D} \geq \max\left(2n + 1, \frac{2}{3}(3p + n + 9)\right)$, *then* $\mathcal{D}$ *satisfies the condition (c5) in the coordinates* $(d, r; \theta_1, \theta_2, \ldots, \theta_{s_p^n})$.

Proof: From Remark 3.4.7 it follows that it is enough to check (c5) for $r = 2(x_p(d, n) + 1)$. This becomes:

$$d \geq 2(p - 1)(x_p(d, n) + 1) + 2. \tag{3.5.12}$$

If we denote by $E_p(d, n) := d - 2(p-1)(x_p(d, n) + 1) - 2$, then (3.5.12) becomes

$$E_p(d, n) \geq 0. \tag{3.5.13}$$

We can see that, if (3.5.13) is true for integers d so that $x_p(d,n) = x$, then it is true for any integer d' so that $x_p(d',n) = y$, $(\forall)y \geq x$. Indeed, if $E_p(d,n) \geq 0$ for all d so that $E_p(d,n) = y$, let d' be so that $x_p(d',n) = y+1$ and $d := d'-(n+p-1)$. Then $E_p(d',n) = E_p(d,n)+(n-p+1) \geq E_p(d,n) \geq 0$ (use Lemma 3.3.1 a)). The previous statement follows then by induction on $x_p(d,n)$.

Now, if $x_p(d,n) = 2$, the relation (3.5.12) becomes

$$d \geq 6p - 4. \tag{3.5.14}$$

Because $x_p(d,n) = 2$ we have $d \geq (5n+3p-3)/2$. But this implies (3.5.14) in the situation (R2) (use $3p \leq n+5$).

So, (3.5.12) is true for any d such that $x_p(d,n) \geq 2$.

If $x_p(d,n) = 1$, (3.5.12) becomes $d \geq 4p-2$. But this results from the suplementary condition $d \geq \frac{2}{3}(3p+n+9)$ in (R2) (use $3p \leq n+5$).

* * *

Because the conditions (c1) and (c2) were verified during the Construction B in the proof Step 2 of Proposition 3.4.1 and (c3), (c4), (c5) were verified in Lemmas 3.5.6, 3.5.7, 3.5.8, from Proposition 3.2.3 we deduce the following

Preliminary Conclusion 3.5.9: *If $(d,g) \in A_p^n$ with (p,n) in situation (R2) and $d \geq \max\left(2n+1, \frac{2}{3}(3p+n+9)\right)$, then there is a (smooth, irreducible) curve $C \subset X_p^n$, non-degenerate in P^n, with $(\deg(C), g(C)) = (d,g)$.*

* * *

Lemma 3.5.10: *If $n \geq 8$ and $\frac{n}{3} \leq p \leq n-4$, $n,p \in Z$, $(d,g) \in A_p^n$ and $2n+1 \leq d < \frac{2}{3}(3p+n+9)$ then it exists a (smooth, irreducible) curve $C \subset X_k^n$ if $n \equiv 0$ (mod 3) and $C \subset X_{k+1}^n$ if $n \equiv 1,2$ (mod 3), non-degenerate in P^n, with $(\deg(C), g(C)) = (d,g)$ (here $k = [n/3]_*$ as usual).*

Proof: We use the criterion given by Proposition 3.2.3.

We need curves in the range

$$\alpha_{p+1}(d-1,n) \leq g \leq \alpha_p(d,n), \quad 2n+1 \leq d < \frac{2}{3}(3p+n+9).$$

Then $\alpha_{p+1}(d-1,n) = [(3d-4n-2)/2]_*$, $\alpha_p(d,n) = [(3d-4n+1)/2]_*$. Because the interval $[\alpha_{p+1}(d-1,n), \alpha_p(d,n)] \cap \mathbf{Z}$ does not depend on p we construct the necessary curves on X_k^n if $n = 3k$ and on X_{k+1}^n if $n = 3k+1, 3k+2$. We remark that $F_d^{p,n}(4) = (3d-4n+2)/2$ (see (3.2.3)).

To be more precise, we consider linear systems associated to invertible sheaves $\mathcal{L} \in Pic(X_p^n)$, $p \in \{k, k+1\}$, $k = [n/3]_*$, which in the Gruson-Peskine coordinates are $(d, 4; 0^{s_p^n - t}, 1^t)$ with $2n+1 \leq d < \frac{2}{3}(3p+n+9)$, $t \in \{1,2,3,4,5\}$ (see (3.2.1) and (3.2.2)). Hence $r = 4$, $\theta_i = 0$, $i = \overline{1, s_p^n - t}$, $\theta_{s_p^n - j+1} \in \{0,1\}$, $j = \overline{2,5}$, $\theta_{s_p^n} = 1$, $s_p^n \in \{5,6,7\}$. We can see immediately that the conditions (c1)-(c5) are satisfied (use $d \geq 2n+1$ and (3.2.2)).

* * *

The *proof* of Proposition 3.5.1 results immediately from the Preliminary Conclusions 3.5.5, 3.5.9 and Lemma 3.5.10.

* * *

Now, from Proposition 3.5.1, using Lemma 3.3.1 b) we obtain the following

Theorem 3.5.11: *If* $n, p \in Z$, $n \geq 8$ *and* $(d,g) \in \bigcup_{\frac{n}{3} \leq p \leq n-4} A_p^n$ *then there is a (smooth, irreducible) non-degenerate curve* $C \subset P^n$ *with* $(\deg(C), g(C)) = (d,g)$.

3.6 The absence of gaps in D_1^n

In this section we finish proving the Main Theorem. For this, we need two more theorems, namely:

Theorem 3.6.1: *If* $n \in Z$, $n \geq 8$ *and* $(d,g) \in \tilde{A}_k^n$ *(see (2.12)) then there is a (smooth, irreducible) curve* $C \subset X_k^n$*, non-degenerate in* P^n *so that* $(\deg(C), g(C)) = (d,g)$.

Theorem 3.6.2: *If* $n \in Z$, $n \geq 8$ *and* $(d,g) \in A'^n_{n-3}$ *(see (2.10)) then there is a (smooth, irreducible) non-degenerate curve* $C \subset P^n$*, so that* $(\deg(C), g(C)) = (d,g)$.

The Theorem 3.6.1 is, essentially, covered by Theorem 1.1 from [Ci]. But, because our statement, which is necessary here, differs from Ciliberto statement, it is necessary to prove it. Here we'll give only a sketch of proof (for details see [P3]). The proof consists in a comparison between the *upper bound* of genus from our Theorem 3.5.11 and the *lower bound* of genus from Ciliberto's Theorem. This comparison is possible due to the Lemma 3.3.2. It will be divided in 3 parts, according to $n \equiv 0 \pmod 3$, $n \equiv 1 \pmod 3$, $n \equiv 2 \pmod 3$.

The main part of the domain A'^n_{n-3} is covered by our Theorem 1 from [P1] and by Ciliberto-Sernesi Main Theorem from [CS]. Again, because the statement of Theorem 3.6.2, which is necessary here, differs from the previous two statements, it is necessary to prove the Theorem. We'll give only a sketch of proof (for details see [P3]). The proof consists in a comparison between the *lower bound* of genus from our Theorem 3.5.11 and the *upper bound* of genus from Ciliberto Sernesi's Theorem (for $\delta = 4$) and a filling of the distance between them when it exists.

Proof of Theorem 3.6.1 (sketch): We'll consider here only the case $n \equiv 1 \pmod 3$ (so $n = 3k+1$, $k \in \mathbf{Z}$, $k \geq 3$) the others being (more or less) similar.

1) **Claim:** *If $d, g \in Z$, there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g, in the domain from the (d, g)-plane given by:*

$$F_d^{k,n+1}(2(x_{k+1}(d-1,n)+1)) \leq g \leq F_d^{k,n}(2(x_{k+1}(d-1,n)+1)), \; d \geq 2n+1$$

(for $F_d^{k,n}(r)$ see (3.2.3); for $x_{k+1}(d,n)$ see (2.2)).

Proof: Let $r := 2(x_{k+1}(d-1,n)+1)$. Now we express $m := 2(F_d^{k,n}(r) - g) \in \mathbf{Z}$, $m \geq 0$ as a sum of 4 squares $m = \sum_{i=1}^{4} \theta_i^2$, $\theta_i \in \mathbf{Z}$, $0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$. Then, we can check that $(d, r; \theta_1, \theta_2, \theta_3, \theta_4)$ satisfy the conditions (c1)-(c4) of Proposition 3.2.3 ((c5) being automatically satisfied). So, we can find $\mathcal{D} \in Pic(X_k^n)$ such that $[\mathcal{D}]$ contains (smooth, irreducible) curves C with $(\deg(C), g(C)) = (d, g)$ in the domain of the Claim.

2) **Claim:** *If $d, g \in Z$, there is a (smooth, irreducible) curve $C \subset X_k^n$, non-degenerate in $\mathbf{P}^n$, of degree d and genus g, in the domain from the (d, g)-plane given by:*

$$F_d^{k,n}(2(x_{k+1}(d-1,n)+1)) < g \leq B(d,n) = \pi_k(d,n), \; d \geq d_1(n).$$

Proof: We recall that in [Ci], section 1.g a positive integer $r_0 = r_0(d,n)$ has been defined, namely $r_0(d,n) = \min\{r \in \mathbf{Z} | r \geq s(d,n)\}$, where $s(d,n) = -2(2k+1) + 2\sqrt{d + 2(k-1) + (2k+1)^2}$ (see [Ci], Lemma 1.11). In order to prove the Claim it's enough to show that (see (2.6) for $d_1(n)$):

$$(3.6.1) \qquad d \geq d_1(n) \Rightarrow F_d^{k,n}(r_0(d,n)-1) \leq F_d^{k,n}(2(x_k(d,n)+1)).$$

Indeed, then the Claim follows from [Ci] - Theorem 1.1 (take $\varphi(d,n) = F_d^{k,n}(r_0(d,n)-1)$), [Ci]-Lemma 1.11 and section 1.d, [Ci] -Lemma 1.18 (use that $[\max\{F_d^{k,n}(r) | r \in \mathbf{Z}\}]_* = \max\{[F_d^{k,n}(r)]_* | r \in \mathbf{Z}\}$), using the fact that we constructed curves of degree $d \geq 2n+1$ and genus $g = [F_d^{k,n}(2(x_{k+1}(d-$

$1, n) + 1))]_*$ in the Claim from 1). We remark that $d_1(n) \geq d_0(n)$ (for $d_0(n)$ see [Ci], section 1,g), both of them being functions of degree 3/2 in n.

In order to prove (3.6.1) write $d - k - 2 = x_{k+1}(d-1, n) + \varepsilon$, $0 \leq \varepsilon \leq 4k$. Doing computations we can see that

$$(3.6.2)\quad 2(x_{k+1}(d-1,n)+1) \leq \rho_k(d,n) \Leftrightarrow \max(0, (8k^2-k+1-d)/(4k)) \leq \varepsilon \leq 4k$$

$$(3.6.3)\quad 2(x_{k+1}(d-1,n)+1) > \rho_k(d,n) \Leftrightarrow 0 \leq \varepsilon \leq (8k^2-k-d)/(4k) \text{ (if } d \leq 8k^2-k)$$

where $\rho_k(d,n) = 2(d+k-1)/(n+k-1)$ is the point where the function $F_d^{k,n}(r)$ achieves its maximum.

a) If $\max(0, (8k^2 - k + 1 - d)/(4k)) \leq \varepsilon \leq 4k$, by (3.6.2) it's enough to have $2(x_{k+1}(d-1, n) + 1) + 1 \geq s(d, n)$, implied by

$$(3.6.4)\qquad (2d + 2k - 1)/(4k + 1) \geq s(d, n)$$

(indeed, $\varepsilon \leq 4k \Rightarrow 2x_{k+1}(d-1, n) + 3 \geq (2d + 2k - 1)/(4k + 1)$).

b) If $0 \leq \varepsilon \leq (8k^2 - k - d)/(4k)$ (if $d \leq 8k^2 - k$), by (3.6.3) it's enough to have $[2(\rho_k(d, n) - x_{k+1}(d-1, n) - 1)]_* + 1 \geq s(d, n)$. But, it can be seen (computing this integer part) that's enough to have

$$(3.6.5)\qquad \frac{(4k+1)d - 4k}{2k(4k+1)} \geq s(d, n)$$

if $0 \leq \varepsilon \leq (4k^2 - 2k - d)/(4k)$ (when the previous integer part is $2x_{k+1}(d - 1, n)$) and

$$(3.6.6)\qquad \frac{(4k+1)d + 4k^2 - 3k}{2k(4k+1)} \geq s(d, n)$$

if $\max(0, (4k^2-2k+1-d)/(4k)) \leq \varepsilon \leq (8k^2-k-d)/(4k)$ (when the previous integer part is $2x_{k+1}(d-1, n) + 1$).

Doing computations, we can see that (3.6.4), (3.6.5) and (3.6.6) hold exactly for $d \geq d_1(n)$. This proves (3.6.1).

3) The Theorem 3.6.1 (for $n = 3k + 1$) follows immediately from 1) and 2), using Lemma 3.3.2 b) ii).

* * *

Proof of Theorem 3.6.2 (sketch): We *define* the following numerical functions (for $d, r \in \mathbf{Z}$, $\delta \in \{2, 3, 4\}$):

$$(3.6.7)\qquad G_d^\delta(r, n) := r\left(d - nr + \frac{r-1}{2}\delta\right).$$

(3.6.8) $$\alpha''_{n-\delta+1}(d,n) := G^{\delta}_{d}(y_\delta(d,n)),$$

where

(3.6.9) $$y_\delta(d,n) := [(d+n-\delta)/(2n-\delta)]_*.$$

Lemma 3.6.3:
a) $G^{\delta}_{d+(2n-\delta)}(r+1) = G^{\delta}_{d}(r) + (d+n-\delta)$;
b) $y_\delta(d+(2n-\delta),n) = y_\delta(d,n)+1$;
c) $\alpha''_{n-\delta+1}(d+(2n-\delta),n) = \alpha''_{n-\delta+1}(d,n) + (d+n-\delta)$.
Proof: easy.

* * *

We'll use the following

Theorem 3.6.4: *Let there be* $\delta \in \{2,3,4\}$, $n \in Z$, $n \geq 2\delta - 1$, $d, g \in Z$, $d \geq 2n+1$ *and* $0 \leq g \leq \begin{cases} \alpha''_{n-\delta+1}(d,n), & \delta = 2 \\ \alpha''_{n-\delta+1}(d,n) - 1, & \delta \in \{3,4\}. \end{cases}$ *Then there is a (smooth, irreducible) non-degenerate curve* $C \subset P^n$ *of degree d and genus g.*

Proof: This is, actually, the bound resulting by the arguments from [P1], §1 (if $\delta = 2$) and from [CS] (if $\delta \in \{3,4\}$). See [CS], Remark 1, p.312.

* * *

Remark 3.6.5: $\alpha''_{n-3}(d,n) \leq \alpha_{n-3}(d,n) - 1 \Leftrightarrow d \neq \tau + t(2n-4)$, $\tau \in \{0,1,2,3,4\}, t \in \mathbf{Z}$.

Theorem 3.6.2 follows from Theorem 3.6.4 ($\delta = 4$) and the following

Lemma 3.6.6: *Let there be* $d, g \in \mathbf{Z}$, $n \in \mathbf{Z}$, $n \geq 7$, $d \geq 2n+1$, *so that* $\alpha''_{n-3}(d,n) \leq g \leq \alpha_{n-3}(d,n) - 1$. *Then:*

i) if $d \geq 3n-5$ *there is a (smooth, irreducible) curve* $C \subset X^n_{n-3}$, *non-degenerate in* P^n, *with* $(\deg(C), g(C)) = (d,g)$;

ii) if $2n+1 \leq d \leq 3n-6$ *there is a (smooth, irreducible) curve* $C \subset X^n_p$ ($p = [(2n-5)/3]_*$), *non-degenerate in* P^n, *with* $(\deg(C), g(C)) = (d,g)$.

Proof: i) Since $\alpha_{n-3}(d+(2n-4),n) = \alpha_{n-3}(d,n) + (d+n-4)$ (Lemma 3.3.1 a), $p = n-3$) and $\alpha''_{n-3}(d+(2n-4),n) = \alpha''_{n-3}(d,n) + (d+n-4)$ (Lemma 3.6.3 c), $\delta = 4$) we can use an argument similar to the construction of curves given in sections 3.4 and 3.5. Hence, for $x_{n-3}(d,n) = 0$ take sheaves $\mathcal{D}_0 \in Pic(X^n_{n-3})$ of the form $\mathcal{D}_0 = (a+2; a, 1^u, 0^{2n-4-u})$, then add succesively $\mathcal{H}^n_{n-3}$. So, we cover the range of i) by degree and arithmetical genus with sheaves $\mathcal{D} \in Pic(X^n_{n-3})$ of the form $\mathcal{D} = \mathcal{D}_0 + t\mathcal{H}^n_{n-3}$, $t \geq 0$. Then use the criterion (3.2.5) from Lemma 3.2.4 in order to prove that $[\mathcal{D}]$ contains (smooth, irreducible) curves.

ii) Because $2n+1 \leq d \leq 3n-6$, we have $\alpha''_{n-3}(d,n) = d-n$ and $\alpha_{n-3}(d,n) = [(3d-4n+1)/2]_*$. So, we need to cover the domain

$$(3.6.10) \qquad d-n \leq g \leq [(3d-4n-1)/2]_*, \quad 2n+1 \leq d \leq 3n-6.$$

Let there be (d,g) in the domain (3.6.10). Put $\mathcal{D} := \mathcal{D}_0 + \mathcal{H}_p^n \in Pic(X_p^n)$ ($\mathcal{H}_p^n = (p+2; p, 1^{s_p^n})$), where $\mathcal{D}_0 := (g_0+2; g_0, 1^u, 0^{s_p^n-u})$ and $d_0 := d-(n+p-1)$, $g_0 := g-(d_0+p-1) = g-d+n$, $u := 2g_0-d_0+2p+4 = 2g-3d+3p+3n+3$. Then $0 \leq u \leq s_p^n = 3p-n+5$, $\deg \mathcal{D}_0 = (\mathcal{D}_0 \cdot \mathcal{H}_p^n) = d_0$, $p_a(\mathcal{D}_0) = g_0$. So, $p_a(\mathcal{D}) = g$, $\deg(\mathcal{D}) = (\mathcal{D} \cdot \mathcal{H}_p^n) = d$. Applying now the criterion (3.2.5) from Lemma 3.2.4 for $\mathcal{D}(t=1)$, we conclude that $[\mathcal{D}]$ contains a (smooth, irreducible) curve C, non-degenerate by degree reasons.

* * *

* * *

Now, we are, finally, ready to prove the Main Theorem. This proof is an immediate consequence of equality (2.14), of Lemma 2.2 and of Theorem 3.5.11, 3.6.1 and 3.6.2.

* * *

4 Comments and further developments

1) It is easy to see that our Main Theorem remains true over an algebraically closed field of *arbitrary characteristic*. This follows replacing the Bertini theorem (which we used several times during our proof) - true in characteristic zero - by the Hartshorne's Bertini-type theorem ([Ha2], théorème 5.1) - true in arbitrary characteristic. The verifications are similar to Rathmann's verifications for curves in $\mathbf{P}^4$ and $\mathbf{P}^5$ ([Ra]).

2) We can consider other topics from Halphen-Castelnuovo theory (see §1) also. For instance, we can consider the property $\mathcal{P}$ = linear normality of (smooth, irreducible) curves from $\mathbf{P}^n$. We recall that a non-degenerate (smooth, irreducible) curve $C \subset \mathbf{P}^n$ is called *linearly normal* if it is not a projection from a bigger $\mathbf{P}^m$ containing the given $\mathbf{P}^n$. Using the Dolcetti-Pareschi ideas ([DP], §1), we can add to the curves C from our Proprosition 3.5.1 i) one hyperplane section H_p^n of X_p^n in order to obtain (smoothing $C+H_p^n$ using Bertini theorem) linearly normal curves on X_p^n in $\mathbf{P}^n$. Putting together all these curves for $n/3 \leq p \leq n-4$ (and using [CS], §4, 2)) we can obtain a big range of existence of linearly normal curves in $\mathbf{P}^n$, $n \geq 8$ (see [P4]).

3) Moreover, concerning the study of families of non-degenerate (smooth, irreducible) curves in $\mathbf{P}^n$, we can consider the Hilbert scheme $H^n_{d,g}$ (see §1). As Kleppe remarked ([Kl], §6) our results from Proposition 3.5.1 i) can be used in order to stand out "good" components of $H^n_{d,g}$ in a big range on (d,g) (here "good" means generically smooth).

4) Finally, few words about the *lacunar* domain D^n_2 from (d,g)-plane (see (2.9)). If $n \geq 7$ and $k := [n/3]_*$ (we recall that $HC(n)$ is completely solved for $3 \leq n \leq 6$) we consider the subdomains of D^n_2 given by:
$D'^n_2 : A(d,n) < g \leq \pi_0(d,n),\ d \geq 2n+1$

$$D''^n_2 : \begin{cases} \alpha_{k+1}(d,n), & \text{if } n \equiv 1,2 \pmod 3 \\ \alpha_k(d,n), & \text{if } n \equiv 0 \pmod 3 \end{cases} < g \leq A(d,n),\ 2n+1 \leq d < d_1(n)$$

(for $A(d,n)$ see (1.5), for $\alpha_p(d,n)$ see (2.1), for $d_1(n)$ see (2.6), for $\pi_p(d,n)$ see (1.1)).

Then $D^n_2 = D'^n_2 \bigcup D''^n_2$. If $(d,g) \in D'^n_2$, inspired from Harris-Eisenbud Conjecture ([H], true for $d \geq 2^{n+1}$) and using the Horrowitz results from [Ho], §1, we can see that a (smooth, irreducible) non-degenerate curve $C \subset \mathbf{P}^n$ of degree d and genus g lies on a surface X^n_p, $1 \leq p < \frac{n}{3}$ or on a scroll (possibly singular). This implies the existence of a relation between d and g, generating gaps (see also [Ci], section 2. g). If $(d,g) \in D''^n_2$, inspired again from Harris-Eisenbud conjecture and using Horrowitz results, we can see that a curve of degree d and genus g lies on a scroll or on the surface X^n_k (if $n = 3k$) or X^n_{k+1} (if $n = 3k+1,\ 3k+2$). In this case there are also gaps on X^n_k (or X^n_{k+1}) and these can be, in principle, classified trying to use a similar argument as in the proof of Théorème 2.5 from [GP2]. However, we do not insist here on D^n_2.

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