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A general method for constructing the generalized oblique projection matrices on the Diagonal Weighting algorithm

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Abstract. In the paper [1] C. Byrne and Y. Censor proposed and analysed the Diagonal Weighting algorithm for least-squares formulations of linear systems. Based on it, in [2] the authors analysed a particular version called Component Averaging method. All these are iterative algorithms based on "generalized oblique projections" generated with diagonal positive semi-definite real matrices, having as desired effect an improvement of convergence of the classical versions (based on orthogonal projections). In the present paper we analyse a general method for constructing the generalized oblique projection matrices. We show that our method includes as a particular case the way proposed in [2] and we also offer another possibility of construction. With respect to the numerical experiments described in the last section of the paper we may conclude that our method is better than the one mentioned in the above cited paper.

1. Generalized oblique projections and the diagonal weighting algorithm

Let A be an $m \times n$ (sparse) real matrix, $b \in I\!R^m, a_i = (a_{i1}, .., a_{in})^t \in I\!R^n$ the i-th row of A and $b_i \in R$ the i-th component of b. We shall denote by $< \cdot, \cdot >$ and $\| \cdot \|$ the Euclidean scalar product and the associated norm , respectively. With these notations we shall define the hyperplane $H_i = \{x \in I\!R^n, < x, a_i >= b_i\}$ and subspace $S_i = \{x \in I\!R^n, < x, a_i >= 0\}$ associated to the i-th equation of linear system

$$Ax = b. \tag{1}$$

We shall suppose through the whole paper that the system (1) is consistent. If G is a diagonal positive semi-definite $n \times n$ matrix $G = diag(g_1, g_2, \dots, g_n)$, $g_j \geq 0, \ \forall \ j = 1, .., n$ we shall denote by G^{-1} the diagonal matrix

$$(G^{-1})_{ij} = \begin{cases} \frac{1}{g_j}, & \text{if} \quad i = j, \ g_j \neq 0 \\ 0, & \text{else} \end{cases} \tag{2}$$

and by $< \cdot, \cdot >_G$, $\| \cdot \|_G$ the scalar semi-product and the associated semi-norm defined by

$$< x, y >_G = \sum_{j=1}^{n} g_j x_j y_j, \ \| x \|_G^2 = < x, x >_G, \ \forall \ x, y \in I\!R^n. \tag{3}$$

With these notations we can define the **generalized oblique projection** of a point $x \in I\!R^n$ onto H_i or S_i with respect to G by

$$P_{H_i}^G(x) = x + \frac{b_i - < x, a_i >}{< a_i, a_i >_{G^{-1}}} G^{-1} a_i \tag{4}$$

and, respectively

$$P_{S_i}^G(x) = x - \frac{< x, a_i >}{< a_i, a_i >_{G^{-1}}} G^{-1} a_i. \tag{5}$$

Remark 1 *If $g_j \neq 0, \ \forall \ j = 1, \ldots, n$ then G^{-1} is the classical inverse of G and the elements defined in (3) the well known* ***energy scalar product*** *and the corresponding* ***energy norm*** *(which are similarly defined for any symmetric and positive definite matrix G). If $G = I$, then (4) and (5) represent the orthogonal projections of x onto H_i and S_i, respectively and we shall denote them as follows*

$$P_{H_i}(x) = x + \frac{b_i - < x, a_i >}{\| a_i \|^2} a_i, \quad P_{S_i}(x) = x - \frac{< x, a_i >}{\| a_i \|^2} a_i. \tag{6}$$

A family $\{G_i\}_{i=1,\ldots,m}$ of real diagonal $n \times n$ matrices such that

$$G_i = diag(g_{i1}, g_{i2}, \ldots, \ g_{in}), \ g_{ij} \geq 0, \ \sum_{i=1}^{m} G_i = I \tag{7}$$

(with I the unit matrix) will be called **sparsity pattern oriented** (**SPO**, for short) with respect to the matrix A if for every $i = 1, \ldots, m, j = 1, \ldots, n$, we have $g_{ij} = 0$ if and only if $a_{ij} = 0$.
With the above constructed elements the authors considered in [2] the **Diagonal Weighting** algorithm (**DW**, for short) for the system (1) defined by: let $x^0 \in I\!R^n$ be the initial approximation and for $k = 0, 1, \ldots$ do

$$x^{k+1} = x^k + \lambda_k \sum_{i=1}^{m} G_i (P_{H_i}^{G_i}(x^k) - x^k), \tag{8}$$

where $\lambda_k \in (0, 2)$ are relaxation parameters.

Remark 2 *If the matrices G_i are defined by*

$$G_i = diag(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}),\ \forall\ i = 1, \ldots, m \tag{9}$$

*then the **(DW)** algorithm (8) becomes the Cimmino's method (see [2]) and has the following form*

$$x^{k+1} = x^k + \frac{\lambda_k}{m} \sum_{i=1}^{m} (P_{H_i}(x^k) - x^k). \tag{10}$$

Remark 3 *If for $j = 1, \ldots, n$ the numbers $s_j \in \{0, 1, \ldots, m\}$ are defined by*

$$s_j = card(\{i \in \{1, .., m\}, a_{ij} \neq 0\}) \tag{11}$$

and the elements of G_i by

$$g_{ij} = \begin{cases} \frac{1}{s_j}, & if \quad a_{ij} \neq 0 \\ 0, & if \quad a_{ij} = 0 \end{cases} \tag{12}$$

*then the method (8) will be called the **Component Averaging** algorithm (**CAV**, for short) (see [2]).*

Remark 4 *An oblique projections version of the classical Jacobi projections method (see [3]) (similar with (10), but with $P_{H_i}^{G_i}$ instead of P_{H_i}, with G an arbitrary symmetric and positive definite matrix) was analysed in [4] as a particular case of the preconditioned algorithms from [5].*

In [2] the following result is proved (particular statement).

Theorem 1 *With all the above defined elements, if the system (1) is consistent and $\lambda_k = 1$, $\forall\ k \geq 0$ then the sequence $(x^k)_{k \geq 0}$ generated with the **(DW)** algorithm (8) converges to one of its solutions.*

Cimmino's (simultaneous) algorithm (10), for the case of large systems of the form (1) with a sparse matrix A, exhibits a very slow convergence rate, due to the fact that the change between succesive iterates is relatively small. This was the reason in [2] for considering the (**CAV**) algorithm. But, from the theoretical and practical considerations presented there, we can not conclude why the use of generalized oblique projections (4) instead the classical ones (6) will ensure a better behaviour of the **(CAV)** algorithm (11)-(12) "against" the classical Cimmino's one (10). We shall present a clear and simple explanation of this fact in the next section of the paper. It is based on "strenghtened" Cauchy inequalities involving the elements of the matrices A and $G_i, i = 1, \ldots, m$. We also constructed a family of matrices G_i as in (7) for which these inequalities are satisfied. In the last section of the paper we describe some numerical experiments on a finite-differences discretization of the one-dimensional steady state heat transfer equation. These experiments show that our choice for the G_i generalized oblique projection matrices gives better results than the choice (11)-(12) used by the authors in [2].

2. Strenghtened Cauchy inequalities

We shall start this section of the paper by observing that, exactly as in [2], Lemma 4.3 we can prove that for any $i \in \{1, \ldots, m\}, z \in S_i$ and any $y \in I\!R^n$ we have

$$\| P_{S_i}^{G_i}(y) - y \|_{G_i}^2 \leq \| z - y \|_{G_i}^2 - \| z - P_{S_i}^{G_i}(y) \|_{G_i}^2 . \tag{13}$$

Let us now denote by $(\tilde{x}^k)_{k\geq 0}, (\tilde{x}^0 = x^0)$ the sequence generated with the algorithm (8) and by $(x^k)_{k\geq 0}$ the sequence from (10). We shall suppose that $\lambda_k = 1, \ \forall \ k \geq 1$ and that at the k-th iteration the error vectors $\tilde{e}^k, e^k \in I\!R^n$, defined by

$$\tilde{e}^k = \tilde{x}^k - x^*, \ \ e^k = x^k - x^* \tag{14}$$

are equal (as it happens for $k = 0$): $\tilde{e}^k = e^k = e$, where x^* is a solution of the consistent system (1). Then, because of the obvious equalities $P_{H_i}^{G_i}(x^*) = P_{H_i}(x^*) = x^*$ and by using (8) and (10) we obtain

$$\tilde{e}^{k+1} = \frac{1}{m} \sum_{i=1}^{m} P_{S_i}(e), \ \ e^{k+1} = \sum_{i=1}^{m} G_i P_{S_i}^{G_i}(e). \tag{15}$$

Then, by taking norms and using the properties of the matrices G_i from (7) we obtain

$$\| \tilde{e}^{k+1} \| \leq \max_{1\leq i\leq m} \| P_{S_i}(e) \|, \ \ \| e^{k+1} \| \leq \max_{1\leq i\leq m} \| P_{S_i}^{G_i}(e) \| . \tag{16}$$

In order to obtain an acceleration by using the **(DW)** algorithm (8) instead of (10) we would like to have

$$\| e^{k+1} \| < \| \tilde{e}^{k+1} \|, \tag{17}$$

but this inequality is very hard to be analysed. Then, going back to (16) we observe that $\tilde{e}^{k+1}$ and e^{k+1} lie in two balls with origin as center and radii $\max_{1\leq i\leq m} \| P_{S_i}(e) \|$ and $\max_{1\leq i\leq m} \| P_{S_i}^{G_i}(e) \|$, respectively. Thus, it seems to be quite natural to firstly compare $\| P_{S_i}(e) \|$ and $\| P_{S_i}^{G_i}(e) \|$. This will be done in the next result.

Proposition 2 *In the above hypothesis we have*

$$\| P_{S_i}^{G_i}(e) \|^2 - \| P_{S_i}(e) \|^2 \leq -2 < e - P_{S_i}^{G_i}(e), P_{S_i}^{G_i}(e) > \leq 0. \tag{18}$$

PROOF. Let $f^i, f^{i,G_i} \in R^n$ be defined by

$$f^i = e - P_{S_i}(e), \;\; f^{i,G_i} = e - P_{S_i}^{G_i}(e). \tag{19}$$

Firstly we shall observe that, because $P_{S_i}(e)$ is the ortogonal projection of e onto S_i and $P_{S_i}^{G_i}(e) \in S_i$ we always have

$$\| f^{i,G_i} \| = \| e - P_{S_i}^{G_i}(e) \| \geq \| e - P_{S_i}(e) \| = \| f^i \| . \tag{20}$$

Thus, using (19), (20) and the following obvious equalities

$$\| P_{S_i}^{G_i}(e) \|^2 = \| e \|^2 - \| f^{i,G_i} \|^2 - 2 < P_{S_i}^{G_i}(e), f^{i,G_i} >, \tag{21}$$

$$\| P_{S_i}(e) \|^2 = \| e \|^2 - \| f^i \|^2 \tag{22}$$

we obtain

$$\| P_{S_i}^{G_i}(e) \|^2 - \| P_{S_i}(e) \|^2 =$$

$$-2 < f^{i,G_i}, P_{S_i}^{G_i}(e) > - (\| f^{i,G_i} \|^2 - \| f^i \|^2) \leq -2 < f^{i,G_i}, P_{S_i}^{G_i}(e) >, \tag{23}$$

i.e. the first inequality in (18). For the second one, by replacing z and y in (13) with $e - f^{i,G_i} \in S_i$ and $-f^{i,G_i}$, respectively and using the obvious equality

$$P_{S_i}^{G_i}(f^{i,G_i}) = 0 \tag{24}$$

we get

$$\| e \|_{G_i}^2 \geq \| P_{S_i}^{G_i}(e) \|_{G_i}^2 + \| f^{i,G_i} \|_{G_i}^2 . \tag{25}$$

From here, by summing following i and using the equality (see [2], formula (4.6))

$$\sum_{i=1}^{m} \| z \|_{G_i}^2 = \| z \|^2 \tag{26}$$

we obtain

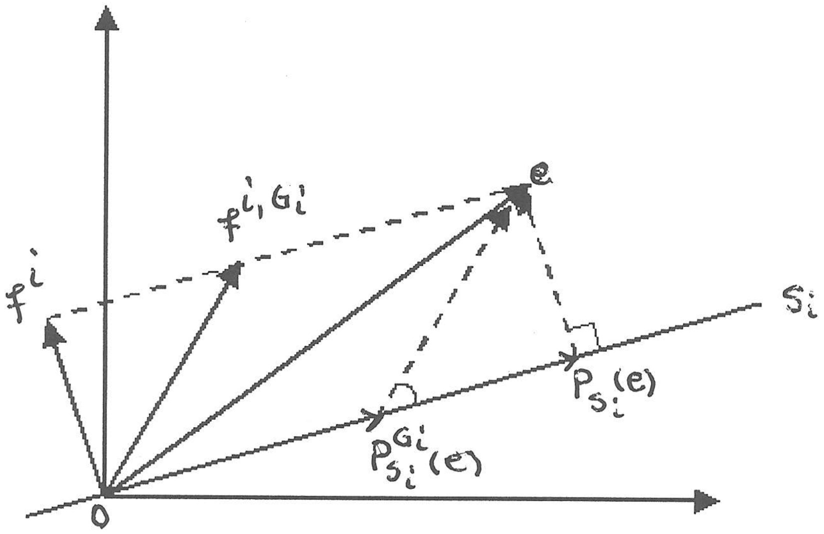
$$\| e \|^2 \geq \| P_{S_i}^{G_i}(e) \|^2 + \| f^{i,G_i} \|^2 \tag{27}$$

from which the inequality

$$< f^{i,G_i}, P_{S_i}^{G_i} > \leq 0 \tag{28}$$

obviously holds and the proof is complete.

Remark 5 *From a geometrical view point, the relation (28) tells us that the generalized oblique projection made with G_i has a "projection angle" less than $90°$ (see also the figure below).*



Thus, what we would like to have would be to get "strenghtened" (strict) inequalities in (18). This will happen if the term $\| f^{i,G_i} \|^2 - \| f^i \|^2$ in (23) will be strictly greater than 0. Thus, by taking into account (20) we would like to exist a constant $\beta > 1$ such that

$$\| f^{i,G_i} \| \geq \beta \| f^i \|, \ \forall\, i = 1, \ldots, m. \tag{29}$$

For this, let $i \in \{1, \ldots, m\}$ be arbitrary fixed and $I(i) \subset \{1, \ldots, m\}$ defined by

$$I(i) = \{j \in \{1, \ldots, m\}, a_{ij} \neq 0\}. \tag{30}$$

Then using (5), (6) and (19) we can write the inequality (29) as follows

$$\beta < a_i, a_i >_{G_i^{-1}} \leq \| a_i \| \| \| G_i^{-1} a_i \|$$

or equivalently

$$\sum_{j\in I(i)} \frac{a_{ij}^2}{g_{ij}} \leq \gamma \sqrt{\sum_{j\in I(i)} a_{ij}^2} \sqrt{\sum_{j\in I(i)} \frac{a_{ij}^2}{g_{ij}^2}}, \tag{31}$$

where $\gamma = \frac{1}{\beta} \in [0, 1)$, i.e. a "strenghtened" Cauchy ineguality. It will hold (independently on i) for the (minimal) value of γ given by

$$\gamma = \max_{1\leq i\leq m} \frac{\sum_{j\in I(i)} \frac{a_{ij}^2}{g_{ij}}}{\sqrt{\sum_{j\in I(i)} a_{ij}^2} \sqrt{\sum_{j\in I(i)} \frac{a_{ij}^2}{g_{ij}^2}}}. \tag{32}$$

Then, taking into account all the above considerations we can conclude that for a choice of the matrices $G_i = diag(g_{i1}, ..., g_{in})$ for which the constant γ from (32) is strictly less then 1 (i.e. $\beta = \frac{1}{\gamma} > 1$) we may expect better convergence properties of the (**DW**) algorithm (8). In the next section of the paper we shall numerically analyse these aspects for the G_i 's choice proposed in [2] and another one suggested by the author.

3. Numerical experiments

We considered in our experiments the one-dimensional stady state heat transfer equation

$$\begin{cases} u''(x) - \alpha u'(x) = 0, & \text{if } \ x \in (0, 1) \\ u(0) = u_0, \ u(1) = u_1 \end{cases} \tag{33}$$

discretized by a classical (centered) finite defferences scheme. Thus, for $n \geq 2, h = \frac{1}{n}$ (the mesh size) and after we eliminated the boundary conditions we obtained the $(n-1) \times (n-1)$ sparse, nonsingular, nonsymmetric system

$$Ax = b, \tag{34}$$

with A and b given by

$$A = \begin{bmatrix} 2 & a_h & & & & \\ c_h & 2 & a_h & & & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & c_h & 2 & a_h \\ & & & & c_h & 2 \end{bmatrix}, \tag{35}$$

where $a_h = -1 + \frac{\alpha h}{2}$, $c_h = -1 - \frac{\alpha h}{2}$, $b = \left(1 + \frac{\alpha h}{2}, 0, \ldots, 0, 1 - \frac{\alpha h}{2}\right)$. It is well known that, in order to get a stable solution by solving (34) the Pèclet number $Pe = \alpha h$ must satisfy

$$Pe \in [0, 2). \tag{36}$$

For solving (34) we used the **(DW)** algorithm (8) with the following two choices for G_i:
Case I $G_i^1 = diag(g_{i1}^1, \ldots, g_{in}^1)$ and

$$g_{ij}^1 = \begin{cases} \frac{1}{s_j}, & \text{if} \quad a_{ij} \neq 0 \\ 0, & \text{if} \quad a_{ij} = 0 \end{cases} \tag{37}$$

and s_j from (11) (i.e. the **(CAV)** algorithm (11)-(12))
and
Case II $G_i^2 = diag(g_{i1}^2, \ldots, g_{in}^2)$ with

$$g_{ij}^2 = \frac{|a_{ij}|}{\sum_{k=1}^{m} |a_{kj}|}. \tag{38}$$

We denoted by $\gamma_i(n, Pe)$, $i = 1, 2$ the corresponding constants calculated as in (32) for different values of $Pe \in [0, 2)$. The results are described in Table 1 bellow (they are independent on n). We observe that in all tests we obtained

$$0 < \gamma_2(n, Pe) < 1 = \gamma_1(n, Pe). \tag{39}$$

After these evaluations we considered the corresponding **(DW)** algorithms which we respectively denoted by **(DW1)** and **(DW2)**. We applied them for solving the sistem (34) for different values of n and Pe with the stopping test

$$\| Ax^k - b \| \leq h. \tag{40}$$

The results described in Table 2 confirm the fact that a relation like (39) determines a better behaviour of the **(DW2)** algorithm than **(DW1)** one.

Final remarks 1. The stopping test (40) was choosen such the error between x^k and the exact solution of (33) was 10^{-1} (which is good enough for many practical problems).
2. We observe in Table 1 that even for values of Pe closer to 2 (i.e. "unstable numerical solution") the behaviour of the **DW2** algorithm rests as for the other values of $Pe \in [1, 2)$.
3. The numerical experiments were made with the numerical linear algebra programs package OCTAVE (free available on Internet).

Table 1. The values of $\gamma_1(n, Pe)$ and $\gamma_2(n, Pe)$		
Pe	$\gamma_1(n, Pe)$	$\gamma_2(n, Pe)$
0.1	1	0.95961
0.5	1	0.96761
0.8	1	0.96861
1	1	0.96716
1.2	1	0.96587
1.4	1	0.96012
1.6	1	0.95502
1.8	1	0.94917
1.9	1	0.94782
1.95	1	0.94827

Table 2. The behaviour of the algorithms (DW1) and (DW2)			
n	Pe	Nr. iter. (**DW1**)	Nr. iter. (**DW2**)
32	0.1	36	34
32	0.5	43	39
32	0.8	55	47
32	1	61	54
32	1.2	76	62
32	1.4	88	69
32	1.6	101	76
32	1.8	115	83
32	1.9	122	86
32	1.95	126	88
64	1	166	138
64	1.8	292	208
64	1.95	320	220

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