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ELENA ALINA SUCIU*

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* Polytechnical University of Bucharest, Chair Mathematics I, Calea Independentei 313, Bucharest, Romania.
E–mail: esuciu@miron.imar.ro

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NAGY-FOIAŞ DIAGRAM FOR HILBERT MODULES OVER THE POLYDISK ALGEBRA *

ELENA ALINA SUCIU

To every Agler type analytic operator-valued function Θ on $\mathbb{D}^N$ we associate a unique Nagy-Foiaş diagram. We show that the modeling morphism coresponding to this diagram coincides with Θ. This is a a generalization to the case of the polydisk algebra of the Nagy-Foiaş model for contractions developed in the case of the disk algebra.

The Nagy-Foiaş diagram for a contractive Hilbert module M over a function algebra A describes in case it exists the geometry of the minimal spectral dilation of the module M, cf. [6]. There is a one-to-one corespondence between Nagy-Foiaş diagrams and a special class of $A-$module maps called modeling morphisms, cf. [6], [7]. This rather abstract description of Nagy-Foiaş diagrams can be expressed in terms of functions in special cases. When A is the disk algebra there is a way to attach to every purely contractive analytic function on $\mathbb{D}$ a unique Nagy-Foiaş diagram for a Hilbert module generated by a contraction T. To this diagram coresponds a unique modeling morphism which turns out to be exactly the characteristic function of T, cf. [4]. We look for such a description in terms of functions in the case of the polydisk algebra. More precisely, we start with a function from the Agler-Schur class, cf. [1], i.e. an analytic function on $\mathbb{D}^N$ which has a certain factorization and we associate to it a unique Nagy-Foiaş diagram, hence a unique modeling morphism. We show that this modeling morphism coincides with Θ in a certain sense. Our work is based on papers [1] and [3]. J. Agler used in [1] the factorization of Θ to construct N contractions $A_1, \cdots, A_N$. They do not commute, but they generate N commuting contractions $\mathcal{A}_1, \cdots, \mathcal{A}_N$ which have a minimal spectral dilation, see [3]. Contractions $\mathcal{A}_1, \cdots, \mathcal{A}_N$ still depend on the factorization of Θ. The interesting thing is that the Nagy-Foiaş diagram they generate is unique. This diagram is constructed imposing a purity condition on Θ.

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1 Preliminaries.

Let M be a contractive Hilbert module over a function algebra A. Denote by M_* the adjoint Hilbert module of M with the multiplication given by $\overline{f}$ for $f \in A$. We say that M admits Nagy-Foiaş diagram if we can construct the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_1 & \longrightarrow & S_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_1 & \longrightarrow & K & \longrightarrow & R_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & R_1 & = & R_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1.1}
$$

where K is a minimal spectral dilation of M, S_0, R_0 are the minimal subspectral dilations of M and M_* adiacent to K, respectively K_*, and S_1, R_1 are the orthogonal complements of M in S_0, respectively R_0. Arrows in the above diagram are inclusions or orthogonal projections $A-$module maps. The necessary and sufficient condition for M to admit Nagy-Foiaş diagram is that $S_0 \vee R_0 = K$. In case A is the disk algebra, then Nagy-Foiaş diagram connects the minimal unitary and isometric dilations of T and T^* respectively, where T is the contraction that defines the multiplication on M.

To every Nagy-Foiaş diagram one can associate a special class of $A-$ module maps, called modeling morphisms. Namely, we take $\Phi = P^K_{\widetilde{R_1}}|S_1$, where $\widetilde{R_1}$ is the minimal spectral extension of R_1. In the case of the disk algebra Φ coincides with the characteristic function of the contraction T.

We want to construct the Nagy-Foiaş diagram and its coresponding modeling morphism for the case of the polydisk algebra. To do this we need some results, definitions and notations from [1] and [3] which we present below.

Consider the Agler-Schur class of analytic functions $\Theta(\lambda)$ on $\mathbb{D}^N$ whose values are bounded operators from a Hilbert space E to a Hilbert space F, both separable, for which there exist Hilbert spaces H_i with $i = 1, \cdots, N$ and analytic functions F_i defined on $\mathbb{D}^N$ with values bounded operators from E to H_i such that

$$I_E - \Theta(\lambda)^*\Theta(z) = \sum_{i=1}^{N}(1 - \overline{\lambda}_i z_i)F_i(\lambda)^*F_i(z), \ \lambda, z \in \mathbb{D}^N. \tag{1.2}$$

Denote $H = \oplus_{i=1}^N H_i$. For a $N-$tuple of operators $T_1, \cdots, T_N$ we shall use notation $\mathbf{T} = (T_1, \cdots, T_N)$ and for $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{T}^N$ we denote $\xi\mathbf{T} = \sum_{i=1}^{N} \xi_i T_i$.

J. Agler showed in [1] that $\Theta(\lambda)$ can be factorized as (1.2) if and only if there exist N

unitary operators

$$G_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} : \begin{matrix} H \\ \oplus \\ E \end{matrix} \to \begin{matrix} H \\ \oplus \\ F \end{matrix}$$

such that $\xi\mathbf{G} = \sum_{i=1}^N \xi_i G_i$ is a unitary operator for any $\xi \in \mathbb{T}^N$. Moreover

$$\Theta(\lambda) = \lambda\mathbf{D} + \lambda\mathbf{C}(I_H - \lambda\mathbf{A})^{-1}\lambda\mathbf{B}, \ \ \lambda \in \mathbb{D}^N.$$

For $t \in \mathbb{Z}^N$ denote $|t| = \sum_{i=1}^N t_i$, $\widetilde{\mathbb{Z}}_+^N := \{t \in \mathbb{Z}^N : |t| \geq 0,\}$, $\widetilde{\mathbb{Z}}_0^N := \{t \in \mathbb{Z}^N : |t| = 0\}$, $\widetilde{\mathbb{Z}}_-^N := \{t \in \mathbb{Z}^N : |t| \leq 0\}$ and $\mathbb{Z}_+^N := \{t \in \mathbb{Z}^N : t_i \geq 0, \ i = 1, \cdots, N\}$. Consider $s = (s_1, s_2, \cdots, s_N) \in \mathbb{Z}_+^N$ and let σ be a permutation of the set

$$(\underbrace{1, \ \cdots, \ 1}_{s_1}, \underbrace{2, \ \cdots, \ 2}_{s_2}, \cdots, \underbrace{N, \ \cdots, \ N}_{s_N}).$$

Denote by $P_{|s|}$ the set of all this permutations. Then the number of elements of $P_{|s|}$ is

$$c_s := \frac{|s|!}{|s_1|!|s_2|! \cdots |s_N|!}, s \in \mathbb{Z}_N^+.$$

The symmetrized multipower of the $N-$tuple $\mathbf{A} = (A_1, \cdots, A_N)$ is

$$\mathbf{A}^s := c_s^{-1} \sum_{\sigma \in P_{|s|}} A_{\sigma(1)} \cdots A_{\sigma(|s|)}, \ \ s \in \mathbb{Z}_N^+ - \{0\}.$$

In this notations if one component s_i of s is equal to zero, then operator A_i does not appear in any terms of $\mathbf{A}^s$. In the case of a commutative $N-$tuple $\mathbf{A} = (A_1, \cdots, A_N)$ we obtain the usual multipower of $\mathbf{A}$, namely $\mathbf{A}^s = A_1^{s_1} \cdots A_N^{s_N}$. Also, we introduce the following notations:

$$(\mathbf{A}\#\mathbf{B})^s := c_s^{-1} \sum_{\sigma \in P_{|s|}} A_{\sigma(1)} \cdots A_{\sigma(|s|-1)} B_{\sigma(|s|)}, \ \ s \in \mathbb{Z}_N^+ - \{0\},$$

$$(\mathbf{C}\&\mathbf{B})^s := c_s^{-1} \sum_{\sigma \in P_{|s|}} C_{\sigma(1)} B_{\sigma(2)}, \ \ |s| = 2, \ s \in \mathbb{Z}_N^+,$$

$$(\mathbf{C}\flat\mathbf{A}\#\mathbf{B})^s := c_s^{-1} \sum_{\sigma \in P_{|s|}} C_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(|s|-1)} B_{\sigma(|s|)}, \ \ |s| \geq 3, \ s \in \mathbb{Z}_N^+.$$

Then, from [3] we know that $\Theta(\lambda)$ has the power expansion

$$\Theta(\lambda) = \lambda\mathbf{D} + \sum_{\substack{|s| = 2 \\ s \in \mathbb{Z}_+^N}} \lambda^s (\mathbf{C}\&\mathbf{B})^s + \sum_{\substack{|s| \geq 3 \\ s \in \mathbb{Z}_+^N}} \lambda^s (\mathbf{C}\flat\mathbf{A}\#\mathbf{B})^s, \lambda \in \mathbb{D}^N. \tag{1.3}$$

We want to associate to $\Theta(\lambda)$ a contractive Hilbert module. For this consider Hilbert spaces

$$\mathcal{H} = l^2(\widetilde{\mathbb{Z}}_0^N, \mathcal{H}),\ \mathcal{E} = l^2(\widetilde{\mathbb{Z}}_+^N, E),\ \mathcal{F} = l^2(\widetilde{\mathbb{Z}}_-^N, F),\ \mathcal{E}_\sim = l^2(\widetilde{\mathbb{Z}}_0^N, E),\ \mathcal{F}_\sim = l^2(\widetilde{\mathbb{Z}}_0^N, F).$$

For an arbitrary Hilbert space H consider the space of square integrable functions $L^2(\mathbb{T}^N, H)$. Then the Fourier transform is

$$\Phi^H : l^2(\mathbb{Z}^N, H) \to L^2(\mathbb{T}^N, H),\ \ (\Phi^H y)(\xi) = \sum_{t \in \mathbb{Z}^N} y(t)\xi^t,\ \ \xi \in \mathbb{T}^N.$$

Φ^H is a unitary and its restrictions to $l^2(\widetilde{\mathbb{Z}}_0^N, H)$ takes values to the subspace of $L^2(\mathbb{T}^N, H)$ of functions with vanishing Fourier coefficients for multiindeces t which are not in $\widetilde{\mathbb{Z}}_0^N$.

As in [3] we define operators

$$\mathcal{A}_i : \mathcal{H} \to \mathcal{H},\ \ \mathcal{A}_i x(t) = \sum_{j=1}^N A_j x(t + e_i - e_j).$$

$$\mathcal{B}_i : \mathcal{E}_\sim \to \mathcal{H},\ \ \mathcal{B}_i v(t) = \sum_{j=1}^N B_j x(t + e_i - e_j).$$

$$\mathcal{C}_i : \mathcal{H} \to \mathcal{F},\ \ \mathcal{C}_i x(t) = \sum_{j=1}^N C_j x(t + e_i - e_j).$$

$$\mathcal{D}_i : \mathcal{E}_\sim \to \mathcal{F},\ \ \mathcal{D}_i v(t) = \sum_{j=1}^N D_j x(t + e_i - e_j),$$

for $t \in \mathbb{Z}_0^N$ and

$$e_i = (0, \cdots, \underset{i}{1}, \cdots, 0).$$

We show that $\mathcal{A}_1,\ \mathcal{A}_2,\ \cdots,\ \mathcal{A}_N$ are contractions. Since

$$\xi\mathbf{G} = \begin{pmatrix} \xi\mathbf{A} & \xi\mathbf{B} \\ \xi\mathbf{C} & \xi\mathbf{D} \end{pmatrix} : \begin{matrix} H \\ \oplus \\ E \end{matrix} \to \begin{matrix} H \\ \oplus \\ F \end{matrix}$$

is a unitary operator for any $\xi \in \mathbb{T}^N$, it results that $\xi\mathbf{A} = P_H \xi\mathbf{G}|H$ is a contraction. Then we have

$$||\mathcal{A}_i x||^2 = ||\Phi^{\mathcal{H}} \mathcal{A}_i x||^2 = \sum_{t \in \widetilde{\mathbb{Z}}_0^N} ||\mathcal{A}_i x(t)\xi^t||^2 = \sum_{t \in \widetilde{\mathbb{Z}}_0^N} ||\sum_{j=1}^N A_j x(t + e_i - e_j)\xi^t||^2 =$$

$$= \sum_{t \in \widetilde{\mathbb{Z}}_0^N} ||\xi_i^{-1} \sum_{j=1}^N \xi_j A_j x(t)\xi^t||^2 = \sum_{t \in \widetilde{\mathbb{Z}}_0^N} ||(\xi A)x(t)\xi^t||^2 \leq \sum_{t \in \widetilde{\mathbb{Z}}_0^N} ||x(t)\xi^t||^2 = ||x||^2.$$

It is easy to see that we have $\mathcal{A}_i\mathcal{A}_j = \mathcal{A}_j\mathcal{A}_i$, for $i \neq j$. In [3] D. S. Kalyuzhniy constructed a unitary dilation $\mathcal{W}_1,\ \mathcal{W}_2,\ \cdots \mathcal{W}_N$ of the contractions $\mathcal{A}_1,\ \mathcal{A}_2,\ \cdots \mathcal{A}_N$ in the following way. Denote $\mathcal{K} = \mathcal{F} \oplus \mathcal{H} \oplus \mathcal{E}$ and set $\mathcal{W}_i : \mathcal{K} \to \mathcal{K},\ \ \mathcal{W}_i(v, x, u) = (v_i, x_i, u_i)$ with

$$v_i(t) = \begin{cases} v(t+e_i), & \text{for } |t| \leq -1 \\ \mathcal{C}_i x(t) + \mathcal{D}_i u(t), & \text{for } |t| = 0 \end{cases},$$

$$x_i(t) = \mathcal{A}_i x(t) + \mathcal{B}_i u(t), \text{for } |t| = 0,$$

$$u_i(t) = u(t+e_i), \text{for } |t| \geq 0.$$

The adjoint of $\mathcal{W}_i$ is $\mathcal{W}_i^* : \mathcal{K} \to \mathcal{K},\ \ \mathcal{W}_i^*(v, x, u) = (v_i, x_i, u_i)$ with

$$v_i(t) = v(t-e_i), \text{for } |t| \leq 0,$$

$$x_i(t) = \mathcal{A}_i^* x(t) + \mathcal{C}_i^* v(t), \text{for } |t| = 0,$$

$$u_i(t) = \begin{cases} u(t-e_i), & \text{for } |t| \geq 1 \\ \mathcal{B}_i^* x(t) + \mathcal{D}_i^* v(t), & \text{for } |t| = 0 \end{cases}$$

Then $\mathcal{W}_i$ are unitary operators which pairwise commute, hence they generate a $\mathbb{T}^N-$spectral $A(\mathbb{D}^N)-$module structure on $\mathcal{K}$. Since $\mathcal{W}_1,\ \mathcal{W}_2,\ ,\cdots \mathcal{W}_N$ is a unitary dilation of $\mathcal{A}_1,\ \mathcal{A}_2,\ \cdots \mathcal{A}_N$ we conclude that contractions $\mathcal{A}_i$ satisfy the von Neumann inequality, hence they generate a contractive Hilbert module structure on $\mathcal{H}$. Clearly, $\mathcal{K}$ is the spectral dilation of $\mathcal{H}$.

2 The Nagy-Foiaş diagram.

In this section we construct the unique Nagy-Foiaş diagram generated by Θ.

Proposition 1 *If $\mathcal{F}_\sim = \vee_{i=1}^N \mathcal{C}_i\mathcal{H}$ and $\mathcal{E}_\sim = \vee_{i=1}^N \mathcal{B}_i^*\mathcal{H}$, then $\mathcal{K} = \cdots \oplus \mathcal{F}_\sim \oplus \mathcal{F}_\sim \oplus \mathcal{H}_\sim \oplus \mathcal{E}_\sim \oplus \mathcal{E}_\sim \cdots$ is the minimal spectral dilation of $\mathcal{H}$.*

Proof. For conventional notation we give the proof for $N = 2$. Denote $\mathcal{K}_+ = \mathcal{F} \oplus \mathcal{H}$. We show that $\overline{A \cdot_\mathcal{K} \mathcal{H}} = \mathcal{K}_+$. For the inclusion $\overline{A \cdot_\mathcal{K} \mathcal{H}} \subseteq \mathcal{K}_+$ it is sufficient to show that $z_1^m z_2^n(0, x, 0) \in \mathcal{K}_+$, for any $m, n \in \mathbb{Z}_+$ and any $x \in \mathcal{H}$. We have

$$z_1^m(0, x, 0) = \mathcal{W}_1^m(0, x, 0) = (v', x', 0)$$

with

$$v'(t) = \begin{cases} 0, & |t| \leq -m \\ \mathcal{C}_1 x(t), & |t| = -(m-1) \\ \mathcal{C}_1\mathcal{A}_1 x(t), & |t| = -(m-2) \\ \mathcal{C}_1\mathcal{A}_1^2 x(t), & |t| = -(m-3) \\ \cdots & \\ \mathcal{C}_1\mathcal{A}_1^{m-1} x(t), & |t| = 0 \end{cases},$$

$$x'(t) = \mathcal{A}_1^m x(t), \quad |t| = 0.$$

In the same way

$$z_1^m z_2^n (0, x, 0) = \mathcal{W}_1^m \mathcal{W}_2^n (0, x, 0) = \mathcal{W}_2^n \mathcal{W}_1^m (0, x, 0) = (v', x', 0)$$

with

$$v'(t) = \begin{cases} 0, & |t| \leq -(m-n-2) \\ \mathcal{C}_2 x(t), & |t| = -(m-n-1) \\ \mathcal{C}_2 \mathcal{A}_2 x(t), & |t| = -(m-n) \\ \cdots & \\ \mathcal{C}_2 \mathcal{A}_2^{n-1} x(t), & |t| = -m \\ \mathcal{C}_1 \mathcal{A}_2^n x(t), & |t| = -(m-1) \\ \mathcal{C}_1 \mathcal{A}_1 \mathcal{A}_2^n x(t), & |t| = -(m-2) \\ \mathcal{C}_1 \mathcal{A}_1^2 \mathcal{A}_2^n x(t), & |t| = -(m-3) \\ \cdots & \\ \mathcal{C}_1 \mathcal{A}_1^{m-1} \mathcal{A}_2^n x(t), & |t| = 0 \end{cases},$$

$$x'(t) = \mathcal{A}_1^m \mathcal{A}_2^n x(t), \quad |t| = 0.$$

Hence, it is clear that $z_1^m z_2^n (0, x, 0) \in \mathcal{K}_+$. To prove the reverse inclusion, namely $\mathcal{K}_+ \subseteq \overline{A \cdot_{\mathcal{W}} \mathcal{H}}$, note that if we take $(v, 0, 0) \in \mathcal{K}_+$ with

$$v(t) = \begin{cases} \mathcal{C}_1 x(t), & |t| = -(m-1) \\ 0, & |t| \neq -(m-1) \end{cases},$$

then we have

$$(v, 0, 0) = \mathcal{W}_1^m (0, x, 0) - \mathcal{W}_1^{m-1} \mathcal{A}_1 (0, x, 0) = z_1^m \cdot_{\mathcal{K}} (0, x, 0) - z_1^{m-1} \cdot_{\mathcal{K}} z_1 \cdot_{\mathcal{H}} (0, x, 0) \in \overline{\mathcal{A} \cdot_{\mathcal{K}} \mathcal{M}}.$$

Also, for $(v, 0, 0) \in \mathcal{K}_+$ with

$$v(t) = \begin{cases} \mathcal{C}_2 x(t), & |t| = -(n-1) \\ 0, & |t| \neq -(n-1) \end{cases},$$

we have

$$(v, 0, 0) = \mathcal{W}_2^n (0, x, 0) - \mathcal{W}_2^{n-1} \mathcal{A}_2 (0, x, 0) = z_2^n \cdot_{\mathcal{K}} (0, x, 0) - z_2^{n-1} \cdot_{\mathcal{K}} z_2 \cdot_{\mathcal{H}} (0, x, 0) \in \overline{\mathcal{A} \cdot_{\mathcal{K}} \mathcal{M}}.$$

The minimality condition given in Proposition 1 can be also expressed in terms of Θ. To do this we need some notations.

Let $\Theta(\lambda) \in \mathcal{L}(E, F)$ for $\lambda \in \mathbb{D}^N$. For $i = 1, \cdots, N$ we define then $\Theta_i(\lambda) \in \mathcal{L}(\mathcal{E}_\sim, \mathcal{F}_\sim)$ by

$$\Theta_i(\lambda) u(t) = \sum_{j=1}^{N} \Theta(0, \cdots, \lambda_j, \cdots, 0) u(t + e_i - e_j), \lambda \in \mathbb{D}, u \in \mathcal{E}_\sim, \ t \in \widetilde{\mathbb{Z}}_0^N.$$

Also, denote by $\Theta^\sharp(\lambda) \in \mathcal{L}(F, E)$ the function defined by $\Theta^\sharp(\lambda) = \Theta(\overline{\lambda})^*$.

Definition 1 *We say that Θ is pure provided*

$$\mathcal{F}_{\sim} = \vee_{i=1}^{N}\Theta_i(\lambda)\mathcal{E}_{\sim},$$

$E_{\sim} = \vee_{i=1}^{N}\Theta_i^{\#}(\lambda)\mathcal{F}_{\sim}$.

This purity condition allows us to construct the Nagy-Foiaş diagram generated by Θ.

Theorem 1 *If Θ is pure, then Hilbert module $\mathcal{H}(\Theta)$ admits a unique Nagy-Foiaş diagram.*

Proof.

We show that the fact that Θ is pure implies that

$$\mathcal{F}_{\sim} = \vee_{i=1}^{N}\mathcal{C}_i\mathcal{H}.$$

Suppose first that we have $v \in \mathcal{F}_{\sim}$ of the form $v = \sum_{i=1}^{N}\Theta_i(\lambda)u_i$, with $u_i \in \mathcal{E}_{\sim}$. In [1] is described the way to construct operators C_j using functions F_j which appear in the factorization (1.2) of Θ. More precisely, for $t \in \widetilde{\mathbb{Z}}_0^N$ we have

$$C_j\begin{pmatrix}\lambda_1F_1(\lambda)\\ \cdots\\ \lambda_NF_N(\lambda)\end{pmatrix}u_i(t) = \Theta(0,\cdots,\lambda_j,\cdots,0)u_i(t),\ \ i,\ j=1,\ \cdots,\ N.$$

We denote

$$\begin{pmatrix}\lambda_1F_1(\lambda)\\ \cdots\\ \lambda_NF_N(\lambda)\end{pmatrix}u_i(t) = x_i(t),\ i=1,\ \cdots,\ N.$$

Hence,

$$v(t) = \sum_{i=1}^{N}\Theta_i(\lambda)u_i(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\Theta(0,\cdots,\lambda_j,\cdots,0)u_i(t+e_i-e_j) =$$

$$= \sum_{i=1}^{N}(\sum_{j=1}^{N}C_jx_i(t+e_i-e_j)) = \sum_{i=1}^{N}\mathcal{C}_ix_i(t) \in \vee_{i=1}^{N}\mathcal{C}_i\mathcal{H}.$$

In the same way one proves that from the fact that $\mathcal{E}_{\sim} = \vee_{i=1}^{N}\mathcal{B}_i^*\mathcal{H}$. The purity condition imposed on Θ in Definition 1 it results then that $\mathcal{K}$ is the minimal spectral extension of $\mathcal{H}$. To construct the Nagy-Foiaş diagram (1.1) associated to $\mathcal{H}$ take $S_0 = \mathcal{F}\oplus\mathcal{H}$, $S_1 = \mathcal{F}$, $R_0 = \mathcal{H}\oplus\mathcal{E}$, $R_1 = \mathcal{E}$. It is easy to see that $\mathcal{W}_iS_0 \subset S_0$, $\mathcal{W}_iS_1 \subset S_1$, $\mathcal{W}_i^*R_0 \subset R_0$, $\mathcal{W}_i^*R_1 \subset R_1$, for $i = 1,\cdots,N$. Hence, Hilbert modules S_0, S_1, R_0, R_1 are subspectral. Since we have $S_0 \vee R_0 = \mathcal{K}$ it results that $\mathcal{H} = \mathcal{H}(\Theta)$ admits Nagy-Foiaş diagram.

The minimality condition imposed on Θ in Definition 1 does not depend on the Agler factorization (1.2). Hence, the Nagy-Foiaş diagram generated by $\mathcal{H}(\Theta)$ is unique.

3 The modeling morphism.

For contractive Hilbert modules which are pure there is a one-to-one corespondence between Nagy-Foiaş diagrams and modeling morphisms, see [7]. In this section we prove that the contractive Hilbert module $\mathcal{H}(\Theta)$ is pure and we construct the modeling morphism coresponding to $\mathcal{H}(\Theta)$. We indicate a way to identify this modeling morphism with Θ.

Proposition 2 *If Θ is minimal, then $\mathcal{H}(\Theta)$ is pure.*

Proof. First, we show that if Θ is minimal, then $\xi\mathbf{A}$, $\xi \in \mathbb{T}^N$ is completely non-unitary. By this we mean that there is no proper subspace H in H_0 reducing $\xi\mathbf{A}$ for each $\xi \in \mathbb{T}^N$ such that $\xi\mathbf{A}|H_0$ consists of unitary operators. Indeed, suppose that there exists $H_0 \subseteq H$ such that $\xi\mathbf{A}|H_0$ is unitary for any $\xi \in \mathbb{T}^N$. Since operators $\xi\mathbf{G}^*$ are also unitaries for any $\xi \in \mathbb{T}^N$, it results that $\xi\mathbf{C}^*|H_0 = 0$, hence $\overline{Im\xi\ \mathbf{C}} \neq F$. Consider $f \in F$, $f \neq 0$ such that $f \perp \overline{Im\ \xi\mathbf{C}}$. Define $v \in \mathcal{F}_\sim$ by $v(t) = f$ for all $t \in \widetilde{\mathbb{Z}}_0^N$. Then, for $x_i \in \mathcal{H}$ we have

$$< v, \sum_{i=1}^N \mathcal{C}_i x_i >= \sum_{t\in\widetilde{\mathbb{Z}}_0^N} < v(t), \sum_{i=1}^N \mathcal{C}_i x_i(t) >= \sum_{t\in\widetilde{\mathbb{Z}}_0^N} < u(t), \sum_{i,j=1}^N C_j x_i(t+e_i-e_j) >=$$

$$= \sum_{t\in\widetilde{\mathbb{Z}}_0^N} < f, \sum_{j=1}^N \xi_j C_j \sum_{i=1}^N \xi_i^{-1} x_i(t) >= \sum_{t\in\widetilde{\mathbb{Z}}_0^N} < f, \xi\mathbf{C} \sum_{i=1}^N \xi_i^{-1} x_i(t) >= 0$$

Then, u is orthogonal on $\vee_{i=1}^N \mathcal{C}_i\mathcal{H}$ which is a contradiction because we showed in Theorem 1 that if Θ is minimal, then $\vee_{i=1}^N \mathcal{C}_i\mathcal{H} = \mathcal{F}_\sim$.

We show now that if $\xi\mathbf{A}$ is c.n.u., then the Hilbert module $\mathcal{H} = \mathcal{H}(\Theta)$ generated by contractions $\mathcal{A}_1, \cdots, \mathcal{A}_N$ is pure. The canonical decomposition of $\mathcal{H}$ is $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_p$ with $\mathcal{H}_s$ the spectral part and $\mathcal{H}_p$ the pure part of $\mathcal{H}$. We know from [5] that the characterization of the spectral part $\mathcal{H}_s$ is

$$\mathcal{H}_s = \{x \in \mathcal{H}|\ \ \|\mathcal{A}^s\mathcal{A}^{*t}x\| = \|x\|,\ \ s,t \in \mathbb{Z}_+^N\} = \{x \in \mathcal{H}|\ \ \|\mathcal{A}^{*s}\mathcal{A}^t x\| = \|x\|,\ \ s,t \in \mathbb{Z}_+^N\},$$

where $\underline{\mathcal{A}^s = \mathcal{A}_1^{s_1}, \cdots \mathcal{A}_N^{s_N}}$, for $s = (s_1, \cdots, s_N) \in \mathbb{Z}_+^N$. Suppose that $\mathcal{H}_s \neq 0$. Denote $H_s = \{x(t)|t \in \widetilde{\mathbb{Z}}_0^N, x \in \mathcal{H}_s\}$. We show that in this case $\xi\mathbf{A}|H_s$ is unitary for any $\xi \in \mathbb{T}^N$, which contradicts the fact that $\xi\mathbf{A}$ is c.n.u.

Consider $x \in \mathcal{H}_s$ and take $s = t = (1, 0, \cdots, 0) \in \mathbb{Z}_+^N$. Then

$$(\mathcal{A}^s\mathcal{A}^{*t}x)(t) = \mathcal{A}_1\mathcal{A}_1^* x(t) = \sum_{i,j=1}^N A_i A_j^* x(t+e_i-e_j).$$

Since $x \in \mathcal{H}_s$ we have $\|\mathcal{A}_1\mathcal{A}_1^* x\|^2 = \|x\|^2$, which implies that

$$\sum_{t\in\widetilde{\mathbb{Z}}_0^N} \| \sum_{i,j=1}^N A_i A_j^* x(t+e_i-e_j)\xi^t\|^2 = \sum_{t\in\widetilde{\mathbb{Z}}_0^N} \|x(t)\xi^t\|^2,$$

hence

$$\|\xi\mathbf{A}(\xi\mathbf{A})^*x(t)\|^2 = \|x(t)\|^2, \quad t \in \widetilde{\mathbb{Z}}_0^N.$$

It results that

$$\xi\mathbf{A}(\xi\mathbf{A})^*|H_s = I_{H_s}, \quad \xi \in \mathbb{T}^N.$$

In the same way one shows that

$$(\xi\mathbf{A})^*\xi\mathbf{A}|H_s = I_{H_s}, \quad \xi \in \mathbb{T}^N.$$

We conclude that $\mathcal{H}(\Theta)$ is pure.

In the remaining of the section we denote by $\mathcal{H}^*(\Theta)$ the Hilbert module generated by $\mathcal{A}_1^*, \cdots \mathcal{A}_N^*$.

Theorem 2 *The modeling morphism associated to $\mathcal{H}^*(\Theta)$ coincides with Θ.*

Proof. We showed in Theorem 1 that $\mathcal{H}(\Theta)$ admits Nagy-Foiaş diagram. Then $\mathcal{H}^*(\Theta)$ also admits Nagy-Foiaş diagram with $S_0 = \mathcal{E} \oplus \mathcal{H},\ S_1 = \mathcal{E},\ R_0 = \mathcal{F} \oplus \mathcal{H},\ R_1 = \mathcal{F}$.

We identify the elements $e \in E$ with the constant functions $v \in E$ defined by

$$v(t) = \begin{cases} e\,, & \text{for } t = (0, \cdots, 0) \\ 0, & \text{for } t \in \widetilde{\mathbb{Z}}_0^N - \{(0, \cdots, 0)\} \end{cases}$$

and we denote them simply by e. Then, by $\mathcal{W}^{*s}E$ we denote $\{\mathcal{W}_1^{*s_1} \cdots \mathcal{W}_N^{*s_N}e,\ s = (s_1, \cdots, s_N) \in \widetilde{\mathbb{Z}}_+^N, e \in E\}$. In the same way one defines $\mathcal{W}^{*s}F$. We have then

$$S_1 = \mathcal{E} = \oplus_{s \in \widetilde{\mathbb{Z}}_+^N} \mathcal{W}^{*s}E, \quad R_1 = \mathcal{F} = \oplus_{s \in \widetilde{\mathbb{Z}}_+^N} \mathcal{W}^{s}F,$$

hence

$$\widetilde{R}_1 = \oplus_{s \in \widetilde{\mathbb{Z}}^N} \mathcal{W}^{s}E = \oplus_{s \in \widetilde{\mathbb{Z}}^N} \mathcal{W}^{*s}E$$

and

$$\widetilde{R}_1 \ominus R_1 = \oplus_{s \in \widetilde{\mathbb{Z}}_-^N - \widetilde{\mathbb{Z}}_0^N} \mathcal{W}^{s}E = \oplus_{s \in \widetilde{\mathbb{Z}}_+^N - \widetilde{\mathbb{Z}}_0^N} \mathcal{W}^{*s}E.$$

Let $P = P_{\widetilde{R}_1}^K|S_1$ be the modeling morphism associated to the Nagy-Foiaş diagram coresponding to $\mathcal{H}(\Theta)$. Then, it is known from [4] that P takes value in $\widetilde{R}_1 \ominus R_1$, hence for $e \in E$ we have

$$Pe = \sum_{s \in \widetilde{\mathbb{Z}}_+^N - \widetilde{\mathbb{Z}}_0^N} \mathcal{W}^{*s} f_s, \ f_s \in F.$$

Since P is a projection we have

$$\sum_{s \in \widetilde{\mathbb{Z}}_+^N - \widetilde{\mathbb{Z}}_0^N} \|f_s\|^2 = \| \sum_{s \in \widetilde{\mathbb{Z}}_+^N - \widetilde{\mathbb{Z}}_0^N} \mathcal{W}^{*s} f_s\|^2 = \|Pe\|^2 \le \|e\|^2 < \infty.$$

We are lead then to define a sequence of contractions $(\Phi_s)_{s\in\widetilde{\mathbb{Z}}_+^N-\widetilde{\mathbb{Z}}_0^N}$ from E to F setting

$$\Phi_s e = f_s, \quad s \in \widetilde{\mathbb{Z}}_+^N - \widetilde{\mathbb{Z}}_0^N .$$

Consider then the function $\Phi(\lambda) \in \mathcal{L}(E,F)$ given by the formal power serie

$$\Phi(\lambda) = \sum_{s\in\widetilde{\mathbb{Z}}_+^N-\widetilde{\mathbb{Z}}_0^N} \lambda^s \Phi_s, \quad \lambda \in \mathbb{D}^N .$$

We show that $\Phi = \Theta$, i.e. Φ is a function from the Agler-Schur class.

First we show that

(3.1)
$$\Phi_s = P_F^{\mathcal{F}} \mathcal{W}^s | E, \quad s \in \widetilde{\mathbb{Z}}_+^N - \widetilde{\mathbb{Z}}_0^N .$$

Indeed, we have

$$< Pe, \mathcal{W}^{*s} f >_{\widetilde{R}_1\ominus R_1} = \sum_{s\in\widetilde{\mathbb{Z}}_+^N-\widetilde{\mathbb{Z}}_0^N} < \mathcal{W}^{*s} f_s, \mathcal{W}^{*s} f >_{\widetilde{R}_1\ominus R_1} =$$

$$=< \mathcal{W}^{*s} f_s, \mathcal{W}^{*s} f >_{\widetilde{R}_1\ominus R_1} =< f_s, f >_F =< \Phi_s e, f >_F .$$

Hence,

$$< \Phi_s e, f >_F =< Pe, \mathcal{W}^{*s} f >_{\widetilde{R}_1\ominus R_1} =< e, \mathcal{W}^{*s} f >_{\widetilde{R}_1\ominus R_1} =< \mathcal{W}^s e, f >_{\widetilde{R}_1\ominus R_1} =< P_F^{\mathcal{F}} \mathcal{W}^s e, f >_F .$$

We show now that we have

$$\Phi_s = \Theta_s, \quad s \in \mathbb{Z}_+^N ,$$

where the coefficients Θ_s of the power expansion of Θ are given by (1.3).

For $e \in E$ the constant function we denote by $(0,0,e)$ the element (v,x,u) from $\mathcal{K}$ defined by $v(t) = 0,\ t \in \widetilde{\mathbb{Z}}_-^N,\ x(t) = 0,\ t \in \widetilde{\mathbb{Z}}_0^N$ and

$$u(t) = \begin{cases} e\,, & \text{for } t = (0,\cdots,0) \\ 0, & \text{for } t \in \widetilde{\mathbb{Z}}_+^N - \{(0,\cdots,0)\} \end{cases}$$

Also denote $(v^{(s)}, x^{(s)}, u^{(s)}) = \mathcal{W}^s(0,0,e)$. Then $u^{(s)}(t) = 0,\ t \in \widetilde{\mathbb{Z}}_+^N$.

We compute now $x^{(s)}(t)$ for $s \in \mathbb{Z}_+^N$ and $t \in \widetilde{\mathbb{Z}}_0^N$, with $-s_i \le t_i \le |s| - s_i$. For $|s| = 1$ we have

$$x^{(s)}(t) = \mathbf{B}^{(s_1+t_1,\cdots,s_N+t_N)} e = \mathbf{B}^{s+t} e.$$

We prove by induction on the components of s that for $|s| \ge 2$ we have

(3.2)
$$x^{(s)}(t) = (\mathbf{A}\#\mathbf{B})^{(s+t)} e.$$

Indeed,

$$x^{(s_1,\cdots,s_N+1)}(t) = \sum_{j=1}^N A_j x^{(s_1,\cdots,s_N)}(t + e_N - e_j) =$$

$$= \sum_{j=1}^{N} A_j x^{(s_1,\cdots,s_N)}(t_1,\cdots,t_j-1,\cdots,t_N+1) = \sum_{j=1}^{N} A_j(\mathbf{A}\#\mathbf{B})^{(s_1+t_1,\cdots,s_j+t_j-1,\cdots,s_N+t_N+1)} =$$

$$= (\mathbf{A}\#\mathbf{B})^{(s_1+t_1,\cdots,s_j+t_j,\cdots,s_N+t_N+1)}.$$

It is easy to see that $x^{(s)}(t) = 0$ for other $t \in \widetilde{\mathbb{Z}}_0^N$.

We compute now $v^{(s)}(t)$, for $s \in \mathbb{Z}_+^N$ and $t \in \widetilde{\mathbb{Z}}_-^N$ with $-s_i \le t_i \le |s| - s_i$. For $|s| = 1$ we have

$$v^{(s)}(t) = \mathbf{D}^{(t+s)}e.$$

Also, for $|s| = 2$ we have

$$v^{(s)}(t) = (\mathbf{C}\&\mathbf{B})^{(t+s)}e.$$

We prove by induction that for $|s| \ge 3$ we have

$$v^{(s)}(t) = (\mathbf{C}\flat\mathbf{A}\#\mathbf{B})^{(t+s)}e.$$

Indeed, using (3.2) we have

$$v^{(s_1,\cdots,s_N+1)}(t) = \mathcal{C}_N x^{(s_1,\cdots,s_N)}(t) = \sum_{j=1}^{N} C_j x^{(s_1,\cdots,s_N)}(t+e_N-e_j) = \sum_{j=1}^{N} C_j x^{(s_1,\cdots,s_N)}(t_1,\cdots,t_j-1,\cdots,t_N+1) =$$

$$= \sum_{j=1}^{N} C_j(\mathbf{A}\#\mathbf{B})^{(t_1+s_1,\cdots,t_j+s_j-1,\cdots,t_N+s_N+1)}e = (\mathbf{C}\flat\mathbf{A}\#\mathbf{B})^{(t_1+s_1,\cdots,t_j+s_j,\cdots,t_N+s_N+1)}e.$$

For $t = (0,\cdots,0)$ we obtain then $v^{(s)}(0,\cdots,0)$, with $s \in \mathbb{Z}_+^N$. Using (3.1) we deduce that $v^{(s)}(0,\cdots,0) = P_F^{\mathcal{F}}\mathcal{W}^s(0,0,e) = \Phi_s e$, hence

$$\Phi_s = \Theta_s, \quad s \in \mathbb{Z}_+^N.$$

We show now that $P_F^{\mathcal{F}}\mathcal{W}^s E = 0$, for $s \in \widetilde{\mathbb{Z}}_+^N - (\widetilde{\mathbb{Z}}_0^N \cup \mathbb{Z}_+^N)$, which means that

$$\Theta_s = 0, \quad s \in \widetilde{\mathbb{Z}}_+^N - (\widetilde{\mathbb{Z}}_0^N \cup \mathbb{Z}_+^N).$$

We have $\widetilde{\mathbb{Z}}_+^N - (\widetilde{\mathbb{Z}}_0^N \cup \mathbb{Z}_+^N) = \cup_{i=1}^N \{(s_1,\cdots,s_N) \in \widetilde{\mathbb{Z}}_+^N - \mathbb{Z}_+^N | s_i < 0\}$. Take $s \in \widetilde{\mathbb{Z}}_+^N$ with $s_1 < 0$. Since $\mathcal{W}_i$ commute we can suppose that $s_1,\cdots,s_k \le 0$ and $s_{k+1},\cdots,s_N \ge 0$. For $t \in \widetilde{\mathbb{Z}}_+^N$ denote

$$u_{s_1,\cdots,s_k}(t) = \begin{cases} e, & t = (-s_1,\cdots,-s_k,0,\cdots,0) \\ 0, & t \ne (-s_1,\cdots,-s_k,0,\cdots,0) \end{cases},$$

Then $\mathcal{W}_1^{s_1}\cdots\mathcal{W}_k^{s_k}(0,0,e) = (0,0,u_{s_1,\cdots,s_k})$. Since we have $s_1+\cdots+s_N > 0$, it results that $-s_1-\cdots-s_k < s_{k+1}+\cdots+s_N$. Then, there exists $k < l \le N$ and $0 \le \mu < s_{l+1}$ such that $-s_1-\cdots-s_k+s_{k+1}+\cdots+s_l+\mu = 0$. It is easy to see that at step $s_1+\cdots+s_l+\mu$, by applying $\mathcal{W}_{k+1}^{s_{k+1}}\cdots\mathcal{W}_{l+1}^{\mu}$ we obtain $v = 0$, $x = 0$ and

$$u_0(t) = \begin{cases} e, & t_0 = (-s_1,\cdots,-s_k,s_{k+1},\cdots,s_l,\mu,0,\cdots,0) \\ 0, & t \ne t_0 \end{cases}$$

We showed before that if we have

$$u(t) = \begin{cases} e, & t = (0,0,\cdots,0) \\ 0, & t \neq (0,0,\cdots,0) \end{cases},$$

then for $s = (s_1,\cdots,s_n) \in \mathbb{Z}_N^+$ we have $\mathcal{W}_1^{s_1}\cdots\mathcal{W}_N^{s_N}(0,0,u) = (v^{(s)}, x^{(s)}, 0)$, with $v^{(s)}(t) = 0$ for $t \in \widetilde{\mathbb{Z}}_-^N - \{t : -s_i \leq t_i \leq |s| - s_i\}$. In the same way one can see that if we take

$$u(t) = \begin{cases} e, & t = t_0 \\ 0, & t \neq t_0 \end{cases},$$

then we have $v^{(s)}(t) = 0, \quad t \in \widetilde{\mathbb{Z}}_-^N - \{t_{0i} - s_i \leq t_i \leq t_{0i} + |s| - s_i\}$. Hence, if we apply $\mathcal{W}_{l+1}^{s_{l+1}-\mu}\cdots\mathcal{W}_N^{s_N}$ to $(0,0,u_0)$ we obtain $(v^{(s)}, x^{(s)}, 0)$ with $v^{(s)}(t) = 0$ for $t \in \widetilde{\mathbb{Z}}_+^N$ with the first component $t_1 \neq -s_1,\cdots,-s_1+|s|$. Since this are all strictly positive it results that $v^{(s)}(0,\cdots,0) = 0$ i.e., $P_F^{\mathcal{F}}\mathcal{W}^s e = 0$.

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Polytechnical University of Bucharest
Chair Mathematics I
Calea Independenţei 313, Bucharest
Romania
e-mail: esuciu@miron.imar.ro