INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

THE DIRICHLET PROBLEM FOR THE MONGE–AMPERE EQUATION AND THE NONLINEAR RESOLVENTS

by

CORNELIU UDREA

Preprint nr. 6/2001

BUCUREȘTI

THE DIRICHLET PROBLEM FOR THE MONGE–AMPERE EQUATION AND THE NONLINEAR RESOLVENTS

by

CORNELIU UDREA*

May, 2001

* University of Pitesti, Department of Applied Mathematics, Str. Targu din Vale, Nr.1, 0300 Pitesti, Jud. Arges, Romania
E–mail: cudrea@linux.math.upit.ro

The Dirichlet Problem for the Monge-Ampère Equation and the Nonlinear Resolvents

Corneliu UDREA
University of Piteşti, Department of the Applied Mathematics,
Str. Târgu din Vale, Nr.1, 0300 Piteşti, Jud.Argeş, ROMANIA
e-mail: cudrea@linux.math.upit.ro.

Abstract. For the Monge-Ampère equation $\nu_u = f \cdot \lambda$ (where $f \in L^1(U)$ and λ is the Lebesgue measure on the strictly convex bounded and open set $U \subset \mathbb{R}^k$) we shall consider the Dirichlet problem $u|_{\partial U} = \varphi$ (where $\varphi \in C(\partial U)$). We shall define a nonlinear operator V^φ on the space $L^1(U)$ which is associated with the solutions of the above Dirichlet problem and moreover we shall define a nonlinear resolvent which has V^φ as its initial operator. Afterwards we shall study the supermedian functions with respect to the above resolvent and we shall prove that these functions are completely determined by a class of concave real functions on U.

1 Preliminaries

We shall make a short review of the knowledges of the theory of convex functions (in conformity with [1], [2], [6] or [8]) and also of the theory of nonlinear operators (according to [5], [9] or [10]) which will be used in this work. Throughout this text $U \subset \mathbb{R}^k$ is a nonvoid open bounded and strictly convex set, $V \subset \mathbb{R}^k$ is a non void convex and open set and λ is the Lebesgue measure on $\mathbb{R}^k$. Also the spaces $L^p(U)$ $(p \in \{1, \infty\})$ are defined with respect to the measure $\lambda|_U$.

All functions are defined λ a.e. and all inequalities (and so that all equalities) are acomplished λ a.e.

Definition 1.1 *(i). For all $A \subset \mathbb{R}^k$ non void convex set we shall use the following notation: $\mathcal{U}(A) := \{u \in \mathbb{R}^A : u$ is a convex function $\}$. Obviously $\mathcal{U}(V) \subset C(V)$.*

(ii). Similarly, for all $\varphi \in C(\partial V)$, $\mathcal{U}_\varphi(V) := \{u \in \mathbb{R}^{\bar{V}} : u \in \mathcal{U}(\bar{V}) \cap C(\bar{V})$ and $u|_{\partial V} = \varphi\}$.

Proposition 1.2 *Let V be a bounded set and $H \subset \mathcal{U}(\bar{V}) \cap C(\bar{V})$ such that there exists $h \in C(\bar{V})$ with properties $h|_{\partial V} = 0$ and for all $u, v \in H$, $|u - v| \leq h$. We shall have that H is uniformly equicontinuous on $\bar{V}$.*

Definition 1.3 ([6]). *(i). If $u \in \mathcal{U}(V)$ and $a \in V$, then*

$$\partial_u(a) := \left\{p \in \mathbb{R}^k : u(x) - u(a) \geq < p, x - a > \text{ for all } x \in V\right\}.$$

is called the subdifferential of u in a. For all $A \subset V$, $\partial_u(A) := \bigcup_{a \in A} \partial_u(a)$.

(ii). The map $(K \longmapsto \lambda(\partial_u(K)))$ is defined on the compact sets of V and it is a Radon measure on V. This measure is denoted by ν_u and it is called the curvature of the convex function u (on the set V).

Proposition 1.4 ([6]). *(i). If $u, v \in \mathcal{U}(V)$ and $D \subset V$ are such that D is a non void open bounded set, $u|_D \leq v|_D$ and $(sci_D u)|_{\partial D} \geq (sci_D v)|_{\partial D}$, then it follows that $\partial_v(D) \subset \partial_u(D)$ (where $sci_D u : \bar{D} \to \bar{\mathbb{R}}$, $sci_D u(x) := \liminf_{D \ni y \to x} u(y)$, for all $x \in \bar{D}$).*

(ii). If $u, v \in \mathcal{U}(V)$ and $\alpha \in \mathbb{R}_+$, then we have that: (a) $\nu_{u+v} \geq \nu_u + \nu_v$. (b) $\nu_{\alpha u} = \alpha^k \nu_u$.

(iii). Let $(u_n)_n \subset \mathcal{U}(V)$ be such that $(u_n)_n$ converges locally uniformly on V to the map u. We shall have that $(\nu_{u_n})_n$ is vaguely convergent to the measure ν_u.

Proposition 1.5 ([6]). *(i).* **(The minimum principle for the convex functions.)** *If V is a bounded set, and $u, v \in \mathcal{U}(V)$ are such that $\nu_u \leq \nu_v$ and $(sci_V u)|_{\partial V} \geq (sci_V v)|_{\partial V}$ if follows that $u \geq v$.*

(ii). **(The minimum principle for the locally convex functions.)** *Let $G \subset \mathbb{R}^k$ be a non void open bounded set and $f, g : \bar{G} \to \mathbb{R}$ be locally convex functions on G and continuous functions on $\bar{G}$ such that $\nu_f \leq \nu_g$ and $f|_{\partial G} \geq g|_{\partial G}$. We have that $f \geq g$. (Here for all $D \subset G$ non void convex set $(\nu_f)|_D = \nu_{f|_D}$).*

(iii). **(The boundedness of the convex functions.)** *If V is bounded, $u \in \mathcal{U}(V)$ and $m \in \mathbb{R}$ are such that $(sci_V u)|_{\partial V} \geq m$, then it follows that:*

$$u \geq m - (diam V)\sqrt[k]{\frac{\nu_u(V)}{\omega_k}} \quad (\text{where } \omega_k := \lambda(B(0_k, 1))).$$

Theorem 1.6 ([6]). *Let μ be a bounded Radon measure on U and $\varphi \in C(\partial U)$. There exists one and only one convex and continuous map $u : \bar{U} \to \mathbb{R}$ such that $\nu_u = \mu$ and $u|_{\partial U} = \varphi$. (The map u what is defined in this theorem will be denoted by $M(\mu, \varphi)$.)*

Proposition 1.7 ([6]). *The following assertions hold:*

(i). For all $\varphi_1, \varphi_2 \in C(\partial U)$ and μ_1, μ_2 bounded Radon measures on U we have:

(a). $M(\mu_1 + \mu_2, \varphi_1 + \varphi_2) \geq M(\mu_1, \varphi_1) + M(\mu_2, \varphi_2)$.

(b). If $\mu_1 \leq \mu_2$ and $\varphi_1 \geq \varphi_2$, then $M(\mu_1, \varphi_1) \geq M(\mu_2, \varphi_2)$.

(ii). For all $\varphi \in C(\partial U)$, $\alpha \in \mathbb{R}_+$ and μ bounded Radon measure on U we find that:

(a). $M(\alpha^k \mu, \alpha\varphi) = \alpha M(\mu, \varphi)$.

(b). $M(\mu, \varphi) \geq \inf \varphi - (diam U) \sqrt[k]{\frac{\mu(U)}{\omega_k}}$.

Definition 1.8 ([5]). *(i). An increasing map $T : L^1(U) \to L^1(U)$ (respectively $T : L^\infty(U) \to L^\infty(U)$) is called operator on $L^1(U)$ (respectively on $L^\infty(U)$).*

(ii). We shall say that $T : L^1(U) \to L^1(U)$ is a sub-Markov operator on $L^1(U)$ iff for all $f, g \in L^1(U)$ and $\alpha \in \mathbb{R}_+$ such that $f \leq g + \alpha$ it follows that $Tf \leq Tg + \alpha$.

(iii). It is obvious that if a map $T : L^1(U) \to L^1(U)$ has the previous property, then T is an operator. Moreover if T satisfies the property of (ii) for all function $f, g \in L^\infty(U)$, then $\|Tf - Tg\|_\infty \leq \|f - g\|_\infty$ for all $f, g \in L^\infty(U)$ (in conformity with [5] or [9]).

(iv).([5]). We shall say that $T : L^1(U) \to L^1(U)$ satisfies the weak complete maximum principle iff for all $f, g \in L^1(U)$ and $\alpha \in \mathbb{R}_+$ such that

$$f + Tf \leq g + Tg + \alpha$$

on the set $\{f > g\} := \{x \in U : f(x) > g(x)\}$, if follows that

$$Tf \leq Tg + \alpha.$$

We remark that if T satisfies the weak complete maximum principle then:

(a). T is an increasing map (that is T is an operator).

(b). $I + T : L^1(U) \to L^1(U)$ is an one to one map (where I is the identity map of $L^1(U)$).

(v).([5]) If $T, N : L^1(U) \to L^1(U)$ are such that

$$(I - N)(I + T) = I = (I + T)(I - N)$$

then (T, N) is called a pair of conjugated maps (on $L^1(U)$).

Proposition 1.9 (similarly to [5] or [9]). *Let $T, N : L^1(U) \to L^1(U)$ be such that (T, N) is a pair of conjugated operators. The following statements are equivalent:*

(i). The function N is a sub-Markov operator and T is an operator.

(ii). The map T satisfies the weak complete maximum principle.

(iii). T is an operator such that for all $f, g \in L^1(U)$ and $\alpha \in \mathbb{R}_+$ if $f + Tf \leq g + Tg + \alpha$ then $Tf \leq Tg + \alpha$.

Proof. (i)$\Rightarrow$(ii). Let $f, g \in L^1(U)$ and $\alpha \in \mathbb{R}_+$ be such that $f + Tf \leq g + Tg + \alpha$ on the set $\{f > g\}$ and $v := \inf\{f + Tf, g + Tg + \alpha\}$. We have that $v \in L^1(U)$, $Nv \leq Tf$, $Nv \leq Tg + \alpha$ and $v = f + Tf$ on the set $\{f > g\}$. If $j := v - Nv$, then $j \in L^1(U)$, $f \leq j$ and $f + Tf \leq j + Tj = v$ hence $Tf = N(f + Tf) \leq Nv \leq Tg + \alpha$.

(ii)$\Rightarrow$(iii). It is obvious.

(iii)$\Rightarrow$(i). Let $f, g \in L^1(U)$ and $\alpha \in \mathbb{R}_+$ be such that $f \leq g + \alpha$. We have that $f = (I+T)(I-N)f = (f-Nf)+T(f-Nf) \leq (g-Ng)+T(g-Ng)+\alpha = (I+T)(I-N)g + \alpha = g + \alpha$.

By the hypothesis we shall find that $Nf = T(f-Nf) \leq T(g-Ng)+\alpha = Ng + \alpha$, that is N is a sub-Markov operator. ∎

Definition 1.10 ([5]). *(i). The family of functions $\mathcal{V} = (V_p)_{p\in(0,\infty)}$ where, for all $p \in (0,\infty)$, $V_p : L^1(U) \to L^1(U)$ is called resolvent (on $L^1(U)$) iff, for all $p, q \in (0,\infty)$ it follows that*

$$(I - (p-q)V_p)(I + (p-q)V_q) = I.$$

(ii). The resolvent $\mathcal{V} = (V_p)_{p\in(0,\infty)}$ is called the resolvent associated with the map $V : L^1(U) \to L^1(U)$ iff, for all $p \in (0,\infty)$, we have that:

$$V = V_p(I + pV) \text{ and } V_p = V(I - pV_p).$$

(iii). If, for all $p \in (0,\infty)$, pV_p is a sub-Markov operator (on $L^1(U)$), then the resolvent $\mathcal{V} = (V_p)_{p\in(0,\infty)}$ is called a sub-Markov resolvent (on $L^1(U)$).

2 Nonlinear Resolvent Associated with the Solutions of the Dirichlet Problem for the Monge-Ampère Equation.

If $\varphi \in C(\partial U)$ then we shall define a sub-Markov resolvent associated with the solutions of the Dirichlet problem $\nu_u = f \cdot \lambda$ and $u|_{\partial U} = \varphi$, where $f \in L^1(U)$ and we shall present its important properties.

Throughout this section $\varphi \in C(\partial U)$.

Definition 2.1 *(i). For all $f \in L^1(U)$, $V^{\varphi} f := -M(f^+ \cdot \lambda, -\varphi)$. In particular $Vf := -M(f^+ \cdot \lambda, 0)$ is the operator what is defined in [9] or [10]. (Here $f^+ := \sup\{f, 0\}$). Obviously $V^{\varphi} f = V^{\varphi}(f^+)$.*

(ii). For all $f \in L^1(U)$, $V^{\varphi} f : \bar{U} \to \mathbb{R}$ is a continuous concave function such that $(V^{\varphi} f)\,|_{\partial U} = \varphi$ and $\nu_{-V^{\varphi} f} = f^+ \cdot \lambda$.

(iii). $V^{\varphi} : L^1(U) \to -\mathcal{U}_{-\varphi}(U) \subset L^1(U)$.

Proposition 2.2 (Algebraic and order properties). *Let $f_1, f_2 \in L^1(U)$. The following assertions hold:*

(i). If $V^{\varphi} f_1 = V^{\varphi} f_2$, then $f_1^+ = f_2^+$ (obviously λ a.e.).

(ii). Let $f_1 \le f_2$. It follows that $V^{\varphi} f_1 \le V^{\varphi} f_2$, so that V^{φ} is an operator on $L^1(U)$.

(iii). The following inequalities hold:

(a). $V^{\varphi}(f_1 + f_2) \le V^{\varphi} f_1 + V f_2$.

(b). $|V^{\varphi} f_1 - V^{\varphi} f_2| \le V(f_1 - f_2)$.

(iv). If $\varphi \ge 0$, then V^{φ} is a subadditive map.

Proof. (i). Since $f_1^+ \cdot \lambda = f_2^+ \cdot \lambda$, it is obvious that $f_1^+ = f_2^+$.

(ii). Whereas $(V^{\varphi} f_1)\,|_{\partial U} = (V^{\varphi} f_2)\,|_{\partial U}$ and $\nu_{-V^{\varphi} f_1} \le \nu_{-V^{\varphi} f_2}$, the assertion holds by the minimum principle for the convex functions (Proposition 1.5 (i)).

(iii). (a) Since $(V^{\varphi}(f_1 + f_2))\,|_{\partial U} = (V^{\varphi} f_1 + V f_2)\,|_{\partial U}$ and $\nu_{-V^{\varphi} f_1 - V f_2} \ge (f_1^+ + f_2^+) \cdot \lambda \ge \nu_{-V^{\varphi}(f_1 + f_2)}$, we apply again the minimum principle for the convex functions.

(b). The inequalities $f_i \le f_j + (f_1 - f_2)^+$, $i, j = 1, 2$, $i \ne j$ and the point (a) involve the assertion.

(iv). We use again the minimum principle for the convex functions. ■

Theorem 2.3 (Topological properties.) *We have the following claims:*

(i). For all $f \in L^1(U)$, $V^{\varphi} f \le \sup \varphi + (diam\, U) \sqrt[k]{\frac{\|f\|_1}{\omega_k}}$.

(ii). $V^{\varphi} : (L^1(U), \|\cdot\|_1) \to (L^1(U), \|\cdot\|_1)$ is an $\frac{1}{k}$-Hölder map.

(iii). Let $\mathcal{F} \subset L^1(U)$ be such that either (a) there exists $h \in L^1(U)$ so as to, for all $f \in \mathcal{F}$, $|f| \le h$, or (b) $\mathcal{F}$ is bounded in $(L^1(U), \|\cdot\|_1)$, and $\mathcal{F}$ is increasing (i.e. for all $f, g \in \mathcal{F}$ there exists $h \in \mathcal{F}$ such that $\sup\{f, g\} \le h$).

We have that $V^{\varphi}(\mathcal{F})$ is relatively compact in the space $(C(\bar{U}), \|\cdot\|_{\infty})$ and accordingly it is relatively compact in $(L^1(U), \|\cdot\|_1)$.

Proof. (i). We apply the Proposition 1.5.(iii). and we find the assertion since $\nu_{-V^{\varphi} f}(U) = \int_U f^+ d\lambda \le \|f\|_1$.

(ii). The previous point and the Proposition 2.(iii). involve that, for all $f_1, f_2 \in L^1(U)$:

$$|V^\varphi f_1 - V^\varphi f_2| \leq V(f_1 - f_2) \leq (\text{diam}U)\sqrt[k]{\frac{\|f_1 - f_2\|_1}{\omega_k}} \text{ and so that}$$

$$\|V^\varphi f_1 - V^\varphi f_2\|_1 \leq \lambda(U)(\text{diam}U)\sqrt[k]{\frac{\|f_1 - f_2\|_1}{\omega_k}}.$$

(Moreover we have that $\|V^\varphi f_1 - V^\varphi f_2\|_\infty \leq (\text{diam}U)\sqrt[k]{\frac{\|f_1-f_2\|_1}{\omega_k}}$).

(iii). In any case let $c \in \mathbb{R}_+$ be such that, for all $f \in \mathcal{F}$, $\|f\|_1 \leq c$. If follows that, for all $f \in \mathcal{F}$:

$$\inf \varphi \leq V^\varphi f \leq \sup \varphi + (\text{diam}U)\sqrt[k]{\frac{c}{\omega_k}},$$

hence $V^\varphi(\mathcal{F})$ is bounded in the space $(C(\bar{U}), \|\cdot\|_\infty)$.

(a). For all $f, g \in \mathcal{F}$, $|V^\varphi f - V^\varphi g| \leq V(f-g) = V((f-g)^+) \leq V(2h)$, hence by the Proposition 1.2 it follows that $V^\varphi(\mathcal{F})$ is uniformly equicontinuous on $\bar{U}$.

(b). Since $\mathcal{F}$ is increasing and bounded (in $\|\cdot\|_1$), the set $\mathcal{F}^+ := \{f^+ : f \in \mathcal{F}\}$ has the same properties. Accordingly we shall find that there exists $h \in L^1(U)$ such that, for all $f \in \mathcal{F}$, $f^+ \leq h$. It follows, similarly to the case (a), that $V^\varphi(\mathcal{F})$ is uniformly equicontinuous on $\bar{U}$.

In any case by the Ascoli theorem we have that $V^\varphi(\mathcal{F})$ is relatively compact in $(C(\bar{U}), \|\,\|_\infty)$. ■

Corollary 2.4 $V^\varphi : (L^1(U), \|\cdot\|_1) \to (L^1(U), \|\cdot\|_\infty)$ *is a continuous map.*

Proof. It is obvious. ■

Remark 2.5 *Let $(f_n)_n \subset L^1(U)$ and $f \in L^1(U)$.*

(i). If $(f_n)_n$ converges to f in $L^1(U)$, then $(V^\varphi f_n)_n$ converges to $V^\varphi f$ uniformly on $\bar{U}$.

(ii). If $(f_n)_n$ converges monotonely to f (λ a.e.), then $(V^\varphi f_n)_n$ converges uniformly and monotonely to $V^\varphi f$ (so that, in particular, V^φ is increasingly continuous on $L^1(U)$).

(iii). By the proof of Theorem 3.(ii). it follows that

$$V^\varphi|_{L^\infty(U)} : (L^\infty(U), \|\cdot\|_\infty) \to (L^\infty(U), \|\cdot\|_\infty)$$

is also $\frac{1}{k}$-Hölder and so that continuous map.

Theorem 2.6 *Let $u \in -\mathcal{U}(U)$, $u \geq 0$ and $f, g \in L^1(U)$ be such that*

$$V^\varphi f \leq V^\varphi g + u \text{ on the set } \{f > g\}.$$

It follows that $V^\varphi f \leq V^\varphi g + u$.

Proof. If $D := \{V^\varphi f > V^\varphi g + u\}$ and $\bar{u} : \bar{D} \to \mathbb{R}$ is the map $\bar{u} := \mathrm{scs}_{\bar{D}} u$ (where $\mathrm{scs}_{\bar{D}} u = -\mathrm{sci}_{\bar{D}}(-u)$) it follows that $D \subset \{f \leq g\}$, D is an open set and $\bar{u}$ is a locally concave function on D.

For all $x \in (\partial D) \cap U$, by the continuity of the above functions we have that

$$V^\varphi f(x) = V^\varphi g(x) + u(x) = V^\varphi g(x) + \bar{u}(x).$$

If $x \in (\partial D) \cap (\partial U)$ if follows that:

$$\begin{aligned} \varphi(x) &\leq \varphi(x) + (\mathrm{sci}_D u)(x) = \mathrm{sci}_D (V^\varphi g + u)(x) \\ &\leq \mathrm{scs}_D (V^\varphi g + u)(x) \leq \mathrm{scs}_D V^\varphi f(x) = \varphi(x). \end{aligned}$$

Therefore $\bar{u}$ is continuous on $(\partial D) \cap (\partial U)$ and

$$(V^\varphi f)|_{\partial D} = (V^\varphi g + \bar{u})|_{\partial D}.$$

On the other hand we have that

$$\begin{aligned} (\nu_{-V^\varphi f})|_D &= (f^+|_D) \cdot \lambda \leq (g^+|_D) \cdot \lambda = (\nu_{-V^\varphi g})|_D \\ &\leq (\nu_{-V^\varphi g - \bar{u}})|_D. \end{aligned}$$

By the minimum principle for the locally convex functions (Proposition 1.5.(ii).) we shall have that $V^\varphi f \leq V^\varphi g + u$ on D what is contrary to the definition of D if D is non void. It follows that $D = \emptyset$, so that $V^\varphi f \leq V^\varphi g + u$ (everywhere on U). ■

Remark 2.7 *(i). It is obvious that V^φ satisfies the weak complete maximum principle and hence $I + V^\varphi : L^1(U) \to L^1(U)$ is an one to one map.*

(ii). For all $p \in (0, \infty)$ pV^φ has the property of the previous theorem and so that pV^φ also satisfies the weak complete maximum principle.

Theorem 2.8 *For all $p \in (0, \infty)$ there exists one and only one map $V_p^\varphi : L^\infty(U) \to L^\infty(U)$ such that*

$$(I - pV_p^\varphi)(I + pV^\varphi) = I = (I + pV^\varphi)(I - pV_p^\varphi).$$

Proof. For all $f \in L^\infty(U)$ we shall define $L_f : L^\infty(U) \to L^\infty(U)$, where $L_f(g) := V^\varphi(f - pg)$, for all $g \in L^\infty(U)$ and $p \in (0,\infty)$ a fixed number. By the Theorem 3.(ii). it follows that, for all $h, g \in L^\infty(U)$,

$$\|L_f h - L_f g\|_\infty \leq (\mathrm{diam} U) \sqrt[k]{\frac{\lambda(U)}{\omega_k}} \cdot \sqrt[k]{\|h-g\|_\infty}$$

and so that ([9]), there exists $r > 0$ such that for all $g \in L^\infty(U)$, if $\|g\|_\infty \leq r$ then $\|L_f g\|_\infty \leq r$.

Since $L_f : (L^\infty(U), \|\cdot\|_\infty) \to (L^\infty(U), \|\cdot\|_\infty)$ is a compact map (the proof is similar to that of the Theorem 3.(iii).), we can apply the Schauder's fixed point theorem: there exists $u_f \in L^\infty(U)$ such that $L_f u_f = u_f$, that is $V^\varphi(f - pu_f) = u_f$. But $I + pV^\varphi$ is an one to one map and accordingly u_f is the unique map $u \in L^\infty(U)$ with property $L_f u = u$.

Let us define $V_p^\varphi : L^\infty(U) \to L^\infty(U)$, $V_p^\varphi f := u_f$, it follows that

$$(I + pV^\varphi)\left(I - pV_p^\varphi\right) = I \text{ and } I - pV_p^\varphi = (I + pV^\varphi)^{-1}. \blacksquare$$

Remark 2.9 *Since for all $f \in L^\infty(U)$ we have that*

$$V_p^\varphi f = V^\varphi\left(f - pV_p^\varphi f\right) \text{ and } V^\varphi f = V_p^\varphi\left(f + V^\varphi f\right),$$

the following assertions hold:

(i). $V_p^\varphi f : \bar{U} \to \mathbb{R}$ is a concave and continuous function on $\bar{U}$ such that $\left(V_p^\varphi f\right)|_{\partial U} = \varphi$ and $\nu_{-V_p^\varphi f} = \left(f - pV_p^\varphi f\right)^+ \cdot \lambda$. If $f \in L^\infty(U)$ is such that $V_p^\varphi(pf) \leq f$ then it follows that $\nu_{-V_p^\varphi(pf)} = p\left(f - V_p^\varphi(pf)\right) \cdot \lambda$.

(ii). For all $p \in (0,\infty)$, $\left(V_p^\varphi\right)^{-1} = V^\varphi(I + pV^\varphi)^{-1}$ (the equality holds on the space $L^\infty(U)$) and, for all $p, q \in (0,\infty)$, $V_p^\varphi = V_q^\varphi\left(I + (q-p)V_p^\varphi\right)$, i.e. $\left(V_p^\varphi\right)_{p\in(0,\infty)}$ is a resolvent on $L^\infty(U)$ and this resolvent is associated with $V^\varphi|_{L^\infty(U)}$.

Corollary 2.10 *The map V_p^φ (which is defined above for all $p \in (0,\infty)$) has the following properties:*

(i). $V_p^\varphi : (L^\infty(U), \|\cdot\|_\infty) \to (L^\infty(U), \|\cdot\|_\infty)$ is a sub Markov operator, accordingly it is a continuous map.

(ii). For all $f, g \in L^\infty(U)$ we have that $V_p^\varphi(f + g) \leq V_p^\varphi f + Vg$.

Proof. (i). Since pV^φ satisfies the weak complete maximum principle and $\left(pV^\varphi, pV_p^\varphi\right)$ is a pair of conjugated maps on $L^\infty(U)$, we can apply the Proposition 1.9 and the Definition 1.8.(iii).

(ii). It is similar to the proof of the Theorem 6. If

$$D := \{V_p^\varphi(f+g) > V_p^\varphi f + Vg\}$$

then $D \Subset U$ is an open set, $\left(V_p^\varphi(f+g)\right)|_{\partial D} = \left(V_p^\varphi f + Vg\right)|_{\partial D}$ and since $Vg \geq 0$ on the set D we have the inequalities

$$\begin{aligned}
\nu_{-V_p^\varphi(f+g)} &= \left(f+g-pV_p^\varphi(f+g)\right)^+ \cdot \lambda \\
&\leq \left(f+g-pV_p^\varphi f - pVg\right)^+ \cdot \lambda \\
&\leq \left(f-pV_p^\varphi f\right)^+ \cdot \lambda + g^+ \cdot \lambda \leq \nu_{-V_p^\varphi f - Vg}.
\end{aligned}$$

By the minimum principle for the locally convex functions if follows that $D = \emptyset$ and accordingly $V_p^\varphi(f+g) \leq V_p^\varphi f + Vg$. ■

Remark 2.11 *(i). By the previous corollary it follows that for all* $f, g \in L^\infty(U)$, $\left|V_p^\varphi f - V_p^\varphi g\right| \leq V(f-g)$ *and so that*

$$\left\|V_p^\varphi f - V_p^\varphi g\right\|_\infty \leq (diamU)\sqrt[k]{\frac{\|f-g\|_1}{\omega_k}} \leq (diamU)\sqrt[k]{\frac{\lambda(U)}{\omega_k}}\sqrt[k]{\|f-g\|_\infty} \text{ and}$$

$$\left\|V_p^\varphi f - V_p^\varphi g\right\|_1 \leq \lambda(U)(diam\, U)\sqrt[k]{\frac{\|f-g\|_1}{\omega_k}}.$$

(ii). Let $f \in L^1(U)$ *and, for all* $n \in \mathbb{N}$, $f_n := \sup\{-n, \inf\{f, n\}\}$.

(a). It is obvious that for all $n \in \mathbb{N}$, $|f_n| \leq |f|$ *and* $f_n \in L^\infty(U)$. *Moreover* $(f_n)_n$ *converges* λ *a.e. (on* U*) to the map* f *and* $(f_n)_n$ *also converges in* $(L^1(U), \|\cdot\|_1)$ *to* f.

(b). According to (i) it follows that, for all $m, n \in \mathbb{N}$

$$\left\|V_p^\varphi f_n - V_p^\varphi f_m\right\|_\infty \leq (diam\, U)\sqrt[k]{\frac{\|f_n - f_m\|_1}{\omega_k}}.$$

Since $(f_n)_n$ *is a Cauchy sequence in* $(L^1(U), \|\cdot\|_1)$, *we shall have that* $\left(V_p^\varphi f_n\right)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $\left(C(\bar{U}), \|\cdot\|_\infty\right)$, *and so that* $\left(V_p^\varphi f_n\right)_n$ *is uniformly convergent on* $\bar{U}$.

Definition 2.12 *(i). For all* $p \in (0,\infty)$ *and* $f \in L^1(U)$ *we shall define*

$$V_p^\varphi f := \lim_{n\to\infty} V_p^\varphi f_n,$$

where $f_n := \sup\{-n, \inf\{f, n\}\}$, *for all* $n \in \mathbb{N}$.

(ii). By the definition, for all $f \in L^1(U)$, *it follows that:*

(a). $V_p^\varphi f : \bar{U} \to \mathbb{R}$ *is a concave and continuous function on* $\bar{U}$ *such that* $(V_p^\varphi f)\restriction_{\partial U} = \varphi$.

(b). $\nu_{-V_p^\varphi f} = \lim\limits_{n\to\infty} \nu_{-V_p^\varphi f_n} = \lim\limits_{n\to\infty} (f_n - pV_p^\varphi f_n)^+ \cdot \lambda = (f - pV_p^\varphi f)^+ \cdot \lambda$.

(iii). Similarly to the proof of the Corollary 10.(ii). we have that for all $f, g \in L^1(U)$, $V_p^\varphi(f+g) \leq V_p^\varphi f + Vg$ *and so that* $|V_p^\varphi f - V_p^\varphi g| \leq V(f-g)$.

Theorem 2.13 *The function family* $\mathcal{V}(\varphi) := (V_p^\varphi)_{p\in(0,\infty)}$ *is a (nonlinear) sub-Markov resolvent which is associated with* V^φ *(on* $L^1(U)$*).*

Proof. Let $f \in L^1(U)$ and $(f_n)_n$ be the sequence of the Remark 11(ii). By the Theorem 8 it follows that $V_p^\varphi f_n = V^\varphi(f_n - pV_p^\varphi f_n)$, for all $n \in \mathbb{N}$, and by the Definition 12 and the Corollary 4. we have that

$$V_p^\varphi f = \lim_{n\to\infty} V_p^\varphi f_n = \lim_{n\to\infty} V^\varphi(f_n - pV_p^\varphi f_n) = V^\varphi(f - pV_p^\varphi f)$$

and

$$(I + pV^\varphi)(I - pV_p^\varphi) = I.$$

It is obvious that $(pV^\varphi, pV_p^\varphi)$ is a pair of conjugated maps and by the Proposition 1.9 and the Remark 7, it follows that pV_p^φ is a sub-Markov operator on the space $L^1(U)$.

Since $V_p^\varphi = V^\varphi(I + pV^\varphi)^{-1}$, for all $p \in (0,\infty)$, $\mathcal{V}(\varphi)$ is a resolvent on $L^1(U)$. ■

Proposition 2.14 *We have the following claims:*

(i). For all $p \in (0,\infty)$, $V_p^\varphi : (L^1(U), \|\cdot\|_1) \to (C(\bar{U}), \|\cdot\|_\infty)$ *is a continuous operator.*

(ii). For all set $\mathcal{F} \subset L^1(U)$ *such that* $\mathcal{F}$ *satisfies the property (a) of the Theorem 3.(iii). it follows that* $V_p^\varphi(\mathcal{F})$ *is a relatively compact set in* $(C(\bar{U}), \|\cdot\|_\infty)$ *(for all* $p \in (0,\infty)$*).*

Proof. (i). By the Definition 12.(iii). it follows that V_p^φ is continuous.

(ii) Since $V_p^\varphi(\mathcal{F}) = \{V^\varphi(f - pV_p^\varphi f) : f \in \mathcal{F}\}$, we have that the set $\{f - pV_p^\varphi f : f \in \mathcal{F}\}$ satisfies the condition (a) of the Theorem 3.(iii). and so that $V_p^\varphi(\mathcal{F})$ is a relatively compact set in $(C(\bar{U}), \|\cdot\|_\infty)$. ■

3 The Supermedian Functions

We shall define and we shall study the $\mathcal{V}(\varphi)$- supermedian functions and afterwards we shall compare these supermedian functions to the concave functions.

Throughout this section $\mathcal{V}(\varphi) = \left(V_p^\varphi\right)_{p\in(0,\infty)}$ is the resolvent what was defined in the previous section. Also we shall consider the resolvent $\mathcal{V}(0) = \left(V_p\right)_{p\in(0,\infty)}$ i.e. the resolvent what is defined for the map $\varphi = 0$. We shall define the extension of the operators $\left(V_p^\varphi\right)_{p\in[0,\infty)}$ (where $V_0^\varphi = V^\varphi$) to the following set of functions:

$\mathcal{F}(\varphi) := \left\{f \in \bar{R}^U : f \text{ is } \lambda \text{ measurable and } f \geq V^\varphi 0\right\}$. Obviously

$$\mathcal{F}(0) := \left\{f \in \bar{R}^U : f \text{ in } \lambda \text{ measurable and } f \geq 0\right\}.$$

Definition 3.1 *Let $f \in \mathcal{F}(\varphi)$.*

(i). For all $p \in (0,\infty)$, $V_p^\varphi f := \sup_{n\in\mathbb{N}} V_p^\varphi(\inf\{f,n\})$ and since V_p^φ is an increasing map if follows that $V_p^\varphi f = \lim_{n\to\infty} V_p^\varphi(\inf\{f,n\})$.

(ii). If $f \in L^1(U)$, then it is obvious that $V_p^\varphi f$ (the map defined here) is the function what is defined in previous section.

Remark 3.2 *(i). By the minimum principle for the convex functions it follows that for all $p \in [0,\infty)$ $V_p^\varphi(V^\varphi 0) \geq V^\varphi 0$, hence $V_p^\varphi : \mathcal{F}(\varphi) \to \mathcal{F}(\varphi)$.*

(ii). Since, for all $p \in [0,\infty)$, $V_p^\varphi : L^1(U) \to L^1(U)$ is an increasingly continuous operator we shall have that for all $f \in \mathcal{F}(\varphi)$ and $p \in [0,\infty)$

$$V_p^\varphi f = \sup\left\{V_p^\varphi g : g \in L^\infty(U) \text{ and } g \leq f\right\}$$

and if $(g_n)_n \subset L^\infty(U)$ is such that $(g_n)_n$ is increasing to f, then

$$V_p^\varphi f = \sup_{n\in\mathbb{N}} V_p^\varphi g_n = \lim_{n\to\infty} V_p^\varphi g_n.$$

(iii). It is obvious that $V_p^\varphi : \mathcal{F}(\varphi) \to \mathcal{F}(\varphi)$ is an increasing map (that is V_p^φ is an operator on $\mathcal{F}(\varphi)$) and moreover V_p^φ is increasingly continuous (i.e. if $(f_n) \subset \mathcal{F}(\varphi)$ is such that $(f_n)_n$ is increasing then

$$V_p^\varphi\left(\sup_{n\in\mathbb{N}} f_n\right) = \sup_{n\in\mathbb{N}} V_p^\varphi f_n = \lim_{n\in\mathbb{N}} V_p^\varphi f_n, \text{ for all } p \in [0,\infty).)$$

(iv). For all $f \in \mathcal{F}(\varphi)$ and $p \in [0,\infty)$, $V_p^\varphi f : \bar{U} \to (-\infty,\infty]$ and $V_p^\varphi f$ is a concave and lower semicontinuous function on $\bar{U}$ such that $(V_p^\varphi f)|_{\partial U} = \varphi$.

Moreover either $\{V_p^\varphi f = \infty\} = \phi$ *or* $\{V_p^\varphi f = \infty\} = U$*, hence if* $V_p^\varphi f < \infty$ *it follows that* $\left(V_p^\varphi(\inf\{f,n\})\right)_n$ *converges to* $V_p^\varphi f$ *uniformly on the compact set of* U *and* $\nu_{-V_p^\varphi f} = \lim_{n\to\infty}(\inf\{f,n\}) - pV_p^\varphi(\inf\{f,n\})^+ \cdot \lambda$.

Definition 3.3 *(i). The function* $u \in \mathcal{F}(\varphi)$ *is called* $\mathcal{V}(\varphi)$*- supermedian iff for all* $p \in (0,\infty)$ *we have that* $V_p^\varphi(pu) \leq u$.

(ii). We shall use the notation:

$$\mathcal{S}(\varphi) := \{u \in \mathcal{F}(\varphi) : u \text{ is } \mathcal{V}(\varphi)\text{-supermedian}\}.$$

Lemma 3.4 *If* $f, g \in \mathcal{F}(\varphi)$ *and* $p \in (0,\infty)$ *it follows that* $V_p^\varphi(f+g) \leq V_p^\varphi f + V_p g$.

Proof. For all $f, g \in L^1(U)$ the inequality $V_p^\varphi(f+g) \leq V_p^\varphi f + V_p g$ can be proved similarly to the Corollary 2.10.

Let $f, g \in \mathcal{F}(\varphi)$; since $f+g = \lim_{n\to\infty}(\inf\{f,n\}+\inf\{g,n\}) = \sup_{n\in\mathbb{N}}(\inf\{f,n\}+\inf\{g,n\})$ by the Remark 2 (iii) it follows that:

$$\begin{aligned} V_p^\varphi(f+g) &= \lim_{n\to\infty} V_p^\varphi(\inf\{f,n\} + \inf\{g,n\}) \\ &\leq \lim_{n\to\infty} \left(V_p^\varphi(\inf\{f,n\}) + V_p(\inf\{g,n\}\right) = V_p^\varphi f + V_p g. \ \blacksquare \end{aligned}$$

Corollary 3.5 *The following assertion holds:*

$$\{u + V^\varphi 0 : u \in \mathcal{S}(0)\} =: \mathcal{S}(0) + V^\varphi 0 \subset \mathcal{S}(\varphi).$$

Proof. For all $u \in \mathcal{S}(0)$ and $p \in (0,\infty)$, we have that:

$$V_p^\varphi(pu + pV^\varphi 0) \leq V_p^\varphi(pV^\varphi 0) + V_p(pu) = V^\varphi 0 + V_p(pu) \leq u + V^\varphi 0. \ \blacksquare$$

Theorem 3.6 *If* $u \in -\mathcal{U}(U)$ *is such that* $\varphi \leq (\text{sci}_U u)|_{\partial U}$*, then* $u \in \mathcal{S}(\varphi)$.

Proof. Let $p \in (0,\infty)$, $u \in -\mathcal{U}(U)$ such that $\varphi \leq (\text{sci}_U u)|_{\partial U}$, $D := \{V_p^\varphi(pu) > u\}$. It follows that $D \subset U$, D is an open set and $\left(V_p^\varphi(pu)\right)|_{\partial D} \leq (\text{scs}_D u)|_{\partial D}$ accordingly $\nu_{-u}(D) \leq \nu_{-V_p^\varphi(pu)}(D)$ (Proposition 1.4.(i).). Moreover $u \in L^\infty(U)$ and $\nu_{-V_p^\varphi(pu)}(D) = 0$, so that $(\nu_{-u})|_D = \left(\nu_{-V_p^\varphi(pu)}\right)|_D$.

Since $V_p^\varphi(pu)$ and $\text{scs}_D u$ are continuous functions on $\bar{D}$, we apply the minimum principle for the locally convex functions and we shall find that $\left(V_p^\varphi(pu)\right)|_D \leq u|_D$, therefore $D = \emptyset$ and $V_p^\varphi(pu) \leq u$ for all $p \in (0,\infty)$. $\blacksquare$

Remark 3.7 *(i). By the definition the map* $V1 : \bar{U} \to \mathbb{R}$, *$V1$ is a continuous and concave function such that* $(V1)|_{\partial U} = 0$ *and* $\nu_{-V1} = \lambda$. *Moreover for all* $x \in U$ *we have that* $(V1)(x) > 0$.

(ii). By the Corollary 5, for all $n \in \mathbb{N}$, $nV1 + V^{\varphi}0 \in \mathcal{S}(\varphi)$ *whereas* $nV1 + V^{\varphi}0$ *is a concave continuous function on* $\bar{U}$ *such that* $(nV1 + V^{\varphi}0)|_{\partial U} = \varphi$. *and we apply the previous theorem.*

(iii). Let us denote for all $n \in \mathbb{N}$, $e_n := nV1 + V^{\varphi}0$ *and let us remark that* $(e_n)_n$ *is an increasing sequence of concave continuous functions on* $\bar{U}$ *such that for all* $x \in U$ $\lim_{n\to\infty} e_n(x) = \sup_{n\in\mathbb{N}} e_n(x) = \infty$.

(iv). For all $f \in \mathcal{F}(\varphi)$, $f = \sup_{n\in\mathbb{N}}(\inf\{f, e_n\})$ *and so that*

$$V_p^{\varphi} f = \lim_{n\to\infty} V_p^{\varphi}(\inf\{f, e_n\}) = \sup_{n\in\mathbb{N}} V_p^{\varphi}(\inf\{f, e_n\}).$$

We shall use the following notation: for all $n \in \mathbb{N}$ *and* $f \in \mathcal{F}(\varphi)$, $f^{(n)} := \inf\{f, e_n\}$.

Proposition 3.8 *The following assertions hold:*

(i). If $u \in \mathcal{F}(\varphi)$ *then the following sentences are equivalent:*

(a). The map u *is* $\mathcal{V}(\varphi)$*-supermedian.*

(b). For all $n \in \mathbb{N}$, *the function* $u^{(n)}$ *is* $\mathcal{V}(\varphi)$*-supermedian (*$u^{(n)}$ *is the function what is defined in the previous remark).*

(ii). Let $(u_n)_n \subset \mathcal{S}(\varphi)$.

(a). The function $\inf_{n\in\mathbb{N}} u_n \in \mathcal{S}(\varphi)$.

(b). If $(u_n)_n$ *is increasing, then* $\sup_{n\in\mathbb{N}} u_n \in \mathcal{S}(\varphi)$.

(iii). We have that $\mathcal{S}(0) + \mathcal{S}(\varphi) \subset \mathcal{S}(\varphi)$.

(iv). For all $u \in \mathcal{S}(\varphi)$ *the map* $(p \mapsto V_p^{\varphi}(pu)) : (0, \infty) \to \mathcal{F}(\varphi)$ *is an increasing map.*

Proof. (i). (a)$\Rightarrow$(b). By the previous remark we have that for all $n \in \mathbb{N}$ $e_n \in \mathcal{S}(\varphi)$ so that for all $p \in (0, \infty)$.

$$\begin{aligned} V_p^{\varphi}(pu^{(n)}) &= V_p^{\varphi}(p\inf\{u, e_n\}) \le \inf\left\{V_p^{\varphi}(pu), V_p^{\varphi}(pe_n)\right\} \\ &\le \inf\{u, e_n\} = u^{(n)}. \end{aligned}$$

(b)$\Rightarrow$(a). We have that (for all $p \in (0, \infty)$):

$$V_p^{\varphi}(pu) = \sup_n V_p^{\varphi}(pu^{(n)}) \le \sup_{n\in\mathbb{N}} u^{(n)} = u \text{ accordingly } u \in \mathcal{S}(\varphi).$$

(ii). (a). It is immediate since $\inf_{n\in\mathbb{N}} u_n \in \mathcal{F}(\varphi)$ and, for all $n \in \mathbb{N}$,

$$V_p^{\varphi}\left(p \inf_{n\in\mathbb{N}} u_n\right) \le V_p^{\varphi}(pu_n) \le u_n$$

(b). It is similar to the claim (b)$\Rightarrow$(a) from (i).

(iii). It is a consequence of the Lemma 4.

(iv). For all $p, q \in (0, \infty)$ such that $q < p$ and for all $n \in \mathbb{N}$ it follows:

$$\begin{aligned} V_p^{\varphi}(pu^{(n)}) &= V_q^{\varphi}\left(pu^{(n)} + (q-p)V_p^{\varphi}(pu^{(n)})\right) \\ &= V_q^{\varphi}(qu^{(n)} + (p-q)\left(u^{(n)} - V_p^{\varphi}(pu^{(n)})\right)) \\ &\geq V_q^{\varphi}\left(qu^{(n)}\right) \end{aligned}$$

and $V_p^{\varphi}(pu) = \lim_{n\to\infty} V_p^{\varphi}\left(pu^{(n)}\right) \geq \lim_{n\to\infty} V_q^{\varphi}(qu^{(n)}) = V_q^{\varphi}(qu)$. ■

Proposition 3.9 *For all $f \in \mathcal{F}(\varphi)$ the following sentences are equivalent:*

(i). $V^{\varphi} f \in -\mathcal{U}(U)$.

(ii). For all $p \in (0, \infty)$, $V_p^{\varphi} f \in -\mathcal{U}(U)$.

Moreover if one of the previous claims holds then $f \in L^1_{loc}(U)$, $\nu_{-V^{\varphi}f} = f^+ \cdot \lambda$ and $\nu_{-V_p^{\varphi}f} = \left(f - pV_p^{\varphi} f\right)^+ \cdot \lambda$, for all $p \in (0, \infty)$.

Proof. Let $\varepsilon_n := f^{(n)} = \inf\{f, nV1 + V^{\varphi}0\}$.

(i)$\Rightarrow$(ii). For all $n \in \mathbb{N}$ we have that

$$V_p^{\varphi}\varepsilon_n = V^{\varphi}\left(\varepsilon_n - pV_p^{\varphi}\varepsilon_n\right) \leq V^{\varphi}\left(f - pV_p^{\varphi}(V^{\varphi}0)\right) \leq V^{\varphi}\left(f - pV^{\varphi}0\right)$$

and according to Proposition 2.2.(iii). it follows that

$$V_p^{\varphi} f = \sup_{n\in\mathbb{N}} V_p^{\varphi}\varepsilon_n \leq V^{\varphi} f + V(-pV^{\varphi}0) < \infty.$$

(ii)$\Rightarrow$(i). Similarly to the previous proof we have that for all $n \in \mathbb{N}$

$$\begin{aligned} V^{\varphi}\varepsilon_n &\leq V^{\varphi}\left(\varepsilon_n - pV_p^{\varphi}\varepsilon_n\right) + V\left(pV_p^{\varphi}\varepsilon_n\right) \\ &= V_p^{\varphi}\varepsilon_n + V(pV_p^{\varphi}\varepsilon_n) \leq V_p^{\varphi} f + V(pV_p^{\varphi} f) < \infty \end{aligned}$$

whereas $V_p^{\varphi} f$ is a concave real function on $\bar{U}$.

For the suplemental sentences we remark that $(V^{\varphi}\varepsilon_n)_n$ converges to $V^{\varphi} f$ uniformly on the compact sets of U, hence

$$\nu_{-V^{\varphi}f} = \lim_{n\to\infty}(\varepsilon_n^+ \cdot \lambda) \leq f^+ \cdot \lambda$$

and moreover $\lim_{n\to\infty}(\varepsilon_n^+ \cdot \lambda) = \left(\sup_{n\in\mathbb{N}} \varepsilon_n^+\right) \cdot \lambda = f^+ \cdot \lambda$.

We have that $\nu_{-V^{\varphi}f} = f^+ \cdot \lambda$ and since $\nu_{-V^{\varphi}f}$ is a Radon measure it follows that f^+ is λ-locally integrable and f is also λ-locally integrable.

By the Lebesgue convergence theorem we have that:

$$\begin{aligned} \nu_{-V_p^{\varphi}f} &= \lim_{n\to\infty} \nu_{-V_p^{\varphi}\varepsilon_n} = \lim_{n\to\infty} \left(\varepsilon_n - pV_p^{\varphi}\varepsilon_n\right)^+ \cdot \lambda \\ &= \left(f - pV_p^{\varphi} \cdot f\right)^+ \cdot \lambda. \blacksquare \end{aligned}$$

Corollary 3.10 *For all $f \in \mathcal{F}(\varphi)$ the following sentences are equivalent:*

(i). $f \in L^1(U)$.

(ii). $V^\varphi f \in -\mathcal{U}_{-\varphi}(U)$ and $\nu_{-V^\varphi f}$ is a bounded measure on U.

(iii) For all $p \in (0,\infty)$, $V_p^\varphi f \in -\mathcal{U}_{-\varphi}(U)$ and $\nu_{-V_p^\varphi f}$ is a bounded measure on U.

Proof. It is obvious. ■

Definition 3.11 *Let $u \in \mathcal{S}(\varphi)$.*

(i). We shall define $\hat{u} := \hat{u}_\varphi := \sup_{p\in(0,\infty)} V_p^\varphi(pu) = \lim_{p\to\infty} V_p^\varphi(pu)$ and the map $\hat{u}$ will be called $\mathcal{V}(\varphi)$-excessive regularization of u.

(ii). It is obvious that:

(a). $\hat{u} : \bar{U} \to (-\infty,\infty]$ is a concave function such that $\hat{u}|_{\partial U} = \varphi$.

(b). $\hat{u} \leq u$.

Theorem 3.12 *For all $u \in \mathcal{S}(\varphi)$ it follows that $\hat{u} = u$ (obviously λ a.e.) on U.*

Proof. If $\{\hat{u} = \infty\} = U$, then $\hat{u} = u = \infty$.

Let $\{\hat{u} = \infty\}$ be the void set. For all $p \in (0,\infty)$ we have that:

$$\nu_{-V_p^\varphi(pu)} = p\big(u - V_p^\varphi(pu)\big)\cdot\lambda, \quad \frac{1}{p}\nu_{-V_p^\varphi(pu)} = \nu_{-\frac{1}{\sqrt[k]{p}}V_p^\varphi(pu)}$$

and

$$\frac{1}{\sqrt[k]{p}}\inf\varphi \leq \frac{1}{\sqrt[k]{p}}V_p^\varphi(pu) \leq \frac{1}{\sqrt[k]{p}}\hat{u},$$

accordingly $\lim_{p\to\infty}\frac{1}{\sqrt[k]{p}}V_p^\varphi(pu) = 0$ uniformly on the compact sets of U. It follows that $\left(\frac{1}{p}\nu_{-V_p^\varphi(pu)}\right)_{p\in(0,\infty)}$ converges (vaguely) to the zero measure when p converges to ∞ and by the Beppo-Levi theorem we have that for all $f \in C_c(U)$

$$\begin{aligned}\int_U f(u-\hat{u})d\lambda &= \lim_{p\to\infty}\int_U f\left(u - V_p^\varphi(pu)\right)d\lambda \\ &= \lim_{p\to\infty}\frac{1}{p}\int_U f d\nu_{-V_p^\varphi(pu)} = 0,\end{aligned}$$

hence $u = \hat{u}$ on U. ■

Corollary 3.13 *We have that $\mathcal{S}(\varphi) = \{u \in (-\infty,\infty]^U : \exists v : U \to \bar{\mathbb{R}}$ concave function such that $(sci_{\bar{U}}v)|_{\partial U} \geq \varphi$ and $u = v$ (λ a.e.) on $U\}$.*

Proof. It follows by the Theorem 6 and the Theorem 12. ■

Corollary 3.14 *The following assertions hold:*
(i). $\mathcal{S}(\varphi)^+ + \mathcal{S}(\varphi) \subset \mathcal{S}(\varphi)$.
(ii). $\mathcal{S}(\varphi)^+ + \mathcal{S}(\varphi)^+ \subset \mathcal{S}(\varphi)^+$ *(where* $\mathcal{S}(\varphi)^+ := \{u \in \mathcal{S}(\varphi) : u \geq 0\}$*).*

Proof. (i). Let $u \in \mathcal{S}(\varphi)^+$ and $v \in \mathcal{S}(\varphi)$. Since $u = \hat{u}$ (λ a.e.) on U and $\hat{u}$ is concave function on U, it follows that $\hat{u} \in \mathcal{S}(0)$ (Theorem 6).

By the Proposition 8.(iii). we have that $\hat{u}+v \in \mathcal{S}(\varphi)$, hence $u+v \in \mathcal{S}(\varphi)$.

(ii). It is obvious. ■

Definition 3.15 *Let* $u \in \mathcal{S}(\varphi)$
(i) If $u = \hat{u}$ *everywhere on* U *and* $u < \infty$, *then* u *is called* $\mathcal{V}(\varphi)$*-excessive.*
(ii). We shall use the following notation:

$$\mathcal{E}(\varphi) := \{u \in \mathcal{S}(\varphi) : u \text{ is } \mathcal{V}(\varphi)\text{-excessive}\}.$$

(iii). Obviously if u *is* $\mathcal{V}(\varphi)$*-excessive then* u *is a real concave function on* U *such that* $(sci_U u)|_{\partial U} \geq \varphi$, *hence we have the following lemma.*

Lemma 3.16 $\mathcal{E}(\varphi) = \{u \in -\mathcal{U}(U) : (sci_U u)|_{\partial U} \geq \varphi.\}$

Proof. It is obvious by the Definition 15 and the Theorems 6. and 12. ■

Corollary 3.17 *The following assertions hold:*
(i). $\mathcal{E}(\varphi)^+ + \mathcal{E}(\varphi) \subset \mathcal{E}(\varphi)$.
(ii). $\mathcal{E}(\varphi)^+ + \mathcal{E}(\varphi)^+ \subset \mathcal{E}(\varphi)^+$ *(where* $\mathcal{E}(\varphi)^+ := \{u \in \mathcal{E}(\varphi) : u \geq 0\}$*).*

Proof. It is obvious. ■

References

[1] Bertin,E.M.J., *Fonctions convexes et théorie du potential*, Preprint nr. 89, sept. 1978, Univ. Utrecht., Dep. of Math.

[2] Bertin,E.M.J., *Convex Potential Theory*, Preprint nr. 489, dec. 1987, Univ. Utrecht., Dep. of Math.

[3] Boboc,N., Bucur, Gh., Cornea, A. *Order and Convexity in Potential Theory: H-Cones*, Lect. Notes in Math. 853, Berlin, 1981.

[4] Deimling, K., *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[5] Dellacherie,C., *Une version non linéaire du théorème de Hunt*, Proc. of I.C.P.T., Japan, 1990, Walter de Gruyter, Berlin, 1992.

[6] Gool, Frans van, *Topics in Nonlinear Potential Theory*, Ph.D. thesis, Utrecht, 1992.

[7] Meyer,P.A., *Probability and Potentials*, Blaisdell Publishing Company, 1966.

[8] Rockafellar,T., *Convex analysis*, Princeton Univ. Press, 1970.

[9] Udrea,C., *Nonlinear Resolvents*, Rev. Roum. de Math. Pures et Appl., 7-8, 1995.

[10] Udrea,C., *Supermedian Functions with Respect to a Nonlinear Resolvent*, Math. Rep., Ed. Acad. de Roumanie 49, 1997.

[11] Udrea,C., *Nonlinear Operators: Boundedness and Maximum Principles*, Rev. Roum. de Math. Pures et Appll., vol. XLVI, 2001.