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DANIEL BELTITA

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DANIEL BELTITA*

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* Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1–764, Ro–70700 Bucharest, Romania
E–mail address: dbeltita@imar.ro

Spectra for solvable Lie algebras of bundle endomorphisms

Daniel Beltiţă

Abstract. The aim of the paper is to investigate spectral properties of the Lie algebras corresponding to the symmetry groups of certain flags of vector bundles over a compact space. Under natural hypotheses, such Lie algebras are solvable, but they are in general infinite dimensional. The spectral theory of finite-dimensional solvable Lie algebras of operators is extended to this natural class of infinite-dimensional solvable Lie algebras. The discussion uses the language of continuous fields of C^*-algebras. The flag manifolds in C^*-algebraic framework are naturally involved here, they providing the basic method for obtaining flags of vector bundles.

Introduction

The background of the present paper is twofold: first, the classical theorem of Lie concerning simultaneous triangularizability for Lie algebras of matrices; second, there is the close relation that is generally expected to exist between the spectrum of a multiplication operator and the range of the multiplier. Both types of questions can be thought about in the unifying framework of algebras of bundle endomorphisms naturally acting on spaces of bundle sections. In fact, our approach will make use of the language of continuous fields of C^*-algebras. The connection is that, roughly speaking, *the endomorphisms of the fibers of a Hermitian vector bundle constitute a continuous field of C^*-algebras.*

To describe the contents of the paper in some more detail, let us begin by recalling that various versions of Lie's theorem were discovered to hold for Lie algebras of Banach space operators. Some references in this connection are [Sa71], [Sa83], [Va72], [Gu80], [Fr82], [Be01b]. We refer to Chapter III in [BS01] for a systematic treatment of this topic. We are mainly interested here in the spectral theoretic aspects of Lie's theorem, taking into consideration the set of diagonal coefficients thought of as a set of characters of a Lie algebra of triangular matrices. This set of characters allows the computation of the spectrum for each matrix belonging to the considered solvable (and triangularized) Lie algebra. It turns out that such a set of characters, which we call Cartan-Taylor spectrum, exists for every finite-dimensional solvable Lie algebra of Banach space operators (cf. [Be99]; see also [Be01a] for a gentle introduction and [BS01] for more details).

In §1 of the present paper we show that the Cartan-Taylor spectrum can be constructed in the case of Hilbert space operators in a C^*-algebraic manner. This C^*-algebraic approach to joint spectra was initiated in [Va82], where joint spectra for *commuting* systems of operators were considered. That approach is based on the fact that, if δ is a closed densely defined operator in some Hilbert space such that $\delta^2 = 0$ (i.e., $\operatorname{Ran}\delta \subseteq \operatorname{Ker}\delta$), then $\operatorname{Ran}\delta = \operatorname{Ker}\delta$ if and only if $\delta + \delta^*$ has a bounded inverse. This result is proved in full generality in Lemma 3.1 in [Va79] and it turns out to be a most convenient tool in order to handle Hilbert space complexes (see [GV82]).

In §2 we prove the main result of the paper, that is Theorem 2.5. Roughly speaking, it asserts that if $\mathcal{A} = ((A(t))_{t\in T}, \Theta)$ is a continuous field of C^*-algebras over some compact space T and $\mathfrak{E}$ is a complex finite-dimensional Lie algebra, then the spectrum $\sigma(\rho)$ (see Definition 1.3) of every Lie algebra morphism $\rho : \mathfrak{E} \to \Theta$ can be computed pointwise. In the case when $A(t) = M_n(\mathbb{C})$ for every $t \in T$, this theorem allows the computation of spectrum of multiplication operators by matrix-valued continuous functions acting on L^2-spaces of $\mathbb{C}^n$-valued functions. (See also Lemma 1.2 in [AR96] for a related result.) In the final of §2 we show that a most convenient Cartan-Taylor spectrum exists for certain infinite-dimensional Lie algebras that are *pointwise finite-dimensional solvable* in a suitable sense (see Definition 2.7).

The aim of §3 is to provide a method for obtaining examples of the pointwise finite-dimensional solvable Lie algebras just referred to above (see Example 3.18). The basic ingredient here is the notion of flag manifold, because on such a manifold is naturally "living" a flag of vector bundles. We use a version of flag manifolds arising in the framework of W^*-algebras (cf. [Sk71]). The definition we are using has the advantage that the corresponding flag manifold is naturally acted on by a certain complex Banach-Lie group (see Proposition 3.3). Our basic references for Banach manifolds and Banach-Lie groups are [Up85] and [Ne00].

1. Cartan-Taylor spectrum in C^*-algebras

1.1. Notation. If $\mathfrak{E}$ is a complex finite-dimensional Lie algebra, then

$$\widehat{\mathfrak{E}} = \{\lambda : \mathfrak{E} \to \mathbb{C} \mid \lambda \text{ is linear and } \lambda|_{[\mathfrak{E},\mathfrak{E}]} = 0\}$$

stands for the set of characters of $\mathfrak{E}$. If B is a unital associative complex algebra and $\rho : \mathfrak{E} \to B$ is a Lie algebra morphism, then for every $\lambda \in \mathfrak{E}$ we denote by $\rho - \lambda$ the Lie algebra morphism defined by

$$\rho - \lambda : \mathfrak{E} \to B, \quad e \mapsto \rho(e) - \lambda(e)1. \quad \blacksquare$$

1.2. Remark. For every finite-dimensional complex vector space $\mathfrak{E}$ let us consider the exterior algebra

$$\wedge\mathfrak{E} = \bigoplus_{p=0}^{\dim \mathfrak{E}} \wedge^p \mathfrak{E}.$$

We shall always think of the unital associative algebra $\mathcal{B}(\wedge\mathfrak{E})$ endowed with its unique C^*-algebra structure (i.e., norm) given by an arbitrary choice of a scalar product on the finite-dimensional vector space $\mathfrak{E}$. (Recall that, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with the same underlying finite-dimensional vector space $\mathcal{V}$, then the identity mapping $\mathrm{id}_{\mathcal{V}} : \mathcal{H}_1 \to \mathcal{H}_2$ induces a natural isometric $*$-isomorphism of the unital C^*-algebra $\mathcal{B}(\mathcal{H}_2)$ onto $\mathcal{B}(\mathcal{H}_1)$.) $\blacksquare$

Now we can introduce a C^*-algebraic version of the notion of spectrum for a representation of a Lie algebra (cf. Definition 1 in §25 from [BS01]). In the case of Abelian Lie algebras, this definition agrees with Definition 1.3 in [Va82].

1.3. Definition. Let $\mathfrak{E}$ be a complex finite-dimensional Lie algebra. If B is a complex associative unital $*$-algebra, then for every Lie algebra morphism $\rho : \mathfrak{E} \to B$ consider the element

$$\delta_\rho \in B \otimes \mathcal{B}(\wedge\mathfrak{E}) = \mathcal{B}(\wedge\mathfrak{E}, B \otimes \wedge\mathfrak{E})$$

defined by

$$\delta_\rho(\underline{u}) = \sum_{i=1}^{p} (-1)^{i-1} \rho(u_i) \otimes \overset{i}{\widehat{\underline{u}}} + \sum_{1 \le i < j \le p} (-1)^{i+j-1} 1 \otimes [u_i, u_j] \wedge \overset{i,j}{\widehat{\underline{u}}}$$

for $0 \le p \le \dim \mathfrak{E}$, $\underline{u} = u_1 \wedge \cdots \wedge u_p \in \wedge^p \mathfrak{E}$, where $\overset{i}{\widehat{\underline{u}}}$ denotes as usually the omission of the i-th factor in $\underline{u}$ etc.. We further define

$$\sigma(\rho) := \{\lambda \in \widehat{\mathfrak{E}} \mid \delta_{\rho-\lambda} + \delta^*_{\rho-\lambda} \text{ invertible in } B \otimes \mathcal{B}(\wedge\mathfrak{E})\}. \quad \blacksquare$$

1.4. Remark (cf. Proposition 1.4 in [Va82]). Let $\dim \mathfrak{E} = 1$ and $e \in \mathfrak{E} \setminus \{0\}$. Denote $a := \rho(e) \in B$. We have $\wedge\mathfrak{E} = \wedge^0\mathfrak{E} \oplus \wedge^1\mathfrak{E} = \mathbb{C}1 \oplus \mathbb{C}e$ and $\delta_\rho(1) = 0$, $\delta_\rho(e) = a$. Thus

$$\delta_\rho = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in B \otimes M_2(\mathbb{C}) \cong B \otimes \mathcal{B}(\wedge\mathfrak{E}).$$

Furthermore we have a natural bijection

$$\widehat{\mathfrak{E}} \to \mathbb{C}, \quad \tilde{\lambda} \mapsto \tilde{\lambda}(e) =: \lambda. \tag{1}$$

On the other hand, for every $\tilde{\lambda} \in \widehat{\mathfrak{E}}$ it is easily seen that

$$\delta_{\rho-\tilde{\lambda}} + \delta^*_{\rho-\tilde{\lambda}} = \begin{pmatrix} 0 & a - \lambda \\ (a-\lambda)^* & 0 \end{pmatrix}.$$

It is straightforward to check that the last matrix is invertible in the unital $*$-algebra $B \otimes M_2(\mathbb{C})$ if and only if $a - \lambda$ is invertible in B. Thus the bijection (1) leads to a bijection between $\sigma(\rho)$ and the spectrum of a in B. $\blacksquare$

In order to put Definition 1.3 in a proper perspective, recall the Taylor joint spectrum for commuting systems of operators introduced in [Ta70] by making use of Koszul complexes. A C^*-algebraic version of this joint spectrum was studied in [Va82]. On the other hand, joint spectra for noncommuting systems of operators were introduced as well, extending the notion from [Ta70] by means of Lie algebra homology and cohomology (see e.g. [Fa93], [BL93], [Ott97]; a systematic treatment of this topic, as well as historical notes, can be found in [BS01]). Note that the commuting n-tuples of operators are in one-to-one correspondence with the representations of the Abelian Lie algebra $\mathbb{C}^n$; in the same way, the representations of a finite-dimensional Lie algebra where a certain basis was fixed lead to systems of operators that are in general noncommutative. Now, the Definition 1.3. above should be thought of as giving the C^*-algebraic version of joint spectrum for noncomutative systems of operators.

1.5. Remark. Let us prove that

$$(\delta_\rho)^2 = 0$$

in order to suggest the connection between Definition 1.3 and Lie algebra homology and cohomology. Let $\rho : \mathfrak{C} \to B$ be as in Definition 1.3. (In fact, the involution of B will not be involved in the reasoning below.) First recall that there exist natural mappings

$$\begin{array}{ccc} B & \xrightarrow{L_0} & \mathcal{B}(B) \\ & & \downarrow M_0 \\ & & B \end{array}$$

such that L_0 is morphism of unital associative algebras, M_0 is just linear and $M_0 \circ L_0 = \mathrm{id}_B$, so L_0 is in particular one-to-one. (More precisely, for $b \in B$, $L_0(b)$ is the left multiplication operator $c \mapsto bc$, while $M_0(\psi) = \psi(1)$ for every $\psi \in \mathcal{B}(B)$.) More generally, we can consider the diagram

$$\begin{array}{ccc} B \otimes \mathcal{B}(\wedge\mathfrak{C}) & \xrightarrow{L_{\mathfrak{C}}} & \mathcal{B}(B \otimes \wedge\mathfrak{C}) \\ & & \downarrow M_{\mathfrak{C}} \\ & & B \otimes \mathcal{B}(\wedge\mathfrak{C}). \end{array}$$

Here $L_{\mathfrak{C}}$ is the composition of the morphisms of unital algebras

$$B \otimes \mathcal{B}(\wedge\mathfrak{C}) \xrightarrow{L^1_{\mathfrak{C}}} \mathcal{B}(B) \otimes \mathcal{B}(\wedge\mathfrak{C}) \xrightarrow{L^2_{\mathfrak{C}}} \mathcal{B}(B \otimes \wedge\mathfrak{C}),$$

where $L^1_{\mathfrak{C}} = L_0 \otimes \mathrm{id}_{\mathcal{B}(\wedge\mathfrak{C})}$, while $L^1_{\mathfrak{C}}$ is given by $(L^2_{\mathfrak{C}}(\psi \otimes \phi))(b \otimes v) = \psi(b) \otimes \phi(v)$ for $\psi \in \mathcal{B}(B)$, $\phi \in \mathcal{B}(\wedge\mathfrak{C})$, $b \in B$, $v \in \wedge\mathfrak{C}$, and $M_{\mathfrak{C}}$ is the restriction mapping

$$\mathcal{B}(B \otimes \wedge\mathfrak{C}, B \otimes \wedge\mathfrak{C}) \to \mathcal{B}(1 \otimes \wedge\mathfrak{C}, B \otimes \wedge\mathfrak{C}) \cong B \otimes \mathcal{B}(\wedge\mathfrak{C}).$$

(Note the vector space isomorphisms $\mathcal{B}(1 \otimes \wedge\mathfrak{C}, B \otimes \wedge\mathfrak{C}) \cong \mathcal{B}(\wedge\mathfrak{C}, B \otimes \wedge\mathfrak{C}) \cong (B \otimes \wedge\mathfrak{C}) \otimes (\wedge\mathfrak{C})^* \cong B \otimes (\wedge\mathfrak{C} \otimes (\wedge\mathfrak{C})^*) \cong B \otimes \mathcal{B}(\wedge\mathfrak{C})$.) Remark that $M_{\mathfrak{C}} \circ L_{\mathfrak{C}} = \mathrm{id}_{B \otimes \mathcal{B}(\wedge\mathfrak{C})}$, thus the morphism of unital algebras $L_{\mathfrak{C}}$ is one-to-one.

Now consider the representation $L_0 \circ \rho : \mathfrak{C} \to \mathcal{B}(B)$ of the Lie algebra $\mathfrak{C}$ on B and denote its Koszul complex by

$$\mathrm{Kos}(L_0 \circ \rho) : 0 \leftarrow B \xleftarrow{\alpha_1} B \otimes \wedge^1\mathfrak{C} \xleftarrow{\alpha_2} \cdots$$

(see [BS01]). Then the operator

$$\alpha = \bigoplus_{p=0}^{\dim \mathfrak{C}} \alpha_p : B \otimes \wedge\mathfrak{C} \to B \otimes \wedge\mathfrak{C}$$

has the properties

$$\alpha^2 = 0 \text{ and } \alpha = L_{\mathfrak{C}}(\delta_\rho).$$

Now, since the morphism of unital associative algebras $L_{\mathfrak{C}}$ is one-to-one, these properties imply $(\delta_\rho)^2 = 0$. ∎

1.6. Remark. Assume that B is a unital C^*-algebra of operators on some complex Hilbert space $\mathcal{H}$. Then every Lie algebra morphism $\rho : \mathfrak{E} \to B$ can be viewed as a representation of $\mathfrak{E}$ and it is easily seen by the above Remark 1.5 and by Lemma 3.1 in [Va79] that $\sigma(\rho)$ from Definition 1.3 above agrees with the spectrum of ρ introduced in Definition 1 in §25 from [BS01]. (When the Lie algebra $\mathfrak{E}$ is Abelian, this fact reduces to Proposition 2.2 in [Va82].) ∎

The following theorem is concerned with the basic properties of the spectrum $\sigma(\cdot)$.

1.7. Theorem. *Let $\mathfrak{E}$ be a complex finite-dimensional Lie algebra. If B is a unital C^*-algebra and $\rho : \mathfrak{E} \to B$ is a Lie algebra morphism then the following assertions hold.*

(i) *The set $\sigma(\rho)$ is compact in the finite-dimensional vector space $\widehat{\mathfrak{E}}$.*

(ii) *If $\mathfrak{I}$ is a one-codimensional ideal of $\mathfrak{E}$, then*

$$\sigma(\rho)|_{\mathfrak{I}} = \sigma(\rho|_{\mathfrak{I}}). \tag{2}$$

(iii) *If $\mathfrak{E}$ is solvable, then* (2) *holds for every ideal $\mathfrak{I}$ of $\mathfrak{E}$ and $\sigma(\rho)$ is solvable.*

(iv) *The Lie algebra $\mathfrak{E}$ is nilpotent if and only if* (2) *holds for every subalgebra $\mathfrak{I}$ of $\mathfrak{E}$. In this case for every $e \in \mathfrak{E}$ the spectrum of $\rho(e) \in B$ equals $\{\widetilde{\lambda}(e) \mid \widetilde{\lambda} \in \sigma(\rho)\}$.*

Proof. In view of Remark 1.6, the assertions (i)–(iv) are consequences of the corresponding properties of the spectrum for a representation. More precisely, (i) follows by Theorem 1 in §25 of [BS01]. For (ii) and (iii) see Theorem 2 and its proof as well as Theorem 3 in §25 of [BS01]. Finally, the first part of (iv) follows by Corollary 1 and Theorem 5 in §25 of [BS01]. To obtain the last part of (iv), use (2) for the subalgebra $\mathfrak{I} = \mathbb{C}e$ of $\mathfrak{E}$, and then apply Remark 1.4 above. ∎

1.8. Definition. Let $\mathfrak{G}$ be a complex finite-dimensional Lie algebra. Consider a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ and denote by $\mathcal{C}_{\mathfrak{H}}$ the sum of all root spaces of $\mathfrak{G}$ (with respect to $\mathfrak{H}$) corresponding to the non-zero roots, so that $\mathfrak{G} = \mathfrak{H} \oplus \mathcal{C}_{\mathfrak{H}}$. Then for every complex unital associative $*$-algebra B and every Lie algebra morphism $\rho : \mathfrak{G} \to B$ define the following set of linear functionals on $\mathfrak{G}$:

$$\Sigma_{\mathfrak{H}}(\rho) := \{\widetilde{\lambda} \mid \widetilde{\lambda}|_{\mathfrak{H}} \in \sigma(\rho|_{\mathfrak{H}}) \text{ and } \widetilde{\lambda}|_{\mathcal{C}_{\mathfrak{H}}} = 0\}. \quad ∎$$

1.9. Theorem. *Assume that $\mathfrak{G}$ is a complex finite-dimensional solvable Lie algebra. If B is a unital C^*-algebra and $\rho : \mathfrak{G} \to B$ is a Lie algebra morphism, then the set $\Sigma_{\mathfrak{H}}(\rho)$ is independent of the choice of the Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$; let us denote this set by $\Sigma(\rho)$. Then $\Sigma(\rho)$ is a compact non-empty subset of the finite-dimensional vector space $\widehat{\mathfrak{G}}$ and for every subalgebra $\mathfrak{L}$ of $\mathfrak{G}$ we have*

$$\Sigma(\rho|_{\mathfrak{L}}) = \Sigma(\rho)|_{\mathfrak{L}}.$$

Proof. First note that (in view of Remark 1.6) if B is a C^*-algebra of operators on some Hilbert space $\mathcal{H}$, then $\Sigma_{\mathfrak{H}}(\rho)$ agrees with the Cartan-Taylor spectrum of the representation $\rho : \mathfrak{G} \to B \subseteq \mathcal{B}(\mathcal{H})$ as introduced in [Do01]. Now the assertions follow by the corresponding properties of the Cartan-Taylor spectrum of a representation (see [Do01]; cf. also [BS01]). ∎

1.10. Definition. The set $\Sigma(\rho)$ in Theorem 1.9 above is called the *Cartan-Taylor spectrum* of ρ. If $\mathfrak{G}$ is a complex finite-dimensional solvable Lie subalgebra of B, then the *Cartan-Taylor spectrum* of $\mathfrak{G}$ is just the Cartan-Taylor spectrum of the inclusion $\mathfrak{G} \to B$. ∎

1.11. Remark. The name "Cartan-Taylor" is intended to point out that the underlying construction heavily leans on the Cartan subalgebras in order to produce an extension of the Taylor joint spectrum. Cf. also the comments preceding Remark 1.5. ∎

2. Continuous fields of C^*-algebras

Throughout this section we denote by T a topological space and by $\mathcal{A} = ((A(t))_{t\in T}, \Theta)$ a continuous field of unital C^*-algebras on T (see [Di64]) such that Θ contains the application associating to each $t \in T$ the unit element of $A(t)$. This application is a unit element of Θ which is denoted by 1. Let A be the unital $*$-subalgebra of Θ consisting of all $x \in \Theta$ such that $\|x(\cdot)\|$ is bounded. We recall that A is a C^*-algebra with respect to the natural norm

$$\|x\| = \sup_{t\in T} \|x(t)\|.$$

For every $t \in T$ we have a natural morphism of $*$-algebras

$$v_t : \Theta \to A(t), \quad x \mapsto x(t),$$

whose restriction to A is onto.

2.1. Remark. If $\mathfrak{E}$ is a complex finite-dimensional Lie algebra and $\rho : \mathfrak{E} \to \Theta$ is a Lie algebra morphism, then for every $t \in T$ the unital $*$-algebra morphism

$$v_t \otimes \mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})} : \Theta \otimes \mathcal{B}(\wedge\mathfrak{E}) \to A(t) \otimes \mathcal{B}(\wedge\mathfrak{E})$$

takes the element $\delta_\rho \in \Theta \otimes \mathcal{B}(\wedge\mathfrak{E})$ into $\delta_{v_t\circ\rho} \in A(t) \otimes \mathcal{B}(\wedge\mathfrak{E})$. ∎

The following fact is a first step towards the main result of the present paper (Theorem 2.5).

2.2. Proposition. *If $\mathfrak{E}$ is a complex finite-dimensional Lie algebra and $\rho : \mathfrak{E} \to \Theta$ is a Lie algebra morphism, then*

$$\bigcup_{t\in T} \sigma(v_t \circ \rho) \subseteq \sigma(\rho).$$

Proof. Consider $\lambda \in \widehat{\mathfrak{E}}$ arbitrary such that $\lambda \notin \sigma(\rho)$. We are to prove that for every $t \in T$ we have $\lambda \notin \sigma(v_t \circ \rho)$.

Since $\lambda \notin \sigma(\rho)$, the element $\delta_{\rho-\lambda} + \delta^*_{\rho-\lambda}$ is invertible in the unital $*$-algebra $\Theta \otimes \mathcal{B}(\wedge\mathfrak{E})$. Then the element

$$(v_t \otimes \mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\delta_{\rho-\lambda} + \delta^*_{\rho-\lambda})$$

is invertible in $A(t) \otimes \mathcal{B}(\wedge\mathfrak{E})$, and thus Remark 2.1 implies that

$$\delta_{v_t\circ(\rho-\lambda)} + \delta^*_{v_t\circ(\rho-\lambda)}$$

is invertible. Since $v_t \circ (\rho - \lambda) = (v_t \circ \rho) - \lambda$, it then follows that $\lambda \notin \sigma(v_t \circ \rho)$, as desired. ∎

2.3. Corollary. *Under the hypothesis of Proposition 2.2, if $\rho(\mathfrak{E}) \subseteq A$, then*

$$\overline{\bigcup_{t\in T} \sigma(v_t \circ \rho)} \subseteq (\rho).$$

Proof. Since A is a C^*-algebra, Theorem 1.6(i) shows that $\sigma(\rho)$ is a closed set. Then the conclusion follows by Proposition 2.2. ∎

It should be noted that the inclusion in Corollary 2.3 might be strict. Let us see an example in this sense.

2.4. Example (cf. Solution 98 in [Ha82]). Take $T = \{1, 2, 3, \dots\}$ endowed with the discrete topology and for every $t \in T$ take $A(t) = \mathcal{B}(\mathbb{C}^t)$, where $\mathbb{C}^t$ is viewed as a Hilbert space with the usual scalar product. Furthermore take

$$\Theta = \prod_{t=1}^{\infty} A(t) = \{(S_t)_{t\in T} \mid S_t \in \mathcal{B}(\mathbb{C}^t),\ t = 1, 2, \dots, \}$$

$$A = \{S = (S_t)_{t\in T} \in \Theta \mid \|S\| := \sup_{t\in T} \|S_t\| < \infty\}.$$

For every $t \in T$ pick

$$S_t = \begin{pmatrix} 0 & & & \mathbf{0} \\ 1 & 0 & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^t)$$

and note that $\|(S_t)^j\| = 1$ for $1 \le j \le t-1$, while $(S_t)^t = 0$. In particular we have $S = (S_t)_{t\in T} \in A$ and $\|S^j\| = 1$ for every $j \ge 1$. Thus for the spectral radius of S we have $r(S) = \lim\limits_{j\to\infty} \|S^j\|^{1/j} = 1$, which implies that the spectrum of S contains at least one complex number of modulus 1. On the other hand, the spectrum of S_t reduces to $\{0\}$ for each $t \in T$, hence Remark 1.4 implies that, for $\mathfrak{E} = \mathbb{C}e$ and $\rho : \mathfrak{E} \to A$ given by $\rho(e) = S$, we have

$$\overline{\bigcup_{t\in T} \sigma(v_t \circ \rho)} = \overline{\bigcup_{t\in T} \{0\}} = \{0\} \ne \sigma(\rho).$$

Consequently the inclusion in Corollary 2.3 is strict in this situation. ∎

Despite of Example 2.4, the following theorem shows that the inclusion in Corollary 2.3 is always an equality when T is a compact space. Before stating the theorem we recall that, if T is compact, then $\Theta = A$.

2.5. Theorem. *If $\mathfrak{E}$ is a complex finite-dimensional Lie algebra, $\rho : \mathfrak{E} \to A$ is a Lie algebra morphism and the space T is compact, then*

$$\bigcup_{t\in T} \sigma(v_t \circ \rho) = \sigma(\rho).$$

The proof of Theorem 2.5 will be preceded by the following auxiliary fact.

2.6. Lemma. *Consider a family of unital C^*-algebras $(B_t)_{t\in T}$ indexed by the compact space T and let B be a unital sub-C^*-algebra of their direct product C^*-algebra. Assume that for every $b = b^* = (b_t)_{t\in T} \in B$ the function $t \mapsto \|b_t\|$ is continuous at the points of T where it vanishes. Then for every $b = b^* = (b_t)_{t\in T} \in B$ the set*

$$\bigcup_{t\in T} \sigma(b_t)$$

is compact. In particular, if b_t is invertible for every $t \in T$, then $\sup\limits_{t\in T} \|b_t^{-1}\| < \infty$.

Proof. The proof will consist of three steps.

1° (Cf. Proposition 10.3.6 and its proof in [Di64].) Let $t_0 \in T$ and D be an open subset of $\mathbb{R}$ such that $\sigma(b_{t_0}) \subseteq D$. We show that there exists a neighborhood U of t_0 such that $\sigma(b_t) \subseteq D$ for each $t \in U$. To this end denote $M = \|b\| = \sup\limits_{t\in T} \|b_t\|$ and let f be a continuous real-valued function on $[-M, M]$ such that $f|_{\sigma(b_{t_0})} \equiv 0$ and $f|_{[-M,M]\setminus D} \equiv 1$. Then $f(b_{t_0}) = 0$. But $f(b_{t_0}) = f(b)_{t_0}$, hence $\|f(b)_{t_0}\| = 0$. Since $f(b) = f(b)^*$, the hypothesis implies that there exists a neighborhood U of t_0 such that $\|f(b)_t\| < 1$ for every $t \in U$. Then for each $t \in U$ we have $\|f(b_t)\| < 1$, thus $\sup f(\sigma(b_t)) < 1$, which in turn implies $\sigma(b_t) \subset D$, as desired.

2° In view of the compactness of T, what we have just proved at 1° implies the compactness of $\bigcup\limits_{t\in T} \sigma(b_t)$. Indeed let $\{D_i\}_{i\in I}$ be an arbitrary family of open subsets of $\mathbb{R}$ such that $\bigcup\limits_{t\in T} \sigma(b_t) \subseteq \bigcup\limits_{i\in I} D_i$. Then for every $t \in T$ there exists a finite subset F_t of I such that $\sigma(b_t) \subseteq \bigcup\limits_{i\in F_t} D_i$. The conclusion of 1° further shows that for some open neighborhood U_t of t we have $\sigma(b_s) \subseteq \bigcup\limits_{i\in I} D_i$ for each $s \in U_t$. Now, since T is compact, there exist finitely many points $t_1, \dots, t_N \in T$ such that $T = \bigcup\limits_{j=1}^{N} U_{t_j}$, so $\bigcup\limits_{s\in T} \sigma(b_s) \subseteq \bigcup\limits_{j=1}^{N} \bigcup\limits_{i\in F_{t_j}} D_i$.

3° For the last assertion of the theorem, note that $\bigcup_{t\in T}\sigma(b_t)$ is a compact set which does not contain 0, so $\inf\{|\lambda| \mid \lambda\in\bigcup_{t\in T}\sigma(b_t)\}>0$. Thus

$$\sup_{t\in T}\|b_t^{-1}\| = \sup_{t\in T}\sup_{\lambda\in\sigma(b_t)}\frac{1}{|\lambda|} = \frac{1}{\inf\{|\lambda| \mid \lambda\in\bigcup_{t\in T}\sigma(b_t)\}} < \infty$$

and we are done. ∎

Proof of Theorem 2.5. In view of Corollary 2.3, it suffices to prove that the complementary of the left hand side is contained in the complementary of the right hand side. Now taking λ arbitrary in the complementary set of the left hand side of the desired equality and replacing ρ by $\rho-\lambda$ we see that what we have to prove is the following assertion: if $0\notin\sigma(v_t\circ\rho)$ for every $t\in T$, then also $0\notin\sigma(\rho)$. Now, if we denote $\alpha:=\delta_\rho+\delta_\rho^*$ and take into account Remark 2.1, this assertion eventually reduces to the following fact:

$$\text{if } (v_t\otimes \mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\alpha)\in A(t)\otimes\mathcal{B}(\wedge\mathfrak{E}) \text{ is invertible for every } t\in T, \text{ then } \alpha\in A\otimes\mathcal{B}(\wedge\mathfrak{E}) \text{ is invertible.} \quad (3)$$

On the other hand, let us denote by $\mathcal{A}$ the direct product C^*-algebra of the family $(A(t))_{t\in T}$. Recall that $A\subseteq\mathcal{A}$ and for every $t_0\in T$ the $*$-morphism $v_{t_0}:A\to A(t_0)$ is the restriction of the canonical projection of $\mathcal{A}$ onto $A(t_0)$. In particular, if $b\in A$ and $v_t(b)=0$ for every $t\in T$, then $b=0$. Since $\dim\mathcal{B}(\wedge\mathfrak{E})<\infty$, it then easily follows that if $\beta\in A\otimes\mathcal{B}(\wedge\mathfrak{E})$ and $(v_t\otimes\mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\beta)=0$ for every $t\in T$, then $\beta=0$. Consequently, if $\mathcal{P}$ denotes the direct product C^*-algebra of the family $(A(t)\otimes\mathcal{B}(\wedge\mathfrak{E}))_{t\in T}$, then the unital $*$-morphism

$$\gamma: A\otimes\mathcal{B}(\wedge\mathfrak{E})\to\mathcal{P},\quad \beta\mapsto((v_t\otimes\mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\beta))_{t\in T},$$

is one-to-one. Then Proposition 1.5.3 in [Di64] applied for the self-adjoint element $\alpha\in A\otimes\mathcal{B}(\wedge\mathfrak{E})$ easily implies that α is invertible if and only if $\gamma(\alpha)$ is invertible. The invertibility of $\gamma(\alpha)$ is further equivalent to

$$\sup_{t\in T}\|((v_t\otimes\mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\alpha))^{-1}\|<\infty, \quad (4)$$

since all of the components of $\gamma(\alpha)$ are invertible by the very hypothesis of (3).

Let us note that (4) will follow by the last assertion of Lemma 2.6, provided the hypothesis of that lemma is verified with the sub-C^*-algebra $\gamma(A)$ of $\mathcal{P}$ in the role of B. To prove this last fact, consider $\beta=\beta^*\in A\otimes\mathcal{B}(\wedge\mathfrak{E})$ and $t_0\in T$ such that $(v_{t_0}\otimes\mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\beta)=0$. We show that the function

$$u:T\to[0,\infty),\quad t\mapsto\|(v_t\otimes\mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\beta)\|$$

is continuous at t_0. Denoting $m=\dim\mathcal{B}(\wedge\mathfrak{E})$ we have $A\otimes\mathcal{B}(\wedge\mathfrak{E})\cong M_m(A)$ ($m\times m$ matrices with entries in A). If for $t\in T$ we denote $(v_t\otimes\mathrm{id}_{\mathcal{B}(\wedge\mathfrak{E})})(\beta)=(\beta_{ij}(t))_{1\le i,j\le m}$, then $\|\beta_{ij}(\cdot)\|$ are continuous functions on T and there exist positive constants C_1 and C_2 such that

$$C_1 g(t)\le u(t)\le C_2 g(t) \text{ for every } t\in T,$$

where $g(t)=\max_{1\le i,j\le m}\|\beta_{ij}(t)\|$. Since the function g is a continuous and vanishes at t_0 (because $u(t_0)=0$), the above inequalities imply that u is also continuous at t_0 and we are done. ∎

We now make a definition which reduces to Definition 1.10 in the case when the index set of the continuous field of C^*-algebras contains only one point.

2.7. Definition. Let $\mathfrak{G}$ be a complex Lie subalgebra of Θ. We say that $\mathfrak{G}$ is *pointwise finite dimensional solvable* if $v_t(\mathfrak{G})$ is a finite dimensional solvable Lie subalgebra of $A(t)$. In this case we define the *Cartan-Taylor spectrum* of $\mathfrak{G}$ by

$$\Sigma(\mathfrak{G})=\bigcup_{t\in T}\{\lambda\circ(v_t|_{\mathfrak{G}}) \mid \lambda\in\Sigma(v_t(\mathfrak{G}))\}.$$

2.8. Remark. If $A(t)$ is finite dimensional for every $t \in T$, then each solvable Lie subalgebra of A is pointwise finite dimensional solvable.

2.9. Theorem. *Assume that the space T is compact. If the complex Lie subalgebra $\mathfrak{G}$ of A is pointwise finite dimensional solvable, then every complex Lie subalgebra $\mathfrak{F}$ of $\mathfrak{G}$ is pointwise finite dimensional solvable and*

$$\Sigma(\mathfrak{G})|_{\mathfrak{F}} = \Sigma(\mathfrak{F}).$$

If $\mathfrak{E}$ is a finite-dimensional nilpotent Lie subalgebra of A, then $\Sigma(\mathfrak{E}) = \sigma(\mathrm{id}_{\mathfrak{E}})$, where $\mathrm{id}_{\mathfrak{E}} : \mathfrak{E} \to A$ is the embedding of $\mathfrak{E}$ into A.

Proof. Since $\mathfrak{F} \subseteq \mathfrak{G}$, we have $v_t(\mathfrak{F}) \subseteq v_t(\mathfrak{G})$ for every $t \in T$, which implies that $\mathfrak{F}$ is pointwise finite dimensional solvable, too. Then

$$\begin{aligned}
\Sigma(\mathfrak{G})|_{\mathfrak{F}} &= \bigcup_{t\in T} \{(\lambda \circ (v_t|_{\mathfrak{G}}))|_{\mathfrak{F}} \mid \lambda \in \Sigma(v_t(\mathfrak{G}))\} \\
&= \bigcup_{t\in T} \{\lambda \circ (v_t|_{\mathfrak{F}}) \mid \lambda \in \Sigma(v_t(\mathfrak{G}))\} \\
&= \bigcup_{t\in T} \{\lambda|_{v_t(\mathfrak{F})} \circ (v_t|_{\mathfrak{F}}) \mid \lambda \in \Sigma(v_t(\mathfrak{G}))\} \\
&= \bigcup_{t\in T} \{\mu \circ (v_t|_{\mathfrak{F}}) \mid \mu \in \Sigma(v_t(\mathfrak{G}))|_{v_t(\mathfrak{F})}\} \\
&= \bigcup_{t\in T} \{\mu \circ (v_t|_{\mathfrak{F}}) \mid \mu \in \Sigma(v_t(\mathfrak{F}))\} \\
&= \Sigma(\mathfrak{F}),
\end{aligned}$$

where Theorem 1.8 is used at the last but one equality.

For the second assertion of the theorem, use Theorem 2.5 to get

$$\sigma(\mathrm{id}_{\mathfrak{E}}) = \bigcup_{t\in T} \sigma(v_t \circ \mathrm{id}_{\mathfrak{E}}) = \bigcup_{t\in T} \sigma(v_t|_{\mathfrak{E}}).$$

On the other hand,

$$\Sigma(\mathfrak{E}) = \{\lambda \circ (v_t|_{\mathfrak{E}}) \mid \lambda \in \Sigma(v_t(\mathfrak{E}))\} = \{\lambda \circ (v_t|_{\mathfrak{E}}) \mid \lambda \in \sigma(\mathrm{id}_{v_t(\mathfrak{E})})\}.$$

Now, in view of Remark 1.5, the desired equality follows by Proposition 2.6 in [Fa93] (cf. also Remark 3 in §25 of [BS01]). ∎

2.10. Remark. Theorem 2.9 above should be compared with Theorem 2 in §27 of [BS01].

3. Flag manifolds and bundle endomorphisms

In the present section we denote by M a W^*-algebra (see [Sk71] and also section C5.3 in [SZ79]), by $\mathcal{P}_M$ the complete lattice of projections in M, that is

$$\mathcal{P}_M = \{p \in M \mid p = p^2 = p^*\},$$

(cf. [SZ79]) and by $\mathcal{E}_M$ the set of idempotents in M, i.e.,

$$\mathcal{E}_M = \{e \in M \mid e = e^2\}.$$

We further denote by $G(M)$ the Banach-Lie group of all invertible elements of M.

We recall (cf. section 2.13 in [SZ79]) the left- and right-support maps

$$l : M \to \mathcal{P}_M, \quad a \mapsto l(a),$$
$$r : M \to \mathcal{P}_M, \quad a \mapsto r(a).$$

By definition, if $a \in M$ then $l(a)$ is the smallest $p \in \mathcal{P}_M$ such that $pa = a$, while $r(a)$ is the smallest $p \in \mathcal{P}_M$ with $ap = a$. In the case when M is a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, $l(a)$ is just the orthogonal projection onto the closure of the range of the operator a, while $r(a)$ is the orthogonal projection onto the orthogonal complement of the kernel of a. Let us also recall that in this case, if $p = p^2 = p^* \in \mathcal{B}(\mathcal{H})$ then the relation

$$ap = pap$$

expresses that the range of the orthogonal projection p is invariant to the operator a.

3.1. Remark. In the case $M = \mathcal{B}(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$, the set $\mathcal{P}_M$ can be naturally identified to the Graßmann manifold of all closed subspaces of $\mathcal{H}$. Thus $\mathcal{P}_M$ has a structure of complex Banach manifold (see e.g. [Up85]). This last fact also holds in the case of an arbitrary C^*-algebra (see [MS98] and section 2.4 in [MS97]). More precisely, $\mathcal{E}_M$ and $\mathcal{P}_M$ have natural structures of complex Banach manifolds such that $\mathcal{E}_M$ is a closed complex submanifold of M and the left-support map restricted to $\mathcal{E}_M$,

$$l|_{\mathcal{E}_M} : \mathcal{E}_M \to \mathcal{P}_M$$

is a submersion, and it can be given by $l(e) = e(1 - e^* + e)^{-1}$ for every $e \in \mathcal{E}_M$.

The following lemma collects a few elementary facts that are needed in order to introduce some natural actions of $G(M)$ on $\mathcal{E}_M$ and $\mathcal{P}_M$ (see Proposition 3.3 below). Here we make use of the order relation $\leq$ on $\mathcal{E}_M$ ($e_1 \leq e_2$ if and only if $e_1e_2 = e_2e_1 = e_1$) as well as of the equivalence relation $\sim$ on $\mathcal{P}_M$ ($p_1 \sim p_2$ if and only if there exists $v \in M$ such that $p_1 = v^*v$ and $p_2 = vv^*$).

3.2. Lemma. *If $a, b \in M$, $g, g_1, g_2 \in G(M)$, $e \in \mathcal{E}_M$ and $p, p_1p_2 \in \mathcal{P}_M$, then the following assertions hold.*

(i) $l(a\,l(b)) = l(ab)$.
(ii) $l(g_1 l(g_2 p)) = l(g_1 g_2 p)$.
(iii) $r(gp) = p$ *and* $l(gp) \sim p$.
(iv) *If p is a finite projection, then $l(gp) = p$ if and only if $gp = pgp$.*
(v) *If $p_1 \leq p_2$, then $l(gp_1) \leq l(gp_2)$.*
(vi) $l(geg^{-1}) = l(g\,l(e))$.

Proof. (i) It suffices to prove that, if $q \in \mathcal{P}_M$, then we have

$$q\,(a\,l(b)) = a\,l(b) \Longleftrightarrow q\,(ab) = ab. \tag{5}$$

If $q\,(a\,l(b)) = a\,l(b)$, then $q\,(a\,l(b))\,b = a\,l(b)\,b$. Since $l(b)\,b = b$ we get $q\,(ab) = ab$. Conversely, assume $q\,(ab) = ab$. Then $(qa - a)\,b = 0$, hence $(qa - a)\,bb^* = 0$. This last relation easily implies $(qa - a)(bb^*)^{1/2} = 0$, that is $(qa - a)|b^*| = 0$. Since M is closed under Borel functional calculus, by Corollary 2.22 in [SZ79] we immediately get $(qa - a)\,s(|b^*|) = 0$, which means $(qa - a)\,l(|b^*|) = 0$ (see section 2.15 in [SZ79]) and (5) is completely proved.

(ii) Use (i).

(iii) We have

$$r(gp) = l((gp)^*) = l(pg^*) \overset{\text{(i)}}{=} l(p\,l(g^*)) = l(p \cdot 1) = l(p) = p.$$

Here $l(g^*) = 1$ as an immediate consequence of the invertibility of g^*. Now $l(gp) \sim p$ because $l(x) \sim r(x)$ for every $x \in M$ (see Theorem 4.3 in [SZ79]).

(iv) Recall that by the very definition of a finite projection (cf. section 4.8 in [SZ79]), if $q \in \mathcal{P}_M$, $q \leq p$ and $q \sim p$, then $q = p$. Now note that $gp = p\,(gp)$ implies $l(gp) \leq p$. We have $l(gp) \sim p$ by (iii), hence $l(gp) = p$.

(v) Denote $q = l(gp_2)$. Then $qgp_2 = gp_2$, hence $qgp_2p_1 = gp_2p_1$. Since $p_1 \leq p_2$, it then follows $qgp_1 = gp_1$, so $q \geq l(gp_1)$.

(vi) Since g^{-1} is invertible, we have $l(g^{-1}) = 1$, hence

$$l(geg^{-1}) \overset{\text{(i)}}{=} l(ge\,l(g^{-1})) = l(ge) \overset{\text{(i)}}{=} l(g\,l(e))$$

and we are done. ∎

The following proposition shows in particular that in a W^*-algebra the set $\mathcal{P}_M$ is naturally acted on by the whole complex Banach-Lie group $G(M)$, not only by its real subgrup consisting of the unitary elements, as it is the case in a general C^*-algebra (cf. [MS95]).

3.3. Proposition. *If we define*

$$\begin{aligned} \alpha &: G(M) \times \mathcal{E}_M \to \mathcal{E}_M, \quad \alpha(g,e) = geg^{-1}, \\ \beta &: G(M) \times \mathcal{P}_M \to \mathcal{P}_M, \quad \beta(g,p) = l(gp), \end{aligned}$$

then α and β are holomorphic actions of the complex Banach-Lie group $G(M)$ on the complex Banach manifolds $\mathcal{E}_M$ and $\mathcal{P}_M$, respectively. These actions are compatible in the sense that the diagram

$$\begin{array}{ccccc} G(M) & \times & \mathcal{E}_M & \xrightarrow{\alpha} & \mathcal{E}_M \\ \mathrm{id}_{G(M)} \downarrow & & \downarrow l|_{\mathcal{E}_M} & & \downarrow l|_{\mathcal{E}_M} \\ G(M) & \times & \mathcal{P}_M & \xrightarrow{\beta} & \mathcal{P}_M \end{array}$$

is commutative.

Proof. It is obvious that α is a holomorphic action of $G(M)$ on $\mathcal{E}_M$. The fact that β is a group action follows by Lemma 3.2 (ii). Now since $l|_{\mathcal{E}_M} : \mathcal{E}_M \to \mathcal{P}_M$ is a submersion (see Remark 3.1), the fact that β is holomorphic will follow by Corollary 8.4(i) in [Up85] as soon as we prove that the diagram is commutative. To see this last fact, take $g \in G(M)$ and $e \in \mathcal{E}_M$ arbitrary. Then

$$(l|_{\mathcal{E}_M} \circ \alpha)(g,e) = l(\alpha(g,e)) = l(geg^{-1})$$

and

$$(\beta \circ (\mathrm{id}_{G(M)} \times l|_{\mathcal{E}_M}))(g,e) = \beta(g, l(e)) = l(g\, l(e))$$

and the commutativity of the diagram follows by Lemma 3.2 (vi). ∎

Now with Proposition 3.3 at hand we can define the Graßmann and flag manifolds suited for our purposes. When $M = \mathcal{B}(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$, one gets certain constructions carried out in [HH94a], [HH94b], [Ne00] and [Ne01].

3.4. Definition. For every $p \in \mathcal{P}_M$ we define

$$\begin{aligned} \mathrm{Gr}_M(p) &= \{l(gp) \mid g \in G(M)\}, \\ G(p) &= \{g \in G(M) \mid l(gp) = p\}. \end{aligned}$$ ∎

3.5. Remark. We have $p \in \mathrm{Gr}_M(p) \subseteq \mathcal{P}_M$. Taking into account Proposition 3.3 above, $\mathrm{Gr}_M(p)$ is the orbit of p under the action α of $G(M)$ on $\mathcal{P}_M$, and $G(p)$ is the stabilizer of p. This implies in particular that $G(p)$ is a closed subgroup of $G(M)$. ∎

The following proposition contains essential information concerning the closed subgroup $G(p)$ of the Banach-Lie group $G(M)$.

3.6. Proposition. *If $p \in \mathcal{P}_M$, then $G(p)$ is a split Lie subgroup of the Banach-Lie group $G(M)$ and its Lie algebra is*

$$\mathfrak{g}(p) = \{a \in M \mid ap = pap\}.$$

If p is moreover a finite projection, then

$$G(p) = \{p \in G(M) \mid gp = pgp\},$$

hence $G(p)$ is even an algebraic subgroup of $G(M)$.

Proof. It is obvious that $\mathfrak{g}(p)$ is a closed Lie (in fact associative) subalgebra of M and the exponential map $\exp : M \to G(M)$ is a homeomorphism of a neighborhood of $0 \in \mathfrak{g}(p)$ onto a neighborhood of $1 \in G(p)$. To see that $\mathfrak{g}(p)$ is a split subspace of M, use the well-known topological isomorphism of Banach spaces

$$\Psi : M \to pMp \oplus (1-p)Mp \oplus pM(1-p) \oplus (1-p)M(1-p), \quad a \mapsto (pap, (1-p)ap, pa(1-p), (1-p)a(1-p)).$$

Since $\Psi(\mathfrak{g}(p)) = pMp$, it immediately follows that $\mathfrak{g}(p)$ has a closed complement in M. If p is a finite projection use Lemma 3.2(iv) to deduce $G(p) = \{p \in G(M) \mid gp = pgp\}$, hence the results of [HK77] can be applied to $G(p)$. ∎

3.7. Remark. Proposition 3.6. above should be compared with Lemma IV.12 in [Ne00]. ∎

3.8. Corollary. *For every projection $p \in \mathcal{P}_M$ there exists a natural structure of complex Banach manifold on $\mathrm{Gr}_M(p)$ such that the transitive action of $G(M)$ on $\mathrm{Gr}_M(p)$ is holomorphic and the natural projection*

$$\pi_p : G(M) \to G(M)/G(p) (\cong \mathrm{Gr}_M(p))$$

is a submersion.

Proof. Use Proposition 3.6 above and Theorem 8.19 in [Up85]. ∎

3.9. Proposition. *Let $p \in \mathcal{P}_M$ and for every $t \in \mathrm{Gr}_M(p)$ define $A_p(t) = tMt$. Endow $\mathrm{Gr}_M(p)$ with the manifold structure given by Corollary 3.5 and define*

$$\Theta_p = \{x : \mathrm{Gr}_M(p) \to M \mid x \textit{ continuous and } x(t) \in tMt \textit{ for every } t \in \mathrm{Gr}_M(p)\}.$$

Then

$$\mathcal{A}_p = ((A_p(t))_{t \in \mathrm{Gr}_M(p)}, \Theta_p)$$

is a continuous field of C^*-algebras on $\mathrm{Gr}_M(p)$ satisfying the hypotheses from the beginning of §2 .

Proof. First note that for $t \in \mathcal{P}_M$ the closed sub-C^*-algebra tMt of M is not a unital subalgebra (since $1 \notin tMt$ if $t \neq 1$), but it is a unital C^*-algebra in its own right with the unit t.

Now we show that $\mathcal{A}_p = ((A_p(t))_{t \in \mathrm{Gr}_M(p)}, \Theta_p)$ is indeed a continuous field of C^*-algebras on $\mathrm{Gr}_M(p)$. What we have to prove is that, if $x : \mathrm{Gr}_M(p) \to M$ is such that $x(t) \in tMt$ for every $t \in \mathrm{Gr}_M(p)$ and for each $\varepsilon > 0$ and each $t \in \mathrm{Gr}_M(p)$ there exists a neighborhood $V_{t,\varepsilon}$ of t such that $\sup\limits_{q \in V_{t,\varepsilon}} \|x(q) - x'(q)\| < \varepsilon$ for some $x' \in \Theta_p$, then $x \in \Theta_p$. To prove this let $t \in \mathrm{Gr}_M(p)$ arbitrary and $g_0 \in G(M)$ with $\pi_p(g_0) = t$. Since $\pi_p : G(M) \to \mathrm{Gr}_M(p)$ is a submersion (cf. Corollary 3.8), there exist open neighborhoods U, V of g_0 and p respectively, and a holomorphic mapping $\psi : V \to G(M)$ such that $\pi_p \circ \psi = \mathrm{id}_V$ (see Corollary 8.30 in [Up85]). But we can suppose $V_{t,\varepsilon} \subseteq V$ for each ε, and then by the assumption on x it easily follows that $x \circ \pi_p : G(M) \to M$ is continuous at g_0. Then x $(= x \circ \pi_p \circ \psi)$ is continuous on V and we are done.

Finally, note that the application associating to each $t \in \mathrm{Gr}_M(p)$ the unit element t of $A_p(t)$ is just the inclusion map $\mathrm{Gr}_M(p) \to M$, whose continuity follows by using as above the fact that π_p is submersion. ∎

Now we introduce the flag manifolds. In the following definition it is not necessary to assume any order relations between the corresponding projections.

3.10. Definition. For $p_1, \ldots, p_n \in \mathcal{P}_M$ define

$$\mathrm{Fl}(p_1, \ldots, p_n) = \{(l(gp_1), \ldots, l(gp_n)) \mid g \in G(M)\},$$
$$G(p_1, \ldots, p_n) = \{g \in G(M) \mid l(gp_k) = p_k \text{ for } k = 1, \ldots, n\}.$$ ∎

3.11. Remark.

(a) There is a transitive action of $G(M)$ on $\mathrm{Fl}(p_1, \ldots, p_n)$,

$$G(M) \times \mathrm{Fl}(p_1, \ldots, p_n) \to \mathrm{Fl}(p_1, \ldots, p_n),$$
$$(g, (t_1, \ldots, t_n)) \mapsto (l(gt_1), \ldots, l(gt_n)),$$

(cf. Proposition 3.3) and $G(p_1, \ldots, p_n)$ is the stabilizer of $(p_1, \ldots, p_n)$ with respect to this action. Thus $\mathrm{Fl}(p_1, \ldots, p_n)$ can be thought of as a homogeneous space

$$G(M)/G(p_1, \ldots, p_n) \cong \mathrm{Fl}(p_1, \ldots, p_n)$$

and under natural hypotheses it also possesses a most convenient manifold structure (see Corollary 3.8 as well as Proposition 3.12 below).

(b) We have $G(p_1, \ldots, p_n) = G(p_1) \cap \cdots \cap G(p_n)$.

(c) For $k = 1, \ldots, n$ there exists a natural mapping

$$\mathrm{pr}_k : \mathrm{Fl}(p_1, \ldots, p_n) \to \mathrm{Gr}_m(p_k), \quad (t_1, \ldots, t_n) \mapsto t_k, \tag{7}$$

that is onto. In fact, we have

$$\mathrm{Fl}(p_1, \ldots, p_n) \subseteq \mathrm{Gr}_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n) \tag{8}$$

and the maps pr_k are just the natural projections.

(d) In view of (a) and (b) above, the mapping (7) can be viewed as the natural onto mapping

$$G(M)/(G(p_1) \cap \cdots \cap G(p_n)) \to G(M)/G(p_k).$$

Thus the inclusion (8) takes the form of the natural embedding

$$G(M)/(G(p_1) \cap \cdots \cap G(p_n)) \to (G(M)/G(p_1)) \times \cdots \times (G(M)/G(p_n)),$$
$$g \cdot (G(p_1) \cap \cdots \cap G(p_n)) \mapsto (g \cdot G(p_1), \ldots, g \cdot G(p_n)). \quad \blacksquare$$

3.12. Proposition. *If $p_1, \ldots, p_n \in \mathcal{P}_M$ and $p_1 \leq \cdots \leq p_n$, then $\mathrm{Fl}(p_1, \ldots, p_n)$ possesses a structure of complex Banach manifold such that the following assertions hold.*

(i) *The natural transitive action of $G(M)$ on $\mathrm{Fl}(p_1, \ldots, p_n)$ is holomorphic and the natural projection*

$$G(M) \to G(M)/G(p_1, \ldots, p_n) (\cong \mathrm{Fl}(p_1, \ldots, p_n))$$

is a submersion.

(ii) *$\mathrm{Fl}(p_1, \ldots, p_n)$ is a complex submanifold of $\mathrm{Gr}_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n)$.*

Proof. For (i) we use the method of proof of Corollary 3.8. More precisely, we show that $G(p_1, \ldots, p_n)$ is a split Lie subgroup of the Banach-Lie group $G(M)$. It is obvious that

$$\mathfrak{g}(p_1, \ldots, p_n) = \{a \in M \mid ap_k =_k ap_k \text{ for } k = 1, \ldots, n$$

is a closed Lie (in fact associative) subalgebra of M and the exponential map $\exp M \to G(M)$ is a homeomorphism of a neighborhood of $0 \in \mathfrak{g}(p_1, \ldots, p_n)$ onto a neighborhood of $1 \in G(p_1, \ldots, p_n)$. To see that $\mathfrak{g}(p_1, \ldots, p_n)$ is a split subspace of M, we make use of the assumption $p_1 \leq \cdots \leq p_n$ to construct the topological isomorphism of Banach spaces

$$\Psi : M \to \bigoplus_{0 \leq i,j \leq n} (p_{i+1} - p_i) M (p_{j+1} - p_j), \quad a \mapsto \oplus_{0 \leq i,j \leq n} (p_{i+1} - p_i) a (p_{j+1} - p_j),$$

where we have denoted $p_0 = 0$ and $p_{n+1} = 1$. Since

$$\Psi(\mathfrak{g}(p_1, \ldots, p_n)) = \bigoplus_{0 \leq i \leq j \leq n} (p_{i+1} - p_i) M (p_{j+1} - p_j)$$

it immediately follows that $\mathfrak{g}(p_1, \ldots, p_n)$ has a closed complement in M.

For (ii) use what we have just proved as well as Remark 3.11(d). $\blacksquare$

3.13. Remark. Let $p_1, \ldots, p_n \in \mathcal{P}_M$ such that $p_1 \leq \cdots \leq p_n$. making use of Lemma 3.7 we can construct on $\mathrm{Fl}(p_1, \ldots, p_n)$ continuous fields of C^*-algebras satisfying the hypotheses from the beginning of §2. To this end fix $k \in \{1, \ldots, n\}$. For every $t = (t_1, \ldots, t_n) \in \mathrm{Gr}_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n)$ denote

$$A_k(t) := A_{p_k}(t_k) = t_k M t_k.$$

Then define

$$\Theta_k = \{x : Gr_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n) \to M \mid x \text{ continuous and } x(t) \in t_k M t_k$$
$$\text{for every } t = (t_1, \dots, t_n) \in \mathrm{Gr}_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n)\}.$$

Then it is easily checked that

$$\mathcal{A}_k = ((A_k(t))_{t \in \mathrm{Gr}_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n)}, \Theta_k)$$

is a continuous field of C^*-algebras on $Gr_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n)$. By restricting it (see section 10.1.7 in [Di64]) to the subset $\mathrm{Fl}(p_1, \dots, p_n)$ of $\mathrm{Gr}_M(p_1) \times \cdots \times \mathrm{Gr}_M(p_n)$, we get a continuous field of C^*-algebras on $\mathrm{Fl}(p_1, \dots, p_n)$, which we also denote by

$$\mathcal{A}_k = ((A_k(t))_{t \in \mathrm{Fl}(p_1, \dots, p_n)}, \Theta_k). \quad \blacksquare$$

3.14. Remark. Let $p_1, \dots, p_n \in \mathcal{P}_M$ be such that $p_1 \leq \cdots \leq p_n \in \mathcal{P}_m$ and all of the projections $p_1, p_2 - p_1, \dots, p_n - p_{n-1}$ are Abelian. (Recall that every Abelian projection is finite, hence Proposition 4.15 in [SZ79] implies that all of the projections $p_1, \dots, p_n$ must be finite.) Then the associative subalgebra of $p_n M p_n$

$$L(p_1, \dots, p_n) = \mathfrak{g}(p_1, \dots, p_n) \cap p_n M p_n$$

is solvable when considered as Lie algebra in the usual way. This follows by making use of the topological isomorphism Ψ from the proof of Proposition 3.12. In fact one obtains that the $(n+1)$-th term in the derived series of the Lie algebra $L(p_1, \dots, p_n)$ vanishes. ∎

We are now going to look at the preceding constructions from the point of view of vector bundles, instead of continuous fields of C^*-algebras. In particular we introduce the tautological bundles on the previously constructed Graßmann manifolds.

3.15. Remark. Let $\psi : M \to \mathcal{B}(\mathcal{H})$ be a unital $*$-representation on the complex Hilbert space $\mathcal{H}$ and pick $p \in \mathcal{P}_M$. Denote

$$\mathrm{Ran}\, \psi(p) = \mathcal{V}.$$

Then the closed subspace $\mathcal{V}$ of $\mathcal{H}$ is invariant to $\psi(g)$ for every $g \in G(p)$ (see Definition 3.4), hence there exists a holomorphic (uniformly continuous) representation

$$\psi_p : G(p) \to \mathcal{B}(\mathcal{V}).$$

This representation further induces a holomorphic vector bundle

$$\pi_{p,\psi} : \mathcal{T}_p(M) \to \mathrm{Gr}_M(p)$$

which is naturally associated to the principal bundle

$$\pi_p : G(M) \to G(M)/G(p) (\cong \mathrm{Gr}_M(p)).$$

We recall that the fiber of π_p is the Banach-Lie group $G(p)$, while the fiber of $\pi_{p,\psi}$ is the Hilbert space $\mathcal{V}$. ∎

3.16. Definition. Let $p_1, \dots, p_n \in \mathcal{P}_M$ be such that $p_1 \leq \cdots \leq p_n$. If $\psi : M \to \mathcal{B}(\mathcal{H})$ is a unital $*$-representation on the complex Hilbert space $\mathcal{H}$, then for every $k \in \{1, \dots, n\}$ we define the vector bundle

$$\phi_k = (\mathrm{pr}_k)^*(\pi_{p_k,\psi}) : \mathcal{T}_k^* \to \mathrm{Fl}(p_1, \dots, p_n),$$

which is the pull-back of $\pi_{p_k,\psi}$ by the map $\mathrm{pr}_k : \mathrm{Fl}(p_1, \dots, p_n) \to \mathrm{Gr}_M(p_k)$. ∎

3.17. Remark. For every $t = (t_1, \ldots, t_n) \in \mathrm{Fl}(p_1, \ldots, p_n)$ and every $k \in \{1, \ldots, n\}$ we have

$$\phi_k^{-1}(t) \cong \operatorname{Ran} \psi(p_k)$$

and

$$\phi_1^{-1}(t) \subseteq \cdots \subseteq \phi_n^{-1}(t).$$

Thus

$$\mathcal{T}_1^* \subseteq \cdots \subseteq \mathcal{T}_n^*$$

and $\phi_n|_{\mathcal{T}_k^*} = \phi_k$ for $k = 1, \ldots, n$. ∎

Finally we can put at work the constructions of the present section in order to obtain examples of situations where Theorem 2.9 applies.

3.18. Example. As above, consider a W^*-algebra M, a unital $*$-representation $\psi : M \to \mathcal{B}(\mathcal{H})$ and $p_1, \ldots, p_n \in \mathcal{P}_M$ such that $p_1 \leq \cdots \leq p_n$ and all of the projections $p_1, p_2 - p_1, \ldots, p_n - p_{n-1}$ are Abelian. Then we have the vector bundles $\phi_k : \mathcal{T}_k^* \to \mathrm{Fl}(p_1, \ldots, p_n)$ for $k = 1, \ldots, n$, hence for every compact space T and every continuous mapping $\tau : T \to \mathrm{Fl}(p_1, \ldots, p_n)$ we obtain the vector bundles

$$\tau^*(\phi_k) : \tau^*(\mathcal{T}_k^*) \to T$$

with the fiber $\operatorname{Ran} \psi(p_k)$. We have

$$\tau^*(\mathcal{T}_1^*) \subseteq \cdots \subseteq \tau^*(\mathcal{T}_n^*),$$

these inclusions being compatible with the bundle projections $\tau^*(\phi_k)$. Since the basis T is compact $\tau^*(\mathcal{T}_n^*)$ can be endowed with a Hermitian structure. Then $\mathcal{A} = ((A(t))_{t \in T}, \Theta)$ is a continuous field of C^*-algebras where $A(t)$ is the set of all continuous endomorphisms of the fiber of $\tau^*(\phi_k)$ over $t \in T$, while Θ is the set of all continuous sections in the (continuous) endomorphism bundle of $\tau^*(\phi_k)$.

Now consider the set $\mathfrak{G}$ of all $x \in \Theta$ such that the corresponding endomorphism of $\tau^*(\mathcal{T}_n^*)$ leaves invariant each of the subbundles $\tau^*(\mathcal{T}_1^*), \ldots, \tau^*(\mathcal{T}_n^*)$. Then Remark 3.14 implies that $\mathfrak{G}$ is a pointwise finite-dimensional solvable Lie subalgebra of Θ provided the fiber of $\tau^*(\mathcal{T}_n^*)$ (that is $\operatorname{Ran} \psi(p_n)$) is *finite dimensional*. Thus $\mathfrak{G}$ falls under the hypothesis of Theorem 2.9.

We further note that, if μ is a Radon measure on T with $\operatorname{supp} \mu = T$, then Θ (and in particular $\mathfrak{G}$) has a natural representation by "multiplication" operators on the Hilbert space of square-integrable sections of $\tau^*(\phi_n)$ with respect to μ. ∎

Now we consider a simplest instance of Example 3.18.

3.19. Example. With the notation of Example 3.18, take $\mathcal{H} = \mathbb{C}^n$ with the usual scalar product, $M = \mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$, ψ the natural representation of $M_n(\mathbb{C})$ on $\mathbb{C}^n$ given by multiplication of column vectors by matrices. Consider the projections

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, p_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \ldots, p_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Then take $T = [0,1]$ and $\tau : [0,1] \to \mathrm{Fl}(p_1, \ldots, p_n)$ be the constant mapping with $\tau(t) = (p_1, \ldots, p_n)$ for every $t \in T$. Then $\tau^*(\phi_n)$ is the trivial bundle over $[0,1]$ with the fiber $\mathbb{C}^n$, that is

$$\tau^*(\mathcal{T}_n^*) = [0,1] \times \mathbb{C}^n.$$

More generally, $\tau^*(\phi_k)$ is the trivial bundle over $[0,1]$ with total space

$$\tau^*(\mathcal{T}_k^*) = [0,1] \times p_k \cdot \mathbb{C}^n.$$

Thus $\mathfrak{G}$ is the Lie algebra of all upper-triangular matrix valued continuous functions on $[0,1]$,

$$\mathfrak{G} = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ \mathbf{0} & & a_{nn} \end{pmatrix} \mid a_{ij} : [0,1] \to \mathbb{C} \text{ continuous, } 1 \le i \le j \le n \right\}.$$

This implies that for every $t \in T$ we have

$$v_t(\mathfrak{G}) = \left\{ \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ & \ddots & \vdots \\ \mathbf{0} & & \alpha_{nn} \end{pmatrix} \mid \alpha_{ij} \in \mathbb{C},\ 1 \le i \le j \le n \right\}$$

hence Corollary 6 in §27 and Theorem 1 in §26 from [BS01] show that

$$\Sigma(v_t(\mathfrak{G})) = \{\lambda_i \mid 1 \le i \le n\},$$

where $\lambda_i : v_t(\mathfrak{G}) \to \mathbb{C}$ is the character given by

$$\lambda(\alpha) = \alpha_{ii} \text{ for every } \alpha = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ & \ddots & \vdots \\ \mathbf{0} & & \alpha_{nn} \end{pmatrix} \in v_t(\mathfrak{G}).$$

Now Theorem 2.10 implies that

$$\Sigma(\mathfrak{G}) = \{\lambda_{i,t} \mid 1 \le i \le n,\ t \in [0,1]\},$$

where $\lambda_{i,t} : \mathfrak{G} \to \mathbb{C}$ is the character given by

$$\lambda_{i,t}(a) = a_{ii}(t) \text{ for every } a = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ \mathbf{0} & & a_{nn} \end{pmatrix} \in \mathfrak{G}.$$

If we consider $[0,1]$ endowed with the Lebesgue measure, then the natural representation referred to in the final of Example 3.18 associates to every continuous function

$$a = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} : [0,1] \to M_n(\mathbb{C})$$

(which is a bundle endomorphism of the trivial bundle over $[0,1]$ with the fiber $\mathbb{C}^n$) the multiplication operator on $L^2([0,1], \mathbb{C}^n)$ given by

$$\begin{pmatrix} f_1 \\ \dots \\ f_n \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \dots \\ f_n \end{pmatrix}. \quad \blacksquare$$

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Institute of Mathematics "Simion Stoilow"
of the Romanian Academy
P.O. Box 1-764
RO 70700 Bucharest
Romania
E-mail address: `dbeltita@imar.ro`