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July, 2001

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Simple Proofs of Extensions of Kuzmin's Theorem with Exponential Convergence for a Class of Fibred Systems

ADRIANA BERECHET

Abstract. Let $\mathcal{B}$ be a compact and convex subset of R^d. One considers a fibred system $(\mathcal{B},T)$ in a class defined by conditions near to that used in [6] but weaker, finite range condition included. Let $\mathcal{H}$ be the partition of $\mathcal{B}$ induced by the finite range structure. One denotes Â the Perron-Frobenius operator associated with T under an arbitrarily fixed probability equivalent to Lebesgue measure. One also considers the Banach space $\mathcal{C}(\mathcal{H})$ (and resp. $\mathcal{L}(\mathcal{H})$) containing continuous (and resp.Lipschitz continuous functions on every cell of $\mathcal{H}$ and one proves for Â acting on each of these spaces a Kuzmin type theorem with exponential convergence in its norm. Then one also proves that Â is weak-mixing with respect to $\mathcal{C}(\mathcal{H})$ (resp. $(\mathcal{H})$).

Introduction.Content. Let $\mathcal{B}$ be a compact convex subset of R^d, Σ the Borel algebra in $\mathcal{B}$, λ the Lebesgue measure on Σ, $T: \mathcal{B} \to \mathcal{B}$ a map such that $(\mathcal{B}, T)$ is a fibred system.

In [4] by condition (A),...,(F),(G) one defines a class of fibred systems for which the basic theorems known for piecewise monotone transformations (abr: p.m.t.) also hold. Let A be the Perron-Frobenius operator associated with T under λ. One proves that the conclusion of Kuzmin's theorem (in the supremum norm) remains valid when A is acting on a collection denoted $\mathcal{L}$ of strictly positive functions; the rate of convergence is $\sqrt{q}$ for a $q \in (0,1)$.

In [4] one considers a class of fibred systems delimited by conditions (A),...,(E) from [6] and a stronger than (F) condition, (F)'. Following [1] one defines the Banach spaces $\mathcal{C}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ where $\mathcal{H}$ is the partition of $\mathcal{B}$ induced by the finite range condition and one proves, using the ergodic theorem Ionescu Tulcea and Marinescu (named ITM for brievity) a Kuzmin type theorem with exponential convergence for the operator A acting on $\mathcal{L}(\mathcal{H})$ in the norm of this space.

In [6] for the case of multidimensional continued fractions a Kuzmin theorem with exponential convergence is given,using ITM. It is proved on the same lines as ITM, but, being formulated in a special with respect to ITM case and using particular properties valid in that case to obtain a more direct one it has small relation with the original proof and is interesting in its own.

In this paper we consider the class of fibred systems defined by conditions (A),...,(F) from [6]. We describe this class in Section 2. That section also contains Propositions 1 – 3 which provide basic properties of T; most of them are known for different classes of fibred systems (see [1] and [4] – [6]). To make easier the lecture of this article we repeat in Section 2 the conditions(A),...,(F) and some definitions given in [4].

Section 3 contains the main results. Theorem 4 is the intermediate result concerning the asymptotic behaviour of the sequence of the iterates of the operator A acting on $\mathcal{C}(\mathcal{H})$; it is the only ergodic theorem used in the proofs of main results.

Then we consider the Perron-Frobenius operator Â associated to T under λ where $\hat{\lambda}$ is an arbitrary fixed probability on Σ equivalent to λ and we prove extension of Kuzmin's theorem for the operator Â acting on $(\mathcal{C}(\mathcal{H}),|.|)$ (Theorem6) and acting on $(\mathcal{L}(\mathcal{H}),|||.|||)$ (Theorem7).

Because of the special nature of the spaces $\mathcal{C}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$, proofs using only the apparatus of classical mathematical analysis, mainly convergence properties of sequences of continuous functions on compacts and Arzelà-Ascoli theorem and only fundamental probability theory notions (Hahn decomposition of $\mathcal{B}$ with respect to sign measures included) are possible and presented here. We mention that also a proof of the uniqueness of the endomorphism of $\mathcal{B}$ avoiding functional techniques needing additional prerequisites was possible (see Lema 3). Imitating the title of the article [3] we use the qualificative "simple" to mention this characteristic of proofs.

At the end, to complete the description of properties of Â on $\mathcal{C}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ we prove that Â is weak-mixing with respect to these spaces (Theorem 8). Then all conclusions proved in [4] for A on $\mathcal{L}(\mathcal{H})$ follow for A considered on either $\mathcal{C}(\mathcal{H})$ or $\mathcal{L}(\mathcal{H})$.

The Appendix contains completions of propositions from [6] needed in this paper.

The class of fibred systems considered in this paper contains the Jacobi-Perron algorithm, the nearest integer continued fraction expansion g-adic expansions.

We indicated by [X] Proposition y the Proposition y reference [X].

The notations follow mainly [6]

2. Description of a class of ergodic fibred system

Let $\|\bullet\|$ be a norm in $\mathbf{R}^d$, Σ be the Borel algebra in $(B, \|\bullet\|)$.We assume that $\lambda(B)=1$.Let $T:B\to B$ be a measurable and nonsingular map such that (B,T) is a measurable with respect to Σ fibred system following the definition given in [4].Let X be the denumerable set of digits and $\{[\ell], \ell\in X\}$ be the collection of fundamental intervals (more usually called cylinders) of first order from this definition .

We assume that $[\ell]$ is connex and $\lambda(\mathrm{Fr}\,[\ell])=0$, $\ell \in X$; here for any $E\subset B$, Fr E indicates frontier of E in $(B, \|\bullet\|)$.

For any $\ell^{(j)}=(\ell_1,\dots,\ell_j)\in X^j$, $j>1$ one defines recursively the set $[\ell^{(j)}]\subset B$ by $[\ell^{(j)}]=[\ell^{(j-1)}]\cap T^{-j+1}[\ell_j]$. With T we associate the label sequence $(a_n(\cdot))_{n\in N}$ defined by $a_1(w)=\ell$ if $w\in[\ell]$, $a_1(w) = a_1(T^{j-1}w)$, $j>1$, whenever this is possible. If $a_1(w),\dots, a_j(w)$ exist and $a^{(j)}(w) = \ell^{(j)}$ for some $\ell^{(j)}\in X^j$, then $[\ell^{(j)}].=\{w \mid a^{(j)}(w) = \ell^{(j)}\}\neq 0$. The sets $[\ell^{(j)}]$, $\ell^{(j)}\in X^j$ such that $\lambda([\ell^{(j)}])>0$ form a partition (mod 0) of B ; we call them the fundamental intervals of order j of the fibred system. The set $X_{(j)}\subset X^j$ containing every $\ell^{(j)}$ for which $\lambda([\ell^{(j)}])>0$ will be called the set of admissible sequences of length j, that is with j terms of the fibred system.

We denote $\mathbf{N} = \{1,2,\dots\}$; let Ω be the set of points $w\in B$ for which $a_j(w)$ exists for any $j\geq 1$.For each $w\in\Omega$ we define the $X^{\mathbf{N}_+}$valued map φ called the representation map of the fibred system by $\varphi(w) = (a_j(w))_{j\geq 1}$. The subset $\varphi(\Omega)\subset X^{\mathbf{N}_+}$will be called the set of admissible infinite sequences of the fibred system. Under our assumptions φ is well defined a.s. and $\lambda(\Omega)=1$.

A fundamental interval $[\ell^{(j)}]$ is said to be proper or full when $B = T^j[\ell^{(j)}]$.

Let , for any $\ell^{(j)}\in X^j$ and $j\geq 1$, be the map $V(\ell^{(j)}) = (T^j\,|_{[(\ell^{(j)})]})^{-1}$, where $h|_E$ indicates the restriction to $E\in\Sigma$ of the map h on B .

Let $\hat{\lambda}$ be any probability on Σ ; as usual , $L^1(B,\Sigma,\hat{\lambda})$ means the collection of complex random variables φ on B for which $\int_B |\varphi|\, d\hat{\lambda} < \infty$ under the norm $\|\varphi\|_{1,\hat{\lambda}} = \int_B |\varphi|\, d\hat{\lambda}$. We shall use the shorthand notations $L_{\hat{\lambda}}^1$, L^1 and resp $\int_B \varphi\, d\lambda$ for spaces $L^1(B,\Sigma,\lambda)$, L_λ^1 and resp the quantity $\int_B \varphi\, d\lambda$.

The defining equation of Perron – Frobenius operator $\tilde{A}$ associated with T under $\tilde{\lambda}$ is

$$\int_{T^{-1}E} h\,d\tilde{\lambda} = \int_E \tilde{A}h\,d\tilde{\lambda}\,, \qquad E \in \Sigma,\ h \in L^1_{\tilde{\lambda}}$$

We denote A, resp $\hat{A}$ the Perron - Frobenius operator associated to T under λ, resp $\tilde{\lambda}$, where $\tilde{\lambda}$ is a arbitrarily chosed probability on Σ equivalent to λ; we mentain unchanged this probability throughout the whole paper. dn

Let χ be the function defined as $\chi\,\{U;\,w\} = 1$ when $w \in U$; $=0$ when $w \notin U$, for any $U \subset B$ and $w \in B$. For every $\ell^{(j)} \in X^j$ and $j \geq 1$ we define a.s. in B the function $\omega(\ell^{(j)}) = A^j \chi\,\{[\ell^{(j)}];\}$. Then $\omega(\ell^{(j)})$ can be nonnull only in $T^j\,[\ell^{(j)}]$. $\ell^{(j)}$

In this paper we deal with the class of fibred systems satisfying conditions (A),...,(F) stated below; they are the same as that identically labeled in [6].

(A) $\sigma(j) := \max\ \mathrm{diam}\ [\ell^{(j)}] \to 0 \quad$ as $j \to 0$
where the maximum is taken over all $\ell^{(j)} \in X_{(j)}$.

(B) (finite range condition). There are finitely many subsets of B, $\mathcal{U}_j$, $1 \leq j \leq \gamma$ such that for any fundamental interval of any order m, there exists $j = j(\ell^{(m)}) \leq \gamma$ such that the equality $T^m[\ell^{(m)}] = \mathcal{U}_j$ holds. $\ell^{(m)}$

(C) (condition of Renyi). There is a constant $C \geq 1$ for which, whatever $\ell^{(n)} \in X_{(n)}$ and $n \in \mathbf{N}$, the inequality

$$\omega(\ell^{(n)}, u) \leq \omega(\ell^{(n)}, t)\, C$$

holds a.e. in $T^n[\ell^{(n)}]$. $\ell^{(n)}$

(D) Every set $\mathcal{U}_j$, $1 \leq j \leq \gamma$ contains a proper fundamental interval.

For each $n \in \mathbf{N}$, let $\Sigma^{(n)}$ be the σ algebra generated by the collection of fundamental intervals of order n, $\mathrm{V}^{\infty}_{n=1}\Sigma^{(n)}$ be the σ algebra generated by the collection $\Sigma^{(n)}$, $n \in \mathbf{N}$ of σ algebras. Condition (A) implies that $\mathrm{V}^{\infty}_{n=1}\Sigma^{(n)} = \Sigma$ (see [1], § 9.1.5).

By conditions (B) and (D) the sets $\mathcal{U}_j$, $1 \leq j \leq \gamma$ cannot have null λ-measure; therefore $\min_{1 \leq j \leq \gamma} \lambda\,(\mathcal{U}_j) = 1 / L_0 \leq 1$ for a $L_0 \geq 1$.
Since the collection $\mathcal{U}_j$, $1 \leq j \leq \gamma$ contains B, it is a covering of B. We denote $\mathcal{H}$ the finite partition of B induced by this covering.

Let $(w,r) := \{\ell^{(r)} \in X_{(r)}$ such that $w \in T^r\ [\ell^{(r)}]\}$. When $[\ell^{(r)}]$ is proper, $\ell^{(r)} \in (w,r)$ for any $w \in B$. Since by Lemma 5 from [6], for any $r \geq 1$ one has proper intervals $[\ell^{(r)}]$, $(w,r) \neq \emptyset$ for any $w \in B$ and $r \geq 1$.

By the general definition of Perron – Frobenius operators , using the previous notation on has

$$A^n f(u) = \sum_{(u,n)} f(V(\ell^{(n)})\, u)\ \omega(\ell^{(n)}, u) \qquad \text{a.s.},$$

$n \in \mathbf{N}$, $f \in L^1$.

Hence the function $A^n f$ has the same expression in all points belonging to the same cell of $\mathscr{H}$ and has in general different expressions on different cells.

Summing over $\ell^{(n)} \in (u,n)$ the inequality

$$\lambda([\ell^{(n)}])\, C^{1} \le \omega(\ell^{(n)}, u) \le \lambda([\ell^{(n)}])\, C \qquad \text{a.s. in } T^n\ [(\ell^{\,n})]$$

which holds by condition (C)we obtain

$$C^{-1} \le A^n 1 \le L_0\, C, \qquad \text{a.s} .\ n \ge 1 \qquad (1)$$

Like in [6] we denote $N_3 := L_0\, C$ (see (2.4)).Then $N_3 \ge 1$.

We denote formulas in the current section with one number ; to formulas from other sections we add the number of the section.

The last conditions are :

(E) There is a constant $N_1>0$ such that whatever $\ell^{(n)} \in X_{(n)}$ and $n \in \mathbf{N}$

$$|\ \omega(\ell^{(n)}, w) - \omega(\ell^{(n)}, u)\ | \le \|\ w - u\ \|\ \ N_1 \lambda([\ell^{(n)}]),$$

$w, u \in T^n\ [(\ell^{\,n})]$.

(F) There is a constant $N_2>0$ such that whatever $\ell^{(s)} \in X_{(s)}$ we have

$$\| V(\ell^{(s)})w - V(\ell^{(s)})u \| \le \| w - u \|\ N_2 \ , \qquad w, u \in T^s\ [(\ell^{\,s})].$$

When condition (F) holds the assumption $\lambda(\mathrm{Fr}[\ell]) = 0$, $\ell \in X$ also holds.

Conditions (E) and (F) imply that formula (1) and condition (C) hold every where in B and that the restricted map $T^n|[\ell^{(n)}]$ which is bijective $[\ell^{(n)}] \leftrightarrow T^n\, [\ell^{(n)}]$ is then bicontinuous.

When conditions (A), ...,(D) hold , by [4] Proposition 1 , T is ergodic with respect to λ and the endomorphism (between measurable spaces) (T ,μ) is exact. We observe that its proof is simple in the sense specified in Section 1.

In the sequel we assume that all conditions (A), ...,(F) are fulfilled.

Let $L_\infty(B, \Sigma, \lambda)$ be the collection of functions $f \in L^1$ with $|f|_\infty :=$ ess sup$_{u \in B} |f(u)| < \infty$ under the norm $|\cdot|_\infty$. We shall write simply L_∞ instead $L_\infty(B, \Sigma, \lambda)$.

We denote $\mathscr{C}$ ($\mathscr{H}$), simply $\mathscr{C}$ for brevity the collection of complex bounded functions f defined on B continuous on every cell $E \in \mathscr{H}$ (to mean that when $\varepsilon > 0$ there exist $\sigma(\varepsilon) = \sigma > 0$ such that if $w,u \in E$ and $\| w-u \| < \sigma, |f(w) - f(u)| < \varepsilon$). Endowed with the norm $|\cdot|$, $|f| := \sup_{w \in B} |f(w)|$, this is a Banach space.

We denote $\mathscr{L}(\mathscr{H})$, simply $\mathscr{L}$ for brevity the collection of complex functions f belonging to $\mathscr{C}(\mathscr{H})$, Lipschitz continuous on every cell of $\mathscr{H}$. We endow it with the norm $|||f||| := |f| + s(f)$, where $s(f) := \max_{J \subset \mathscr{H}} s(f;J)$ and $s(f;J) = \sup\ [|f(w) - f(u)| / \| w-u \|]$, where sup is considered over $w,u \in \mathscr{W}$, $w \neq u$. Then $\mathscr{L}$ is a Banach space and we have $\mathscr{L} \subset \mathscr{C} \subset L_\infty$.

By Proposition A1, the operator A takes $\mathscr{L}$ (and resp $\mathscr{C}$) into itself. Also, if $f \in \mathscr{L}$, by this proposition the sequences $\{s(A^n f)\}_{n\geq 1}$ and $\{|||A^n f|||\}_{n \geq 1}$ are bounded. Specifically $|||A^n f||| \leq \check{N} |||f|||$, $n \in \mathbf{N}$, where $\tilde{N} = \max\{N_1 + N_3, N_2 N_3\}$.

Remark that the upper bound of $s(A^n f)$, $n \geq 1$ given in Proposition $A1$ (hence under condition (F)) is the analogous of the bound obtained in [4] Proposition 3 (hence under the stronger condition (F)'). It does not imply condition labeled 2 in [6] from ergodic theorem ITM which is an immediate consequence of [6] Proposition 3. Also, by Proposition A_1 the property of A on $\mathscr{L}$ to be Doeblin – Forbet operator is not implied, like it is by [4] Proposition 3 under condition (F)'.

On the same lines, if we define conformly [1] the random system $((B, \Sigma), (X, \mathscr{P}(X)), V(\cdot), U\chi\{[\cdot]:\})$ where $\mathscr{P}(X)$ is the collection of all subsets of X and U is the Perron – Frobenius operator associated with T under the probability μ defined by (7) below, the bounds given in Proposition A1 do not entail its contraction property. Our proof of main results does not use these properties explicitely or implicitely.

$$A_r = \frac{1}{r} \sum_{n=0}^{r-1} A^n \qquad (2)$$

For any $r \in \mathbf{N}$ we denote A_r the linear bounded operator on L^1

Proposition 1. There exist in $\mathscr{L}$ a solution of functional equation

$$Af = f. \qquad (3)$$

Proof . The sequence $A_r 1$, $r\in \mathbf{N}$ contains a subsequence which converges to a function belonging to $\mathscr{L}$, h say (see Proposition A2).
Let $r(t)$, $t\in \mathbf{N}$ be the indices of a converging subsequence of $A_r 1$, $r\in \mathbf{N}$; hence $r(t) \to \infty$ as $t \to \infty$ and we have

$$|A_{r(t)}1 - h| \to 0 \qquad \text{as } t \to \infty . \tag{4}$$

Let $n,s,t_1,t_2\in \mathbf{N}$, $t_1 < t_2$ be such that $r(t_1) \le s \le r(t_1+1)$, $r(t_2) \le n \le r(t_2+1)$.
Then $|A_s 1 - A_n 1| \le |A_{s-r(t_1)}(A_{r(t_1)}1 - h)| + |A_{n-r(t_2)}(A_{r(t_2)}1 - h)|$.

Using (1) we have

$$|A\ | = \sup\frac{|A_r f|}{|f|} \le \sup\frac{|A_r f||f|}{|f|} = |A_r 1| \le N_3 \qquad r\in\mathbf{N} \tag{5}$$

where the supremum is considered over $f\in\mathscr{L}$, $f\neq 0\in\mathscr{L}$ .Hence
$|A_s 1 - A_n 1| \le N_3(|A_{r(t_1)}1 - h| + |A_{r(t_2)}1 - h|) \to 0$ as $n,s \to \infty$; we conclude that the sequence $A_r 1$, $r\in \mathbf{N}$ is fundamental in $\mathscr{C}$.

Then by (4) we get

$$|A_r 1 - h| \to 0 \qquad \text{as } r \to \infty ,\ r\in \mathbf{N}. \tag{6}$$

We also have

$$|AA_r 1 - A_r 1| = \left|\frac{1}{r}A^r 1 - \frac{1}{r}A^0 1\right| \le \frac{1}{r}(1+N_3) \to 0 \qquad \text{as } r\to\infty$$

Using the last two limit relations and the identity

$$|Ah - h| = |Ah - AA_r 1 + AA_r 1 - A_r 1 + A_r 1 - h| ,$$

it is immediate that the equality $0 = Ah - h$ follows. □

Let μ be the measure on Σ defined as

$$\mu(E) = \int_E h\, d\lambda , \qquad E\in\Sigma \tag{7}$$

The equivalence relation between the measures ν_1 and ν_2 on Σ will be indicated by $\nu_1 \equiv \nu_2$; the relation „ ν_1 is absolutely continuous with respect to ν_2 " will be indicated by $\nu_1 \ll \nu_2$.

Proposition 2 . The measure μ is preserved by T and $\mu \equiv \nu$.

Proof. Integrating with respect to λ the equality $h = Ah$ on arbitrary $E \in \Sigma$ we have

$$\mu(E) = \int_E h\, d\lambda = \int_E Ah\, d\lambda = \mu(T^{-1}E).$$

Hence T is μ preserving.

Convergence in the norm $|\cdot|$ is equivalent to weak convergence in $\mathscr{C}$, hence with punctual convergence. Therefore by (6) we also proved that

$$h(u) = \lim_{n\to\infty} (A_n 1)(u) \qquad , u \in W\,,\ W \in \mathscr{H} \tag{8}$$

By [4] Proposition A3 , h belongs to $\mathscr{L}$ and clearly (8) implies that h is a solution of (3) . We note that in (8) the convergence is uniform on cells. Integrating (8) with respect to λ on B we get

$$\mu(B) = \int h\, d\lambda = \lim_{n\to\infty} \int A_n 1 d\lambda = 1.$$

It follows that μ is a probability on Σ.

By (1) , (2) and (8) we have h^{-1}, $h \le N_3$; then by integrating the last inequalities with respect to λ on arbitrary $E \in \Sigma$, we conclude that $\mu \equiv \lambda$.

Lemma 3. The probability μ on Σ defined by (7) is the unique one absolutely continuous with respect to λ which is preserved by T.

Proof. Assume for a contradition that is exists a probability $\tilde{\mu}$ on Σ, $\tilde{\mu} \neq \mu$, $\tilde{\mu} \ll \lambda$ which is preserved by T. Let $\tilde{\alpha}$ be ~~the~~ its λ - density. Then $\tilde{\alpha} \in L_\infty$ and satisfies equation (3).

Let $\hat{\mu}$ be the sign measure on Σ defined as $\hat{\mu} = \mu - \tilde{\mu}$ and let $\hat{\mu} = \hat{\mu}^+ - \hat{\mu}^-$ be the Jordan decomposition of $\hat{\mu}$ .We claim that as $\hat{\mu}$, also $\hat{\mu}^+$ and $\hat{\mu}^-$ are absolutely continuous with respect to λ .Indeed , if $N \in \Sigma$ is such that $\lambda(N) = 0$, then $\int_N h\, d\lambda = \int_N \tilde{\alpha} d\lambda = 0$, hence $\mu(N) = \hat{\mu}(N) = 0$, so that $\hat{\mu}^+(N) = \hat{\mu}^-(N) = 0$.

Since by [4] Proposition 1, T is ergodic with respect to λ it results that it is ergodic with respect to $\hat{\mu}^+$ (resp $\hat{\mu}^-$) too.

For any $g \in L^1$ we denote $g^+(t) = \max\{ g(t) , 0 \}$ and $g^-(t) = \max(-g(t) , 0)$, $t \in B$, the nonnegative and nonpositive parts of g . We denote B^+ and B^- the sets of a Hahn decomposition of B with respect to $\hat{\mu}$.

Assume that $0 < \hat{\mu}(B^-) < 1$.The real function on B defined as $\hat{\alpha} = h - \tilde{\alpha}$ satisfies (3). It is immediate that the equality $A\hat{\alpha} = \hat{\alpha}$ implies $A\hat{\alpha}^+ = \hat{\alpha}^+$ a.s. in B^+ (and resp $A\hat{\alpha}^- = \hat{\alpha}^-$ a.s. in B^-). Integrating this equality with respect to λ on B^+ (resp on B^-) we see that B^+ (resp B^-) is a set invariant to T with respect to $\hat{\mu}^+$ (resp $\hat{\mu}^-$). Under our assumptions on $\hat{\mu}$ (B^-) this contradicts the fact that T is ergodic with respect to $\hat{\mu}^+$ (resp $\hat{\mu}^-$).This in turn contradicts its ergodicity with respect to $\hat{\mu}$.

Assume that $\hat{\mu}(B^-) = 0$.Then $\mu(V) \ge 0\,, \quad V \in \Sigma$ (9)

Let W $\in\Sigma$ be such that $0 < \hat{\mu}$ (W) < 1 .Then denoting W ' the complement of W in B, we have $\hat{\mu}$ (W ')< 0 and this contradicts (9) .Hence in (9) only equality can hold and this contradicts the assumption $\tilde{\mu} \neq \mu$.

Similarly one proves that the assumption $\hat{\mu}$ (B^+) = 0 leads to a contradiction , too.

Assume that $\hat{\mu}$ (B^-) = 1.Then since $\hat{\mu}$ (B^-) = $\hat{\mu}$ (B^+)-1 has to be nonnegative only $\hat{\mu}$ (B^+) = 0 in the last equality is possible and this leads to a contradiction by the above alineat. □

One can observe that the proofs of Propositions A1 and 2 below and of [4] Propositions A1 and A3 used in this section are simple in the sense specified in Section 1. Then also the proofs of propositions from this section are simple in the same sense.

We note that propositions 2 and 3 can be proved in weaker assumptions using another approach but much longer preliminaries are necessary and the proof is no more simple (see F. Schweiger The Metrical Theory of Jacobi – Perron Algorithm LNM 334 Springer Verlag 1973).

3.The ergodic behaviour of Perron – Frobenius operator $\hat{A}$ on $\mathscr{L}(\mathscr{H})$ and on $\mathscr{C}(\mathscr{H})$.

We begin this section with Theorem 4 by which one proves in fact that the operator A is ergodic with respect to $\mathscr{C}$. Then we consider the more general Perron – Frobenius operator $\hat{A}$ of T under an arbitrarily probability $\tilde{\lambda}$ equivalent to λ and we prove the Kuzmin type Theorems 6 and 7.

Using that [4] Theorem 6 extends to operator A on $\mathscr{C}$ and on $\mathscr{L}$ under the assumptions from this paper at the end also prove Theorem 8.

We record that by assumptions from previous section , conditions (A),...,(F) are supposed to hold.

Theorem 4. For any function f belonging to $\mathscr{C}$ one has

$$| A_r f - h\int f d\lambda | \to 0 \qquad \text{as } r \to \infty \qquad (1)$$

Proof. Fix arbitrary function f belonging to $\mathscr{C}$.Applying Proposition A2 we conclude that there exists a subsequence $(A_{r'} f)_{r'}$ where $\{r'\} \subset \mathbf{N}$ (depending on f) and a function $\mathscr{A}_0(f) = \mathscr{A}_0 \in \mathscr{C}$ for which

$$| A_{r'} f - \mathscr{A}_0 | \to 0 \qquad \text{as } r' \to \infty \qquad (2)$$

Parallelizing the argument used in the proof of Proposition 1 (in fact simply replacing with $A_r f$,resp $h\int f\mathrm{d}\lambda$ the functions $A_r f$, resp h) to derive starting with relation (2.4) relation (2.6) one obtains from preceeding formula

$$| A_r f - \mathcal{A}_0 | \to 0 \qquad\qquad \text{as } r\to\infty ,\ r\in \mathrm{N} \qquad (3)$$

Hence $(A_r f)_{r\in \mathrm{N}}$ converges in the norm $|\cdot|$ to $\mathcal{A}_0$.

Parallelizing the argument used to show in the mentioned proof starting with (2.6) that $Ah = h$,one obtains starting with (3) that $A\mathcal{A}_0 = \mathcal{A}_0$. Hence [6] Proposition 2 applies to $\mathcal{A}_0 = \mathcal{A}_0\ (f)$; we can conclude that there exist $\xi\in \mathbf{C}$ such that $\mathcal{A}_0 = \xi\ h$ a.s. (actually the equality holds everywhere in the present case).

Integrating this equality with respect to λ on B we obtain

$$\xi = \int\xi\, h\, \mathrm{d}\lambda = \int \mathcal{A}_0\, \mathrm{d}\lambda.$$

By (3) we know that the punctual convergence

$$\lim_n A_n f = \mathcal{A}_0 \qquad\qquad \text{as } n\to\infty ,\ n\in\mathbf{N} \qquad (4)$$

holds and is uniform on every cell. Integrating (4) with respect to λ on B we get

$$\int \mathcal{A}_0\, \mathrm{d}\lambda = \lim_n \int A_n f\, \mathrm{d}\lambda = \int f\mathrm{d}\lambda$$

Hence $\xi = \int f\, \mathrm{d}\lambda$ and then

$$\mathcal{A}_0 = h\, \int f\mathrm{d}\lambda \qquad\qquad (5)$$

The relation (1) follows by (3) and (5). □

Justified by (5) we define the linear bounded operator A^{∞} as

$$A^{\infty} f := h\, \int f\mathrm{d}\lambda , \qquad\qquad f\in L^1 \qquad (6)$$

Remark. Theorem 4 can be restated as : The operator A is ergodic with respect to $\mathscr{C}$ and its limit in the norm $|\cdot|$ is given by (6).

On account of the fact that the proof of Proposition A2 below and that of [6] Proposition 2 are simple in the sense precized in Section 1 it is seen that so is also the proof of the last theorem.

We record that we denoted $\hat{\lambda}$ an arbitrarily fixed probability on Σ equivalent to λ ; let $\hat{\alpha} := \mathrm{d}\hat{\lambda}/\mathrm{d}\lambda$ .Then there is a constant $N_4 \geq 1$ such that $\hat{\alpha}$, $\hat{\alpha}^{-1} \leq N_4$, a.s. . In all next propositions we shall assume at least that $\hat{\alpha}$ belongs to $\mathscr{C}$; then the last written inequality holds everywhere.

We denote $\hat{A}$ the Perron – Frobenius operator associated to T under $\hat{\lambda}$ .Then we have

$$\hat{A}^r f = (\hat{A}^r f\,\hat{\alpha})/\hat{\alpha} , \qquad\qquad r\in \mathbf{N} ,\ f\in L^1_{\hat{\lambda}} \qquad (7)$$

We observe that $\lambda\equiv\hat{\lambda}$ implies $L^1_{\hat{\lambda}} = L^1$.

We denote $\hat{\mathbf{a}}$ the $\hat{\lambda}$ density of μ , $\hat{\mathbf{a}} = \mathrm{d}\mu/\mathrm{d}\hat{\lambda}$.

By the Radon–Nykodim theorem we have $\hat{\mathbf{a}} = h/\hat{\alpha}$. Hence ess inf $\hat{\mathbf{a}} > 0$.

Lemma 5. Assume that $\hat{\alpha} \in \mathscr{L}$ (resp $\mathscr{C}$).Then a function $f \in \mathscr{L}$ (resp $\mathscr{C}$) satisfies functional equation

$$\hat{A}f = f \tag{8}$$

if and only if f equals $\hat{\mathbf{a}}$ multiplied by a constant (complex in general).

Proof. Because of (7) equation (8) is equivalent with $Af\hat{\alpha} = f\hat{\alpha}$ and by [6] Proposition 2' this equality implies $f\hat{\alpha} = \xi\, h$ with a $\xi \in \mathbf{C}$. Since $\inf \hat{\alpha} > 0$ and $\hat{\alpha} \in \mathscr{L}$ (resp $\mathscr{C}$) also $f = \xi\, h/\hat{\alpha} \in \mathscr{L}$(resp $\mathscr{C}$). Hence $f = \hat{\mathbf{a}}\, \xi$.

The converse is obvious and we omit the proof. □

Motivated by formulas (6) and (7) we introduce the linear bounded operator $\hat{A}^{\infty}$ on L^1 defined as

$$\hat{A}^{\infty} f = (\cancel{\hat{A}^{\infty}} f\hat{\alpha}\,)/\,\hat{\alpha} \tag{9}$$

It is easely seen that $\hat{A}\hat{A}^{\infty} = \hat{A}^{\infty}\hat{A} = \hat{A}^{\infty}$ (extension of Kuzmin's theorem in the norm $|\cdot|$).

Theorem 6. Assume that $\hat{\mathbf{a}} \in \mathscr{C}$. Then there exists a positive constant $\theta < 1$ such that

$$|\ \hat{A}^n - \hat{\mathbf{a}} \int \cdot\, d\lambda\ | = \mathcal{O}(\theta^n) \qquad , n \geq 1.$$

Proof. Fix arbitrary function f belonging to $\mathscr{C}$. By Theorem 4 there exists a subsequence $(A^{r'}f)_{r'}$ where $\{r'\} \subset \mathbf{N}$, of $(A^{r}f)_{r\in\mathbf{N}}$ (depending on f) such that

$$|\, A^{r'}f - h\int f d\lambda\, | \to 0 \qquad \text{as } r' \to \infty \ , \{r'\} \subset \mathbf{N} \tag{10}$$

Parallelizing the argument used in the proof of Proposition 1 to derive starting with relation (2.4) relation (2.6) we obtain starting with (10)

$$|\, A^{r}f - h\int f d\lambda\, | \to 0 \qquad \text{as } r \to \infty \ , \cancel{\{r\}} \cancel{\subset} \mathbf{N} \tag{11}$$ /ʅ /ʅ ∠∈

For any $r \geq 1$ we can write

$$|\ \hat{A}^{r}f - \hat{\mathbf{a}}\int f d\hat{\lambda}\ | = |\,(A^{r}f\hat{\alpha} - h\int f d\hat{\lambda})\ \hat{\alpha}^{-1}\,| \leq |\,\hat{A}^{r}f\hat{\alpha} - h\int f\hat{\alpha} d\lambda\,|\, N_4$$

Since $f\hat{\alpha}$ belongs to $\mathscr{C}$ (11) applies so that

$$|\ \hat{A}^{r}f - \hat{\mathbf{a}}\int f d\hat{\lambda}\ | \to 0 \qquad \text{as } r \to \infty \ , r \in \mathbf{N} \tag{12}$$

By (2.1) and by its very definition $\hat{A}^{\infty}$ is a bounded operator on $\mathscr{C}$; then (12) implies that $\hat{A}$ is aperiodic with respect to $\mathscr{C}$.

It is well known (see e.g.[1] Lemma 3.1.19) that if U is a linear bounded operator aperiodic with respect to a Banach space ($\mathscr{R}$, $|\|\bullet\||$) then there exists a positive constant $q < 1$ such that

$$|\| U^s - U^{\infty} \|| = \mathcal{O}(q^s) \qquad ,s \in \mathbf{N} \tag{13}$$

where $U^{\infty} := \lim_s U^s$ as $s \to \infty$ in the norm $|\|\bullet\||$.

Applying this result to $\hat{A}$ on $\mathscr{C}$, by (13) the stated assertion follows. □

Theorem 7.(extension of Kuzmin's theorem in the norm $|||\cdot|||$) . Assume that $\hat{\alpha} \in \mathscr{L}$ .Then there exists a positive constant $\theta < 1$ such that

$$||| \hat{A}^r - \hat{a} \int \cdot d\hat{\lambda} ||| = \mathcal{O}(\theta^r) \qquad \text{as } r \to \infty.$$

Proof. Let V be the linear bounded operator on L^1 defined as $Vf = (\hat{A} - \hat{A}^{\infty})f$. One has then $V^j f = \hat{A}^j f - \hat{a} \int f d\hat{\lambda}$, $j \ge 1$ (see (9)).

Fix arbitrary $f \in \mathscr{L}$ and $W \in \mathscr{H}$; let $u, w \in W$, $u \ne w$. By (12) we have

$$\frac{1}{\|w-u\|} | V^j f(w) - V^j f(u) | \to o \qquad \text{as } j \to \infty \tag{14}$$

Therefore given any $\varepsilon > 0$ there exists $j_0(\varepsilon) = j_0$ such that when $j \ge j_0$ the quantity in the left side of (14) is dominated by ε . Then , $s(V^j f; W) \le \varepsilon, j \ge j_0$. This implies $\lim_{j\to\infty} s(V^j f; W) \le \varepsilon$. Since ε is arbitrary we also have $\lim_{j\to\infty} s(V^j f; W) = 0$. Hence the relation

$$s(V^r f) \to 0 \qquad , \text{as } r \to \infty \tag{15}$$

also holds . Having in view (12) and (15) we obtain

$$||| \hat{A}^n f - \hat{a} \int f d\hat{\lambda} ||| = | V^n f | + s(V^n f) \to 0 \quad \text{as } n \to \infty .$$

Therefore $\hat{A}$ is aperiodic with respect to $\mathscr{L}$ .The assertion stated results by applying (13) once more. □

We note that the proof of [1] Proposition 3.1.19 is a simple one in the sense specified in Section 1. Then so are also the last two proofs.

In the next statement we employ standard notions of ergodic theory (see [1] as basic reference or see [4]). In the sequel $\mathscr{Y}$ stands for either the space $\mathscr{C}$ or $\mathscr{L}$ in the assertions valid for each of these spaces.

By Lemma 5, when $\hat{\mathbf{a}} \in \mathscr{Y}$, 1 is an eigenvalue of operator $\hat{A}$ on $\mathscr{Y}$. We denote $\mathscr{E}(\sigma; \mathscr{Y}) = \{f \in \mathscr{Y} ; \hat{A}f = \sigma f\}$ for any $\sigma \in \mathbf{C}$; hence $\mathscr{E}(\sigma; \mathscr{Y})$ is the subspace of proper functions corresponding to the eigenvalue σ of $\hat{A}$ on $\mathscr{Y}$.

Theorem 8. Assume that $\hat{\alpha}$ belongs to $\mathscr{C}$(resp $\mathscr{L}$). Then $\hat{A}$ is weak –mixing with respect to $\mathscr{C}$(resp $\mathscr{L}$). If $\hat{\alpha} \in \mathscr{L}$, then $\mathscr{E}(1, \mathscr{C}) = \mathscr{E}(1, \mathscr{L})$.

Proof. Since by Lemma 5 dim $\mathscr{E}(1, \mathscr{Y}) = 1$, the multiplicity of the eigenvalue 1 is 1. To prove that $\hat{A}$ is weak-mixing with respect to $\mathscr{Y}$ amounts now to prove the fact that 1 is the unique eigenvalue with modulus 1 of A considered on $\mathscr{Y}$.

We record that by [4] Theorem 6 we proved that A is weak-mixing with respect to $\mathscr{L}(\mathscr{H})$ .We observe that the proof is valid unchanged under our assumptions, that is with condition (F) replacing (F)', for A with domain either $\mathscr{C}$ or $\mathscr{L}$. Thus it results that A is weak-mixing with respect to $\mathscr{Y}$.

The complex number σ is an eigenvalue of $\hat{A}$ on $\mathscr{Y}$ when and only when there exist $\varphi \in \mathscr{E}(\sigma; \mathscr{Y})$, $\varphi \neq 0 \in \mathscr{Y}$ such that $\hat{A}\varphi = \sigma\varphi$ .This equality is equivalent with $A\varphi\hat{\alpha} = \sigma\varphi\cancel{\phi}$. Since A is weak-mixing with respect to $\mathscr{Y}$, $\sigma \neq 1$ when $\varphi \neq 0 \in \mathscr{Y}$ in the last equality is impossible. Hence when $\sigma \neq 1$, $\mathscr{E}(\sigma; \mathscr{Y}) = \{0\}$. $/\hat{\alpha}$

Assume that $\hat{\alpha} \in \mathscr{L}$; then $\hat{\mathbf{a}} \in \mathscr{L}$. Let $\varphi \in \mathscr{E}(1; \mathscr{C})$. By Lemma 5, $\varphi = \hat{\alpha}\xi$ for $\xi \in \mathbf{C}$, hence $\varphi \in \mathscr{L}$. Whence $\mathscr{E}(1; \mathscr{C}) \subset \mathscr{E}(1; \mathscr{L})$.The opposite inclusion also holds since $\mathscr{L} \subset \mathscr{C}$. Therefore the stated equality follows. □

Appendix. Throughout the Appendix we assume that all conditions (A),...,(F) hold.

Proposition A1. The operator A applies the space $\mathscr{C}$(resp $\mathscr{L}$) into itself. If $\varphi \in \mathscr{L}$, then

$$s(\hat{A}^n\varphi) \leq s(\varphi) N_2 N_3 + |\varphi| N_1 \qquad , n \in \mathbf{N} .$$

Proof. Clearly conditions delimiting the class of fibred systems we deal with in this paper is different from that used in [6] only by condition (F).

a. The proof of [6] Proposition A1 remains valid when condition (F) replaces (F)' because in its proof condition (F)' intervenes only once, namely to write relation (A2) and there only its weaker from coinciding with condition (F) is in fact written . Hence also the Remark to [6] Proposition A1 remains valid.

b. In the proof of [6] Proposition 3 condition (F)' is applied only to find a superior bound to the sum denoted there Σ^1.Condition (F) is condition (F)' where $\hat{\sigma}(n)$ is relaxed to 1 . If in the bound of Σ^1 we replace $\hat{\sigma}(n)$ by 1 inequality (3.8) from [6] becomes

$$| A^n\varphi(w) - A^n\varphi(u) | \le \| w - u \| (\mathbf{s}(\varphi) N_2 N_3 + |\varphi| N_1) \ , u \in \mathbf{N} \, , \varphi \in \mathscr{L},$$

for any $w,u \in \mathrm{W}$, any $\mathrm{W} \in \mathscr{H}$. The stated inequality follows.

The Lipschitz continuity of $(A^n\varphi)|_{\overline{W}}$ also holds ; since W was arbitrary from $\mathscr{H}$, $A^n\varphi \in \mathscr{L}$ follows.

Using (2.1) the inequality

$$||| A^n\varphi(w) - A^n\varphi(u) ||| \le \mathbf{s}(\varphi) N_2 N_3 + |\varphi| (N_1 + N_3)$$

immediately follows from the definition of the norm $||| \cdot |||$.

We record that the operator A_r is defined by (2.2).

Proposition A2. Let $f \in \mathscr{L}$ (resp $\mathscr{C}$) . The sequence $\{ A_r f \}_{r>0}$ contains a subsequence which converges in the norm $| \cdot |$ to a function belonging to $\mathscr{L}$(resp $\mathscr{C}$) .

Proof. Fix $f \in \mathscr{L}$. Proposition A1 implies that $\{ A_r f \}_{r \in \mathbf{N}}$ is bounded in the norm $||| \cdot |||$. Hence $Af \in \mathscr{L}$ follows from [6] Proposition A3 whose proof is valid unchanged under present assumptions.

Now assume that f is a real function belonging to $\mathscr{C}$. We denoted in [6] the continuous extension to $\overline{H_j}$ of $(A^r f)_j = (A^r f)|_{H_j}$ by $(\widetilde{A^r f})_j$, $r \in \mathbf{N}$, $j = 1,.., \gamma'$ (where $\gamma' = \mathrm{card}\, \mathscr{H}$). By the Remark to [6] Proposition A1 the sequence of functions $(\widetilde{A^r f})_j$, $r \ge 1$ is equicontinuous for each $j = 1,..,\gamma'$; this implies that also the sequence $(\widetilde{A_r f})_j$, $r \ge 1$ is equicontinuous , $j \le \gamma'$. By (2.1) there sequences are equally bounded in the norm $| \cdot |$; hence theorem Arzelà – Ascoli can be applied.

The proof continues with successive selections of converging subsequences like the proof of [6] Proposition A3 which deals with the (more general) sequence denoted there $\{X_r\}_{r \ge 1}$.

The extension to complex functions f is clear and we omit it.

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