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November, 2002

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ON THE STATISTICS OF THE FIRST EXIT TIME IN THE PERIODIC TWO-DIMENSIONAL LORENTZ GAS

FLORIN P. BOCA, RADU N. GOLOGAN AND ALEXANDRU ZAHARESCU

ABSTRACT. We consider a billiard in the punctured torus obtained by removing a cross-shaped pocket from $\mathbb{T}^2$, with the trajectory starting from the center of the puncture. In this case the phase space is given by the initial velocity ω of the particle. Let $\tilde{\tau}_\varepsilon(\omega)$, and respectively $r_\varepsilon(\omega)$, denote the first exit time (length of the trajectory), and respectively the number of collisions with the side cushions when $\mathbb{T}^2$ is identified with $[0,1)^2$. We prove that the probability measures associated with the random variables $\varepsilon\tilde{\tau}_\varepsilon$ and $\varepsilon r_\varepsilon$ are convergent as $\varepsilon \searrow 0$, providing explicit formulas for the limits.

1. INTRODUCTION AND MAIN RESULTS

Various ergodic and statistical properties of the periodic Lorentz gas were studied during the last decades (see [20], [1], [6], [7], [8], [9], [17], [12], [10], [11], [14], [15], [13], [5] for a non-exhaustive list of references). In the case of uniformly distributed circular obstacles of radius $0 < \varepsilon < \frac{1}{2}$ in $\mathbb{R}^2$, one considers the region

$$Z_\varepsilon = \{x \in \mathbb{R}^2 \, ; \, \mathrm{dist}(x, \mathbb{Z}^2) > \varepsilon\}$$

and the first exit time (sometimes called free path length)

$$\tau_\varepsilon(x,\omega) = \inf\{\tau > 0 \, ; \, x + \tau\omega \in \partial Z_\varepsilon\}, \quad x \in Y_\varepsilon = Z_\varepsilon/\mathbb{Z}^2, \ \omega \in \mathbb{T}.$$

Equivalently, one can consider the free motion of a point-like particle in the billiard table Y_ε obtained by removing pockets of the form of quarters of a circle of radius ε. If we identify $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $[0,1)^2$, then Y_ε can be regarded as a punctured two-torus. The reflection in the side cushions of the table is specular and the trajectory between two such reflections is rectilinear. Assume that the particle has constant speed, say equal to 1, and leaves the table when it reaches one of the four pockets. In this setting $\tau_\varepsilon(x,\omega)$ coincides with the exit time from the table (or equivalently the length of the trajectory) in the case where the initial position is $x \in Y_\varepsilon$ and the initial velocity is $\omega \in \mathbb{T}$. The average

$$l_\varepsilon = \iint_{\Sigma_\varepsilon^+} \tau_\varepsilon(x,\omega) \, d\nu_\varepsilon(x,\omega)$$

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of the exit time τ_ε (which is a Borel function, possibly unbounded) over the phase space $\Sigma_\varepsilon^+ = \{(x,\omega) \in \partial Y_\varepsilon \times \mathbb{T}\,;\ \omega \cdot n(x) > 0\}$, where $n(x)$ is the inward unit normal at $x \in \partial Y_\varepsilon$ and ν_ε the Liouville measure on Σ_ε^+, was evaluated by Chernov in [10] (see also [11], [15]). It was found that

$$l_\varepsilon = \frac{\pi |Y_\varepsilon|}{|\partial Y_\varepsilon|} = \frac{\pi(1-\pi\varepsilon^2)}{2\pi\varepsilon} = \frac{1-\pi\varepsilon^2}{2\varepsilon} = \frac{1}{2\varepsilon} + O(\varepsilon).$$

It is natural to replace the phase space $(\Sigma_\varepsilon^+, \nu_\varepsilon)$ by its suspension $(Y_\varepsilon \times \mathbb{T}, \mu_\varepsilon)$, where $d\mu_\varepsilon(x,\omega) = \frac{dx\, d\omega}{|Y_\varepsilon \times \mathbb{T}|}$, and to study the distribution of τ_ε defined by

$$\phi_\varepsilon(t) = \iint_{Y_\varepsilon \times \mathbb{T}} \chi_{[t,\infty)}\big(\tau_\varepsilon(x,\omega)\big)\, d\mu_\varepsilon(x,\omega) = \mu_\varepsilon\big(\{(x,\omega) \in Y_\varepsilon \times \mathbb{T}\,;\ \tau_\varepsilon(x,\omega) > t\}\big).$$

As proved by Bourgain, Golse and Wennberg [5], there are constants $C_1, C_2 > 0$ such that

$$\frac{C_1}{\varepsilon} \geq \phi_\varepsilon(t) \geq \frac{C_2}{\varepsilon}, \qquad \forall t > \frac{1}{\varepsilon}, \tag{1.1}$$

as $\varepsilon \searrow 0$. In this paper we shall replace the circular obstacles of radius ε by small crosses $C_\varepsilon = [-\varepsilon,\varepsilon] \times \{0\} \cup \{0\} \times [-\varepsilon,\varepsilon]$, situated at all integer lattice points with the exception of the origin. Thus the billiard table $\mathbb{T}_\varepsilon^2$ is obtained from the unit square $[0,1]^2$ by removing pockets of length $\varepsilon > 0$ from each corner. We shall only consider the case where the trajectory starts at $O = (0,0)$ with initial velocity $\omega \in \left[0, \frac{\pi}{2}\right]$ and exit time (length of the trajectory) $\tilde{\tau}_\varepsilon(\omega)$, and we shall average over ω only. In this situation we will give very precise estimates about the average

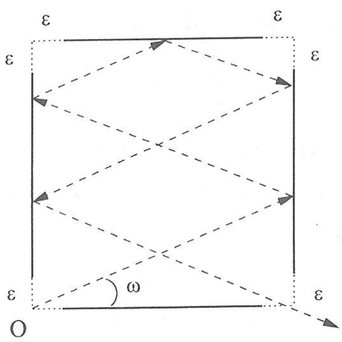

FIGURE 1. The billiard in $\mathbb{T}_\varepsilon^2$

over ω of the exit time and of the number of collisions of the particle with the side cushions. In contrast with the techniques employed to estimate the position-velocity average of the exit time, which are of geometric nature, our computations build on results and ideas concerning the distribution of consecutive Farey fractions ([2],[3]). They ultimately rely on estimates ([16],[19]) of Weil type ([21]) for Kloosterman sums with non-prime modulus. The related problem of evaluating the moments

$$c_r = \int_0^{\pi/2} \tilde{\tau}_\varepsilon(\omega)^r \, d\omega, \qquad r > 0,$$

was raised by Ya. G. Sinai in the case of circular obstacles in a seminar at the Moskow University in 1981. An answer for the model of cross-shaped obstacles described above was given in [4], where it was proved that

$$\int_\alpha^\beta \tilde{\tau}_\varepsilon(\omega)^r \, d\omega = c_{r,\alpha,\beta}\varepsilon^{-r} + O_{r,\delta}(\varepsilon^{-r+\frac{1}{6}-\delta}), \qquad \forall r > 0,\ \forall \delta > 0, \tag{1.2}$$

with

$$c_{r,\alpha,\beta} = \frac{12 D_r}{\pi^2} \int_\alpha^\beta \frac{d\omega}{\cos^r \omega}$$

and

$$D_r = \frac{1 - \frac{1}{2^r} + \ln 2}{r(r+1)} - \frac{1 - \frac{1}{2^r}}{r^2} + \frac{1 - \frac{1}{2^{r+1}}}{(r+1)^2} - \frac{1}{r(r+1)} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)\binom{r+1}{k}}{k 2^k}.$$

In particular this formula shows that the mean of the first exit time is

$$\int_0^{\pi/2} \tilde{\tau}_\varepsilon(\omega) \, d\omega = \frac{6 \ln 2 \ln(2+\sqrt{2})}{\pi^2 \varepsilon} + O_\delta(\varepsilon^{-\frac{5}{6}-\delta}) \approx \frac{0.742792}{\varepsilon}.$$

Since all the probability measures $\mu_{\alpha,\beta,\varepsilon}$ defined by

$$\mu_{\alpha,\beta,\varepsilon}(f) = \frac{1}{\beta-\alpha} \int_\alpha^\beta f\big(\varepsilon \tilde{\tau}_\varepsilon(\omega)\big) \, d\omega, \qquad f \in C_c\big([0,\infty)\big),$$

have the support included into a common compact (cf. Lemma 2.1), the equality (1.2) implies that the measures $\mu_{\alpha,\beta,\varepsilon}$ converge weakly to a probability measure $\mu_{\alpha,\beta}$ with compact support on $[0,\infty)$ as $\varepsilon \searrow 0$. Although the moments of $\mu_{\alpha,\beta}$ are recovered by

$$\int_\alpha^\beta \omega^r \, d\mu_{\alpha,\beta}(\omega) = c_{r,\alpha,\beta}, \qquad r > 0,$$

it is not obvious how to compute explicitly the density of $\mu_{\alpha,\beta}$. In this paper we give a direct proof for this convergence. Besides, the method employed here leads to the computation of the probability measure $\mu_{\alpha,\beta}$ in closed form. To state the main result, we consider the repartition function $F_{\alpha,\beta,\varepsilon}$ of $\mu_{\alpha,\beta,\varepsilon}$ defined by

$$F_{\alpha,\beta,\varepsilon}(t) = \mu_{\alpha,\beta,\varepsilon}([0,t]) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \chi_{[0,t]}\big(\varepsilon\tilde{\tau}_\varepsilon(\omega)\big)\,d\omega = \frac{|\{\omega\in[\alpha,\beta]\,;\,\varepsilon\tilde{\tau}_\varepsilon(\omega)\le t\}|}{\beta-\alpha}.$$

We also consider the function

$$\psi(s) = \frac{1-s}{s}\left(1+\ln\frac{s}{1-s}\right), \tag{1.3}$$

and prove

Theorem 1.1. *For every $0\le\alpha<\beta\le\frac{\pi}{4}$ and every $t\ge 0$, the limit $F_{\alpha,\beta}(t)=\lim\limits_{\varepsilon\searrow 0}F_{\alpha,\beta,\varepsilon}(t)$ exists. Moreover,*

$$F_{\alpha,\beta}(t) = \frac{12}{\pi^2(\beta-\alpha)}\cdot\begin{cases}(\beta-\alpha)t\cos\beta+\displaystyle\int_{t\cos\beta}^{t\cos\alpha}\big(\arccos\tfrac{s}{t}-\alpha\big)\,ds = t(\sin\beta-\sin\alpha) & \text{if } t\in\big[0,\frac{1}{2\cos\alpha}\big],\\[2ex] (\beta-\alpha)t\cos\beta+\displaystyle\int_{t\cos\beta}^{1/2}\big(\arccos\tfrac{s}{t}-\alpha\big)\,ds+\int_{1/2}^{t\cos\alpha}\psi(s)\big(\arccos\tfrac{s}{t}-\alpha\big)\,ds & \\ \quad = t\sin\beta-\frac{\alpha}{2}+\frac12\arccos\frac{1}{2t}-\frac12\sqrt{4t^2-1}+\displaystyle\int_{1/2}^{t\cos\alpha}\psi(s)\big(\arccos\tfrac{s}{t}-\alpha\big)\,ds & \text{if } t\in\big[\frac{1}{2\cos\alpha},\frac{1}{2\cos\beta}\big],\\[2ex] (\beta-\alpha)\Big(\frac12+\displaystyle\int_{1/2}^{t\cos\beta}\psi(s)\,ds\Big)+\int_{t\cos\beta}^{t\cos\alpha}\psi(s)\big(\arccos\tfrac{s}{t}-\alpha\big)\,ds & \text{if } t\in\big[\frac{1}{2\cos\beta},\frac{1}{\cos\alpha}\big],\\[2ex] (\beta-\alpha)\Big(\frac12+\displaystyle\int_{1/2}^{t\cos\beta}\psi(s)\,ds\Big)+\int_{t\cos\beta}^{1}\psi(s)\big(\arccos\tfrac{s}{t}-\alpha\big)\,ds & \text{if } t\in\big[\frac{1}{\cos\alpha},\frac{1}{\cos\beta}\big],\\[2ex] (\beta-\alpha)\Big(\frac12+\displaystyle\int_{1/2}^{1}\psi(s)\,ds\Big) & \text{if } t\in\big[\frac{1}{\cos\beta},\infty\big).\end{cases}$$

According to Lemma 2.1 we have $\sup\limits_\omega\tilde{\tau}_\varepsilon(\omega)\le\sqrt{2}\big[\frac{1}{\varepsilon}\big]\le\frac{\sqrt{2}}{\varepsilon}$. Hence all functions $F_{\alpha,\beta,\varepsilon}$, and thus their limit $F_{\alpha,\beta}$, are constant on the interval $[\sqrt{2},\infty)$, and this constant is equal to 1. As a result we gather

$$\int_{1/2}^{1}\psi(s)\,ds = \frac{\pi^2}{12}-\frac12.$$

With the change of variable $\frac{s}{1-s}=x$ we get the following amusing

Corollary 1.2. $$\int_1^{\infty} \frac{\ln x}{x(1+x)^2}\,dx = \frac{\pi^2}{12} - \ln 2.$$

The continuous function $F_{\alpha,\beta}$ represents the repartition function of an absolutely continuous probability measure $\mu_{\alpha,\beta}$ supported on $\left[0, \frac{1}{\cos\beta}\right]$, with density $f_{\alpha,\beta}$. That is, $F_{\alpha,\beta}(t) = \int_0^t f_{\alpha,\beta}(s)ds$. From Theorem 1.1 we infer by a direct computation

Corollary 1.3. *The probability measures $\mu_{\alpha,\beta,\varepsilon}$ converge weakly to a probability measure $\mu_{\alpha,\beta}$ as $\varepsilon \searrow 0$. Moreover, $\mu_{\alpha,\beta}$ has compact support and is absolutely continuous, with density*

$$f_{\alpha,\beta}(t) = \frac{12}{\pi^2(\beta-\alpha)} \cdot \begin{cases} \sin\beta - \sin\alpha & \text{if } t \in \left[0, \frac{1}{2\cos\alpha}\right], \\ \sin\beta - \frac{\sqrt{4t^2-1}}{2t} + \int\limits_{\alpha}^{\arccos 1/(2t)} \psi(t\cos x)\cos x\,dx & \text{if } t \in \left[\frac{1}{2\cos\alpha}, \frac{1}{2\cos\beta}\right], \\ \int\limits_{\alpha}^{\beta} \psi(t\cos x)\cos x\,dx & \text{if } t \in \left[\frac{1}{2\cos\beta}, \frac{1}{\cos\alpha}\right], \\ \int\limits_{\arccos 1/t}^{\beta} \psi(t\cos x)\cos x\,dx & \text{if } t \in \left[\frac{1}{\cos\alpha}, \frac{1}{\cos\beta}\right], \\ 0 & \text{if } t \in \left[\frac{1}{\cos\beta}, \infty\right) \end{cases}$$

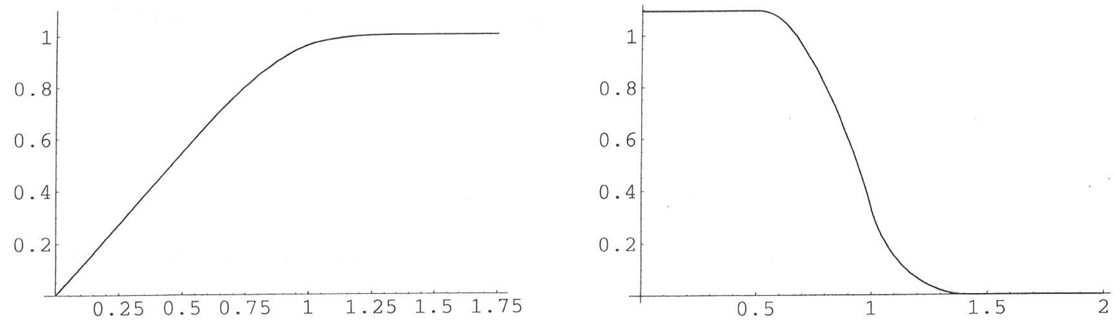

FIGURE 2. The repartition function $F_{0,\frac{\pi}{4}}$ and the density function $f_{0,\frac{\pi}{4}}$

The average of the number of collisions $r_\varepsilon(\omega)$ of the particle with the side cushions was also estimated in [4], where it was proved that

$$\int_\alpha^\beta r_\varepsilon(\omega)d\omega = c_{\alpha,\beta}\varepsilon^{-1} + O_\delta(\varepsilon^{-\frac{5}{6}-\delta}), \qquad \forall \delta > 0,$$

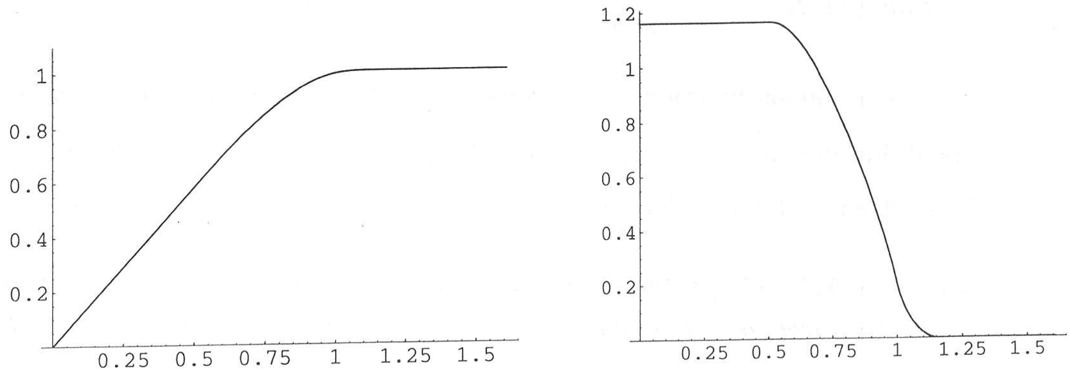

FIGURE 3. The repartition function $F_{0,\frac{\pi}{6}}$ and the density function $f_{0,\frac{\pi}{6}}$

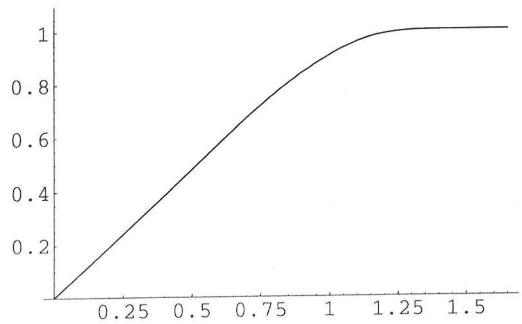

FIGURE 4. The repartition function $F_{\frac{\pi}{6},\frac{\pi}{4}}$ and the density function $f_{\frac{\pi}{6},\frac{\pi}{4}}$

as $\varepsilon \searrow 0$, where

$$c_{\alpha,\beta} = \frac{6\left(\beta - \alpha + \ln \frac{\cos\alpha}{\cos\beta}\right)\ln 2}{\pi^2}.$$

As in the case of the length function, we shall consider the probability measures $\nu_{\alpha,\beta,\varepsilon}$ on $[0,\infty)$ defined by

$$\nu_{\alpha,\beta,\varepsilon}(f) = \frac{1}{\beta-\alpha}\int_\alpha^\beta f\big(\varepsilon r_\varepsilon(\omega)\big)\,d\omega,$$

with repartition function

$$G_{\alpha,\beta,\varepsilon}(t) = \nu_{\alpha,\beta,\varepsilon}\big([0,t]\big) = \frac{|\{\omega \in [\alpha,\beta]\,;\ \varepsilon r_\varepsilon(\omega) \le t\}|}{\beta-\alpha}, \qquad t \in [0,\infty).$$

In this case we prove

Theorem 1.4. *For every $0 \le \alpha < \beta \le \frac{\pi}{4}$ and every $t \ge 0$, the limit $G_{\alpha,\beta}(t) = \lim_{\varepsilon \searrow 0} G_{\alpha,\beta,\varepsilon}(t)$ exists. Moreover,*

$$G_{\alpha,\beta}(t) = \frac{12}{\pi^2(\beta-\alpha)} \cdot \begin{cases} \frac{t}{2}\Big(\beta - \alpha + \ln \frac{\sin\beta+\cos\beta}{\sin\alpha+\cos\alpha}\Big) = t\displaystyle\int_{\alpha}^{\beta} \frac{dx}{1+\tan x} & \text{if } t \in \big[0, \frac{1+\tan\alpha}{2}\big], \\ \frac{-\alpha+(1-t)\arctan(2t-1)+t\beta}{2} + \frac{t}{2}\cdot \ln\Big((\sin\beta+\cos\beta)\sqrt{\frac{1-2t+2t^2}{2t^2}}\Big) & \\ \qquad + \displaystyle\int_{1/2}^{\frac{t}{1+\tan\alpha}} \psi(s)\big(\arctan(\tfrac{\lambda}{s}-1)-\alpha\big)\,ds & \text{if } t \in \big[\frac{1+\tan\alpha}{2}, \frac{1+\tan\beta}{2}\big], \\ \frac{\beta-\alpha}{2} + \beta\displaystyle\int_{1/2}^{\frac{t}{1+\tan\beta}} \psi(s)\,ds - \alpha\int_{1/2}^{\frac{t}{1+\tan\alpha}} \psi(s)\,ds + \int_{\frac{t}{1+\tan\beta}}^{\frac{t}{1+\tan\alpha}} \psi(s)\arctan(\tfrac{t}{s}-1)\,ds & \\ & \text{if } t \in [1+\tan\alpha, 1+\tan\beta], \\ \frac{\beta-\alpha}{2} + \beta\displaystyle\int_{1/2}^{\frac{t}{1+\tan\beta}} \psi(s)\,ds - \alpha\int_{1/2}^{1} \psi(s)\,ds + \int_{\frac{t}{1+\tan\beta}}^{1} \psi(s)\arctan(\tfrac{t}{s}-1)\,ds & \\ & \text{if } t \in [1+\tan\alpha, 1+\tan\beta], \\ (\beta-\alpha)\Big(\frac{1}{2} + \displaystyle\int_{1/2}^{1} \psi(s)\,ds\Big) & \text{if } t \in [1+\tan\beta, \infty). \end{cases}$$

Corollary 1.5. *The probability measures $\nu_{\alpha,\beta,\varepsilon}$ converge weakly to a probability measure $\nu_{\alpha,\beta}$ as $\varepsilon \searrow 0$. Moreover, $\nu_{\alpha,\beta}$ has compact support and is absolutely continuous, with density given by*

$$g_{\alpha,\beta}(t) = \frac{12}{\pi^2(\beta-\alpha)} \cdot \begin{cases} \displaystyle\int_{\alpha}^{\beta} \frac{dx}{1+\tan x} & \text{if } t \in \left[0, \frac{1+\tan\alpha}{2}\right], \\ \displaystyle\int_{\arctan(2t-1)}^{\beta} \frac{dx}{1+\tan x} + \int_{\alpha}^{\arctan(2t-1)} \frac{\psi\left(\frac{t}{1+\tan x}\right)}{1+\tan x} & \text{if } t \in \left[\frac{1+\tan\alpha}{2}, \frac{1+\tan\beta}{2}\right], \\ \displaystyle\int_{\alpha}^{\beta} \frac{\psi\left(\frac{t}{1+\tan x}\right)}{1+\tan x}\, dx & \text{if } t \in \left[\frac{1+\tan\beta}{2}, 1+\tan\alpha\right], \\ \displaystyle\int_{\arctan(t-1)}^{\beta} \frac{\psi\left(\frac{t}{1+\tan x}\right)}{1+\tan x}\, dx & \text{if } t \in [1+\tan\alpha, 1+\tan\beta], \\ 0 & \text{if } t \in [1+\tan\beta, \infty). \end{cases}$$

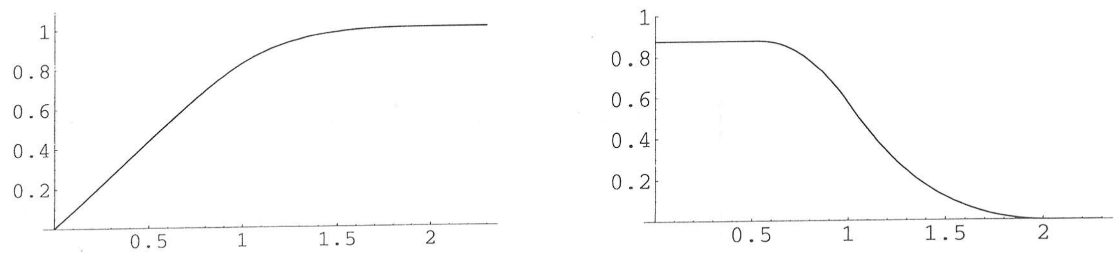

FIGURE 5. The repartition function $G_{0,\frac{\pi}{4}}$ and the density function $g_{0,\frac{\pi}{4}}$

It is easy to see that the use of small diamonds $|x-m|+|y-n| \leq \varepsilon$ centered at lattice points (m,n) does not change anything significant in Theorems 1.1 and 1.4, or in their corollaries. However, the problem becomes more complicated in the case where the disks $(x-m)^2+(y-n)^2 < \varepsilon^2$ are the obstacles.

It would also be interesting to study whether the probability measures μ_ε considered in [5] have a weak limit as $\varepsilon \searrow 0$, sharpening (1.1). If this is the case, then the limit cannot have compact support, as it happens in the case when the initial position is O and one only averages over the initial velocity. Another interesting problem is the study of the distribution of the average over ω of $\tau(x_0,\omega)$ when x_0 is a particular point in $[0,1)^2$. The latter seems to be related to (seemingly non-trivial) inhomogeneous Diophantine approximation topics.

2. Formulas for sectors ending at Farey points

For each integer $Q \geq 1$, let $\mathcal{F}_Q$ denote the set of Farey fractions of order Q, i.e. irreducible rational numbers in the interval $(0,1]$ with denominator $\leq Q$. It is well known that if $\frac{a}{q} < \frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$, then

$$a'q - aq' = 1 \qquad \text{and} \qquad q + q' > Q.$$

Conversely, if $q, q' \in \{1, \ldots, Q\}$, $q + q' > Q$ and $a'q - aq' = 1$ for some $a \in \{1, \ldots, q-1\}$ and $a' \in \{1, \ldots, q'-1\}$, then $\frac{a}{q} < \frac{a'}{q'}$ are consecutive elements in $\mathcal{F}_Q$ (see for instance [18]). We keep throughout the paper $0 \leq \alpha < \beta \leq \frac{\pi}{4}$, $t > 0$ and $0 < \varepsilon < \frac{1}{2}$ fixed, and set

$$Q = \left[\frac{1}{\varepsilon}\right] = \text{the integer part of } \frac{1}{\varepsilon}.$$

A key remark whose proof relies on the basic properties of Farey fractions is that every line of slope between 0 and 1 which pass through O will necessarily intersect the set

$$\mathcal{C}_\varepsilon = C_\varepsilon + \{(q,a)\,;\, a/q \in \mathcal{F}_Q\},$$

which consists in

$$N_Q = \#\mathcal{F}_Q = \sum_{n=1}^{Q} \varphi(n) = \frac{3Q^2}{\pi^2} + O(Q \ln Q)$$

obstacles identical to C_ε, distributed at the points (q,a), with $\frac{a}{q} \in \mathcal{F}_Q$. More precisely, we prove

Lemma 2.1. *For any $\omega \in \left(0, \frac{\pi}{4}\right]$ we have*

$$\{\lambda\omega\,;\, \lambda > 0\} \cap \mathcal{C}_\varepsilon \neq \emptyset.$$

Proof. Let t_P denote the slope of the line OP. We use the inequalities $q + q' \geq Q + 1 > \frac{1}{\varepsilon} \geq Q \geq \max(q, q')$ to infer

$$t_A = \frac{a}{q} \leq t_{S'} = \frac{a' - \varepsilon}{q'} < t_N = \frac{a + \varepsilon}{q} \leq t_{A'} = \frac{a'}{q'},$$

where we set $A = (q,a)$, $A' = (q',a')$, $N = (q, a+\varepsilon)$, $N' = (q', a'+\varepsilon)$, $S = (q, a-\varepsilon)$, $S' = (q', a'-\varepsilon)$, $W = (q-\varepsilon, a)$, $W' = (q'-\varepsilon, a')$. This clearly shows that any line of slope ω that passes through O will necessarily intersect $\mathcal{C}_\varepsilon$. Thus every trajectory will end near a point (q,a) with $\frac{a}{q} \in \mathcal{F}_Q$. Moreover, this point is uniquely determined by ω. $\square$

Remark. We have actually shown that the intervals $I_\gamma = \left[\frac{a-\varepsilon}{q}, \frac{a+\varepsilon}{q}\right]$, $\gamma = \frac{a}{q} \in \mathcal{F}_Q$, cover the interval $[0,1]$, and that I_γ and $I_{\gamma'}$ are disjoint if and only if γ and γ' are consecutive in $\mathcal{F}_Q$. Moreover, in this case we have

$$I_\gamma \cap I_{\gamma'} = \left[\frac{a' - \varepsilon}{q'}, \frac{a + \varepsilon}{q}\right] \subseteq \left[\frac{a}{q}, \frac{a'}{q'}\right].$$

For each $\omega \in \left(0, \frac{\pi}{4}\right]$, we put $l_{\omega,\varepsilon} = (q,a)$ if the half-line $\mathbb{R}_+\omega$ first intersects $C_\varepsilon + (q,a)$ among the components of $\mathcal{C}_\varepsilon$. We need a few more things about consecutive Farey fractions. Suppose

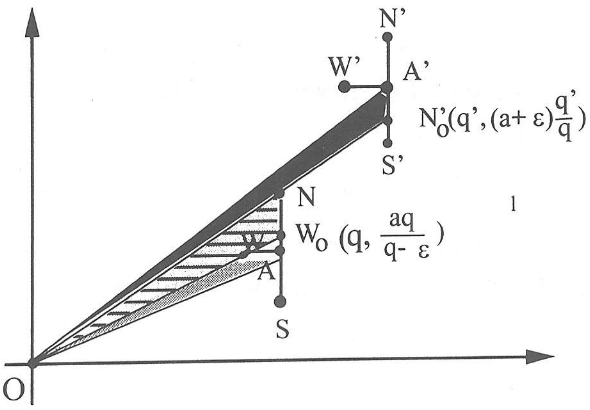

FIGURE 6. The case $q < q'$

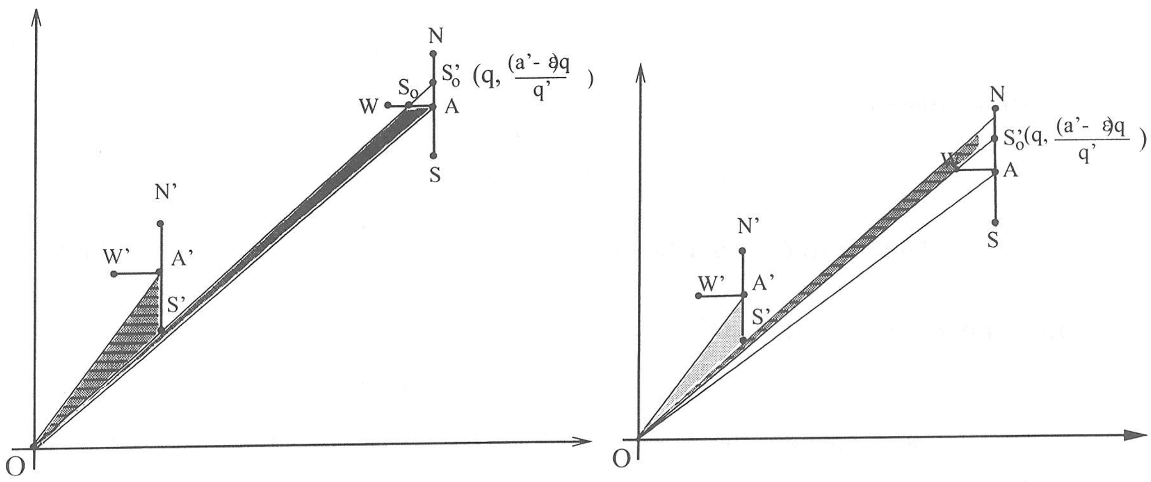

FIGURE 7. The case $q > q'$ and $t_{S'} \leq t_W$, respectively $q > q'$ and $t_{S'} > t_W$

that $\frac{a'}{q'} < \frac{a}{q} < \frac{a''}{q''}$ are three consecutive fractions in $\mathcal{F}_Q$. Then the relation $aq' - a'q = 1$ gives $q' = \bar{a} \pmod{q}$, where $\bar{a}$ denotes the multiplicative inverse of $a \pmod{q}$. Since $q' \in (Q - q, Q]$, then $q' - \bar{a}$ is the unique multiple of q in the interval $(Q - q - \bar{a}, Q - \bar{a}]$. Hence $q' - \bar{a} = q\left[\frac{Q-\bar{a}}{q}\right]$, and therefore

$$q' = q\left[\frac{Q - \bar{a}}{q}\right] + \bar{a}. \tag{2.1}$$

Employing $a''q - aq'' = 1$ and $q'' \in (Q - q, Q]$, we arrive in a similar way at

$$q'' = q\left[\frac{Q + \bar{a}}{q}\right] - \bar{a}. \tag{2.2}$$

Taking into account (2.1) and (2.2), we see that

$$q' > q \iff \bar{a} \leq Q - q$$

and

$$q'' > q \iff \bar{a} \geq 2q - Q,$$

whence

Lemma 2.2. $\min(q', q'') > q \iff Q - q \geq \bar{a} \geq 2q - Q$.

$q' < q < q'' \iff \bar{a} \geq \max(2q - Q, Q - q + 1)$.

$q'' < q < q' \iff \bar{a} \leq \min(2q - Q - 1, Q - q)$.

$q > \max(q', q'') \iff 2q - Q > \bar{a} > Q - q$.

We denote by $\omega_{q,a}$ the angle determined by the trajectories which end near the lattice point (q, a), that is

$$\omega_{q,a} = |\{\omega \in (0, \pi/4)\, ;\, l_{\omega,\varepsilon} = (q, a)\}|.$$

We also consider

$$S_{\alpha,\beta}(t, \varepsilon) = \sum_{\substack{a/q \in \mathcal{F}_Q \cap [\tan\alpha, \tan\beta] \\ q^2 + a^2 < t^2 Q^2}} \omega_{q,a}\, .$$

The proof of Lemma 2.1 and the triangle inequality show that

$$S_{\alpha,\beta}\big(t - \varepsilon t - 2\varepsilon, \varepsilon\big) \leq (\beta - \alpha) F_{\alpha,\beta,\varepsilon}(t) \leq S_{\alpha,\beta}\big(t + \varepsilon t + 2\varepsilon, \varepsilon\big). \tag{2.3}$$

To evaluate the sum $S_{\alpha,\beta}(t, \varepsilon)$ we first need to estimate $\omega_{q,a}$ in each of the four situations from Lemma 2.2. First we analyze the cases I-IV making use of the formulas

$$\begin{aligned} \arctan(x + h) - \arctan x &= \frac{h}{1 + x^2} + O(h^2) \\ &= \frac{h}{1 + (x + h)^2} + O(h^2), \quad x \in [0, 1] \text{ and } h > 0 \text{ small}, \end{aligned}$$

$$\varepsilon = \frac{1}{Q} + O\left(\frac{1}{Q^2}\right) = \frac{1}{Q} + O(\varepsilon^2),$$

and of the inequalities $(q + q')\varepsilon \geq (Q + 1)\varepsilon > 1$ and $(q + q'')\varepsilon > 1$.

Case I: $\min(q', q'') > q$.

In this case $\frac{a - \varepsilon}{q} < \frac{a' + \varepsilon}{q'}$ and $\frac{a'' - \varepsilon}{q''} < \frac{a + \varepsilon}{q}$, thus

$$\begin{aligned} \omega_{q,a} &= \arctan \frac{a + \varepsilon}{q} - \arctan \frac{a - \varepsilon}{q} = \frac{\frac{2\varepsilon}{q}}{1 + \left(\frac{a - \varepsilon}{q}\right)^2} + O\left(\frac{\varepsilon^2}{q^2}\right) \\ &= \frac{2\varepsilon q}{q^2 + (a - \varepsilon)^2} + O\left(\frac{\varepsilon^2}{q^2}\right) = \frac{2\varepsilon q}{q^2 + a^2} + O\left(\varepsilon q \cdot \frac{\varepsilon a}{(q^2 + a^2)^2} + \frac{\varepsilon^2}{q^2}\right) \\ &= \frac{2\varepsilon q}{q^2 + a^2} + O\left(\frac{\varepsilon^2}{q^2}\right) = \frac{2q}{Q(q^2 + a^2)} + O\left(\frac{\varepsilon^2}{q}\right). \end{aligned}$$

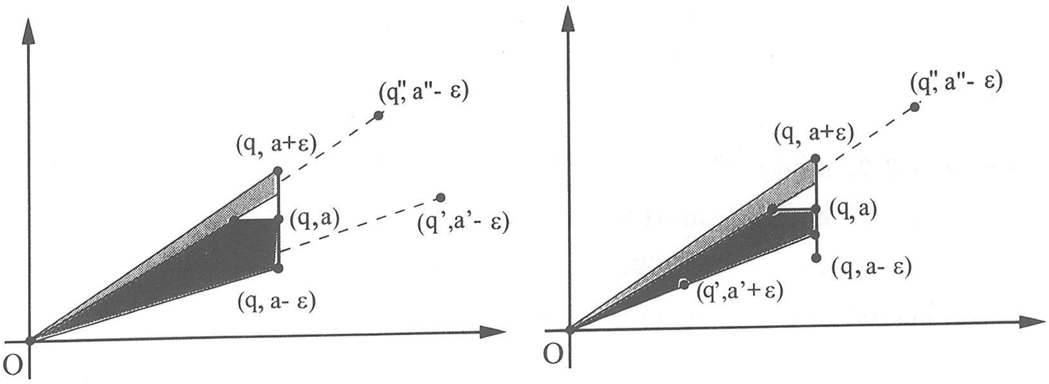

FIGURE 8. Cases I and II respectively

Case II: $q' < q < q''$.

In this case $\frac{a-\varepsilon}{q} < \frac{a'+\varepsilon}{q'} < \frac{a}{q} < \frac{a''-\varepsilon}{q''} < \frac{a+\varepsilon}{q}$. With $h = \frac{a+\varepsilon}{q} - \frac{a'+\varepsilon}{q'} = \frac{1-\varepsilon(q-q')}{qq'}$, we gather

$$\begin{aligned}\omega_{q,a} &= \arctan\frac{a+\varepsilon}{q} - \arctan\frac{a'+\varepsilon}{q'} = \frac{\frac{1-\varepsilon(q-q')}{qq'}}{1+\left(\frac{a+\varepsilon}{q}\right)^2} + O\left(\left(\frac{1-\varepsilon(q-q')}{qq'}\right)^2\right)\\ &= \frac{q(1-\varepsilon(q-q'))}{q'(q^2+(a+\varepsilon)^2)} + O\left(\frac{\varepsilon^2}{q'^2}\right) = \frac{q(1-\varepsilon(q-q'))}{q'(q^2+a^2)} + O\left(\frac{q}{q'}\cdot\frac{\varepsilon a}{(q^2+a^2)^2} + \frac{\varepsilon^2}{q'^2}\right).\end{aligned}$$

Since $q > q'$ and $q + q' > Q$, we have $q > \frac{Q}{2}$. Hence the error in the formula for $\omega_{q,a}$ above is $\ll \frac{\varepsilon}{q^2 q'} + \frac{\varepsilon^2}{q'^2} \ll \frac{\varepsilon^3}{q'} + \frac{\varepsilon^2}{q'^2} \ll \frac{\varepsilon^2}{q'^2}$, and we find

$$\begin{aligned}\omega_{q,a} &= \frac{q(1-\varepsilon(q-q'))}{q'(q^2+a^2)} + O\left(\frac{\varepsilon^2}{q'^2}\right) = \frac{q\left(1-\frac{q-q'}{Q}\right)}{q'(q^2+a^2)} + O\left(\frac{q^2\cdot\frac{1}{Q^2}}{q'q^2} + \frac{\varepsilon^2}{q'^2}\right)\\ &= \frac{q(Q-q+q')}{Qq'(q^2+a^2)} + O\left(\frac{\varepsilon^2}{q'}\right).\end{aligned}$$

Since $q' = \bar{a} \pmod{q}$ and $q' < q$, we have $q' = \bar{a}$, and so

$$\omega_{q,a} = \frac{q(Q-q+\bar{a})}{Q\bar{a}(q^2+a^2)} + O\left(\frac{\varepsilon^2}{q'}\right).$$

Case III: $q' > q > q''$.

In this case $q'' = q - \bar{a}$. Moreover, we have $\frac{a-\varepsilon}{q} < \frac{a'+\varepsilon}{q'} < \frac{a}{q} < \frac{a''-\varepsilon}{q''} < \frac{a+\varepsilon}{q}$. As a result, we may take $h = \frac{a''}{q''} - \frac{a-\varepsilon}{q} = \frac{1-\varepsilon(q-q'')}{qq''}$, gathering

$$\begin{aligned}\omega_{q,a} &= \arctan\frac{a''-\varepsilon}{q''} - \arctan\frac{a-\varepsilon}{q} = \frac{\frac{1-\varepsilon(q-q'')}{qq''}}{1+\left(\frac{a-\varepsilon}{q}\right)^2} + O\left(\frac{\varepsilon^2}{q''^2}\right)\\ &= \frac{q(Q-\bar{a})}{Q(q-\bar{a})(q^2+a^2)} + O\left(\frac{\varepsilon^2}{q''}\right).\end{aligned}$$

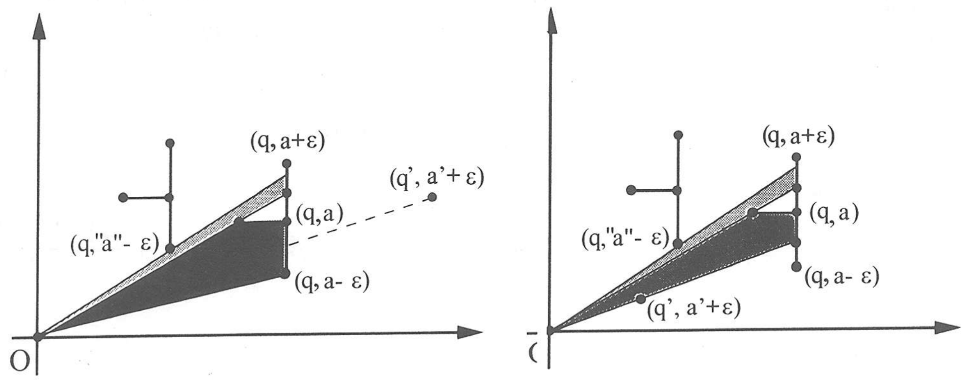

FIGURE 9. Cases III and IV respectively

Case IV: $q > \max(q', q'')$.
In this case $\frac{a-\varepsilon}{q} < \frac{a'+\varepsilon}{q'} < \frac{a}{q} < \frac{a''-\varepsilon}{q''} < \frac{a+\varepsilon}{q}$, and we gather

$$\omega_{q,a} = \arctan \frac{a''-\varepsilon}{q''} - \arctan \frac{a'+\varepsilon}{q'} = \omega^{(1)}_{q,a} + \omega^{(2)}_{q,a},$$

with

$$\omega^{(1)}_{q,a} = \arctan \frac{a''-\varepsilon}{q''} - \arctan \frac{a}{q} = \frac{\frac{1-\varepsilon q}{qq''}}{1+\frac{a^2}{q^2}} + O\left(\left(\frac{1-\varepsilon q}{qq''}\right)^2\right) = \frac{q(Q-q)}{Qq''(q^2+a^2)} + O\left(\frac{\varepsilon^2}{q''}\right),$$

$$\omega^{(2)}_{q,a} = \arctan \frac{a}{q} - \arctan \frac{a'+\varepsilon}{q'} = \frac{\frac{1-\varepsilon q}{qq'}}{1+\frac{a^2}{q^2}} + O\left(\left(\frac{1-\varepsilon q}{qq''}\right)^2\right) = \frac{q(Q-q)}{Qq'(q^2+a^2)} + O\left(\frac{\varepsilon^2}{q'}\right).$$

Since in this case $q'' = q - \bar{a}$ and $q' = \bar{a}$, we arrive at

$$\begin{aligned}\omega_{q.a} &= \frac{q(Q-q)}{Q(q-\bar{a})(q^2+a^2)} + \frac{q(Q-q)}{Q\bar{a}(q^2+a^2)} + O\left(\varepsilon^2\left(\frac{1}{q'}+\frac{1}{q''}\right)\right) \\ &= \frac{q^2(Q-q)}{Q(q^2+a^2)\bar{a}(q-\bar{a})} + O\left(\varepsilon^2\left(\frac{1}{q'}+\frac{1}{q''}\right)\right).\end{aligned}$$

3. THE EXISTENCE AND COMPUTATION OF THE REPARTITION FUNCTIONS $F_{\alpha,\beta}$ AND $G_{\alpha,\beta}$

To estimate the sum $S_{\alpha,\beta}(t,\varepsilon)$ we utilize

Lemma 3.1. *Let $q \geq 1$ be an integer and let $\mathcal{I}, \mathcal{J} \subset \mathbb{R}$ be intervals of length lesser than q. Let $f : \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ be a C^1 function, and let $T > 1$ and $\delta > 0$. Then*

$$\sum_{\substack{a \in \mathcal{I}, b \in \mathcal{J} \\ ab = 1 \ (\mathrm{mod}\ q)}} f(a,b) = \frac{\varphi(q)}{q^2} \iint_{\mathcal{I} \times \mathcal{J}} f(x,y)\, dx\, dy + O_\delta\left(T^2 q^{\frac{1}{2}+\delta} \|f\|_\infty + T q^{\frac{3}{2}+\delta} \|Df\|_\infty + \frac{|\mathcal{I}|\,|\mathcal{J}|\,\|Df\|_\infty}{T} \right),$$

where $\|\cdot\|_\infty = \|\cdot\|_{\infty, \mathcal{I} \times \mathcal{J}}$.

Proof. By [3, Lemma 1.7]) we have

$$N_q(\mathcal{I}, \mathcal{J}) = \frac{\varphi(q)}{q^2} \cdot |\mathcal{I}|\,|\mathcal{J}| + O_\delta(q^{\frac{1}{2}+\delta}), \tag{3.1}$$

where $N_q(\mathcal{I}, \mathcal{J})$ denotes the number of pairs of integers $(x,y) \in \mathcal{I} \times \mathcal{J}$ for which $xy = 1 \ (\mathrm{mod}\ q)$. We partition the intervals $\mathcal{I}$ and $\mathcal{J}$ respectively into T intervals $\mathcal{I}_1, \ldots, \mathcal{I}_T$ and $\mathcal{J}_1, \ldots, \mathcal{J}_T$, of equal length $\frac{|\mathcal{I}|}{T}$ and respectively $\frac{|\mathcal{J}|}{T}$. We wish to approximate $f(x,y)$ by a constant whenever $(x,y) \in \mathcal{I}_i \times \mathcal{J}_j$. For, we choose for each pair of indices (i,j) two points $x_{i,j} \in \mathcal{I}_i$ and $y_{i,j} \in \mathcal{J}_j$ for which

$$\iint_{\mathcal{I}_i \times \mathcal{J}_j} f(x,y)\, dx\, dy = |\mathcal{I}_i|\,|\mathcal{J}_j| f(x_{i,j}, y_{i,j}). \tag{3.2}$$

We have

$$f(x,y) = f(x_{i,j}, y_{i,j}) + O\left(\frac{q}{T} \cdot \|Df\|_\infty \right)$$

whenever $(x,y) \in \mathcal{I}_i \times \mathcal{J}_j$, which gives in turn

$$\begin{aligned} \sum_{\substack{a \in \mathcal{I}, b \in \mathcal{J} \\ ab = 1 (\mathrm{mod}\ q)}} f(a,b) &= \sum_{i,j=1}^{T} \sum_{\substack{(x,y) \in \mathcal{I}_i \times \mathcal{J}_j \\ xy = 1 (\mathrm{mod}\ q)}} f(x,y) \\ &= \sum_{i,j=1}^{T} N_q(\mathcal{I}_i, \mathcal{J}_j) f(x_{i,j}, y_{i,j}) + O\left(\frac{q}{T} \sum_{i,j=1}^{T} N_q(\mathcal{I}_i, \mathcal{J}_j) \|Df\|_\infty \right). \end{aligned} \tag{3.3}$$

Since $|\mathcal{I}_i|, |\mathcal{J}_j| < q$, the estimate (3.1) applies to the intervals $\mathcal{I}_i$ and $\mathcal{J}_j$, providing

$$N_q(\mathcal{I}_i, \mathcal{J}_j) = \frac{\varphi(q)}{q^2} \cdot |\mathcal{I}_i|\,|\mathcal{J}_j| + O_\delta(q^{\frac{1}{2}+\delta}). \tag{3.4}$$

As a result of (3.4) and (3.2), the expression in (3.3) becomes

$$\sum_{i,j=1}^{T} N_q(\mathcal{I}_i,\mathcal{J}_j) f(x_{i,j},y_{i,j}) + O_\delta\big(T^2 q^{\frac12+\delta}\|f\|_\infty\big)$$
$$+ O_\delta\left(\frac{q}{T}\sum_{i,j=1}^{T}\left(\frac{\varphi(q)}{q^2}\cdot|\mathcal{I}_i|\,|\mathcal{J}_j| + q^{\frac12+\delta}\right)\|Df\|_\infty\right)$$
$$= \sum_{i,j=1}^{T} N_q(\mathcal{I}_i,\mathcal{J}_j) f(x_{i,j},y_{i,j}) + O_\delta\big(T^2 q^{\frac12+\delta}\|f\|_\infty\big)$$
$$+ O_\delta\left(\left(\frac{\varphi(q)|\mathcal{I}|\,|\mathcal{J}|}{Tq} + Tq^{\frac32+\delta}\right)\|Df\|_\infty\right)$$
$$= \frac{\varphi(q)}{q^2}\sum_{i,j=1}^{T}\iint\limits_{\mathcal{I}_i\times\mathcal{J}_j} f(x,y)\,dx\,dy + O_\delta\big(T^2 q^{\frac12+\delta}\|f\|_\infty\big)$$
$$+ O_\delta\left(\left(\frac{|\mathcal{I}|\,|\mathcal{J}|}{T} + Tq^{\frac32+\delta}\right)\|Df\|_\infty\right)$$
$$= \frac{\varphi(q)}{q^2}\iint\limits_{\mathcal{I}\times\mathcal{J}} f(x,y)\,dx\,dy + O_\delta\left(T^2 q^{\frac12+\delta}\|f\|_\infty + \left(\frac{|\mathcal{I}|\,|\mathcal{J}|}{T} + Tq^{\frac32+\delta}\right)\|Df\|_\infty\right).$$

□

If we put $I = [q\tan\alpha. q\tan\beta] \cap [0,\sqrt{t^2Q^2-q^2}]$, then

$$I = \begin{cases} [q\tan\alpha, q\tan\beta] & \text{if } q\in[1,tQ\cos\beta],\\ \big[q\tan\alpha,\sqrt{t^2Q^2-q^2}\,\big] & \text{if } q\in[tQ\cos\beta,tQ\cos\alpha],\\ \emptyset & \text{if } q\in[tQ\cos\alpha,\infty). \end{cases}$$

We also set

$$I_1 = [1,q-1]\cap\big(-\infty,\min(Q-q+1,2q-Q)\big),$$
$$I_2 = [1,q-1]\cap\big[\min(Q-q+1,2q-Q),\max(Q-q,2q-Q-1)\big],$$
$$I_3 = [1,q-1]\cap\big(\max(Q-q,2q-Q-1),\infty\big) = q - I_1,$$
$$f_j(x,y) = f_j(Q,q,x,y) = \frac{qF_j(y)}{Q(q^2+x^2)},\qquad \text{where}$$
$$F_1(y) = 2,\quad y\in I_2,\qquad F_2(y) = \frac{Q-q+y}{y},\quad y\in I_3,$$
$$F_3(y) = \frac{Q-y}{q-y} = F_2(q-y),\quad y\in I_1,\qquad F_4(y) = \frac{q(Q-q)}{y(q-y)}\quad y\in I_2.$$

Since $\sum_{q=1}^{Q}\frac{\varphi(q)}{q}\le\frac{1}{\varepsilon}$, the analysis of cases I-IV and Lemma 2.2 provide

$$S_{\alpha,\beta}(t,\varepsilon) = S^{\mathrm{I}}_{\alpha,\beta}(t,\varepsilon) + S^{\mathrm{II}}_{\alpha,\beta}(t,\varepsilon) + S^{\mathrm{III}}_{\alpha,\beta}(t,\varepsilon) + S^{\mathrm{IV}}_{\alpha,\beta}(t,\varepsilon) + O(\varepsilon),$$

where we set

$$S^{\mathrm{I}}_{\alpha,\beta}(t,\varepsilon) = \sum_{\substack{a/q\in\mathcal{F}_Q\cap[\tan\alpha,\tan\beta]\\ q\le 2Q/3\\ q^2+a^2<t^2Q^2\\ \bar{a}\in I_2}} \frac{2q}{Q(q^2+a^2)} = \sum_{1\le q\le Q\min(2/3,t\cos\alpha)} \sum_{\substack{a\in I,\, b\in I_2\\ ab=1 \pmod q}} f_1(a,b),$$

$$S^{\mathrm{II}}_{\alpha,\beta}(t,\varepsilon) = \sum_{\substack{a/q\in\mathcal{F}_Q\cap[\tan\alpha,\tan\beta]\\ q^2+a^2<t^2Q^2\\ \bar{a}\in I_3}} \frac{q(Q-q+\bar{a})}{Q\bar{a}(q^2+a^2)} = \sum_{1\le q\le Q\min(1,t\cos\alpha)} \sum_{\substack{a\in I,\, b\in I_3\\ ab=1 \pmod q}} f_2(a,b),$$

$$S^{\mathrm{III}}_{\alpha,\beta}(t,\varepsilon) = \sum_{\substack{a/q\in\mathcal{F}_Q\cap[\tan\alpha,\tan\beta]\\ q^2+a^2<t^2Q^2\\ \bar{a}\in I_1}} \frac{q(Q-\bar{a})}{Q(q^2+a^2)(q-\bar{a})} = \sum_{1\le q\le Q\min(1,t\cos\alpha)} \sum_{\substack{a\in I,\, b\in I_1\\ ab=1 \pmod q}} f_3(a,b),$$

$$S^{\mathrm{IV}}_{\alpha,\beta}(t,\varepsilon) = \sum_{\substack{a/q\in\mathcal{F}_Q\cap[\tan\alpha,\tan\beta]\\ q>2Q/3\\ q^2+a^2<t^2Q^2\\ \bar{a}\in I_2}} \frac{q^2(Q-q)}{Q(q^2+a^2)\bar{a}(q-\bar{a})} = \sum_{2Q/3<q\le Q\min(1,t\cos\alpha)} \sum_{\substack{a\in I,\, b\in I_2\\ ab=1 \pmod q}} f_4(a,b)$$

Lemma 3.2. *For every* $j\in\{1,2,3,4\}$ *we have*

(i) $\quad \|f_j\|_\infty \ll \dfrac{1}{Qq}$.

(ii) $\quad \|Df_j\|_\infty \ll \dfrac{1}{q^2(Q-q+1)}$.

Proof. (i) In case I it is clear that $F_2(y)=2$.

In case II we have $y>Q-q$, thus $0<\frac{Q-q+y}{y}=F_2(y)<2$.

In case III we have $y<2q-Q$. Hence $Q-y<2q-2y$ and $0<F_3(y)=\frac{Q-y}{q-y}<2$, so $\|f_3\|_\infty \ll \frac{1}{Qq}$.

In case IV we get $2q-Q>y>Q-q$. Hence $q-y>Q-q$, and so $Q-q<\min(y,q-y)$. If $y>\frac{q}{2}$, then $0\le F_4(y)=\frac{q(Q-q)}{y(q-y)}<\frac{q}{y}<2$. If $y<\frac{q}{2}$, then $0\le F_4(y)<\frac{2(Q-q)}{y}<2$, so $\|F_4\|_\infty<2$. Hence $\|f_j\|_\infty \le \frac{2q}{Qq^2}=\frac{2}{Qq}$.

(ii) In case II we have $|F_2'(y)|=\frac{Q-q}{y^2}<\frac{Q-q}{(Q-q)^2}\le\frac{Q-q}{(Q-q+1)^2}<\frac{1}{Q-q+1}$.

In case III we have $q-y\ge Q-q+1$ and $F_3'(y)=\frac{Q-q}{(q-y)^2}\le\frac{Q-q}{(Q-q+1)^2}=\frac{1}{Q-q}$ since $q-y>Q-q$.

In case IV we get $|F_4'(y)|=\left|-\frac{Q-q}{y^2}+\frac{Q-q}{(q-y)^2}\right|<\frac{2(Q-q)}{(Q-q+1)^2}$ since $y\ge Q-q+1$ and $q-y\ge Q-q+1$.

Summarizing, we collect

$$\left|\frac{\partial f_j}{\partial x}\right| = \frac{2qx}{Q(q^2+x^2)^2}\cdot|F_j(y)| \ll \frac{1}{Qq^2}$$

and

$$\left|\frac{\partial f_j}{\partial y}\right| = \frac{q}{Q(q^2+x^2)}\cdot|F_j'(y)| \ll \frac{1}{Qq(Q-q+1)}.$$

Hence

$$\|Df_j\|_\infty \ll \frac{1}{Qq^2} + \frac{1}{Qq(Q-q+1)} \ll \frac{1}{q^2(Q-q+1)}.$$

□

We shall next apply Lemma 3.1 to the functions f_1, f_2, f_3, f_4, to $\mathcal{I} = I$ and $\mathcal{J}$ one of the intervals I_1, I_2, I_3, and to $T = Q^{\frac{1}{6}}$. It is clear that $|\mathcal{I}|$ and $|\mathcal{J}|$ are no greater than q. We first employ Lemma 3.2 to estimate the error term, and note that

$$T^2 \sum_{q=1}^{Q} q^{\frac{1}{2}+\delta} \|f_j\|_\infty \ll \frac{T^2}{Q} \sum_{q=1}^{Q} q^{-\frac{1}{2}+\delta} \ll T^2 Q^{-\frac{1}{2}+\delta} = Q^{-\frac{1}{6}+\delta},$$

$$T \sum_{q=1}^{Q} q^{\frac{3}{2}+\delta} \|Df_j\|_\infty \ll T \sum_{q=1}^{Q} \frac{q^{\frac{3}{2}+\delta}}{q^2(Q-q+1)} = T \sum_{1\le q\le Q/2} \frac{q^{-\frac{1}{2}+\delta}}{Q-q+1} + T \sum_{Q/2<q\le Q} \frac{q^{-\frac{1}{2}+\delta}}{Q-q+1}$$

$$\ll TQ^{-\frac{1}{2}+\delta} + TQ^{-\frac{1}{2}+\delta} \ln Q \ll Q^{-\frac{1}{3}+\delta} \ln Q,$$

$$\frac{1}{T} \sum_{q=1}^{Q} \frac{q^2}{q^2(Q-q+1)} \ll \frac{1}{T} \cdot \ln Q \ll Q^{-\frac{1}{6}+\delta}.$$

So, if we set

$$L(q) = L(q,t) = \int_I \frac{q}{q^2+x^2}\, dx = \begin{cases} \beta - \alpha & \text{if } q \in [1, tQ\cos\beta], \\ \arccos \dfrac{q}{tQ} - \alpha & \text{if } q \in [tQ\cos\beta, tQ\cos\alpha], \\ 0 & \text{if } q \in [tQ\cos\alpha, \infty), \end{cases}$$

$$H_1(q) = 2|I_2| = 2\int_{I_2} dy, \qquad H_2(q) = \int_{I_3} \frac{Q-q+y}{y}\, dy,$$

$$H_3(q) = \int_{I_1} \frac{Q-y}{q-y}\, dy = H_2(q), \qquad H_4(q) = q(Q-q)\int_{I_2} \frac{dy}{y(q-y)} = 2(Q-q)\int_{I_2} \frac{dy}{y},$$

then

$$S^{\mathrm{I}}_{\alpha,\beta}(t,\varepsilon) = \frac{1}{Q} \sum_{1\le q\le Q\min(2/3,t\cos\alpha)} \frac{\varphi(q)}{q^2} \cdot L(q)H_1(q) + O_\delta(Q^{-\frac{1}{6}+\delta}),$$

$$S^{\mathrm{II}}_{\alpha,\beta}(t,\varepsilon) = \frac{1}{Q} \sum_{1\le q\le Q\min(1,t\cos\alpha)} \frac{\varphi(q)}{q^2} \cdot L(q)H_2(q) + O_\delta(Q^{-\frac{1}{6}+\delta}),$$

$$S^{\mathrm{III}}_{\alpha,\beta}(t,\varepsilon) = \frac{1}{Q} \sum_{1\le q\le Q\min(1,t\cos\alpha)} \frac{\varphi(q)}{q^2} \cdot L(q)H_2(q) + O_\delta(Q^{-\frac{1}{6}+\delta}),$$

$$S^{\mathrm{IV}}_{\alpha,\beta}(t,\varepsilon) = \frac{1}{Q} \sum_{2Q/3<q\le Q\min(1,t\cos\alpha)} \frac{\varphi(q)}{q^2} \cdot L(q)H_4(q) + O_\delta(Q^{-\frac{1}{6}+\delta}).$$

If $q \le \frac{Q+1}{2}$, then $I_1 = I_3 = \emptyset$ so $H_2(q) = H_3(q) = 0$, and $I_2 = [1, q-1]$ thus $H_1(q) = 2(q-2)$.

If $\frac{Q+1}{2} < q < \frac{2Q+1}{3}$, then $I_1 = [1, 2q-Q)$ thus $H_2(q) = 2q - Q - 1 + (Q-q)\ln\frac{q-1}{Q-q}$, and $I_2 = [2q-Q, Q-q]$ thus $H_1(q) = 2(2Q-3q)$.

If $\frac{2Q+1}{3} \le q \le Q$, then $I_1 = [1, Q-q+1]$ thus $H_2(q) = Q - q + (Q-q)\ln\frac{q-1}{2q-Q-1}$, and $I_3 = (2q-Q-1, q-1]$ thus $H_4(q) = 2(Q-q)\ln\frac{2q-Q-1}{Q-q+1}$. Summarizing, we find for all $\delta > 0$

$$S_{\alpha,\beta}(t,\varepsilon) = \frac{1}{Q}\sum_{q=1}^{Q}\frac{\varphi(q)}{q}\cdot V(q) + O_\delta(Q^{-\frac{1}{6}+\delta}),$$

where $V(q) = V(q,t) = 2L(q)W(q)$ and we set

$$W(q) = W(q,t) = \begin{cases} 1 - \frac{2}{q} & \text{if } 1 \le q \le \min\left(tQ\cos\alpha, \frac{Q+1}{2}\right), \\ \frac{Q-q-1}{q} + \frac{Q-q}{q}\cdot\ln\frac{q-1}{Q-q} & \text{if } \frac{Q+1}{2} < q \le \min\left(tQ\cos\alpha, \frac{2Q+1}{3}\right), \\ \frac{Q-q}{q} + \frac{Q-q}{q}\cdot\ln\frac{q-1}{Q-q+1} & \text{if } \frac{2Q+1}{3} < q \le Q\min(1, t\cos\alpha), \\ 0 & \text{if } q > Q\min(1, t\cos\alpha). \end{cases} \tag{3.5}$$

To estimate $S_{\alpha,\beta}(t,\varepsilon)$ we shall employ [2, Lemma 2.3], which requires estimates for the supremum and for the variation of V on $[1,Q]$. It is easy to see that $\|L\|_\infty \ll 1$ and $\|W\|_\infty \ll 1$, thus

$$\|V\|_\infty \ll 1. \tag{3.6}$$

To estimate the variation of V, we first write

$$\begin{aligned} \int_1^Q |V'(q)|\,dq &\ll \int_1^Q |L(q)W'(q)|\,dq + \int_1^Q |L'(q)W(q)|\,dq \\ &\ll \int_1^Q |W'(q)|\,dq + \int_1^Q |L'(q)|\,dq \ll 1 + \int_1^Q |W'(q)|\,dq. \end{aligned}$$

But we see that

$$\int_1^{(Q+1)/2} |W'(q)|\,dq = W\left(\frac{Q+1}{2}\right) - W(1) \ll 1,$$

and by a direct computation

$$\sup_{q\ge(Q+1)/2} |W'(q)| \ll \frac{\ln Q}{Q}.$$

Thus

$$\int_1^Q |V'(q)|\,dq \ll 1 + \ln Q \ll \ln Q, \tag{3.7}$$

and we may apply [2, Lemma 2.3], employing (3.5), (3.6) and (3.7), to gather

$$\sum_{q=1}^{Q} \frac{\varphi(q)}{q} \cdot V(q) = \frac{6}{\pi^2} \int_1^Q V(q)\, dq + O(\ln^2 Q).$$

This leads immediately to

$$S_{\alpha,\beta}(t,\varepsilon) = \frac{6}{\pi^2 Q} \int_1^Q V(q)\, dq + O_\delta(\varepsilon^{\frac{1}{6}-\delta}). \tag{3.8}$$

Using now the inequalities

$$0 < \int_{Q/2}^{Q} \frac{Q-q}{q} \Big(\ln \frac{q}{Q-q} - \ln \frac{q-1}{Q-q} \Big)\, dq = \int_{Q/2}^{Q} \frac{Q-q}{q} \cdot \ln \Big(1 + \frac{1}{q-1} \Big)\, dq \ll \int_{Q/2}^{Q} \frac{dq}{q-1} \ll 1$$

and

$$0 < \int_{Q/2}^{Q} \frac{Q-q}{q} \Big(\ln \frac{q-1}{Q-q} - \ln \frac{q-1}{Q-q+1} \Big)\, dq \ll \int_{Q/2}^{Q} \frac{dq}{Q-q}\, dq \ll 1,$$

together with (3.8),(3.5) and $\int_1^Q \frac{dq}{q} = \ln Q$, we arrive at

$$S_{\alpha,\beta}(t,\varepsilon) = \frac{12}{\pi^2 Q} \int_1^Q L(q)\, W_1(q)\, dq + O_\delta(\varepsilon^{\frac{1}{6}-\delta}), \tag{3.9}$$

where we set

$$W_1(q) = \begin{cases} 1 & \text{if } 0 \le q \le Q \min\big(\frac{1}{2}, t\cos\alpha\big), \\ \frac{Q-q}{q} + \frac{Q-q}{q} \cdot \ln \frac{q}{Q-q} & \text{if } \frac{Q}{2} < q \le Q \min(1, t\cos\alpha), \\ 0 & \text{if } q > Q \min(1, t\cos\alpha). \end{cases} \tag{3.10}$$

The change of variable $q = Qs$, together with (3.9) and (3.10) lead to

$$S_{\alpha,\beta}(t,\varepsilon) = \frac{12}{\pi^2} \int_0^1 L_1(s) W_2(s)\, ds + O_\delta(\varepsilon^{\frac{1}{6}-\delta}), \tag{3.11}$$

where

$$L_1(s) = L_1(s,t) = \begin{cases} \beta - \alpha & \text{if } s \in [0, t\cos\beta], \\ \arccos \frac{s}{t} - \alpha & \text{if } s \in [t\cos\beta, t\cos\alpha], \\ 0 & \text{if } s \in [t\cos\alpha, \infty), \end{cases} \tag{3.12}$$

and

$$W_2(s) = W_2(s,t) = \begin{cases} 1 & \text{if } s \in \left[0, \min\left(t\cos\alpha, \frac{1}{2}\right)\right], \\ \psi(s) & \text{if } s \in \left[\frac{1}{2}, \min(1, t\cos\alpha)\right], \\ 0 & \text{if } s \in [\min(1, t\cos\alpha), \infty), \end{cases} \tag{3.13}$$

with ψ as in (1.3). Then (3.11), (3.12), (3.13) and (2.3) prove Theorem 1.1. To estimate $G_{\alpha,\beta}(t)$ we follow literally the proof of Theorem 1.1, replacing only $S_{\alpha,\beta}(t,\varepsilon)$ by

$$R_{\alpha,\beta}(t,\varepsilon) = \sum_{\substack{a/q \in \mathcal{F}_Q \cap [\tan\alpha, \tan\beta] \\ q+a<tQ}} \omega_{q,a},$$

and approximating $(\beta-\alpha)G_{\alpha,\beta,\varepsilon}(t)$ by $R_{\alpha,\beta}(t,\varepsilon)$ as in (2.3). The interval I is being replaced in this instance by $J = [q\tan\alpha, q\tan\beta] \cap [0, tQ-q]$. We clearly have

$$J = [q\tan\alpha . q\tan\beta] \cap [0. tQ - q] = \begin{cases} [q\tan\alpha, q\tan\beta] & \text{if } q \in \left[1, \frac{tQ}{1+\tan\beta}\right], q\tan\alpha, tQ - q > q\tan\alpha, tQ - \\ & \text{if } q \in \left[\frac{tQ}{1+\tan\beta}, \frac{tQ}{1+\tan\alpha}\right], \\ \emptyset & \text{if } q \in \left[\frac{tQ}{1+\tan\alpha}, \infty\right], \end{cases}$$

and arrive using similar arguments at

$$G_{\alpha,\beta,\varepsilon}(t) = \frac{12}{\pi^2 Q} \int_0^Q L_2(s) W_3(s)\, dt + O_\delta(\varepsilon^{\frac{1}{6}-\delta}), \tag{3.14}$$

where

$$L_2(s) = L_2(s,t) = \begin{cases} \beta - \alpha & \text{if } s \in \left[0, \min\left(\frac{t}{1+\tan\beta}, \frac{1}{2}\right)\right], \\ \arctan\left(\frac{t}{s} - 1\right) - \alpha & \text{if } s \in \left[\frac{1}{2}, \min\left(1, \frac{t}{1+\tan\alpha}\right)\right], \\ 0 & \text{if } s > \min\left(1, \frac{t}{1+\tan\alpha}\right), \end{cases} \tag{3.15}$$

and

$$W_3(s) = W_3(s,t) = \begin{cases} 1 & \text{if } s \in \left[0, \min\left(\frac{t}{1+\tan\alpha}, \frac{1}{2}\right)\right], \\ \psi(s) & \text{if } s \in \left[\frac{1}{2}, \min\left(1, \frac{t}{1+\tan\alpha}\right)\right], \\ 0 & \text{if } s > \min\left(1, \frac{t}{1+\tan\alpha}\right). \end{cases} \tag{3.16}$$

By (3.13), (3.14), (3.15) and (3.16), we find that

$$G_{\alpha,\beta}(t) = \frac{12}{\pi^2(\beta-\alpha)} \cdot \begin{cases} \frac{t(\beta-\alpha)}{1+\tan\beta} + \int\limits_{\frac{t}{1+\tan\beta}}^{\frac{t}{1+\tan\alpha}} \left(\arctan(\tfrac{t}{s}-1)-\alpha\right) ds & \text{if } t\in\left[0,\frac{1+\tan\alpha}{2}\right], \\ \frac{t(\beta-\alpha)}{1+\tan\beta} + \int\limits_{\frac{t}{1+\tan\beta}}^{1/2} \left(\arctan(\tfrac{t}{s}-1)-\alpha\right) ds + \int\limits_{1/2}^{\frac{t}{1+\tan\alpha}} \psi(s)\left(\arctan(\tfrac{t}{s}-1)-\alpha\right) ds & \\ \qquad\qquad \text{if } t\in\left[\frac{1+\tan\alpha}{2},\frac{1+\tan\beta}{2}\right], & \\ (\beta-\alpha)\left(\frac{1}{2}+\int\limits_{1/2}^{\frac{t}{1+\tan\beta}} \psi(s)ds\right) + \int\limits_{\frac{t}{1+\tan\beta}}^{\frac{t}{1+\tan\alpha}} \psi(s)\left(\arctan(\tfrac{t}{s}-1)-\alpha\right) ds & \\ \qquad\qquad \text{if } t\in\left[\frac{1+\tan\beta}{2},1+\tan\alpha\right], & \\ (\beta-\alpha)\left(\frac{1}{2}+\int\limits_{1/2}^{\frac{t}{1+\tan\beta}} \psi(s)ds\right) + \int\limits_{\frac{t}{1+\tan\beta}}^{1} \psi(s)\left(\arctan(\tfrac{t}{s}-1)-\alpha\right) ds & \\ \qquad\qquad \text{if } t\in\left[1+\tan\alpha,1+\tan\beta\right], & \\ (\beta-\alpha)\left(\frac{1}{2}+\int\limits_{1/2}^{1} \psi(s)ds\right) & \text{if } t\in[1+\tan\beta,\infty). \end{cases}$$

Theorem 1.4 follows now from the previous equality and some straightforward computations.

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