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A New Class of Optimal Viscosity Profile in Oil Recovery

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Abstract

This paper is devoted to the study of the secondary oil recovery process: the oil contained in a porous medium is obtained by pushing it with a second fluid (water). We consider the Hele-Shaw approximation. If the second fluid is less viscous, the fingering phenomenon appears, first studied by Saffman and Taylor (1959). To minimize this instability, an intermediate polymer-solute region (i.r.), with a variable viscosity μ, is considered between water and oil (see Gorell and Homsy (1983)). This viscosity increases from water to oil. The linear stability of the interfaces are governed by a Sturm-Liouville problem which contains eigenvalues in the boundary conditions. The characteristic values are the growth constants of the perturbations. The stability can be improved by choosing an optimal viscosity profile μ which gives us a smallest growth constant. A finite-difference procedure and the Gerschgorin's localization theorem were used by Carasso and Paşa (1998) to solve the above problem and to get *a formula* of an *exponential* optimal viscosity profile in (i.r.). In the present paper we consider the Rayleigh quotient to estimate the characteristic values of the above Sturm-Liouville problem. We get a *class* of optimal viscosity profiles in (i.r.), including *linear* and *exponential* profiles. The corresponding total amount of polymer and the (i.r.) length are estimated in terms of the limit value of μ on the (i.r.)-oil interface. We give lower estimates of these above parameters for a *given improvement of the stability* , compared with the Saffman-Taylor case. The present results are compared with the previous theoretical optimal viscosity profiles.

Keywords: Flow in porous media, Hele-Shaw model, oil recovery, Rayleigh quotient, Sturm-Liouville eigenproblem.

Mathematics Subject Classification: 34B24, 34L15, 76D27, 76S05.

1 Introduction

We study the secondary oil recovery process: the oil contained in a porous medium is obtained by pushing it with a second immiscible fluid (usually water). We consider a homogeneous porous medium and the Hele-Shaw approximation, then an interface exists between the two immiscible fluids. If the second fluid is less viscous then the fingering phenomenon appears, first studied by Saffman and Taylor (1959) and Chouke *et al.* (1959). An improved stability is obtained by applying a surface tension on the interface.

An intermediate region (i.r.), containing a polymer solute with a variable viscosity μ, is considered between water and oil (see Gorell and Homsy (1983)). Then we consider three immiscible fluids: water, polymer and oil, separated by two interfaces. A surface tension can be considered on the water-(i.r.) and (i.r.)-oil interfaces. The unknown viscosity μ in (i.r.) is a parameter which is used to improve the interfaces stability. It increases from water to oil. We suppose that μ is an invertible function of polymer concentration. Gorell and Homsy considered a continuous viscosity on the interface water-(i.r.). Therefore the viscosity is discontinuous only on the (i.r.)-oil interface and a surface tension acts only on this interface. The three regions are moving due to the water velocity U far upstream. The flow is given by Darcy law, the continuity equation of the velocity and the "conservation" law of the viscosity μ. On the two interfaces, the Laplace's law was used to describe the contact conditions between the immiscible fluids. A steady basic solution, with straight initial interfaces, has been considered by Gorell and Homsy (1983). The interface stability is governed by a Sturm-Liouville problem, containing eigenvalues in the boundary conditions. The characteristic values are the growth constant (in time) of the perturbations. An improved stability of the interfaces means to have a smaller characteristic values in the above problem, compared with the Saffman-Taylor case. A numerical exponential viscosity profile has been obtained, giving an improved stability, according to previous experimental results of Mungan (1971), Pearson (1977), Shah and Schecter (1977) and Uzoigwe *et al.* (1974).

An asymptotic analysis is given in Paşa and Polisevski (1992), in the case of a small quantity of polymer.

An existence theorem for an optimal viscosity in (i.r.) has been obtained by Paşa (1996), by using the Rayleigh quotient.

An explicit formula for an optimal viscosity in (i.r.) has been obtained by Carasso and Paşa (1998). The above Sturm-Liouville problem was discretizated by the finite-difference method. The Gerschgorin's localization theorem was used to estimate the characteristic values. The obtained optimal viscosity profile in (i.r.) is *exponential*, according to the numerical results of Gorell and Homsy (1983). To justify the previous discretization, a convergence theorem has been proved by Carasso and Paşa (2000).

A "very slow" viscosity profile in (i.r.) was obtained in Paşa (2002), by using the result of Carasso and Paşa (1998). This "very slow" *exponential* profile gives us a growth constant which is similar to the corresponding Saffman-Taylor value: the water viscosity was replaced by the limit value of the viscosity on the (i.r.) - oil interface.

In the present paper we consider the Rayleigh quotient to get a *class* of optimal viscosity in (i.r.), including *exponential* and *linear* profiles. In this paper we get an *exact* estimate of the characteristic values of the considered Sturm-Liouville problem. We use the initial stability problem, without any discretization. The obtained exponential profile is coherent with previous theoretical and numerical results. The linear profile is more favorable: a smaller amount of polymer and (i.r.) length are necessary to get the same growth constant. Moreover, this new class of optimal viscosity allows us to consider *variable* coefficients in the Sturm-Liouville equation. The corresponding total amount of polymer and the (i.r.) length are estimated in terms of the above limit value of the viscosity on the (i.r.) - oil interface. We compute the above parameters corresponding to *a given improvement of the stability*, compared with the Saffman-Taylor case. The values of the maximal growth constant, obtained with the *exponential* viscosity profile of Carasso and Paşa (1998) and with the new *linear* and *exponential* viscosity profiles, are compared.

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2 Review of the stability problem

We study here the model introduced by Gorell and Homsy (1983) — see Fig. 1. Numerical results concerning this model were obtained in Daripa *et al.* (1986), Daripa (1987), Daripa *et al.* (1988), Daripa *et al.* (1988). The model was studied also in Paşa and Polisevski (1992), Paşa (1996, 2002), Carasso and Paşa (1998, 2000).

A homogeneous porous medium is considered in the plane x_1Oy and the Hele-Shaw approximation is used. The medium is saturated with three immiscible fluids: water (with the constant viscosity μ_1), polymer (with the unknown variable viscosity μ) and oil (with the constant viscosity μ_2). The three regions are moving due to the water velocity U at infinity upstream in the positive Ox_1 direction. The polymer is contained in the intermediate region (i.r.). We have two sharp interfaces: water-(i.r.) and (i.r.)-oil. In the intermediate region, the expected viscosity μ is an increasing function of the distance. The main point is to consider that the viscosity is an invertible function of the polymer solute.

Some limitations exist if we consider a miscible water-polymer mixture in (i.r.). Petitjeans and Maxvorthy (1996) and Lajeunesse *et al.* (1999) proved that exist a viscosity ratio and a velocity U above which the Hele-Shaw flow becomes unstable and a three-dimensional pattern appears.

In the three regions of the porous medium, the flow is given by the continuity equation for the velocity and the Darcy law.

In the intermediate region (i.r.). we consider a given amount of polymer — denoted by M in the following — then we obtain a "continuity" equation for the viscosity μ.

On the interfaces we consider the Laplace's law: the pressure drop is balanced by the curvature times the surface tension ; moreover the velocity is continuous.

Therefore the flow is governed by the following system:

$$\partial u/\partial x_1 + \partial v/\partial y = 0, \qquad x_1 \in \mathbb{R},\ x_1 \notin \text{interfaces},\ y \in \mathbb{R}, \tag{1}$$

$$\partial P/\partial x_1 = -\mu u, \qquad x_1 \in \mathbb{R},\ x_1 \notin \text{interfaces},\ y \in \mathbb{R}, \tag{2}$$

$$\partial P/\partial y = -\mu v, \qquad x_1 \in \mathbb{R},\ x_1 \notin \text{interfaces},\ y \in \mathbb{R}, \tag{3}$$

$$\partial \mu/\partial t + u\partial \mu/\partial x_1 + v\partial \mu/\partial y = 0, \qquad x_1 \in (\text{i.r.}),\ y \in \mathbb{R}, \tag{4}$$

where (u, v) is the velocity of the fluid and P is the pressure. The above equations admit the following basic solution describing the steady displacement:

$$u = U, \quad v = 0, \quad \mu = \mu_b(x_1 - Ut), \quad P = -U \int \mu_b(s - Ut)ds.$$

We introduce the moving reference system $x = x_1 - Ut$. The above basic solution let us consider an intermediate region (i.r.) with constant length l at the left of the origin in the moving coordinates xOy. We emphasize that μ_b is an arbitrary function which verifies the following properties:

$$\mu_b \in C^1(\text{i.r.}), \quad \mu_1 \leq \mu_b(x) < \mu_2, \quad \mu_b'(x) > 0, \quad x \in (\text{i.r.}). \tag{5}$$

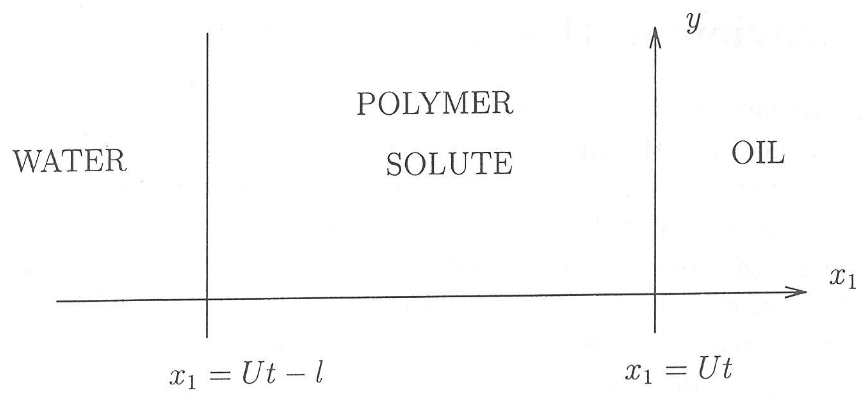

Fig. 1.

We have two sharp straight interfaces in the moving reference xOy: the water-(i.r.) interface, which corresponds to $x = -l$ and the (i.r.)-oil interface, which corresponds to $x = 0$. On the interfaces we consider the above Laplace's law.

We study the linear stability of the interfaces defined above. We consider the small perturbations (u', v', P', μ') and then we get the following system:

$$\partial u'/\partial x + \partial v'/\partial y = 0, \quad x \in \mathbb{R},\ x \notin \{-l, 0\},\ y \in \mathbb{R}, \tag{6}$$
$$\partial P'/\partial x = -\mu' U - \mu_b u', \quad x \in \mathbb{R},\ x \notin \{-l, 0\},\ y \in \mathbb{R}, \tag{7}$$
$$\partial P'/\partial y = -\mu_b v', \quad x \in \mathbb{R},\ x \notin \{-l, 0\},\ y \in \mathbb{R}, \tag{8}$$
$$\partial \mu'/\partial t + u' \cdot d\mu_b/dx = 0, \quad x \in (-l, 0),\ y \in \mathbb{R}. \tag{9}$$

The last relation is proved in Gorell and Homsy (1983) and also in Paşa (2002). As the problem (6)–(9) is linear, we can decompose the perturbations in Fourier components. We start with the horizontal component of the velocity perturbation:

$$u'(x, y, t) = f(x) \exp(iky + \sigma t), \tag{10}$$

where $f(x)$ is the amplitude of the perturbation, k is the wavenumber in the Oy direction and σ is the growth constant in time. The corresponding expressions for v', P' and μ' are given by applying (6), (7) and (9). As in Gorell and Homsy (1983) and Paşa (2002), cross differentiating (7) and (8) let us obtain the second order differential equation for f in the intermediate region:

$$\mu_b[f_{xx} - k^2 f] + (\mu_b)_x f_x + \frac{k^2 U}{\sigma}(\mu_b)_x f = 0, \quad x \in (-l, 0), \tag{11}$$

where f_x refers to df/dx. In the above Sturm-Liouville equation (11), f are the eigenfunctions and $1/\sigma$ are the eigenvalues. We need two boundary conditions to solve the equation (11). We consider that a surface tension S acts on the interface $x = -l$ and a surface tension T acts on $x = 0$. The contact conditions on the interfaces are given by the above Laplace's law. All the details concerning the contact conditions on the interfaces are given in Gorell and Homsy (1983) and also in Paşa (2002). We have:

$$\mu_b^+(-l)f_x^+(-l) = f(-l)\{\mu_1 k + \frac{Uk^2}{\sigma}[\mu_1 - \mu_b^+(-l)] + \frac{Sk^4}{\sigma}\}, \quad (12)$$

$$\mu_b^-(0)f_x^-(0) = f(0)\{-k\mu_2 + \frac{Uk^2}{\sigma}[\mu_2 - \mu_b^-(0)] - \frac{Tk^4}{\sigma}\}, \quad (13)$$

where the superscript $^-$ and $^+$ denote the "left" and the "right" limits.

The Sturm-Liouville problem (11)–(13) can be used to study the case of Saffman and Taylor, where no intermediate region is considered, that is $l = 0$. In this case we have only one interface at $x = 0$ and a surface tension T acts on it. The value of the basic viscosity is μ_1 for $x < 0$ and μ_2 for $x > 0$, therefore the corresponding eigenfunctions will be exponentials. As perturbations must be zero far enough, Gorell and Homsy (1983) got the well-known formula of Saffman and Taylor (1959):

$$\sigma_{ST} := \frac{(\mu_2 - \mu_1)Uk - Tk^3}{\mu_2 + \mu_1}. \quad (14)$$

A maximum value σ_{MST} is obtained for the wavenumber k_{MST}:

$$\sigma_{MST} := \frac{2}{3} \cdot \frac{(\mu_2 - \mu_1)U}{\mu_2 + \mu_1} \cdot k_{MST}, \quad (15)$$

$$k_{MST} := \{(\mu_2 - \mu_1)U/3T\}^{1/2}. \quad (16)$$

We use the above relations to get the following dimensionless quantities, as in Gorell and Homsy (1983):

$$\sigma^* := \frac{2 \cdot \sigma}{3\sqrt{3} \cdot \sigma_{MST}}, \quad \mu^*(x) := \frac{\mu_b(x)}{\mu_1}, \quad k^* := \frac{k}{k_{MST}\sqrt{3}},$$

$$\alpha := \frac{\mu_2}{\mu_1}, \quad x^* := k_{MST} \cdot x \cdot \sqrt{3}, \quad L := k_{MST} \cdot l \cdot \sqrt{3}, \quad (17)$$

$$f^* := \frac{f}{U}, \quad \lambda := \frac{1}{\sigma}, \quad M := \int_{-L}^{0} \mu^*(x^*)dx^*.$$

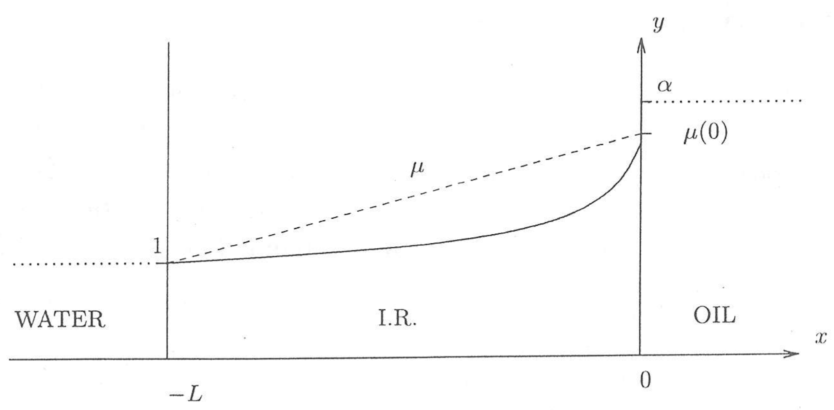

Fig. 2.

We remark that the maximum value σ^*_{MST} of the dimensionless growth constant σ^*_{ST} is given by

$$\sigma^*_{MST} := \frac{2}{3\sqrt{3}}. \tag{18}$$

As in Gorell and Homsy (1983), we consider the simpler case when the viscosity is continuous on $x = -l$, therefore we have a viscosity jump only on the (i.r.)-oil interface. This interface is the only one where a surface tension T acts. Finally, $\mu_b^+(-l) = \mu_1$ and $S = 0$ in (12): then we get a simpler form of the condition (12):

$$f_x^+(-l) = kf(-l). \tag{19}$$

In the following we will omit the superscript $*$ and we will use the notation $df/dx := f'$. Then the relations (11), (12) and (19) give us the problem:

$$\begin{cases} -(\mu f')'(x) + k^2\mu(x)f(x) = \lambda k^2\beta\mu'(x)f(x), & x \in (-L, 0), \\ f'(-L) = (\lambda A + B)f(-L), \\ f'(0) = (\lambda C + D)f(0), \end{cases} \tag{20}$$

where

$$\beta := \frac{\alpha + 1}{\alpha - 1}, \tag{21}$$

and

$$\begin{aligned} & A := 0, \ B := k, \\ & C := \beta k^2[\alpha - \mu(0) - k^2(\alpha - 1)]/\mu(0), \\ & D := -k\alpha/\mu(0). \end{aligned} \tag{22}$$

The main point is to find the optimal viscosity μ which gives us the smallest growth constant σ. In the following section we give a theoretical framework of the above problem.

3 Stability Analysis

This section is devoted to a brief study of the above problem (20).

Let a and b be two real numbers such that $a < b$. Let $X := W_2^1(a, b)$ and μ be an arbitrary *positive strictly increasing* function on (a, b) — accordingly to the property (5) — and we suppose moreover that it is sufficiently regular on this interval. We consider the following Sturm-Liouville eigenvalue problem: find $f \in X$ and $\lambda \in \mathbb{C}$ such that

$$\begin{cases} -(\mu f')'(x) + k^2\mu(x)f(x) = \lambda k^2\beta\mu'(x)f(x), & x \in (a, b), \\ f'(a) = (\lambda A + B)f(a), \\ f'(b) = (\lambda C + D)f(b), \end{cases} \tag{23}$$

where k and β are real positive numbers. We suppose that

$$A \leq 0, \quad D \leq 0, \quad B > 0, \quad \text{and} \quad C > 0. \tag{24}$$

3.1 Weak formulation - Rayleigh quotient

Let define the following bilinear forms $\phi : X \times X \to \mathbb{R}$ and $\psi : X \times X \to \mathbb{R}$ by:

$$\phi(f,g) := \int_a^b \mu f' g' + k^2 \int_a^b \mu f g + B(\mu f g)(a) - D(\mu f g)(b), \tag{25}$$

$$\psi(f,g) := k^2 \beta \int_a^b \mu' f g - A(\mu f g)(a) + C(\mu f g)(b), \tag{26}$$

where $(\mu f g)(a)$ stands for $\mu(a) f(a) g(a)$. Then an equivalent form of the problem (23) is (see Courant and Hilbert (1965)): find $f \in X$ and $\lambda \in \mathbb{C}$ such that

$$\phi(f,g) = \lambda \psi(f,g), \quad \text{for all } g \in X. \tag{27}$$

Let define $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ by

$$\Phi(f) := \varphi(f,f) = \int_a^b \mu (f')^2 + k^2 \int_a^b \mu f^2 + B(\mu f^2)(a) - D(\mu f^2)(b), \tag{28}$$

$$\Psi(f) := \psi(f,f) = k^2 \beta \int_a^b \mu' f^2 - A(\mu f^2)(a) + C(\mu f^2)(b). \tag{29}$$

Then Φ and Ψ satisfy the following properties:

i) Φ is a convex functional (because φ is a bilinear positive form).

ii) Φ is continuous on X.

iii) Φ is Gâteau differentiable and $d\Phi(f)g = 2\varphi(f,g)$.

iv) Φ is coercive.

v) Ψ is weakly continuous on X if $\mu \in W_2^2(a,b)$.

Let consider the following minimization problem: find $u \neq 0$ such that

$$\frac{\Phi(u)}{\Psi(u)} = \inf \left\{ \frac{\Phi(f)}{\Psi(f)}, \ f \in X, \ f \neq 0 \right\}. \tag{30}$$

The properties i)-v) imply (see Céa (1971)) that the above problem admits a solution $u \in X$. Following Courant and Hilbert (1965), u is an eigenfunction for the problem (27) corresponding to the smallest eigenvalue $\lambda_R := \dfrac{\Phi(u)}{\Psi(u)}$, which is a real positive number.

3.2 Upper estimate of the characteristic value

Let λ_R be the smallest eigenvalue of the problem (23) and let define $\sigma_R := \dfrac{1}{\lambda_R}$. The following theorem gives an upper bound for σ_R in terms of the data of the problem.

Theorem 1 *Let u be a solution of* (30) *and $\mu \in C^1(a,b)$ and define*

$$\gamma := \sup_{x\in(a,b)} \mu'(x) = \max_{x\in(a,b)} \mu'(x) < +\infty.$$

Then

$$\sigma_R := \frac{\Psi(u)}{\Phi(u)} \leq \frac{\max[k^2\beta\gamma, \mu(b)\max(-A,C)]}{\min[k^2, \min(B,-D)]\mu(a)}. \tag{31}$$

Proof: Let u be a solution of (30). As μ is a strictly increasing continuous function

$$\begin{aligned}\Phi(u) &\geq \mu(a)(k^2\int_a^b u^2 + Bu^2(a) - Du^2(b)) \\ &\geq \mu(a)\min[k^2, \min(B,-D)](\int_a^b u^2 + u^2(a) + u^2(b)).\end{aligned}$$

Moreover μ' is a continuous function, so

$$\begin{aligned}\Psi(u) &\leq k^2\beta\gamma\int_a^b u^2 + \mu(b)(-Au^2(a) + Cu^2(b)) \\ &\leq \max[k^2\beta\gamma, \mu(b)\max(-A,C)](\int_a^b u^2 + u^2(a) + u^2(b)),\end{aligned}$$

which ends the proof. ∎

4 Optimal viscosity profiles

We now apply the result of the previous section to the second oil recovery problem. The problem (20) corresponds to the problem (23) with data (21)–(22). The instability on the (i.r.)-oil interface only appears when the limit value of the viscosity on the (i.r.) - oil interface is smaller than the viscosity of the oil. So we suppose that

$$1 < \mu(0) < \alpha. \tag{32}$$

Moreover, we are only interested in the case when $C > 0$, i.e. when

$$0 < k < \left(\frac{\alpha - \mu(0)}{\alpha - 1}\right)^{1/2} < 1. \tag{33}$$

Indeed, if (33) does not hold, the above problem (23) is not well-defined.

Our aim is to control the maximum characteristic value σ_R. As we supposed that $C > 0$ and as $A := 0$, we get $\max(-A, C) = C$. Condition (33) implies that $\min(B, -D) = k^2$. Finally, by Theorem 1,

$$\sigma_R \leq \frac{\beta}{\mu(-L)} \max[\max_{x\in(-L,0)} \mu'(x), \alpha - \mu(0) - k^2(\alpha - 1)]. \tag{34}$$

A class of optimal viscosity profile. The previous inequality (34) can be used to define a new *class* of optimal viscosity profiles: indeed, we remark that if μ satisfies the following condition

$$\max_{x\in(-L,0)} \mu'(x) \leq \alpha - \mu(0), \tag{35}$$

then

$$\sigma_R \leq \frac{\beta}{\mu(-L)}(\alpha - \mu(0)). \tag{36}$$

Let define the corresponding new maximum growth constant σ_{MR} by

$$\sigma_{MR} := \frac{\beta}{\mu(-L)}(\alpha - \mu(0)). \tag{37}$$

If condition (35) holds, we can choose $\mu(0)$ which gives us a maximum growth constant less than the Saffman-Taylor — dimensionless — value σ^*_{MST} defined by (18). In the following we give two families of profiles which satisfy condition (35) — see Fig. 2.

4.1 Linear optimal profiles

In this section we build a family of linear profiles which belong to the new class of optimal profiles defined above. Let μ_0 and L be real positive numbers satisfying the following conditions:

$$1 < \mu_0 < \alpha, \quad L \geq \frac{\mu_0 - 1}{\alpha - \mu_0}. \tag{38}$$

Then consider the viscosity linear profile, defined on $(-L, 0)$ by

$$\mu(x) := \frac{\mu_0 - 1}{L}(x + L) + 1. \tag{39}$$

Then $\mu(-L) = 1$ and $\mu(0) = \mu_0$. Moreover, following (38), the inequality (32) holds and, for all $x \in (-L, 0)$,

$$\mu'(x) = \frac{\mu_0 - 1}{L} \leq \alpha - \mu_0 = \alpha - \mu(0).$$

Finally, the condition (35) holds and then, by (36), we get

$$\sigma_R \leq \beta(\alpha - \mu_0). \tag{40}$$

Therefore $\mu_0 \to \alpha$ implies $\sigma_R \to 0$. This means that we can choose μ_0 to get a growth constant smaller than the Saffman-Taylor value. The above relation is used to get a lower estimate for the total amount of polymer M:

$$M := \int_a^b \mu(x)dx = L\left(\frac{\mu(0) - 1}{2} + 1\right) \geq \frac{\mu_0 - 1}{\alpha - \mu_0}\left(\frac{\mu_0 - 1}{2} + 1\right) = \frac{\mu_0^2 - 1}{2(\alpha - \mu_0)}. \tag{41}$$

We can remark that $\mu_0 \to \alpha$ implies $L, M \to \infty$.

4.2 Exponential optimal profile

In this section we build a family of "sub-exponential" profiles which satisfy condition (35). Consider a profile defined on $(-L, 0)$ which verifies the following conditions:

$$\mu \text{ is a strictly increasing function,} \tag{42}$$

$$\mu(-L) = 1 \quad \text{and} \quad 1 < \mu(0) < \alpha, \tag{43}$$

$$\frac{\mu'(x)}{\mu(x)} \leq \frac{\alpha}{\mu(0)} - 1, \quad \text{for all } x \in (-L, 0). \tag{44}$$

Following (42), for all $x \in (-L, 0)$, $\mu(x) < \mu(0)$ and then condition (44) implies that, for all $x \in (-L, 0)$, $\mu'(x) \leq \alpha - \mu(0)$. This implies that $\max\limits_{x \in (-L,0)} \mu'(x) \leq \alpha - \mu(0)$ i.e. condition (35) holds. Finally, we get the same previous result obtained for a linear profile:

$$\sigma_R \leq \beta(\alpha - \mu(0)). \tag{45}$$

Now, we will give some lower-estimations for L and M in terms of $\mu(0)$. Let define $Q := \dfrac{\alpha}{\mu(0)} - 1$; by integrating both sides of inequality (44), we get

$$\forall x \in (-L, 0), \quad \mu(x) \leq \exp[(x + L)Q].$$

That is why the profiles (42)–(44) are called "sub-exponential". We remark also that $\mu(0) \leq \exp(LQ)$ and then

$$L \geq \frac{1}{Q} \ln \mu(0). \tag{46}$$

Moreover, following conditions (42) and (43) we get trivially

$$M := \int_{-L}^{0} \mu(x)dx > \int_{-L}^{0} dx = L. \tag{47}$$

Finally

$$M > L \geq \frac{\mu(0) - 1}{\alpha} \ln \mu(0). \tag{48}$$

The previous inequalities implies that if $\mu(0)$ tends to α then L and M tend to infinity.

Example 1 (exponential profile) *Let μ_0 be such that $1 < \mu_0 < \alpha$ and let define*

$$L := \frac{\mu_0}{\alpha - \mu_0} \ln \mu_0. \tag{49}$$

We can now consider the following profile defined on $(-L, 0)$ by

$$\mu(x) := \exp\left[(x + L)\frac{\alpha - \mu_0}{\mu_0}\right], \tag{50}$$

which satisfies conditions (42)–(44). Then we get

$$M = \frac{\mu_0}{\alpha - \mu_0}(\mu_0 - 1). \tag{51}$$

4.3 Comparison with previous exponential profiles

The exponential optimal viscosity profile in (i.r.) obtained by Carasso and Paşa (1998) depends also on $\mu(0)$, but the condition for the limit value $\mu(0)$ involves L and M. Therefore it was not possible to get some estimations of L and M directly in terms of $\mu(0)$. However, the corresponding maximal growth constant — obtained by the Gerschgorin's localization theorem — and denoted by σ_G in the following was obtained directly in terms of $\mu(0)$:

$$\sigma \le \sigma_G := \frac{2\beta}{3\alpha} \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}, \tag{52}$$

for the following exponential profile:

$$\frac{\mu'(x)}{\mu(x)} \le \frac{2}{3\alpha} \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}. \tag{53}$$

We recall that $\alpha > 1$ and $1 < \mu(0) < \alpha$ and then we get

$$\frac{\sigma_G}{\sigma_{MR}} = \frac{2}{3\sqrt{3}} \frac{\sqrt{\alpha - \mu(0)}}{\alpha\sqrt{\alpha - 1}} < \frac{2}{3\sqrt{3}} \frac{\sqrt{\alpha - \mu(0)}}{\sqrt{\alpha - 1}} < \frac{2}{3\sqrt{3}},$$

i.e.

$$\sigma_G < \frac{2}{3\sqrt{3}} \sigma_{MR} \quad \text{where} \quad \sigma_{MR} = \beta(\alpha - \mu(0)). \tag{54}$$

Therefore the upper bound given by Carasso and Paşa (1998) is smaller than the present upper bound : as we consider a more general class of optimal viscosity profiles — we allow variable coefficients in the problem (23) — this result is coherent. However, we emphasize that (45) and (52) give us $\sigma \to 0$ when $\mu(0) \to \alpha$.

4.4 Comparison between linear and exponential profile

This section is devoted to the comparison between the lengths of the (i.r.) and the corresponding total amounts of polymer obtained with the linear profile (39) and with the exponential profile (50).

Let L_1 (resp. L_2) be the *smallest suitable* length of the (i.r.) obtained with the linear profile (39) (resp. with the exponential profile (50)), and by M_1 (resp. by M_2) the corresponding total amount of polymer. Then, following (38), (41), (49) and (51),

$$L_1 = \frac{\mu_0 - 1}{\alpha - \mu_0}, \qquad L_2 = \frac{\mu_0}{\alpha - \mu_0} \ln \mu_0,$$

$$M_1 = \frac{\mu_0^2 - 1}{2(\alpha - \mu_0)}, \qquad M_2 = \frac{\mu_0}{\alpha - \mu_0}(\mu_0 - 1).$$

We remark that, for all $x > 1$, $1 + x(\ln x - 1) > 0$, and then as $1 < \mu_0 < \alpha$,

$$L_1 \le L_2.$$

Moreover,

$$\frac{M_1}{M_2} = \frac{1}{2}\left(1 + \frac{1}{\mu_0}\right) < 1.$$

Finally, for a *given* μ_0 in $(1, \alpha)$, that is for a *given* maximum growth constant σ_{MR}, the linear profile (39) — with the smallest suitable (i.r.) length — involves a smaller (i.r.) length than the exponential profile (50), and it involves also a smallest total amount of polymer.

4.5 A given improvement of the stability

The aim of this section is to get a *given* improvement of the stability in the (i.r.)-oil interface, by considering above viscosity profiles and by choosing a suitable $\mu(0)$:

Let p be a real number such that $0 < p < 1$ and let define $\mu(0)$ by

$$\mu(0) := \alpha - \frac{2}{\beta 3\sqrt{3}}\, p = \alpha - \frac{2p}{3\sqrt{3}} \frac{\alpha - 1}{\alpha + 1}, \quad 0 < p < 1. \tag{55}$$

As $\alpha > 1 > p > 0$,

$$\frac{2p}{3\sqrt{3}} \frac{1}{\alpha + 1} < 1,$$

and then $\mu(0)$ is such that $1 < \mu(0) < \alpha$. If we consider the linear or exponential profiles defined above, we get

$$\sigma_{MR} = \beta(\alpha - \mu(0)) = \frac{2p}{3\sqrt{3}} = \sigma_{MST} \times p, \tag{56}$$

where $0 < p < 1$, that is we get a *given* improvement of a ratio p of the stability.

Once we defined $\mu(0)$ by (55), we get lower estimates for the (i.r.) length L and for the total amount of polymer M, depending on the choice of the viscosity profile:

- in the linear case we get the estimations (38) and (41).
- in the sub-exponential case, we get either the estimation (48) or the relations (49) and (51).

5 Conclusions

The three regions model (1)–(4) was first studied by Gorell and Homsy (1998).

A theoretical formula for an exponential optimal viscosity profile in (i.r.) has been obtained by Carasso and Paşa (1998), according to the numerical results of Daripa *et al.* (1986, 1987, 1988), Gorell and Homsy (1983), Shan and Sechter (1977) and Uzoigwe *et al.* (1974).

Carasso and Paşa (1998) have obtained the maximal growth constant σ_G given by the relation (52). A finite-difference approximation of the Sturm-Liouville problem (11)–(13)–(19) and the Gerschgorin's localization theorem were used. The value σ_G has been obtained in terms of the limit value of the viscosity on the (i.r.) - oil interface, denoted by $\mu(0)$. The value $\mu(0)$ has been involved in a condition, in terms of the (i.r.) length and the total amount of polymer contained in the (i.r.), denoted respectively by L and M. Therefore it was not possible to estimate L and M directly in terms of $\mu(0)$. The main point of this above paper was the following: if the chosen profile μ satisfy condition (53), then $\mu(0) \to \alpha$ implies that $\sigma_G \to 0$, where α is the ratio between the oil and the water viscosities.

In the present paper we suggest a *new class* of profiles which give the new estimate (36) of the *exact* growth constant σ_R. This estimate is also in terms of the above limit value $\mu(0)$. To prove this result, we use the Rayleigh quotient (30) of the initial Sturm-Liouville problem (11)–(13)–(19) *without any numerical treatment.*

We use the relation (34) in order to obtain *a new class* of theoretical optimal profiles of viscosity in (i.r.), characterized by the condition (35). This condition allows us to obtain the general formula (37) for the maximum growth constant. The condition (35) gives us two particular optimal viscosity profiles: the *linear* profiles defined by (38)–(39) and the *sub-exponential* profiles satisfying (42)–(44).

The linear profiles allows us to consider variable coefficients in the problem (11)–(13)–(19), while the exponential case (50) gives us only constant coefficients, as in Carasso and Paşa (1998).

For both linear and exponential profiles we obtain the same maximal value σ_{MR} of the growth constant, given by (37). We have $\sigma_{MR} \to 0$ when $\mu(0) \to \alpha$, as in Carasso and Paşa (1998). Therefore the formula (37) and a value $\mu(0)$ close enough to α give us an improved stability, compared with the Saffman-Taylor case.

In the sections 4.1 and 4.2 we obtain lower estimations of the (i.r.) length L and of the amount of polymer M, directly in terms of $\mu(0)$, then we generalize the previous result of Carasso and Paşa. We have the relations (38) and (41) for the linear case and the relations (46) and (47) for the sub-exponential case.

In the section 4.3 we give the comparison (54) between the present maximal growth constant σ_{MR} (45) and the previous maximal value σ_G (52). We have $\sigma_G \leq \sigma_{MR}$. This is not surprising, because we consider now a more general case, including variable coefficients in the stability problem (11)–(13)–(19).

In the section 4.4 we prove that the linear case is more favorable: indeed, we need smaller L and M to get the same growth constant σ_R compared to the exponential case.

In the section 4.5 we compute L and M corresponding to *a given improvement of the stability*, compared with the Saffman-Taylor case. For this we define $\mu(0)$ by (55) and then σ_{MR} verifies (56), where σ_{MST} is the dimensionless maximal growth constant of Saffman and Taylor. Once $\mu(0)$ is fixed, some estimations of L and M are given, in terms of α and p — the *given* ratio of the improvement — depending on the considered profile:

- in the linear case we get the estimations (38) and (41).
- in the sub-exponential case, we get either the estimation (48) or the relations (49) and (51).

In conclusion, a new class of optimal viscosity profiles is given in the present paper, by considering the three regions model of Gorell and Homsy (1983). This class contains linear and an exponential profiles. We use the initial stability problem, without any approximation. The formula for the exponential profile is coherent compared to previous theoretical and numerical results. We emphasize that the linear case is more favorable than the exponential one. Finally we compute the (i.r.) length and the amount of polymer corresponding to a given improvement of stability, compared with the Saffman-Taylor case — without (i.r.).

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