INSTITUTION–INDEPENDENT ULTRAPRODUCTS

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Institution-independent Ultraproducts

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Abstract. We generalise the ultraproducts method from classical model theory to an institution-independent (i.e. independent of the details of the actual logic or institution) framework based on a novel very general treatment of the semantics of some important concepts in logic, such as quantification, logical connectives, and ground atomic sentences. Unlike previous categorical abstract model theoretic approaches to ultraproducts, our work makes essential use of concepts central to institution theory, such as signature morphisms and model reducts. The institution-independent fundamental theorem on ultraproducts is presented in a modular manner, different combinations of its various parts giving different results in different logics or institutions. We present applications to institution-independent compactness, axiomatizability, and higher order sentences, and illustrate our concepts and results with examples from four different algebraic specification logics. In the introduction we also discuss the relevance of our institution-independent approach to the model theory of algebraic specification and computing science, but also to classical and abstract model theory.

1. Introduction

The theory of institutions (Goguen and Burstall, 1992) is a categorical abstract model theory which formalises the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them. Institutions constitute the modern level of algebraic specification theory and can be considered its most fundamental mathematical structure. It is already an algebraic specification tradition to have an institution underlying each language or system, in which all language/system constructs and features can be rigorously explained as mathematical entities. Most modern algebraic specification languages follow this tradition, including CASL (Mossakowski, 2001), Maude (Meseguer, 1993), or CafeOBJ (Diaconescu and Futatsugi, 2003). There is an increasing multitude of logics in use as institutions in algebraic specification and computing science. Some of them, such as first order predicate (in many variants), second order, higher order, Horn, type theoretic, equational, modal (in many variants), infinitary logics, etc., are well known or at least familiar to the ordinary logicians, while others such as behavioural or rewriting logics are known and used mostly in computing science.

The original goals of institution theory are to do as much computing science and model theory as possible, independent of what the actual logic may be (Goguen and Burstall, 1992). This mathematical paradigm is often called 'institution-independent' computing science or model theory. While the former goal has been greatly accomplished in the algebraic specification literature, there were only very few and rather isolated attempts towards the latter (Tarlecki, 1986a; Tarlecki, 1986b; Salibra and Scollo, 1996). This situation contrasts with the feeling shared by some researchers that deep concepts and results in model theory can be reached in a significant way via institution theory. This paper can be regarded as a new step towards this goal, part of a coming series of works in institution-independent model theory.

The significance of institution-independent model theory is manifold:

— It provides model-theoretic results and analysis for various logics in a generic way. Only a limited number of model-theoretic properties are usually studied for the logics in use in computing science and algebraic specification, however it is important to have as deep as possible understanding of the model-theoretic properties of the underlying logic because the specification or software engineering properties of the logic depend intimately on the former ones ((Diaconescu et al., 1993) is one of the works that support this argument). We sometimes notice that the failure of some specification properties of a logic is due to the rather subtle wrong definition of some details of the logic. We also notice that often the right definition of a logic can be checked through its model-theoretic properties, otherwise said good model-theoretic properties lead to good specification properties.

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— It exports model-theoretic methods from classical logic to other logics. Classical first-order predicate logic has developed very rich a powerful model-theoretic methods, which exported to an institution-independent framework can become available for the multitude of computing science or algebraic specification logics.

— It provides a new way of doing model theory. While the points we made above have a more application oriented significance, this point has a pure mathematics methodological significance. The institution-independent way of obtaining a model theoretic result, or just viewing a concept, leads to a deeper understanding of *why* a certain model theoretic phenomenon holds. Such top-down understanding is not suffocated by the details of the actual logic, it decompose the model-theoretic phenomenon (in various layers of abstract conditions), and provides a clear picture of its limits.

Although these points are largely valid for any form of abstract model theory, they are especially relevant for the institution-independent abstract model theory. One of the reasons for this is that up to our knowledge, the theory of institutions provide the most complete definition of abstract model theory, the only one including signature morphisms, model reducts, and even mappings (morphisms) between logical systems, as primary concepts. Also, as mentioned above, the current algebraic specification logics and an increasing number of computing science logics are formalised as institutions.

This work exports one of the most important and powerful classical model theory methods, namely the ultraproducts method (C.C.Chang and H.J.Keisler, 1973), to an institution-independent framework. This framework not only clarifies the conditions that are necessary for the development of the ultraproducts method, but also develops a simple but effective institution-independent approach to quantification and logical connectives. In this approach the concept of variable and valuation is presented in a more uniform and much simpler way than in the usual presentations of logic, without the need to distinguish between closed and open formulæ and naturally including higher order variables. We think that this very simple and general institution-independent 'internal logic' is one of the main contributions of this work, reflecting the benefits of the way of doing model theory promoted by the theory of institutions.

Since the categorical definition of the ultraproduct construction, there have been a few abstract model theoretic approaches to ultraproducts, (Andréka and Németi, 1978) being one of the most representative. If we compare it to (Andréka and Németi, 1978), our institution-independent approach to ultraproducts is different in many essential aspects. For example, we work with the given sentences of the institution rather than defining a semantics-oriented concept of sentence and satisfaction as in (Andréka and Németi, 1978) which leads to extremely complex combinatorial definitions and proofs. Besides gains in simplicity and clarity, our approach make the applications much easier and the understanding of the ultraproducts method smoother. This is a direct consequence of the more fundamental difference of using institutions rather than simple categories as the basic framework for the ultraproducts method. By using institutions rather than categories, we are able to make use of essential model theoretic concepts such as signature morphism and model reduct and expansion, which is not possible in other abstract model theoretic approaches (Andréka and Németi, 1978).

1.1. Summary and Contributions of this Work

In the preliminary section, besides briefly revising some terminology, concepts, and notations about filters, categories, and institutions, we introduce the novel institution concept of *representable signature morphism* and explore some of its basic properties. Representable signature morphisms can be regarded as an abstract institution-independent formulation of the concept of first-order signature entities (such as variables or constants).

The next section is devoted to an institution-independent study of logical connectives, quantification (in both existential and universal form), and of basic sentences, which are the simplest sentences matching the model theoretic structure of the institution. We show that in the applications,

all sentences can be obtained by iteration of some of the logical connectives and some quantification over the basic sentences. This decomposition of the satisfaction relation between models and sentences into satisfaction of basic sentences, of logical connectives, and of quantification, is one of the contributions of this section. While the institution-independent concept of logical connectives is obvious and the concept of basic sentence is based upon a simple form of satisfaction via injectivity in the sense of (Andréka and Németi, 1981), the key contribution of this section lies in our approach to quantification. In the applications, the latter includes naturally both first-order and higher-order forms of quantification.

The main section of the paper starts by recalling the categorical definition of reduced products and ultraproducts, then studies the interaction between reduced products and model reducts. The latter plays a crucial role for dealing with quantifiers in our institution-independent approach to the fundamental theorem on ultraproducts.

By following the structure of the internal logic introduced in the previous section, our formulation of the institution-independent fundamental theorem on ultraproducts deconstructs this main result on ultraproducts into parts having individual significance. Depending of the actual institution, these parts can be combined in various different ways and can also be used independently for obtaining weaker preservation properties but for a larger class of sentences. This presentation of the main result has also the benefit of enabling a clear perspective on the semantic limits of an actual logic or institution.

The final section is devoted to some applications, such as institution-independent compactness, elementary axiomatizability, or higher-order quantification. The applications are meant only to illustrate in a rather limited way the institution-independent ultraproducts method, the emphasis of this paper being on the fundamentals. Wider applications is topic for further research based on this work.

The concepts introduced and the results obtained are illustrated with examples from four different institutions: first-order predicate logic (with equality), rewriting logic, partial algebra, and hidden algebra for behavioural logic. All these four logics are very briefly presented in the Appendix, mainly for setting up some notation and terminology. The reader is required to have some familiarity with them or else to study the corresponding literature. Although the examples from these actual institutions serve also as application ground for the results of this paper, they are mainly used for helping the understanding of the concepts introduced by this work.

2. Preliminaries

2.1. FILTERS AND ULTRAFILTERS

In this section we recall the basic concepts and definitions about filters and ultrafilters, restricted to the case of the partial order of the subsets of a subset. Let I be a nonempty set. We denote by 2^I the set of all subsets of I. Recall that a *filter F over I* is defined to be a set $F \subseteq 2^I$ such that

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$, and
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

A filter F is *proper* if and only if F is not 2^I and it is an *ultrafilter* if and only if

$$X \in F \text{ if and only if } (I \setminus X) \notin F$$

for all $X \in 2^I$. Notice that ultrafilters are proper filters.

A set $S \subseteq 2^I$ *has the finite intersection property* if $J_1 \cap J_2 \cap \ldots \cap J_n \neq \emptyset$ for all $J_1, J_2, \ldots, J_n \in S$. The following classical result is known as the 'Ultrafilter Theorem':

THEOREM 1. (C.C.Chang and H.J.Keisler, 1973) If $S \subseteq 2^I$ has the finite intersection property, then there exists an ultrafilter U over I such that $S \subseteq U$. □

2.2. CATEGORIES

This work assumes some familiarity with category theory, and generally uses the same notations and terminology as Mac Lane (MacLane, 1998), except that composition is denoted by ";" and written in the diagrammatic order. The application of functions (functors) to arguments may be written either normally using parentheses, or else in diagrammatic order without parentheses, or, more rarely, by using sub-scripts or super-scripts. The category of sets is denoted as $\mathbb{S}et$, and the category of categories[1] as $\mathbb{C}at$. The opposite of a category $\mathbb{C}$ is denoted by $\mathbb{C}^{op}$. The class of objects of a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$; also the set of arrows in $\mathbb{C}$ having the object a as source and the object b as target is denoted as $\mathbb{C}(a,b)$. The isomorphism of objects in categories is denoted by $\simeq$. A *diagram D* in a category $\mathbb{C}$ is just a functor $J \xrightarrow{D} \mathbb{C}$ when J is a small category.

For any object a, the comma category $a/\mathbb{C}$ has

- arrows $f \in \mathbb{C}(a,b)$, as objects, and
- arrows $h \in \mathbb{C}(b,b')$ such that $f;h = f'$, as arrows between $f \in \mathbb{C}(a,b)$ and $f' \in \mathbb{C}(a,b')$.

2.2.1. *Finiteness*

A category J is *directed* if to any two objects i and j there exist arrows $i \to k \leftarrow j$. A limit (colimit) of a functor $J \xrightarrow{D} \mathbb{C}$ is *directed* if the category J is directed.

There are many approaches on finiteness concepts at the level of category theory, most of them very similar; our categorical treatment to finiteness, has already been used in a serie of papers on the so-called 'category-based equational logic' (Diaconescu, 1994; Diaconescu, 1995; Diaconescu, 2000). We say that an object a in a category $\mathbb{C}$ is *quasi-finite* if and only if for any arrow $a \xrightarrow{f} d$ to the vertex of a colimiting co-cone $\mu\colon D \Rightarrow d$ of a directed diagram $J \xrightarrow{D} \mathbb{C}$ there exists $i \in |J|$ and an arrow $a \xrightarrow{f_i} D(i)$ such that $f = f_i; \mu_i$.

Notice that our concept of finiteness is slightly more general than that defined in (Herrlich and Strecker, 1973) by the fact that $\mathbb{C}(a,-)\colon \mathbb{C} \to \mathbb{S}et$ preserves small directed colimits. Finite sets, finitely presented algebras or relational structures, etc. are all examples of quasi-finite objects in our sense.

2.3. INSTITUTIONS

In this section besides briefly reviewing some of the basic concepts of institution theory, we also introduce some novel concepts necessary for this work. Besides the seminal paper (Goguen and Burstall, 1992), (Diaconescu et al., 1993) contains many results about institutions with direct application to modularisation in algebraic specification languages.

From a logic perspective, institutions are much more abstract than Tarski's model theory, and also have another basic ingredient, namely signatures and the possibility of translating sentences and models across signature morphisms. A special case of this translation is familiar in first-order model theory: if $\Sigma \to \Sigma'$ is an inclusion of first-order signatures[2] and M is a Σ'-model, then we can form the *reduct* of M to Σ, denoted $M\!\upharpoonright_\Sigma$. Similarly, if e is a Σ-sentence, we can always view it as a Σ'-sentence (but there is no standard notation for this).

Institutions formalize the concept of 'logic' from a categorical abstract model-theoretic perspective. The key axiom, called the *satisfaction condition*, says that *truth is invariant under change of notation*, which is surely a very basic intuition for classical logic.

DEFINITION 1. An *institution* $\mathfrak{I} = (\mathbb{S}ign, Sen, \text{MOD}, \models)$ consists of

1. a category $\mathbb{S}ign$, whose objects are called *signatures*,

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[1] We steer clear of any foundational problem related to the "category of all categories"; several solutions can be found in the literature, see, for example (MacLane, 1998).

[2] Called "languages" in (C.C.Chang and H.J.Keisler, 1973).

2. a functor $Sen\colon \mathbb{S}ign \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,

3. a functor $\mathrm{MOD}\colon \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-*model homomorphisms*, and

4. a relation $\models_\Sigma \subseteq |\mathrm{MOD}(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign|$, called Σ-*satisfaction*,

such that for each morphism $\varphi\colon \Sigma \to \Sigma'$ in $\mathbb{S}ign$, the *satisfaction condition*

$$M' \models_{\Sigma'} Sen(\varphi)(e) \ \text{ iff } \ \mathrm{MOD}(\varphi)(M') \models_\Sigma e$$

holds for each $M' \in |\mathrm{MOD}(\Sigma')|$ and $e \in Sen(\Sigma)$. We may denote the reduct functor $\mathrm{MOD}(\varphi)$ by $\_\!\restriction_\varphi$ and the sentence translation $Sen(\varphi)$ by $\varphi(\_)$. Also, we will sometimes say that the signature morphism φ has a certain property 'P' if $\mathrm{MOD}(\varphi)$ has the property 'P'. When $M = M'\!\restriction_\varphi$, we will say that M' is an *expansion of M along* φ. $\square$

DEFINITION 2. Let Σ be a signature in an institution $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$.

  – For each set of Σ-sentences E, let $E^* = \{M \in \mathrm{MOD}(\Sigma) \mid M \models_\Sigma e \text{ for each } e \in E\}$, and
  – For each class $\mathcal{M}$ of Σ-models, let $\mathcal{M}^* = \{e \in Sen(\Sigma) \mid M \models_\Sigma e \text{ for each } M \in \mathcal{M}\}$.

If E is a set of sentences and e is a single sentence, then $e \in E^{**}$ is denoted by $E \models e$.

Two models M and M' of the same signature are *elementarily equivalent* (denoted as $M \equiv M'$) if they satisfy the same set of sentences, i.e. $\{M\}^* = \{M'\}^*$.

Two sentences e and e' of the same signature are *semantically equivalent* (denoted as $e \equiv e'$) if they are satisfied by the same class of models, i.e., $\{e\}^* = \{e'\}^*$. $\square$

DEFINITION 3. In any institution $\mathfrak{I}$, a class $\mathcal{K}$ of models for a signature is *elementary* if it is closed, i.e., $\mathcal{K}^{**} = \mathcal{K}$. $\square$

REMARK 1. Each elementary class of models is closed under elementary equivalence. $\square$

DEFINITION 4. Let $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ be an institution. (Σ, E) is a *theory* when Σ is a signature and E is *closed* set of Σ-sentences, i.e., $E = E^{**}$.

A theory E is *presented by* E_0 if $E_0 \models e$ for each $e \in E$, and is *finitely presented* if there exists a finite E_0 which presents E.
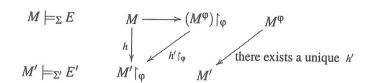
A *theory morphism* $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Let $\mathbb{T}h$ denote the category of all theories in $\mathfrak{I}$. $\square$

REMARK 2. For any institution $\mathfrak{I}$, the model functor MOD extends from the category of its signatures $\mathbb{S}ign$ to the category of its theories $\mathbb{T}h(\mathfrak{I})$, by mapping a theory (Σ, E) to the full subcategory $\mathrm{MOD}(\Sigma, E)$ of $\mathrm{MOD}(\Sigma)$ formed by the Σ-models which satisfy E. $\square$

DEFINITION 5. (Diaconescu et al., 1993) A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is *conservative* if and only if each (Σ, E)-model has an expansion to a (Σ', E')-model, i.e., for each Σ-model M satisfying E, there exists a Σ'-model M' satisfying E' such that $M'\!\restriction_\varphi = M$. $\square$

An important particular case for this work is that of *conservative* signature morphisms. In classical model theory, a signature morphism $\varphi\colon \Sigma \to \Sigma'$ is conservative if and only if φ does not add new operations of sorts that are 'empty' (i.e., without constants) in Σ. Consequently, if Σ has only 'non-empty' sorts, then each signature morphism $\varphi\colon \Sigma \to \Sigma'$ is conservative.

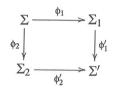DEFINITION 6. A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is *liberal* if and only if the reduct functor $\_\!\restriction_\varphi\colon \mathrm{MOD}(\Sigma', E') \to \mathrm{MOD}(\Sigma, E)$ has a left-adjoint $(\_)^\varphi$.

$$M \models_\Sigma E \qquad\qquad M \longrightarrow (M^\varphi)\restriction_\varphi \qquad M^\varphi$$

$$h\downarrow \qquad \swarrow h'\restriction_\varphi \qquad \swarrow \text{there exists a unique } h'$$

$$M' \models_{\Sigma'} E' \qquad\qquad M'\restriction_\varphi \qquad M'$$

The institution $\mathfrak{I}$ is *liberal* if and only if each theory morphism is liberal. $\square$

DEFINITION 7. An institution $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ is *exact* if and only if the model functor $\mathrm{MOD}\colon \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ preserves finite limits. $\mathfrak{I}$ is *semi-exact* if and only if MOD preserves only pullbacks. $\square$

FACT 1. Consider a semi-exact institution $\mathfrak{I}$, a pushout of signatures

$$\begin{array}{ccc}
\Sigma & \xrightarrow{\phi_1} & \Sigma_1 \\
{\scriptstyle\phi_2}\downarrow & & \downarrow{\scriptstyle\phi_1'} \\
\Sigma_2 & \xrightarrow[\phi_2']{} & \Sigma'
\end{array}$$

and two models, a Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1\restriction_{\phi_1} = M_2\restriction_{\phi_2}$. Then by the semi-exactness, there exists an unique Σ'-model M' such that $M'\restriction_{\phi_1'} = M_1$ and $M'\restriction_{\phi_2'} = M_2$. We call this model the *amalgamation* of M_1 and M_2 and denote it by $M_1 \otimes M_2$.

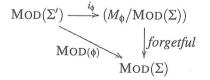A similar amalgamation concept can also be defined for model homomorphisms. $\square$

Exactness properties for institutions formalise the possibility of amalgamating models of different signatures when they are consistent on some kind of 'intersection' of the signatures (formalised as a pushout square).

2.3.1. *Representable signature morphisms*

The topic of this subsection represents a novel categorical generalisation of the concept of (first-order) 'variables' from conventional logic to the framework of institutions.

Let us have a brief look at the conventional concept of variable in general algebra or first-order predicate logic (Appendix A). Given a signature Σ and a set of variables X for Σ, we may consider the extended signature $\Sigma \cup X$ by regarding the variables as constants. Then each $\Sigma \cup X$-model is just a Σ-model M plus an interpretation of the elements of X into M. But the interpretations of X into M are in canonical bijection with the Σ-model homomorphisms $T_\Sigma(X) \to M$, where $T_\Sigma(X)$ is the free Σ-algebra over X. Therefore, a $\Sigma \cup X$-model is the same with a Σ-model homomorphism $T_\Sigma(X) \to M$ with M a Σ-model. This can be regarded as a categorical property of the signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$, suggesting the following institution-independent definition:

DEFINITION 8. Let $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ be an institution. A signature morphism $\phi\colon \Sigma \to \Sigma'$ is *representable* if and only if there exists a Σ-model M_ϕ (called the *representation of* ϕ) and an isomorphism i_ϕ of categories such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{MOD}(\Sigma') & \xrightarrow{\;i_\phi\;} & (M_\phi/\mathrm{MOD}(\Sigma)) \\
& {\scriptstyle \mathrm{MOD}(\phi)}\searrow & \downarrow{\scriptstyle forgetful} \\
& & \mathrm{MOD}(\Sigma)
\end{array}$$

If the representation M_ϕ is quasi-finite in $\mathrm{MOD}(\Sigma)$, we say that ϕ is *finitary representable*. $\square$

Informally, this definition says that each Σ'-model is just a Σ-model plus an interpretation of the representation model into the Σ-model.

EXAMPLE 1. In the institution of first-order predicate logic (Appendix A) each extension of signatures $\phi\colon (\Sigma, \Pi) \to (\Sigma', \Pi)$ only adding constants to Σ is representable by the free (Σ, Π)-model over the added constants. If ϕ adds a finite number of constants, then ϕ is finitary representable.

Similarly, signature morphisms only adding constants are also representable in rewriting logic (Appendix B), partial algebra (Appendix C), and hidden algebra (Appendix D). □

Although the next result is not used anywhere in this paper, it shows some basic and expected composability properties of representable signature morphisms. The informal meaning of the former three items of Proposition 1 is that the 'union of variables' exists, is associative, and has the 'empty set' as identity, while the meaning of the latter fourth item is the this 'union' is commutative too.

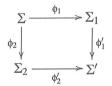PROPOSITION 1. In any institution

1. the composition of representable signature morphisms is representable,
2. the identity signature morphism is representable if and only if the corresponding signature has initial models,
3. if each signature of the institution has initial models then the representable signature morphisms form a subcategory of the category of signatures, and
4. if the institution is semi-exact and its categories of models have finite coproducts, then the subcategory of representable signature morphisms creates pushouts.

□

Proof. 1. Consider the following representable signature morphisms $\Sigma \xrightarrow{\phi} \Sigma' \xrightarrow{\phi'} \Sigma''$. We show that $\phi; \phi'$ is represented by $M_{\phi'}\!\restriction_\phi$ where $M_{\phi'}$ is the representation of ϕ'.

For each Σ''-model M'', we define $i_{\phi;\phi'}(M'') = i_{\phi'}(M'')\!\restriction_\phi$.

On the other hand, given any $m\colon (M_{\phi'})\!\restriction_\phi \to M$, we define $i_{\phi;\phi'}^{-1}(m) = i_{\phi'}^{-1}(i_\phi^{-1}(m))$ (notice that $m\colon h \to h; m$ in $M_\phi/\mathrm{MOD}(\Sigma)$ where $h = i_\phi(M_{\phi'})$).

2. is immediate and 3. follows immediately from 1. and 2.

4. Consider the following pushout of signature morphisms

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{\;\phi_1\;} & \Sigma_1 \\
{\scriptstyle \phi_2}\big\downarrow & & \big\downarrow{\scriptstyle \phi_1'} \\
\Sigma_2 & \xrightarrow[\;\phi_2'\;]{} & \Sigma'
\end{array}
$$

We need to prove that ϕ_1' and ϕ_2' are representable when ϕ_1 and ϕ_2 are representable.

Let the Σ-model M_{ϕ_1} represent ϕ_1 and the Σ-model M_{ϕ_2} represent ϕ_2. Let $M_{\phi_1} \xrightarrow{m_1} M_{\phi_1} + M_{\phi_2} \xleftarrow{m_2} M_{\phi_2}$ be the coproduct of M_{ϕ_1} and M_{ϕ_2}. Then the conclusion follows from the fact the following diagram of forgetful functors is a pullback (we leave this simple calculation as exercise to the reader):

$$
\begin{array}{ccc}
\mathrm{MOD}(\Sigma) & \longleftarrow & M_{\phi_1}/\mathrm{MOD}(\Sigma) \\
\big\uparrow & & \big\uparrow \\
M_{\phi_2}/\mathrm{MOD}(\Sigma) & \longleftarrow\; m_2/(M_{\phi_2}/\mathrm{MOD}(\Sigma)) \simeq (M_{\phi_1} + M_{\phi_2})/\mathrm{MOD}(\Sigma) \simeq m_1/(M_{\phi_1}/\mathrm{MOD}(\Sigma))
\end{array}
$$

3. Internal Logic

In this section we define a method of describing the sentences supporting our results on institution-independent ultraproducts. If we fix the institution, these are sentences of this institution rather

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than being new sentences constructed on top of the sentences of the original institution via some institution-independent building operations such as logical connectives and some form of quantification (such as (Tarlecki, 1986b) does for the sentences defining the quasi-varieties of models in arbitrary institutions). This description of the sentences supporting our results on institution-independent ultraproducts will allow us to notice easily that, in most of the institutions used in algebraic specification or computing science theory, these sentences are in fact *all* the sentences of the institution; this gives a wide range of applications to the results of this paper.

At the basic level we have the *basic* sentences which are the basic constituents for the sentences supporting our results on institution-independent ultraproducts. The complex level is obtained from the basic level by iterations of *logical connectives* and an abstract form of *quantification* (both universally and existentially). Although this description might have a strong first-order flavour, it goes well beyond conventional first order logic because of two reasons. On the one hand, this is done in an *arbitrary* institution, almost without any technical restrictions. On the other hand, the level of generality of our concept of quantification is much higher than the conventional quantification with first-order variables, particular conventional cases including second order quantification, for example.

3.1. BASIC SENTENCES

In the actual institutions, the basic sentences are the simplest sentences matching the structure of the models of the institution, i.e. which are preserved by the model homomorphisms, and they usually constitute the bricks from which the complex sentences are constructed by using logical connectives and quantification.

Notice that the satisfaction of basic sentences is a particular case of 'injectivity' satisfaction in the sense of (Andréka and Németi, 1981).

DEFINITION 9. Given a signature Σ, a Σ-sentence e is *basic* if there exists a Σ-model M_e such that for each Σ-model M, $M \models_\Sigma e$ if and only if there exists a model homomorphism $M_e \to M$.

We say that an basic sentence e is *finitary* if the model M_e is quasi-finite in the category $\mathrm{MOD}(\Sigma)$ of Σ-models. $\square$

REMARK 3. In any institution basic sentences are preserved by model homomorphisms, i.e., $N \models e$ whenever $M \models e$ and there exists a model homomorphism $h\colon M \to N$. $\square$

EXAMPLE 2. *First-order predicate logic.*
In the case of first-order predicate logic with equality (Appendix A), the ground atoms are finitary basic. Recall that a ground atom is either an equality between ground terms or a relation (predicate) with ground terms as arguments.

If we consider a ground equation $(\forall\emptyset)\, t = t'$ for an algebraic signature Σ, then let T_Σ/E be the (quotient) initial Σ-algebra satisfying $(\forall\emptyset)\, t = t'$. In this case E is the congruence generated by the pair (t, t'). Then, an algebra A satisfies $(\forall\emptyset)\, t = t'$ if and only if there exists a homomorphism $T_\Sigma/E \to A$.

If we consider a ground atomic relation $\pi(t_1 \ldots t_n)$ for a first-order predicate logic signature (Σ, Π), where $t_1, \ldots, t_n$ is a list of Σ-terms, then we consider the (Σ, Π)-model T such that as an algebra, T is the initial term Σ-algebra T_Σ, and which interprets all relation symbols as the empty relation except $T_\pi = \{(t_1, \ldots, t_n)\}$. Then $M \models \pi(t_1 \ldots t_n)$ if and only if there exists a homomorphism $T \to M$, for each (Σ, Π)-model M.

Finite conjunctions of ground atoms would also be finitary basic, but in the case of infinitary logic, infinite conjunctions of ground atoms would be only basic. $\square$

EXAMPLE 3. *Rewriting logic.*
In the case of the rewriting logic (see Appendix B), the (atomic) ground equations and the ground transitions are finitary basic.

For example, given a ground transition $(\forall\emptyset)\, t \Rightarrow t'$ for a signature Σ, let $(T_\Sigma, \leq)$ be the preorder model where T_Σ is the initial term Σ-algebra, and $\leq$ is the preorder compatible with the Σ-operations generated by the pair (t, t'). One can notice easily that for each preorder model M, $M \models_\Sigma (\forall\emptyset)\, t \Rightarrow t'$ if and only if there exists a preorder model homomorphism $(T_\Sigma, \leq) \to M$. $\square$

EXAMPLE 4. *Partial algebra.*
In the case of partial algebra (Appendix C), we show that the strong ground equations are basic. Let $(\forall\emptyset)\, t \overset{S}{=} t'$ be a strong ground (Σ,Δ)-equation for a partial algebraic signature (Σ,Δ) with S the set of sorts. By Proposition 3, consider the initial total $\Sigma\cup\Delta\cup\perp$-algebra $T_{\Sigma\cup\Delta\cup\perp,\Gamma\cup\{t=t'\}}$ for the theory $\Gamma\cup\{(\forall\emptyset)\, t = t'\}$. Its corresponding partial (Σ,Δ)-algebra by Proposition 3 is (the total algebra) $T_{\Sigma\cup\Delta,E}$, where E is the $\Sigma\cup\Delta$-congruence generated by (t,t') on the initial $\Sigma\cup\Delta$-algebra $T_{\Sigma\cup\Delta}$.

Notice that, by Proposition 3, for each partial algebra A, $A \models (\forall\emptyset)\, t \overset{S}{=} t'$ if and only if $\overline{A} \models (\forall\emptyset)\, t = t'$ if and only if there exists a total $(\Sigma\cup\Delta\cup\perp)$-homomorphism $\overline{h}\colon T_{\Sigma\cup\Delta\cup\perp,\Gamma\cup\{t=t'\}} \to \overline{A}$ if and only if there exists a partial (Σ,Δ)-homomorphism $h\colon T_{\Sigma\cup\Delta,E} \to A$.

Finally, a strong ground equation is finitary basic if the signature if finite. The reason for this is that in this case Γ is finite and therefore $T_{\Sigma\cup\Delta\cup\perp,\Gamma\cup\{t=t'\}}$ is finitely presented, which means that $T_{\Sigma\cup\Delta,E}$ is finitely presented and thus quasi-finite in the category of partial (Σ,Δ)-algebras. $\square$

3.2. LOGICAL CONNECTIVES

The institution-independent approach to logical connectives is straightforward. We only give here the definitions for negation and conjunction because all other logical connectives can be generated from these.

DEFINITION 10. Given a signature Σ,

for each Σ-sentence e, let $\neg e$ be a new sentence, called the *negation* of e, and
for all Σ-sentences e and e', let $e \wedge e'$ be a new sentence, called the *conjunction* of e and e'.

Given a Σ-model M, we extend the satisfaction to negations and conjunctions by

$M \models_\Sigma \neg e$ if and only if $M \not\models_\Sigma e$, and
$M \models_\Sigma e \wedge e'$ if and only if ($M \models_\Sigma e$ and $M \models_\Sigma e'$).

These naturally determines extensions of the original institution $\mathfrak{I}$ to its

negation closure $\mathfrak{I}^\neg = (\mathbb{S}ign, Sen^\neg, \mathrm{MOD}, \models)$, and
conjunction closure $\mathfrak{I}^\wedge = (\mathbb{S}ign, Sen^\wedge, \mathrm{MOD}, \models)$

where

$Sen^\neg(\Sigma) = Sen(\Sigma) \cup \{\neg e \mid e \in Sen(\Sigma)\}$ and $\varphi(\neg e) = \neg\varphi(e)$, and
$Sen^\wedge(\Sigma) = Sen(\Sigma) \cup \{e \wedge e' \mid e, e' \in Sen(\Sigma)\}$ and $\varphi(e \wedge e') = \varphi(e) \wedge \varphi(e')$.

for each signature Σ and each signature morphism $\varphi\colon \Sigma \to \Sigma'$.
For each set $S \subseteq Sen(\Sigma)$ let

$\neg S = \{e \in Sen(\Sigma) \mid e \equiv \neg e' \text{ for some } e' \in S\}$ be its *negation closure*, and
$\wedge S = \{e \in Sen(\Sigma) \mid e \equiv e' \wedge e'' \text{ for some } e', e'' \in S\}$ be its *conjunction closure*.

The institution $\mathfrak{I}$

admits negation if and only if for each $e \in Sen(\Sigma)$ there exists $e' \in Sen(\Sigma)$ such that $e' \equiv \neg e$, and

admits conjunction if and only if for each $e', e'' \in Sen(\Sigma)$ there exists $e \in Sen(\Sigma)$ such that $e \equiv e' \wedge e''$

for each signature Σ. $\square$

The institution-independent semantics of other logical connectives, such as disjunction, implication, equivalence, etc. can be defined directly in a same way. This can also be extended to infinitary versions of the logical connectives, such as infinitary conjunctions and infinitary disjunctions.

3.3. QUANTIFIERS

DEFINITION 11. Given a signature morphism $\Sigma \xrightarrow{\chi} \Sigma'$, a Σ-sentence e is *universal/existential* χ-*quantification* of a Σ'-sentence e' if and only if for each Σ-model M

$$M \models_\Sigma e \text{ if and only if } (M' \models_{\Sigma'} e' \text{ for all/some } \Sigma'\text{-models } M' \text{ with } M'{\restriction}_\chi = M)$$

$\square$

This is a very abstract and general concept of quantification, which, for example, in the particular case of classical model theory includes the second order quantification. Notice that this internalisation of the quantification does not use the ordinary concepts of open formulæ and valuations (of unbounded variables), but rather considers the "variables" as part of the signature and treats the "valuations" as model expansions along the signature extension defined by the addition of the "variables" to the signature. This is exactly what happens in applications because each valuation of variables into a model can be regarded as an expansion of the model to the signature extended with the variables. Otherwise said, for quantification we need only to mark a part of the signature over which the quantification is done. Although this way of thinking about variables and quantification might be quite alien in the usual presentations of classical logic, it is actually quite common in algebraic specification logics where it underlies the so-called 'theorem of constants' (Goguen, 2002).

DEFINITION 12. Let $S \subseteq Sen(\Sigma)$ be a set of Σ-sentences and let $\Sigma \xrightarrow{\chi} \Sigma'$ be a signature morphism. Then the *universal/existential* χ-*quantification closure of S* is defined as

$$\{e \in Sen(\Sigma) \mid e \text{ is universal/existential } \chi\text{-quantification of } e' \text{ for some } e' \in Sen(\Sigma')\}.$$

and is denoted by $\forall \chi.S$, respectively $\exists \chi.S$. $\square$

EXAMPLE 5. Given a signature (Σ, Π) in first order predicate logic, the ordinary first order quantification by a set X of variables is the same with the χ-quantification, where $\chi \colon (\Sigma, \Pi) \hookrightarrow (\Sigma \cup X, \Pi)$. Notice that in this case χ is representable (Definition 8), and is finitary representable when X is finite.

The cases when $\chi \colon (\Sigma, \Pi) \hookrightarrow (\Sigma', \Pi')$ is any signature inclusion correspond to the second order quantification by the operations $\Sigma' \setminus \Sigma$ and predicates (relations) $\Pi' \setminus \Pi$.

Quantifications higher than second order can be modelled by Definition 11 provided that the classical concept of first order predicate logic signature is extended in order to accommodate symbols denoting higher order structures.

While quantification in rewriting logic (Appendix B) and hidden algebra (Appendix D) are modelled by Definition 11 in the same way as in first-order predicate logic, some special notice is needed for the case of partial algebra.

Given a partial algebra signature (Σ, Δ) (Appendix C), the ordinary first order quantification by a set X of variables is the same with the χ-quantification, where $\chi \colon (\Sigma, \Delta) \hookrightarrow (\Sigma \cup X, \Delta)$ is the signature inclusion. Notice that the variables X are treated as total rather than partial constant symbols because the valuations of the variables in partial algebra are total. This is possible due to having explicit declarations for total operations as part of the partial algebra signatures. $\square$
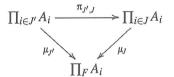
4. Ultraproducts in Institutions

4.1. Categorical reduced products

The reduced product construction from classical model theory (see Chapter 4 of (C.C.Chang and H.J.Keisler, 1973)) has been probably defined categorically for the first time in (Matthiessen, 1978) and has been used in some abstract model theoretic works, such as (Andréka and Németi, 1978). The equivalence between the category theoretic and the set theoretic definitions of the reduced products is shown in (Grätzer, 1979). Let us recall here the category theoretic definition of the reduced products:

DEFINITION 13. Let $\mathbb{C}$ be a category with small products and small directed colimits. Consider a family of objects $\{A_i\}_{i \in I}$. Each filter F over the set of indices I determines a functor $F \xrightarrow{A_F} \mathbb{C}$ such that $A_F(J \subset J') = \prod_{i \in J'} A_i \xrightarrow{\pi_{J',J}} \prod_{i \in J}$ for each $J, J' \in F$ with $J \subset J'$, and with $\pi_{J',J}$ being the canonical projection.

Then the *reduced product of $\{A_i\}_{i \in I}$ modulo F* is the colimit $\mu: A_F \Rightarrow \prod_F A_i$ of the functor A_F.

$$\prod_{i \in J'} A_i \xrightarrow{\pi_{J',J}} \prod_{i \in J} A_i$$
$$\mu_{J'} \searrow \qquad \swarrow \mu_J$$
$$\prod_F A_i$$

If F is ultrafilter then the reduced product modulo F is called an *ultraproduct*. $\square$

REMARK 4. Notice that F is a small directed category because as a filter is a directed poset, hence under the assumptions of Definition 13 the reduced products always exist. $\square$

EXAMPLE 6. For each signature in first-order predicate logic (Appendix A), rewriting logic (Appendix B), partial algebra (Appendix C), or hidden algebra (Appendix D), its category of models has reduced products.

In all these cases, the forgetful functor from the category of models to the category of many-sorted sets mapping each model to its underlying carrier creates small products and small directed colimits. While this observation is obvious in the case of the products, in the case of the directed colimits it is a direct consequence of the finiteness of the arities of the operation or relation symbols of the signature (see Proposition 2, Chapter IX of (MacLane, 1998) for the case of [varieties of] many-sorted algebra).

Notice also that in the case of the partial algebras, this argument is obtained via Proposition 3. $\square$

DEFINITION 14. Let $G: \mathbb{C}' \to \mathbb{C}$ be a functor and F be a filter over a set I. Then

- G *preserves the reduced product* $\mu': B_F \Rightarrow \prod_F B_i$ (for $\{B_i\}_{i \in I}$ a family of objects in $\mathbb{C}'$), if $\mu'G: B_F; G \Rightarrow \prod_F G(B_i)$ is also a reduced product in $\mathbb{C}$ of $\{G(B_i)\}_{i \in I}$, and

- G *lifts the reduced product* $\mu: A_F \Rightarrow \prod_F A_i$ (for $\{A_i\}_{i \in I}$ a family of objects in $\mathbb{C}$), if for each object B in $\mathbb{C}'$ such that $G(B) = \prod_F A_i$, there exists $\{B_i\}_{i \in I}$ a family of objects in $\mathbb{C}'$ such that $G(B_i) = A_i$ for each $i \in I$ and there exists a reduced product $\mu': B_F \Rightarrow B$ such that $\mu'G = \mu$.

Given a class $\mathcal{F}$ of filters, we say that functor *preserves/lifts $\mathcal{F}$-reduced products* if it preserves/lifts all reduced products modulo F for each filter $F \in \mathcal{F}$. $\square$

In general, in the applications, the preservation of reduced products is an easy property that holds naturally without other conditions. On the other hand, the lifting of the reduced products holds only for a restricted class of signature morphisms.

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FACT 2. Any functor preserving small products and small directed colimits preserves reduced products. □

EXAMPLE 7. Any signature morphism in first-order predicate logic (Appendix A), rewriting logic (Appendix B), partial algebra (Appendix C), and hidden algebra (Appendix D), preserve the reduced products of models.
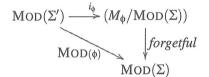
The model products are preserved by the signature morphisms because in all these institutions the signature morphisms are liberal and all limits are preserved by right-adjoint functors (MacLane, 1998).

The directed colimits of models are created, and thus preserved, by the signature morphisms by the generalisation of the argument that the forgetful functors from the categories of models to the categories of many-sorted sets mapping each model to its underlying carrier creates small directed colimits (see Example 6). □

As mentioned above, by contrast to the preservation of the reduced products, in general, only a restricted class of signature morphisms lift the reduced products in the applications. The following result gives a general class of signature morphisms that lift the reduced products in any institution.
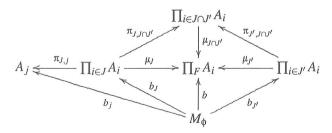
PROPOSITION 2. In any institution the finitary representable conservative signature morphisms lift all reduced products. □

Proof. Consider a finitary representable conservative signature morphism $\phi: \Sigma \to \Sigma'$. Let M_ϕ be the Σ-model representing ϕ. Recall that there exists an canonical isomorphism i_ϕ of categories such that the following diagram commutes:

$$\text{MOD}(\Sigma') \xrightarrow{i_\phi} (M_\phi/\text{MOD}(\Sigma))$$

with $\text{MOD}(\phi)$ and $forgetful$ to $\text{MOD}(\Sigma)$.

Consider a family $\{A_i\}_{i \in I}$ of Σ-models and a filter F over I. Let $\mu: A_F \Rightarrow \prod_F A_i$ be the corresponding reduced product and let B be a Σ'-model such that $B{\restriction_\phi} = \prod_F A_i$.

Let $i_\phi(B) = b: M_\phi \to \prod_F A_i$. Because M_ϕ is quasi-finite, there exists $J \in F$ and $b_J: M_\phi \to \prod_{i \in J} A_i$ such that $b_J; \mu_J = b$. For each $j \in J$, let $b_j = b_J; \pi_{J,j}$, where $\pi_{J,j}: \prod_{i \in J} A_i \to A_j$ is the projection from the product to its j-th component. Then we define $B_j = i_\phi^{-1}(b_j)$ for each $j \in J$ and, because ϕ is conservative, let B_i be an arbitrary expansion of A_i to a Σ'-model if $i \notin J$. Let $b_i = i_\phi(B_i)$ for each $i \in I$, and for each $J' \in F$ let (by the universal property of the product) $b_{J'}: M_\phi \to \prod_{i \in J'} A_i$ be the unique arrow such that $b_{J'}; \pi_{J',i} = b_i$ for each $i \in J'$. If we show that μ is a colimiting co-cone defining the reduced product $b = \prod_F b_i$ in $M_\phi/\text{MOD}(\Sigma)$, then this proposition is proved because of the canonical isomorphism between $M_\phi/\text{MOD}(\Sigma)$ and $\text{MOD}(\Sigma')$.

$$
\begin{array}{c}
\prod_{i \in J \cap J'} A_i \\
\pi_{J, J \cap J'} \nearrow \quad \downarrow \mu_{J \cap J'} \quad \nwarrow \pi_{J', J \cap J'} \\
A_j \xleftarrow{\pi_{J,j}} \prod_{i \in J} A_i \xrightarrow{\mu_J} \prod_F A_i \xleftarrow{\mu_{J'}} \prod_{i \in J'} A_i \\
b_j \nwarrow \quad \uparrow b_J \quad \uparrow b \quad \nearrow b_{J'} \\
M_\phi
\end{array}
$$

We first show that $\mu: b_F \Rightarrow b$ is a co-cone, where $b_F: F \to M_\phi/\text{MOD}(\Sigma)$ is the functor with $b_F(J') = b_{J'}$ for each $J' \in F$ and with $b_F(J' \subseteq J'') = \pi_{J'',J'}$. Consider an arbitrary $J' \in F$. Then $b_{J'}; \mu_{J'} = b_{J'}; \pi_{J',J \cap J'}; \mu_{J \cap J'} = b_{J \cap J'}; \mu_{J \cap J'} = b_J; \pi_{J,J \cap J'}; \mu_{J \cap J'} = b_J; \mu_J = b$. (Notice that here we have used the crucial fact that $J \cap J' \in F$ because both $J, J' \in F$ and each filter is closed under finite intersections.)

Now consider another co-cone $\nu: b_F \Rightarrow b'$ with $b': M_\phi \to A'$. By the forgetful functor $M_\phi/\text{MOD}(\Sigma) \to \text{MOD}(\Sigma)$, $\nu: A_F \Rightarrow A'$ is a co-cone, therefore by the colimit property in $\text{MOD}(\Sigma)$,

there exists an unique $h\colon \prod_F A_i \to A'$ such that $\mu; h = \nu$. All we still have to prove is that $h\colon b \to b'$ in $M_\phi/\mathrm{MOD}(\Sigma)$. But we have that $b; h = b_J; \mu_J; h = b_J; \nu_J = b'$.

EXAMPLE 8. Any signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$ in first-order predicate logic (Appendix A), rewriting logic (Appendix B), partial algebra (Appendix C), and hidden algebra (Appendix D) lifts the reduced products of models where X is a finite set of arbitrary constants and when X does not introduce a constant on a sort which does not have constants in Σ.

Such signature inclusions are finitary representable by the free Σ-model over X, and the fact that X does not introduce a constant on a sort which does not have constants in Σ guarantees that $\Sigma \hookrightarrow \Sigma \cup X$ is conservative too. $\square$

4.2. THE FUNDAMENTAL THEOREM

For this section we assume a fixed institution $\mathfrak{I} = (\mathbb{S}ign, Sen, \mathrm{MOD}, \models)$ such that all its categories of models have small products and small directed colimits.

DEFINITION 15. Let $\mathcal{F}$ be a class of filters. For each signature Σ, a Σ-sentence e is

- *preserved by $\mathcal{F}$-reduced factors* if $\prod_F A_i \models_\Sigma e$ implies $\{i \in I \mid A_i \models_\Sigma e\} \in F$,
- *preserved by $\mathcal{F}$-reduced products* if $\{i \in I \mid A_i \models_\Sigma e\} \in F$ implies $\prod_F A_i \models_\Sigma e$, and

for each filter $F \in \mathcal{F}$ over a set I and for each family $\{A_i\}_{i \in I}$ of Σ-models.

A sentence is a *Łoś-sentence* when is preserved by all ultrafactors and all ultraproducts. $\square$

The following theorem is the fundamental result of this paper.

THEOREM 2. For any class $\mathcal{F}$ of filters,

1. The basic sentences are preserved by all reduced products.
2. The finitary basic sentences are preserved by all reduced products and all reduced factors.
3. The sentences preserved by $\mathcal{F}$-reduced products are closed under existential χ-quantification, when χ is conservative and preserves $\mathcal{F}$-reduced products.
4. The sentences preserved by $\mathcal{F}$-reduced factors are closed under existential χ-quantification, when χ lifts $\mathcal{F}$-reduced products.
5. The sentences preserved by $\mathcal{F}$-reduced factors and the sentences preserved by $\mathcal{F}$-reduced products are both closed under conjunction.
6. The sentences preserved by $\mathcal{F}$-reduced products are closed under infinite conjunctions.
7. The Łoś-sentences are closed under negation.

$\square$

Proof. 1. Let F be any filter over I and let $\{A_i\}_{i \in I}$ be a family of of Σ-models for a signature Σ.

Let e be a basic sentence and consider $J = \{i \in I \mid A_i \models_\Sigma e\}$. There exists a model homomorphism $M_e \to A_i$ for each $i \in J$, therefore by the universal property of the products, there exists a model homomorphism $M_e \to \prod_{i \in J} A_i$. When composing this with $\prod_{i \in J} A_i \xrightarrow{\mu_J} \prod_F A_i$, we get a model homomorphism $M_e \to \prod_F A_i$, which implies that $\prod_F A_i \models e$.

2. Consider a finitary basic Σ-sentence e. By 1. we have to prove only that e is preserved by reduced factors. If $\prod_F A_i \models e$, then there exists a model homomorphism $M_e \to \prod_F A_i$. Since M_e is quasi-finite, there exists a model homomorphism $M_e \to \prod_{i \in J} A_i$ for some nonempty $J \in F$, which, by the product projections, means that $A_i \models e$ for all $i \in J$. Therefore $\{i \in I \mid A_i \models_\Sigma e\} \in F$ because $J \subseteq \{i \in I \mid A_i \models_\Sigma e\}$.

3. Let $\chi\colon \Sigma \to \Sigma'$ be a signature morphism which is conservative and preserves reduced products. Let e' be a Σ'-sentence preserved by reduced products, and let e be an existential χ-quantification of

e'. Consider a filter $F \in \mathcal{F}$ over a set I, and let $\{A_i\}_{i \in I}$ be a family of Σ-models such that $J = \{i \in I \mid A_i \models_\Sigma e\} \in F$. We have to prove that $\prod_F A_i \models_\Sigma e$.

For each $i \in J$ let B_i be a Σ'-model such that $B_i\restriction_\chi = A_i$ and $B_i \models_{\Sigma'} e'$. Because χ is conservative, for each $i \notin J$, let B_i be a Σ'-model such that $B_i\restriction_\chi = A_i$. Because e' is preserved by reduced products and because $J \subseteq \{i \in I \mid B_i \models_{\Sigma'} e'\}$ and J is filter, we have that $\prod_F B_i \models_{\Sigma'} e'$. Because χ preserves reduced products, we have that $(\prod_F B_i)\restriction_\chi = \prod_F A_i$, which implies that $\prod_F A_i \models_\Sigma e$.

4. Let $\chi : \Sigma \to \Sigma'$ be a signature morphism which lifts reduced products. Let e' be a Σ'-sentence preserved by reduced factors, and let e be an existential χ-quantification of e'. Consider a filter $F \in \mathcal{F}$ over a set I, and let $\{A_i\}_{i \in I}$ be a family of Σ-models such that $\prod_F A_i \models_\Sigma e$. We have to prove that $\{i \in I \mid A_i \models_\Sigma e\} \in F$.

Let B be a χ-expansion of $\prod_F A_i$ such that $B \models_{\Sigma'} e'$. Because χ lifts reduced products, for each $i \in I$ there exists a Σ'-model B_i such that $B_i\restriction_\chi = A_i$ and such that $\prod_F B_i = B$. Because e' is preserved by reduced factors, $J = \{i \in I \mid B_i \models_{\Sigma'} e'\} \in F$. But $J \subseteq \{i \in I \mid A_i \models_\Sigma e\}$, therefore $\{i \in I \mid A_i \models_\Sigma e\} \in F$ because F is filter.

5. This follows from the following:

- $\{i \in I \mid A_i \models e\} = \{i \in I \mid A_i \models e'\} \cap \{i \in I \mid A_i \models e''\}$,
- $J' \cap J'' \in F$ if and only if $J', J'' \in F$, and
- $\prod_F A_i \models e$ if and only if $\prod_F A_i \models e'$ and $\prod_F A_i \models e''$.

where the sentence e is the conjunction of e' and e'' in a signature Σ, $F \in \mathcal{F}$ is a filter over a set I, and $\{A_i\}_{i \in I}$ is any family of Σ-models.

6. Given a signature Σ, for each family $\{e_l\}_{l \in L}$ of Σ-sentences preserved by $\mathcal{F}$-reduced products, assume that $\{i \in I \mid A_i \models e_l \text{ for each } l \in L\} \in F$, where $F \in \mathcal{F}$ is any filter over a set I and $\{A_i\}_{i \in I}$ is any family of Σ-models. Then for each $l \in L$, $\{i \in I \mid A_i \models e_l\} \supseteq \{i \in I \mid A_i \models e_l \text{ for each } l \in L\} \in F$, thus $\{i \in I \mid A_i \models e_l\} \in F$, therefore $\prod_F A_i \models e_l$ for each $l \in L$.

7. Let e be the negation of a Łoś-Σ-sentence e' for a signature Σ. Then, for each ultrafilter U over a set I and for any family $\{A_i\}_{i \in I}$ of Σ-models, we have that $\prod_U A_i \models_\Sigma e$ if and only if $\prod_U A_i \not\models_\Sigma e'$ if and only if $\{i \in I \mid A_i \models_\Sigma e'\} \notin U$ and, because U is ultrafilter, if and only if $\{i \in I \mid A_i \models_\Sigma e\} \in U$.

The following Corollary can be regarded as an institution-independent generalisation of the so-called 'Fundamental Theorem on Ultraproducts' for first-order predicate logic (C.C.Chang and H.J.Keisler, 1973), originally due to Łoś (Łoś, 1955).

COROLLARY 1. The Łoś-sentences contain all finitary basic sentences and are closed under logical connectives and any χ-quantification for which χ is conservative and preserves and lifts reduced products. $\square$

Proof. Although this Corollary follows directly from Theorem 2, a special notice is needed for the case when the institution does not admit negations of its sentences. This is needed because the universal quantification can be expressed in terms of existential quantification and negation, and all logical connectives can also be expressed in terms of conjunctions and negation.

For example, consider e' an universal χ-quantification of a Łoś-sentence e. Then $e' \equiv \neg(\exists \chi.\neg e)$ in the negation closure $\mathfrak{S}^\neg$. Therefore e' is a Łoś-sentence in $\mathfrak{S}^\neg$, which implies that it is a Łoś-sentence in the institution $\mathfrak{S}$ too.

Corollary 1 can be specialised by using Proposition 2:

COROLLARY 2. The Łoś-sentences contain all finitary basic sentences and are closed under logical connectives and any χ-quantification for which χ is conservative finitary representable and preserves reduced products. $\square$

DEFINITION 16. The institution $\mathfrak{S}$ is a *Łoś-institution* if and only if all its sentences are Łoś-sentences. $\square$

EXAMPLE 9. First-order predicate logic (Appendix A), rewriting logic (Appendix B), and partial algebra (Appendix C) are Łoś-institutions. This follows from Corollary 2 by noticing that:

- each sentence of first-order predicate and rewriting logic and each strong equation in partial algebra is obtained from the finitary basic sentences (cf. Examples 2,3,4) by conjunction, implication, and universal quantification and, the existential equations in partial algebra are obtained via Proposition 3 from finitary basic sentences by conjunction, negation, and universal quantification, where
- the quantification of all these sentences is finitary representable (cf. Example 5 and Example 8) and preserves the reduced products (cf. Example 7).

The same observation holds when the sentences of those institutions are extended to full first-order sentences built on top of the corresponding basic sentences.

On the other hand, in general, the behavioural sentences in the hidden algebra institution (Appendix D) are not Łoś-sentences. This is due to the infinitary nature of the behavioural satisfaction, because each ground behavioural equation $(\forall \emptyset)\, t \sim t'$ is semantically equivalent to the set of (universally quantified strict) equations $\{(\forall X)\, c[z/t] = c[z/t'] \mid c \text{ visible behavioural context}\}$ where X denotes the set of the variables of c.

However, we can prove that existentially quantified behavioural unconditional equations (also called *behavioural queries*) (Goguen et al., 2002) are preserved by ultraproducts. This is a consequence of the fact that the universally quantified (strict) equations are Łoś-sentences and of 6. and 3. of Theorem 2 for $\mathcal{F}$ the class of ultrafilters. □

5. Some Applications

In this section, the institutions are implicitly assumed to have small products and small directed colimits for their categories of models.

5.1. Σ_1^1 SENTENCES

Recall (C.C.Chang and H.J.Keisler, 1973) that a Σ_1^1-sentence in first-order predicate logic is a second-order sentence all of whose relation and operation quantifiers occur at the beginning and are existential. The following Definition generalises the concept of Σ_1^1-sentence to any institution:

DEFINITION 17. In any institution, e is a Σ_1^1-sentence if it is an existential χ-quantification of a Łoś sentence, where χ is any conservative reduced products preserving signature morphism. □

The following Corollary follows from Theorem 2 and can be regarded as an institution-independent generalisation of the result of (C.C.Chang and H.J.Keisler, 1973) stating that the Σ_1^1-sentences in first-order predicate logic (Appendix A) are preserved by ultraproducts. Notice that, unlike in the particular case of first-order predicate logic, this result follows *directly* from the fundamental theorem 2 due to our general concept of quantification.

COROLLARY 3. In any institution each Σ_1^1-sentence is preserved by ultraproducts. □

We encourage the interested readers to explore the significance of the Σ_1^1-sentences in other institutions of interest other than first-order predicate logic.

5.2. COMPACTNESS

The following result is not only an institution-independent generalisation of the compactness via ultraproducts result of (Frayne et al., 1962) (see also (C.C.Chang and H.J.Keisler, 1973)), but it is also obtained for a more general class of sentences, which for example in the particular case of first-order predicate logic (Appendix A) include a class of second-order sentences (see Section 5.1) or in the case of hidden algebra (Appendix D) include behavioural queries (cf. Example 9).

THEOREM 3. In any institution, let E be a set of sentences preserved by ultraproducts. Let I be the set of all finite subsets of E. Consider a model A_i for each finite subset $i \in I$. Then there exists an ultraproduct $\prod_U A_i$ such that $\prod_U A_i \models E$. $\square$

Proof. Let $S = \{\{i \in I \mid \rho \in i\} \mid \rho \in E\}$. S has the finite intersection property because

$$\{\rho_1, \rho_2, \ldots, \rho_n\} \in \{i \in I \mid \rho_1 \in i\} \cap \{i \in I \mid \rho_2 \in i\} \cap \ldots \cap \{i \in I \mid \rho_n \in i\}$$

By the 'Ultrafilter Theorem' 1, let U be an ultrafilter such that $S \subseteq U$.

For each $\rho \in E$, we have that $\{i \in I \mid \rho \in i\} \subseteq \{i \in I \mid A_i \models \rho\}$. This means that $\{i \in I \mid A_i \models \rho\} \in U$. Because ρ is preserved by ultraproducts, it implies that $\prod_U A_i \models \rho$. Because $\rho \in E$ is arbitrary, it follows that $\prod_U A_i \models E$.

COROLLARY 4. Let E be a set of sentences preserved by ultraproducts, and let e be a sentence preserved by ultrafactors such that $E \models e$. Then there exists a finite subset $E' \subseteq E$ such that $E' \models e$. $\square$

Proof. Let us assume the contrary, i.e., that for each finite $i \subseteq E$, $i \not\models e$. This means that there exist models A_i such that $A_i \models i$ but $A_i \not\models e$.

Let I be the set of all finite subsets of E. By Theorem 3, there exists an ultraproduct such that $\prod_U A_i \models E$. Therefore $\prod_U A_i \models e$. Because e is preserved by ultrafactors, $\{i \in I \mid A_i \models e\} \in U$. But $\{i \in I \mid A_i \models e\} = \emptyset$ which is a contradiction since as ultrafilter U is a proper filter.

DEFINITION 18. An institution is *compact* if for each set of sentences E and each sentence e, if $E \models e$ then there exists a finite subset $E' \subseteq E$ such that $E' \models e$. $\square$

COROLLARY 5. Any Łoś-institution is compact. $\square$

EXAMPLE 10. Cf. Example 9, first-order predicate logic, rewriting logic, and partial algebra are compact. These results are expected because, for example, first-order predicate logic and rewriting logic are complete. Less expected are probably compactness results involving Σ_1^1-sentences (Cf. Corollary 3): "if $E \models e$ where E is a set of Σ_1^1-sentences and e is an ordinary sentence, then there exists a finite subset $E' \subseteq E$ such that $E' \models e$." $\square$

5.3. AXIOMATIZABILITY

The results of this section are institution-independent generalisations of the basic axiomatizability results in first-order predicate logic of (Frayne et al., 1962) (see also (C.C.Chang and H.J.Keisler, 1973)).

THEOREM 4. Let $\mathfrak{I}$ be a Łoś-institution that admits negation and conjunction. Then a class of models is elementary if and only if it is closed under ultraproducts and elementary equivalence. $\square$

Proof. The implication that any elementary class of models is closed under elementary equivalence and ultraproducts follows immediately from Remark 1 and the definition of Łoś-sentences 15.

For the opposite implication, consider a class of models $\mathcal{K}$ closed under ultraproducts and elementary equivalence. Let $E = \mathcal{K}^*$. We prove that $\mathcal{K} = \mathrm{MOD}(\Sigma, E)$.

Let $B \in \mathrm{MOD}(\Sigma, E)$. Consider I the set of the finite subsets of $\{B\}^*$. For each $i \in I$, there exists $A_i \in \mathcal{K}$ such that $A_i \models i$. (Otherwise for all $A \in \mathcal{K}$, $A \models \neg(e_1 \wedge \ldots \wedge e_n)$, where $i = \{e_1, \ldots, e_n\}$, which implies that $\neg(e_1 \wedge \ldots \wedge e_n) \in E$, which further implies that $B \models \neg(e_1 \wedge \ldots \wedge e_n)$ which contradicts the fact that $B \models e_1 \wedge \ldots \wedge e_n$.) By Theorem 3, there exists an ultrafilter U over I such that $\prod_U A_i \models \{B\}^*$. This implies that $\prod_U A_i \equiv B$ (otherwise if there exists a sentence e such that $\prod_U A_i \models e$ but $B \not\models e$, then $B \models \neg e$ and therefore $\prod_U A_i \models \neg e$ which is a contradiction). Because $\mathcal{K}$ is closed under ultraproducts and elementary equivalence, it follows that $B \in \mathcal{K}$.

COROLLARY 6. In a Łoś-institution that admits negation and conjunction, a class of models for a signature is the class of models of a finitely presented theory if and only if both it and its complement are elementary. □

Proof. If $E = \{e_1, \ldots, e_n\}$ is a finite set of Σ-sentences, then the complement of $\text{MOD}(\Sigma, E)$ is $\text{MOD}(\Sigma, \neg(e_1 \wedge \ldots \wedge e_n))$.

For the opposite implication, consider $\text{MOD}(\Sigma, E)$ an elementary class of models such that its complement is also elementary. We show by that there exists $E_0 \subseteq E$ finite such that $\text{MOD}(\Sigma, E) = \text{MOD}(\Sigma, E_0)$. If us assume the opposite, then for each $E_0 \subseteq E$ finite there exists a model A in the complement of $\text{MOD}(\Sigma, E)$ such that $A \models E_0$. Let $I = \{E_0 \subseteq E \mid E_0 \text{ finite}\}$. By Theorem 3, there exists an ultraproduct $\prod_U A_i$ over I such that $\prod_U A_i \models E$ and $A_i \notin \text{MOD}(\Sigma, E)$ and $A_i \models i$ for each $i \in I$. But because the complement of $\text{MOD}(\Sigma, E)$ is closed under ultraproducts, we also get that $\prod_U A_i \notin \text{MOD}(\Sigma, E)$, which is a contradiction.

6. Conclusions and Future Research

We generalised the ultraproducts method from classical model theory to an institution-independent framework based on a very general institution-independent treatment of quantification, logical connectives, and basic sentences (simplest sentences preserved by model homomorphisms). We showed some immediate applications of the fundamental theorem on ultraproducts, such as institution-independent compactness, axiomatizability, and Σ_1^1-sentences. We illustrated the concepts and results of our work with examples from four different logics or institutions.

Our development of the institution-independent ultraproducts method also lead to several novel concepts in the theory of institutions, such as

— representable signature morphisms - used to abstract the concept of first-order variables to institutions, and

— a general institution-independent treatment of quantifiers, naturally including higher-order quantifiers - resulting in a simpler presentation of logics without open formulæ, valuations of free variables, etc.

This work opens up several future research directions:

— extend the area of the institution-independent applications of the ultraproducts method started in this paper by generalising other results from classical model theory,

— study of the model theoretic properties of the various institutions in use in algebraic specification and computing science by applying the institution-independent ultraproducts method,

— further explore the significance of our internal logic, especially our approach to quantification, and apply it for exporting other methods from classical model theory to an institution-independent framework, and

— extend our internal logic approach to other logical features not discussed here, such as various modalities, for example.

References

Andréka, H. and I. Németi: 1978, 'Łoś Lemma Holds in Every Category'. *Studia Scientiarum Mathematicarum Hungarica* **13**, 361–376.

Andréka, H. and I. Németi: 1981, 'A General Axiomatizability Theorem Formulated in Terms of Cone-Injective Subcategories'. In: B. Csakany, E. Fried, and E. Schmidt (eds.): *Universal Algebra*. North-Holland, pp. 13–35. Colloquia Mathematics Societas János Bolyai, 29.

Burmeister, P.: 1986, *A Model Theoretic Approach to Partial Algebras*. Akademie Verlag Berlin.

C.C.Chang and H.J.Keisler: 1973, *Model Theory*. North Holland, Amsterdam.

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Diaconescu, R.: 1994, 'Category-based Semantics for Equational and Constraint Logic Programming'. DPhil thesis, University of Oxford.

Diaconescu, R.: 1995, 'Completeness of Category-based Equational Deduction'. *Mathematical Structures in Computer Science* **5**(1), 9–41.

Diaconescu, R.: 2000, 'Category-based Constraint Logic'. *Mathematical Structures in Computer Science* **10**(3), 373–407.

Diaconescu, R. and K. Futatsugi: 2000, 'Behavioural Coherence in Object-Oriented Algebraic Specification'. *Universal Computer Science* **6**(1), 74–96. First version appeared as JAIST Technical Report IS-RR-98-0017F, June 1998.

Diaconescu, R. and K. Futatsugi: 2003, 'Logical Foundations of CafeOBJ'. *Theoretical Computer Science*. To appear.

Diaconescu, R., J. Goguen, and P. Stefaneas: 1993, 'Logical Support for Modularisation'. In: G. Huet and G. Plotkin (eds.): *Logical Environments*. Cambridge, pp. 83–130. Proceedings of a Workshop held in Edinburgh, Scotland, May 1991.

Frayne, T., A. Morel, and D. Scott: 1962, 'Reduced direct products'. *Fundamenta Mathematica* **51**, 195–228.

Goguen, J.: 2002, *Theorem Proving and Algebra*. MIT. To appear.

Goguen, J. and R. Burstall: 1992, 'Institutions: Abstract Model Theory for Specification and Programming'. *Journal of the Association for Computing Machinery* **39**(1), 95–146.

Goguen, J., G. Malcolm, and T. Kemp: 2002, 'A Hidden Herbrand Theorem: Combining the Object, Logic and Functional Paradigms'. *Journal of Logic and Algebraic Programming*. To appear.

Goguen, J. and J. Meseguer: 1985, 'Completeness of Many-sorted Equational Logic'. *Houston Journal of Mathematics* **11**(3), 307–334.

Goguen, J. and G. Roşu: 1999, 'Hiding more of Hidden Algebra'. In: J. M. Wing, J. Woodcock, and J. Davies (eds.): *FM'99 – Formal Methods*, Vol. 1709 of *Lecture Notes in Computer Science*. pp. 1704–1719.

Grätzer, G.: 1979, *Universal Algebra*. Springer.

Hennicker, R. and M. Bidoit: 1999, 'Observational Logic'. In: A. M. Haeberer (ed.): *Algebraic Methodology and Software Technology*. pp. 263–277. Proc. AMAST'99.

Herrlich, H. and G. Strecker: 1973, *Category Theory*. Allyn and Bacon.

Łoś, J.: 1955, 'Quleques remarques, théorèmes et problèmes sur les classes définissables d'algèbres'. In: *Mathematical Interpretation of Formal Systems*. North-Holland, Amsterdam, pp. 98–113.

MacLane, S.: 1998, *Categories for the Working Mathematician*. Springer, second edition.

Matthiessen, G.: 1978, 'Regular and strongly finitary structures over strongly algebroidal categories'. *Canad. J. Math.* **30**, 250–261.

Meseguer, J.: 1992, 'Conditional rewriting logic as a unified model of concurrency'. *Theoretical Computer Science* **96**(1), 73–155.

Meseguer, J.: 1993, 'A Logical Theory of Concurrent Objects and Its Realization in the Maude Language'. In: G. Agha, P. Wegner, and A. Yonezawa (eds.): *Research Directions in Concurrent Object-Oriented Programming*. The MIT Press.

Mossakowski, T.: 2001, 'Relating CASL with Other Specification Languages: the Institution Level'. *Theoretical Computer Science*. To appear.

Reichel, H.: 1984, *Structural Induction on Partial Algebras*. Akademie Verlag Berlin.

Salibra, A. and G. Scollo: 1996, 'Interpolation and compactness in categories of pre-institutions'. *Mathematical Structures in Computer Science* **6**, 261–286.

Tarlecki, A.: 1986a, 'On the Existence of Free Models in Abstract Algebraic Institutions'. *Theoretical Computer Science* **37**, 269–304. Preliminary version, University of Edinburgh, Computer Science Department, Report CSR-165-84, 1984.

Tarlecki, A.: 1986b, 'Quasi-Varieties in Abstract Algebraic Institutions'. *Journal of Computer and System Sciences* **33**(3), 333–360. Original version, University of Edinburgh, Report CSR-173-84.

Appendix

In the Appendix we give very brief presentations of a number of institutions which are used in this paper as examples for illustrating some of the concepts introduced by this work and some of the applications of the main results. Although we assume some familiarity with these institutions, the reader is encouraged to consult the recommended references for more details. Also, some notations and terminology used in some sections of the Appendix rely on notations and terminology from previous sections.

A. First-order Predicate Logic

The role of this very brief presentation of (many-sorted) first-order predicate logic (with equality) is mainly for fix some notations and conventions. A detailed definition of the first-order predicate logic institution can be found in (Goguen and Burstall, 1992).

Recall that a (many-sorted) signature in first-order predicate logic is a tuple (S, Σ, Π) (often denoted just by (Σ, Π)) where S is the set of sorts, Σ is the set of (S-sorted) operation symbols, and Π is the set of (S-sorted) relation symbols. By $\Sigma_{w \to s}$ we denote the set of operations with arity w and sort s, and by Π_w we denote the set of relations with arity w.

Given a signature (Σ, Π), a *model M* of first-order predicate logic interprets:

— each sort s as a set M_s,
— each operation symbol $\sigma \in \Sigma_{w \to s}$ as a function $M_\sigma \colon M_w \to M_s$, where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ for $w = s_1 \ldots s_n$, and
— each relation symbol $\pi \in \Pi_w$ as a relation $M_\pi \subseteq M_w$.

Any *ground* (i.e., without variables) Σ-term $t = \sigma(t_1 \ldots t_n)$, where σ is an operation symbol and $t_1, \ldots, t_n$ are subterms, gets interpreted as an element M_t in a Σ-model M by $M_t = M_\sigma(M_{t_1} \ldots M_{t_n})$.

A (Σ, Π)-*model homomorphism* $h \colon M \to M'$ is an indexed family of functions $\{h_s \colon M_s \to M'_s\}_{s \in S}$ such that

— h is a Σ-algebra homomorphism $M \to M'$, i.e., $h(M_\sigma(m)) = M'_\sigma(h(m))$ for each $\sigma \in \Sigma_{w \to s}$ and each $m \in M_w$,[3] and
— $h(m) \in M'_\pi$ if $m \in M_\pi$ for each relation $\pi \in \Pi_w$ and each $m \in M_w$.

The sentences are the well-known first-order closed formulæ (including equations), and their satisfaction by the models is the well-known Tarskian satisfaction (see (Goguen and Burstall, 1992; C.C.Chang and H.J.Keisler, 1973) for details). In the many-sorted case we restrict the quantification of the sentences to *conservative* sets of variables for a signature, i.e., which do not have variables for the 'empty' sorts, which are sorts not having constants in the signature. This condition on the quantification in the many-sorted case was noticed for the first time in the context of the completeness of many-sorted equational logic (Goguen and Meseguer, 1985).

A *signature morphism* $\phi = (\phi^{\mathrm{sort}}, \phi^{\mathrm{op}}, \phi^{\mathrm{rel}}) \colon (S, \Sigma, \Pi) \to (S', \Sigma', \Pi')$ consists of a function between the sets of sorts $\phi^{\mathrm{sort}} \colon S \to S'$, a function between the sets of operation symbols $\phi^{\mathrm{op}} \colon \Sigma \to \Sigma'$, and a function between the sets of relation symbols $\phi^{\mathrm{rel}} \colon \Pi \to \Pi'$ such that $\phi^{\mathrm{op}}(\Sigma_{w \to s}) \subseteq \Sigma'_{\phi^{\mathrm{sort}}(w) \to \phi^{\mathrm{sort}}(s)}$ and $\phi^{\mathrm{rel}}(\Pi_w) \subseteq \Pi'_{\phi^{\mathrm{sort}}(w)}$ for any string of sorts $w \in S^*$ and each sort $s \in S$.[4]

Given a signature morphism $\phi \colon (S, \Sigma, \Pi) \to (S', \Sigma', \Pi')$, the *reduct* $M' \!\restriction_\phi$ of a (S', Σ', Π')-model M' is defined by $(M' \!\restriction_\phi)_s = M'_{\phi^{\mathrm{sort}}(s)}$ for each sort $s \in S$, $(M' \!\restriction_\phi)_\sigma = M'_{\phi^{\mathrm{op}}(\sigma)}$ for each operation symbol $\sigma \in \Sigma$, and $(M' \!\restriction_\phi)_\pi = M'_{\phi^{\mathrm{rel}}(\pi)}$ for each relation symbol $\pi \in \Pi$.

The *sentence translation* along ϕ of any sentence is defined inductively on the structure of the sentences by replacing the symbols from (S, Σ, Π) with symbols from (S', Σ', Π') as defined by ϕ.

Notice that by discarding the relational part, we get the many-sorted algebra institution with full first-order equational sentences.

B. Rewriting Logic

Rewriting logic (Meseguer, 1992) is emerging as one of the most important new algebraic specification logics. Here we refer to a simplified variant of rewriting logic which is used for defining the CafeOBJ institution (Diaconescu and Futatsugi, 2003), however this example can be extended to the original definition of rewriting logic without any difficulty.

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[3] By $h(m)$ we mean in fact $h_w(m)$, where $h_w \colon M_w \to M'_w$ is the canonical component-wise extension of h.

[4] For any string of sorts $w = s_1 \ldots s_n$, by $\phi^{\mathrm{sort}}(w)$ we mean the string of sorts $\phi^{\mathrm{sort}}(s_1) \ldots \phi^{\mathrm{sort}}(s_n)$.

Recall (from (Diaconescu and Futatsugi, 2003)) that our rewriting logic signatures are just ordinary (many-sorted) algebraic signatures. The models are *preorder models* which are (algebraic) interpretations of the signatures into $\mathbb{P}re$ (the category of preorders) rather than in $\mathbb{S}et$ (the category of sets) as in the case of ordinary algebras. More precisely, given a signature Σ, a model M interprets:

— each sort s as a preorder M_s, and
— each operation $\sigma \in \Sigma_{w \to s}$ as a preorder functor $M_\sigma \colon M_w \to M_s$, where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ for $w = s_1 \ldots s_n$.

The *sentences* are either ordinary equations or *transitions*, both in their unconditional or conditional form. For example, the unconditional Σ-transitions for a signature Σ, are sentences of the form

$$(\forall X)\, t \; => \; t'$$

where X is a conservative many-sorted set of variables for Σ and t, t' are Σ-terms with variables X. Conditional sentences in rewriting logic are universally quantified implications where the hypotheses are finite conjunctions of transitions or equations and the conclusion is a transition or an equation.

The signature morphisms, the model reducts, and the sentence translations along signature morphisms are defined in the same way with ordinary (many-sorted) algebra (Appendix A).

A preorder model M *satisfies* a transition $M \models (\forall X)\, t \; => \; t'$, if and only if $M'_t \leq M'_{t'}$ for each expansion M' of M along the signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$. The satisfaction of conditional sentences extends the satisfaction of equations and transitions to the conditional case; we leave this as exercise to the reader.

More details of this institution of rewriting logic can be found in (Diaconescu and Futatsugi, 2003), while (Meseguer, 1992) has the details of the institution of full rewriting logic.

C. Partial Algebra

There are many approaches to partial algebra, two classical references being (Burmeister, 1986; Reichel, 1984). Our formalisation of the partial algebra institution is tailored to the needs of this paper but without affecting the logic and model theory of partial algebra.

A *partial algebraic signature* is a pair (Σ, Δ), where Σ is the set of the total operations and Δ is the set of the partial operations.[5] A *partial (Σ, Δ)-algebra* A is just like a $\Sigma \cup \Delta$-algebra but interpreting the operations of Δ as partial functions rather than total functions. A *homomorphism* $h \colon A \to B$ between partial algebras, is a family of partial functions $\{h_s \colon A_s \multimap B_s\}_{s \in S}$ indexed by the set of sorts S of (Σ, Δ) such that either both $h(A_\sigma(a))$ and $B_\sigma(h(a))$ are undefined or they are defined and equal, for each operation $\sigma \in (\Sigma \cup \Delta)_{w \to s}$ and each argument $a \in A_w$.[6]

The *interpretation* A_t of a $\Sigma \cup \Delta$-ground term t in a partial (Σ, Δ)-algebra is defined inductively by

— A_t is undefined if A_{t_k} is undefined for some $k \in \{1, \ldots, n\}$ or $(A_{t_1}, \ldots, A_{t_n})$ does not belong to the definition domain of A_σ, otherwise
— $A_t = A_\sigma(A_{t_1}, \ldots, A_{t_n})$.

where $t = \sigma(t_1 \ldots t_n)$ is a term with σ any (Σ, Δ)-operation and $t_1, \ldots, t_n$ subterms.

Signature morphisms, model reducts, and sentence translations are defined similarly to the case of the total algebra (see Appendix A).

The sentences are either *strong* or *existential* equations, both in their conditional or unconditional form. For any unconditional strong $(\Sigma \cup \Delta)$-equation $(\forall X)\, t \stackrel{s}{=} t'$, where X is a conservative many-sorted set of variables for (Σ, Δ), a partial (Σ, Δ)-algebra A satisfies it if and only if

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[5] In this notation we ignore the set of sorts, which are of course common to the total and the partial operations.
[6] Notice that by convention $h(a)$ is defined if and only if is defined on all components of a.

- A'_t and $A'_{t'}$ are both undefined, or
- A'_t and $A'_{t'}$ are both defined and $A'_t = A'_{t'}$.

for each expansion A' of the partial algebra A along the signature inclusion $(\Sigma, \Delta) \hookrightarrow (\Sigma \cup X, \Delta)$.

For any unconditional existential $(\Sigma \cup \Delta)$-equation $(\forall X)\, t \stackrel{e}{=} t'$, where X is a many-sorted set of variables for (Σ, Δ), a partial (Σ, Δ)-algebra A satisfies it if and only if

- A'_t and $A'_{t'}$ are both defined and $A'_t = A'_{t'}$.

for each expansion A' of the partial algebra A along the signature inclusion $(\Sigma, \Delta) \hookrightarrow (\Sigma \cup X, \Delta)$. These definitions extend without any problems to the conditional case. We leave it as exercise to the reader.

The following result show how this version of partial algebra is equivalent to an equationally defined class (i.e. variety) of total algebras, which is very useful for establishing some properties of partial algebras. We omit here its straightforward proof.

PROPOSITION 3. For any partial algebra signature (Σ, Δ) with S the set of sorts, let $\perp = \{\perp_s\}_{s \in S}$ be an indexed set of new constant symbols and let Γ be set of the equations

$$(\forall x_1 \ldots \forall x_n)\sigma(x_1 \ldots \perp_s \ldots x_n) = \perp_{s'}$$

for all operations $\sigma \in \Sigma \cup \Delta$.

Then the functor mapping each partial (Σ, Δ)-algebra A to the total $(\Sigma \cup \Delta \cup \perp, \Gamma)$-algebra $\overline{A}$ such that

- $\overline{A}_s = A_s \cup \{\perp_s\}$ for each sort $s \in S$,
- for each operation $\sigma \in \Sigma \cup \Delta$, $\overline{A}_\sigma(a) = A_\sigma(a)$ if a belongs to the definition domain of A_σ, and
- $\overline{A}_\sigma(a) = \perp_s$ otherwise, where s is the sort of σ,

and mapping each partial algebra homomorphism $h\colon A \to B$ to the total algebra homomorphism $\overline{h}\colon \overline{A} \to \overline{B}$ such that for each sort s,

- $\overline{h}_s(a) = h_s(a)$ if a belongs to the definition domain of h_s, and
- $\overline{h}_s(a) = \perp_s$ otherwise.

is an isomorphism between the category of partial (Σ, Δ)-algebras and the category of total $(\Sigma \cup \Delta \cup \perp, \Gamma)$-algebras.

Moreover,

$$A \models_{\Sigma, \Delta} (\forall X)\, t \stackrel{s}{=} t' \quad \text{iff} \quad \overline{A} \models_{\Sigma \cup \Delta \cup \perp} (\forall X)\, t = t'$$

for each strong equation $(\forall X)\, t \stackrel{s}{=} t'$, and

$$A \models_{\Sigma, \Delta} (\forall X)\, t \stackrel{e}{=} t' \quad \text{iff} \quad \overline{A} \models_{\Sigma \cup \Delta \cup \perp} ((\forall X)\, t = t' \text{ and } \neg(\exists X)t = \perp)$$

for each existential equation $(\forall X)\, t \stackrel{e}{=} t'$. $\square$

D. Hidden Algebra

Hidden algebra is the institution underlying behavioural specification, which is one of the most important new algebraic specification formalisms. In the literature there are several versions of hidden algebra, with only slight technical differences between them (Diaconescu and Futatsugi, 2000; Hennicker and Bidoit, 1999; Goguen and Roşu, 1999). Here we adopt a slightly modified version of *coherent hidden algebra* (abbreviated *CHA*) of (Diaconescu and Futatsugi, 2000).

A *CHA signature* is a tuple (H, V, Σ, Σ^b), where

- *H* and *V* are disjoint sets of *hidden* sorts and *visible* sorts, respectively,
- Σ is a $H \cup V$-sorted signature,
- $\Sigma^b \subseteq \Sigma$ is a subset of *behavioural operations* such that $\sigma \in \Sigma^b_{w \to s}$ has *exactly* one hidden sort in *w*.

A CHA model *M* for a signature (H, V, Σ, Σ^b) is just an ordinary Σ-algebra.

CHA sentences can be ordinary (strict) equations, *behavioural equations* (both in conditional or unconditional format), or *coherence declarations* (see (Diaconescu and Futatsugi, 2000; Diaconescu and Futatsugi, 2003) for details). Recall ((Diaconescu and Futatsugi, 2000; Diaconescu and Futatsugi, 2003)) that coherence declarations are semantically equivalent to conditional behavioural equations and that the strict equations are treated in the same way as in the case of the ordinary algebra. An unconditional *behavioural equation* is a sentence of the form

$$(\forall X)\, t \sim t'$$

where *X* is a conservative set of variables and t, t' are Σ-terms over *X*.

Recall that a Σ-*context* $c[z]$ is any Σ-term *c* with a marked variable *z* occurring only once in *c*. A context $c[z]$ is *behavioural* iff all operations above[7] *z* are behavioural.

Given a Σ-algebra *A*, two elements (of the same sort *s*) *a* and *a'* are called *behaviourally equivalent*, denoted $a \sim_s a'$ (or just $a \sim a'$), iff $A^a_c = A^{a'}_c$ for each *visible* behavioural context *c*, where A^a and $A^{a'}$ are any expansions of *A* along the signature inclusion $\Sigma \hookrightarrow \Sigma \cup Y$, where *Y* is the set of variables of *c*, and such that $A^a_y = A^{a'}_y$ for each $y \in Y \setminus \{z\}$, $A^a_z = a$, and $A^{a'}_z = a'$.

Then, a Σ-algebra *A* satisfies an (unconditional) behavioural equation $A \models (\forall X)\, t \sim t'$, iff $A'_t \sim A'_{t'}$ for each *A'* expansion of the algebra *A* along the signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$.

This definition extends without any problems to the conditional case. We leave it as exercise to the reader.

Recall also that a *CHA signature morphism* $\phi \colon (H, V, \Sigma, \Sigma^b) \to (H', V', \Sigma', \Sigma'^b)$ is an many-sorted signature morphism $(H \cup V, \Sigma) \to (H' \cup V', \Sigma')$ such that

(M1) $\phi(V) \subseteq V'$ and $\phi(H) \subseteq H'$,

(M2) $\phi(\Sigma^b) = \Sigma'^b$ and $\phi^{-1}(\Sigma'^b) \subseteq \Sigma^b$,

Finally, model reducts and sentence translations along CHA signature morphisms are the same with those from ordinary many-sorted algebra (Appendix A).

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[7] Meaning that *z* is in the subterm determined by the operation.