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by

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Domain decomposition Schwarz method for strongly nonlinear variational inequalities

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Abstract

We prove the convergence for a subspace correction method applied to strongly nonlinear variational inequalities in a general reflexive Banach space, provided that the convex set verifies a certain assumption. In the following we prove that this assumption holds for the Schwarz method in which the convex set is described by constraints on the function values at the points of the domain. Also, this assumption holds for the one and two-level Schwarz method in the finite element space, and we explicitely write the constants in the error estimation depending on the domain decomposition and mesh parameters. Numerical examples are given to illustrate the convergence of the method with both one and two levels, for the two-obstacle problem of a nonlinear elastic membrane.

Keywords: domain decomposition methods, Schwarz method, subspace correction, nonlinear variational inequalities, finite elements, multilevel methods, obstacle problems
AMS subject classification: 65N55, 65N30, 65J15

1 Introduction

The literature on the domain decomposition methods is very large and it is motivated by an increasing need on the solution of large-scale problems since these methods provide numerical solvers which are efficient and parallelizable on multi-processor machines. The multiplicative and additive Schwarz methods for elliptic linear problems have been studied by many researchers, among them Lions [23]–[25], Chan, Hou and Lions [7], P. Le Tallec [22], A. Quarteroni and A. Valli [30], Bramble, Pasciak, Wang and Xu [5], and Badea [1], for the multiplicative methods, and Dryja [9], Dryja and Widlund [10], [11], and Nepomnyaschikh [29], for the additive version. For problems related to variational inequalities, we can cite the papers written by Hoffman and Zou [16], Kuznetsov and Neittaanmäki [19], Kuznetsov, Neittaanmäki and Tarvainen [20]–[21], Lü, Liem and Shih [26], Zeng and Zhou [39], Badea [2], Badea si Wang [3], Tai [32]–[34], and Tai and Tseng [36]. Also, the multilevel and multigrid methods can be viewed as domain decomposition methods and we can cite the results obtained by Kornhuber [18], Mandel [28], and Smith, Bjørstad and Gropp [31]. However, very few papers deal with the application of these methods to nonlinear problems. We can cite in this direction the papers written by Tai and Espedal [35], Tai and Xu [37]

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for nonlinear equations, Hoffmann and Zhou [17], Lui [27], and Zeng and Zhou in [40] for inequalities having nonlinear source terms. Evidently, the above lists of citations is not exhaustive and it can be completed by many other papers.

Almost exclusively, the convergence of the domain decomposition methods for variational inequalities coming from the minimization of a functional is studied in the case when this functional is quadratic. The main goal of this paper is to give the error estimates for the one and two-level Schwarz domain decomposition methods applied to the minimization of the non quadratic functionals over general convex sets. The most of the papers consider the convex set decomposed according to the space decomposition as a sum of subconvexe sets. This is a easy condition when we deal with the obstacle problems. We have tried to extend our analysis to other types of convex sets, and when we use the Sobolev or finite element spaces, we consider convex sets which are described by constraints on the function values at the points of the domain, $K = \{v \in W_0^{1,s}(\Omega) : |v(x)| = \sqrt{v(x)^2} \leq b(x) \text{ a.e. in } \Omega\}$, for instance. Evidently, such a convex set is of the two-obstacle type, but its d dimensional corresponding, $K = \{v = (v_1, \cdots, v_d) \in [W_0^{1,s}(\Omega)]^d : |v(x)| = \sqrt{v_1(x)^2 + \cdots + v_d(x)^2} \leq b(x) \text{ a.e. in } \Omega\}$, can not be easily decomposed as a sum of subconvex sets. Consequently, when we look for solutions in $W^{1,s}(\Omega)$, our convex sets are of one or two-obstacle type, but if they lie in $[W^{1,s}(\Omega)]^d$, then they may be of other types, too. We shall characterize more precisely the convex sets we consider in Section 4. For the writing simplicity, we have considered in the next sections problems having the solution in $W^{1,s}(\Omega)$, but all the obtained results hold reading $[W^{1,s}(\Omega)]^d$ in the place of $W^{1,s}(\Omega)$.

The convergence of a domain decomposition algorithm solving variational inequalities coming from the minimization of quadratic functionals over convex sets defined by constraints on the function values at the points of the domain is proved in [2]. In [37], it is proved that the multiplicative space decomposition method applied to the minimization without constraints of a differentiable and convex functional defined in a reflexive Banach space uniformly converges. In [4], using the subspace correction techniques in [5] and [38], and more general conditions in [35] on the convex functional, it is proved that the convergence rate for the one and two-level domain of the algorithm in [2] is of the same order as the convergence rate of the linear elliptic jump coefficient problems [6]. We generalize in this paper the results in [4] and [37] to the minimization of the non quadratic functionals.

The paper is organized as follows. In Section 2, we state the multiplicative Schwarz method for nonlinear variational inequalities as a subspace correction method in a general reflexive Banach space for the minimization of non quadratic functionals, and we prove the convergence of this algorithm provided that a certain assumption holds. In Section 3, under a little stronger assumption, which essentially introduces a constant depending on the convex set and the space decomposition, we estimate the error of the algorithm. Section 4 is devoted to the convergence of the method in Sobolev spaces, proving that the introduced assumptions hold. For the Sobolev spaces the algorithm is exactly a variant of the Schwarz method. In Section 5, we present an analysis for the one and two-level Schwarz method in the finite element spaces, where the assumptions hold, too. In these cases, we are able to explicitely write the constant introduced in the assumption in Section 3 depending on the mesh and domain decomposition parameters. The proof for the two-level method is based on a lemma which can be viewed as a Friedrichs - Poincaré inequality for the finite element spaces. Finally, in Section 6, we illustrate the convergence of the method with both one and two levels by numerical examples concerning the two-obstacle problem of a nonlinear elastic membrane. Also, we give in this section some details concerning the procedure we have used in the computing code to solve the problems on subdomains.

2 General convergence result

Let V be a reflexive Banach space and $V_1, \cdots, V_m$, be some closed subspaces of V. Also, we consider a non empty closed convex set $K \subset V$, and we make the following

ASSUMPTION 2.1. *For any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K \text{ for } i = 1, \cdots, m, \tag{2.1}$$

$$v - w = \sum_{i=1}^{m} v_i, \tag{2.2}$$

and the application

$$V \times V_1 \times \cdots \times V_m \ni (v - w, w_1, \cdots, w_m) \to (v_1, \cdots, v_m) \in V_1 \times \cdots \times V_m \tag{2.3}$$

is bounded, i.e. it transforms the bounded sets in some bounded sets.

A similar assumption has been introduced in [2] to prove the convergence of the Schwarz method for variational inequalities coming from the minimization of quadratic functionals. It looks to be complicated enough, but, as we shall see in Section 4, it holds for problems in which we use the Sobolev spaces and the convex set K is defined by constraints of the function values at the points of the domain. We consider a Gâteaux differentiable functional $F : K \to R$, which will be assumed to be coercive if K is not bounded. We assume that for any real number $M > 0$, if we write $L_M = \sup_{\substack{||v||, ||u|| \leq M \\ v, u \in K}} ||v - u||$, there exist two functions $\alpha_M, \beta_M : [0, L_M] \to \mathbf{R}^+$, such that

$$\alpha_M \text{ is continuous and strictly increasing on } [0, L_M], \text{ and } \alpha_M(0) = 0, \tag{2.4}$$

$$\beta_M \text{ is continuous at } 0 \text{ and } \beta_M(0) = 0, \tag{2.5}$$

and satisfying

$$< F'(v) - F'(u), v - u > \geq \alpha_M(||v - u||), \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \tag{2.6}$$

and

$$\beta_M(||v - u||) \geq ||F'(v) - F'(u)||_{V'}, \text{ for any } u, v \in K, \ ||u||, ||v|| \leq M, \tag{2.7}$$

where F' is the Gâteaux derivative of F.

We know (see [13], Proposition 5.5, page 25) that if (2.6) holds for any $M > 0$, then the functional F is strictly convex. Also, it is easy to prove that if (2.7) is true for any $M > 0$, then F is continuously differentiable. Reciprocally, we can prove in a similar way to that given in [14] (Lemma 1.1, page 61) for the case of the Euclidean spaces, that if the closed unity sphere is compact in the strong topology of the space Banach V, F' is continuous and F is strictly convex, then for any $M>0$ and for a $\tau \in [0, L_M]$,

$$\alpha_M(\tau) = \inf_{||v-u||=\tau,\ ||v||,||u|| \le M,\ v,u \in K} < F'(v) - F'(u), v-u >$$

exists, and this function α_M satisfies (2.4) and (2.6). Also, if the closed unity sphere is compact in the strong topology of the space Banach V and F' is continuous, then for any $M>0$ and for a $\tau \in [0, L_M]$,

$$\beta_M(\tau) = \sup_{||v-u||=\tau,\ ||v||,||u|| \le M,\ v,u \in K} ||F'(v) - F'(u)||_{V'},$$

exists and satisfies (2.5) and (2.7). We see that if $u, v \in K$, $||v||, ||u|| \le M$, and if for a $\tau \le ||v-u||$ we define $v_\tau = (1 - \frac{\tau}{||v-u||})u + \frac{\tau}{||v-u||}v$, then $v_\tau \in K$, $||v_\tau|| \le M$ and $||v_\tau - u|| = \tau$. Consequently, we get from the definition of L_M that the above functions α_M and β_M can be defined for all $\tau \in [0, L_M]$.

It is evident that if (2.6) and (2.7) hold, then

$$\begin{aligned} &\alpha_M(||v-u||) \le < F'(v) - F'(u), v-u > \le \beta_M(||v-u||)||v-u||, \\ &\text{for any } u, v \in K,\ ||u||, ||v|| \le M. \end{aligned} \tag{2.8}$$

Following the way in [14] (Lemmas 1.1 and 1.2, pages 61–63), we can prove that

$$\begin{aligned} &< F'(u), v-u > + \lambda_M(||v-u||) \le F(v) - F(u) \le \\ &< F'(u), v-u > + \mu_M(||v-u||),\ \text{for any } u, v \in K,\ ||u||, ||v|| \le M, \end{aligned} \tag{2.9}$$

where

$$\lambda_M(\tau) = \int_0^\tau \alpha_M(\theta) \frac{d\theta}{\theta}, \tag{2.10}$$

and

$$\mu_M(\tau) = \int_0^\tau \beta_M(\theta) d\theta. \tag{2.11}$$

Now, we consider the minimization problem

$$u \in K : F(u) \le F(v),\ \text{for any } v \in K. \tag{2.12}$$

It is well known (see [13]) that if V is a reflexive Banach space and F is strictly convex, differentiable, and coercive if K is not bounded, then the above problem has a unique solution, and it is also the unique solution of the problem

$$u \in K :\ < F'(u), v-u > \ge 0,\ \text{for any } v \in K. \tag{2.13}$$

From (2.9) we see that, for a given $M>0$ such that the solution u of (2.13) satisfies $||u|| \le M$, we have

$$\lambda_M(||v-u||) \le F(v) - F(u),\ \text{for any } v \in K,\ ||v|| \le M. \tag{2.14}$$

The proposed algorithm corresponding to the subspaces $V_1, \cdots, V_m$ and the convex set K is written as follows

Algorithm 2.1. *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we compute sequentially for $i = 1, \cdots, m$, $w_i^{n+1} \in V_i$ satisfying*

$$w_i^{n+1} = \arg \min_{\substack{u^{n+\frac{i-1}{m}} + v_i \in K \\ v_i \in V_i}} G(v_i), \text{ with } G(v_i) = F(u^{n+\frac{i-1}{m}} + v_i), \tag{2.15}$$

and then we update

$$u^{n+\frac{i}{m}} = u^{n+\frac{i-1}{m}} + w_i^{n+1}.$$

This algorithm does not assume a decomposition of the convex set K depending on the subspaces V_i, and it has been proposed in [2] in an equivalent form. The above form of this algorithm has been proposed in [4]. As for problem (2.12), since the subspaces V_i are reflexive Banach spaces, problem (2.15) has a unique solution and it also satisfies the variational inequality

$$\begin{aligned} & w_i^{n+1} \in V_i,\ u^{n+\frac{i-1}{m}} + w_i^{n+1} \in K : \ < F'(u^{n+\frac{i-1}{m}} + w_i^{n+1}), v_i - w_i^{n+1} > \geq 0, \\ & \text{for any } v_i \in V_i,\ u^{n+\frac{i-1}{m}} + v_i \in K. \end{aligned} \tag{2.16}$$

We have the following general convergence result.

Theorem 2.1. *We consider that V is a reflexive Banach, $V_1, \cdots, V_m$ are some closed subspaces of V, K is a non empty closed convex subset of V, and F is Gâteaux differentiable functional on K which is assumed to be coercive if K is not bounded. If Assumption 2.1 hold, and for any $M > 0$ there exist two functions α_M and β_M satisfying (2.4)–(2.7), then, for any $i = 1, \cdots, m$, $u^{n+\frac{i}{m}} \to u$, strongly in V, as $n \to \infty$, where u is the solution of problem (2.12) and $u^{n+\frac{i}{m}}$ are given by Algorithm 2.1 starting from an arbitrary given u^0.*

Proof. From (2.16) and (2.9), we have

$$F(u^{n+\frac{i-1}{m}}) - F(u^{n+\frac{i}{m}}) \geq \lambda_M(||w_i^{n+1}||), \text{ for any } n \geq 0 \text{ and } i = 1, \cdots, m, \tag{2.17}$$

and therefore, using (2.12), we get

$$F(u) \leq F(u^{n+\frac{i}{m}}) \leq F(u^{n+\frac{i-1}{m}}) \leq F(u^0), \text{ for any } n \geq 0 \text{ and } i = 1, \cdots, m. \tag{2.18}$$

Taking into account the boundedness of K or the coerciveness of F, it follows that there exists a real constant $M > 0$ such that

$$||u|| \leq M,\ ||u^0|| \leq M,\ ||u^{n+\frac{i}{m}}|| \leq M, \text{ for any } n \geq 0 \text{ and } i = 1, \cdots, m. \tag{2.19}$$

From (2.17) we also get

$$F(u^n) - F(u^{n+1}) \geq \sum_{i=1}^{m} \lambda_M(||w_i^{n+1}||), \text{ for any } n \geq 0. \tag{2.20}$$

Consequently, from (2.18), the series $\sum_{n=1}^{\infty} \lambda_M(||w_i^{n+1}||)$ is convergent for any $i = 1, \cdots, m$, and therefore

$$||w_i^{n+1}|| \to 0, \text{ as } n \to \infty, \text{ for any } i = 1, \cdots, m. \tag{2.21}$$

Applying Assumption (2.1) for $w = u^{n+\frac{i-1}{m}}$, $v = u$, and $w_i = w_i^{n+1}$, we have a decomposition $u_1, \cdots, u_m$ of $u - u^{n+\frac{i-1}{m}}$. From (2.1), we can replace v_i by u_i in (2.16), and we get

$$< F'(u^{n+\frac{i}{m}}) - F'(u^{n+1}), u_i - w_i^{n+1} > + < F'(u^{n+1}), u_i - w_i^{n+1} \geq 0.$$

Using (2.2) we have

$$\sum_{i=1}^{m} < F'(u^{n+\frac{i}{m}}) - F'(u^{n+1}), u_i - w_i^{n+1} > + < F'(u^{n+1}), u - u^{n+1} > \geq 0.$$

Using this inequality, from (2.19), (2.9) and (2.7) we obtain

$$\begin{aligned} & F(u^{n+1}) - F(u) + \lambda_M(||u - u^{n+1}||) \leq < F'(u^{n+1}), u^{n+1} - u > \leq \\ & \sum_{i=1}^{m} < F'(u^{n+\frac{i}{m}}) - F'(u^{n+1}), u_i - w_i^{n+1} > = \\ & \sum_{i=1}^{m} \sum_{j=i+1}^{m} < F'(u^{n+\frac{j-1}{m}}) - F'(u^{n+\frac{j}{m}}), u_i - w_i^{n+1} > \leq \\ & \sum_{i=1}^{m} \sum_{j=i+1}^{m} ||F'(u^{n+\frac{j-1}{m}}) - F'(u^{n+\frac{j}{m}})||_{V'} ||u_i - w_i^{n+1}|| \leq \\ & \sum_{i=1}^{m} \beta_M(||w_i^{n+1}||) \sum_{i=1}^{m} ||u_i - w_i^{n+1}||. \end{aligned} \tag{2.22}$$

From (2.21) and (2.3) we get that the sequence $\{\sum_{i=1}^{m} ||u_i - w_i^{n+1}||\}_n$ is bounded. Also, from (2.21) and (2.5) we have $\sum_{i=1}^{m} \beta_M(||w_i^{n+1}||) \to 0$ as $n \to \infty$. Consequently, $F(u^{n+1}) - F(u) \to 0$ and $\lambda_M(||u - u^{n+1}||) \to 0$ as $n \to \infty$. Now, from (2.4) and (2.10) it is clear that $u^n \to u$ as $n \to \infty$. □

3 Error estimate

The error estimate essentially stands on the convergence order of the functions $\alpha_M(\tau)$ and $\beta_M(\tau)$ to zero as $\tau \to 0$. In the following we take these functions of polynomial form

$$\alpha_M(\tau) = A_M \tau^p, \quad \beta_M(\tau) = B_M \tau^{q-1}, \tag{3.1}$$

where $A_M > 0$, $B_M > 0$, $p > 1$ and $q > 1$ are some real constants. We have marked here that the constants A_M and B_M depend on M, and we see from (2.8) that we must take $p \geq q$. Now, from (2.10) and (2.11) we get

$$\lambda(\tau) = \frac{A_M}{p} \tau^p, \quad \mu(\tau) = \frac{B_M}{q} \tau^q. \tag{3.2}$$

Naturally, the convergence rate will depend on the spaces $V_1, \cdots, V_m$, and we shall consider the following form of Assumption 2.1 having condition (2.3) slightly modified

ASSUMPTION 3.1. *There exists a constant C_0 such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$, there exist $v_i \in V_i$, $i = 1, \cdots, m$, satisfying*

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K \text{ for } i = 1, \cdots, m, \tag{3.3}$$

$$(3.4) \qquad v - w = \sum_{i=1}^{m} v_i,$$

and

$$(3.5) \qquad \sum_{i=1}^{m} ||v_i||^p \le C_0^p \left(||v-w||^p + \sum_{i=1}^{m} ||w_i||^p \right).$$

In the case of the minimization of the quadratic functionals in [4], the above assumption has been introduced for $p = 2$.

The introduction of some parameters $\varepsilon_{ij} \ge 0$, $i, j = 1, \cdots, m$, is useful to obtain some sharper error estimations, especially in the case of minimization of the quadratic forms. Following this way we shall assume that for a given $M > 0$, if $v \in K$, $||v|| \le M$, and $v_i \in V_i$, satisfying $v + v_i \in K$, $||v + v_i|| \le M$, $i = 1, \cdots, m$, then we have

$$(3.6) \qquad < F'(v+v_i) - F'(v), w_j > \le \varepsilon_{ij} B_M ||v_i||^{q-1} ||w_j||$$

for any $w_i \in V_i$, $i = 1, \cdots, m$. Evidently, using (2.7), we may always take $\varepsilon_{ij} = 1$, $i, j = 1, \cdots, m$, in (3.6).

The following theorem is a generalization for nonlinear inequalities of the result in [37] concerning the convergence of the method for nonlinear equations.

Theorem 3.1. *On the conditions of Theorem 2.1 we consider the functions α_M and β_M defined in (3.1) and we make Assumption 3.1. If u is the solution of problem (2.12) and u^n, $n \ge 0$, are its approximations obtained from Algorithm 2.1, then we have the following error estimations:*

(i) if $p = q$ we have

$$(3.7) \qquad \begin{aligned} F(u^n) - F(u) &\le \left(\frac{\hat{C}}{\hat{C}+1}\right)^n \left[F(u^0) - F(u)\right], \\ ||u^n - u||^p &\le \frac{\hat{C}+1}{\bar{C}} \left(\frac{\hat{C}}{\hat{C}+1}\right)^n \left[F(u^0) - F(u)\right]. \end{aligned}$$

(ii) if $p > q$ we have

$$(3.8) \qquad \begin{aligned} F(u^n) - F(u) &\le \frac{F(u^0)-F(u)}{\left[1+n\tilde{C}(F(u^0)-F(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}, \\ ||u - u^n||^p &\le \frac{\hat{C}}{\bar{C}} \frac{\left(F(u^0)-F(u)\right)^{\frac{q-1}{p-1}}}{\left[1+(n-1)\tilde{C}(F(u^0)-F(u))^{\frac{p-q}{q-1}}\right]^{\frac{(q-1)^2}{(p-1)(p-q)}}}. \end{aligned}$$

The constants $\hat{C}$, $\bar{C}$ and $\tilde{C}$ are given in (3.11), (3.14) and (3.16), respectively.

Proof. As in (2.22), using λ_M given in (3.2), (3.6), and (3.5) in which we take $v_i = u_i$,

$v = u$, $w = u^n$ and $w_i = w_i^{n+1}$, we have

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \tfrac{A_M}{p}||u - u^{n+1}||^p \leq \\
&\sum_{i=1}^{m}\sum_{j=i+1}^{m} \varepsilon_{ij} B_M ||w_j^{n+1}||^{q-1}||u_i - w_i^{n+1}|| \leq \\
&B_M \left(\sum_{i=1}^{m} ||u_i - w_i^{n+1}||^p\right)^{\frac{1}{p}} \left[\sum_{i=1}^{m}\left(\sum_{j=i+1}^{m} \varepsilon_{ij}||w_j^{n+1}||^{q-1}\right)^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \leq \\
&B_M|\varepsilon_{ij}| \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}} \left(\sum_{i=1}^{m} ||u_i - w_i^{n+1}||^p\right)^{\frac{1}{p}} \leq \\
&B_M|\varepsilon_{ij}| \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}} \left[\left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m} ||u_i||^p\right)^{\frac{1}{p}}\right] \leq \\
&B_M|\varepsilon_{ij}| \left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{q-1}{p}} \left[(1 + C_0)\left(\sum_{i=1}^{m} ||w_i^{n+1}||^p\right)^{\frac{1}{p}} + C_0||u - u^n||\right],
\end{aligned}
$$

where we have written

$$
|\varepsilon_{ij}| = \left[\sum_{i=1}^{m}\left(\sum_{j=i+1}^{m} \varepsilon_{ij}^{\frac{p}{p-q+1}}\right)^{\frac{p-q+1}{p-1}}\right]^{\frac{p-1}{p}} \leq m^{2-\frac{q}{p}}. \tag{3.9}
$$

Therefore, using (2.14) with $v = u^n$, (2.20), and λ_M given in (3.2), we have

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \tfrac{A_M}{p}||u - u^{n+1}||^p \leq \\
&B_M(\frac{p}{A_M})^{\frac{q}{p}}|\varepsilon_{ij}| \left(F(u^n) - F(u^{n+1})\right)^{\frac{q-1}{p}} \\
&\left[(1 + C_0)\left(F(u^n) - F(u^{n+1})\right)^{\frac{1}{p}} + C_0\left(F(u^n) - F(u)\right)^{\frac{1}{p}}\right] \leq \\
&B_M(\frac{p}{A_M})^{\frac{q}{p}}|\varepsilon_{ij}| \left(F(u^n) - F(u^{n+1})\right)^{\frac{q-1}{p}} \\
&\left[(1 + 2C_0)\left(F(u^n) - F(u^{n+1})\right)^{\frac{1}{p}} + C_0\left(F(u^{n+1}) - F(u)\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

But, for some given $\eta > 0$ and $\zeta > 0$, we have $\zeta x^{\frac{1}{p}} - \eta x \leq (\frac{\zeta^p}{\eta})^{\frac{1}{p-1}}$, for any $x \geq 0$. Consequently, for a $0 < \eta < 1$, subtracting $\eta(F(u^{n+1}) - F(u))$ from both sides of the last inequality, we get

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \tfrac{A_M}{p(1-\eta)}||u - u^{n+1}||^p \leq \\
&\hat{C}\left[F(u^n) - F(u^{n+1})\right]^{\frac{q-1}{p-1}},
\end{aligned} \tag{3.10}
$$

where

$$
\begin{aligned}
\hat{C} = \hat{C}(m, C_0, u^0) = B_M(\tfrac{p}{A_M})^{\frac{q}{p}}|\varepsilon_{ij}| &\left[(1 + 2C_0)\left(F(u^0) - F(u)\right)^{\frac{p-q}{p(p-1)}} + \right. \\
&\left.\left(B_M(\tfrac{p}{A_M})^{\frac{q}{p}}|\varepsilon_{ij}|\right)^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}}/\eta^{\frac{1}{p-1}}\right]/(1-\eta).
\end{aligned} \tag{3.11}
$$

We have marked above that the constant $\hat{C}$ depends on m, C_0 and u^0, and we have used (2.18) to write $F(u^n) - F(u^{n+1}) \leq F(u^0) - F(u)$. From (3.10) we have

$$[F(u^{n+1}) - F(u)] \leq \hat{C}\left[F(u^n) - F(u^{n+1})\right]^{\frac{q-1}{p-1}}. \tag{3.12}$$

Using again (2.18) we have $F(u^n) - F(u^{n+1}) \leq F(u^n) - F(u)$, and from (2.14) and (3.2) we get $\frac{A_M}{p}||u^{n+1} - u||^p \leq F(u^{n+1}) - F(u)$. From these two last inequalities and (3.10) we get

$$||u - u^{n+1}||^p \leq \frac{\hat{C}}{\bar{C}}\left[F(u^n) - F(u)\right]^{\frac{q-1}{p-1}}, \tag{3.13}$$

where

$$\bar{C} = \frac{(2-\eta)A_M}{(1-\eta)p}. \tag{3.14}$$

Now, if $p = q$, we can easily find (3.7) from (3.12) and (3.13). If $p \neq q$, we get from (3.12) that

$$F(u^{n+1}) - F(u) + \frac{1}{\hat{C}^{\frac{p-1}{q-1}}}[F(u^{n+1}) - F(u)]^{\frac{p-1}{q-1}} \leq F(u^n) - F(u),$$

and applying Lemma 3.2 in [37] we get

$$F(u^{n+1}) - F(u) \leq \left[\tilde{C} + (F(u^n) - F(u))^{\frac{q-p}{q-1}}\right]^{\frac{q-1}{q-p}},$$

or

$$F(u^{n+1}) - F(u) \leq \left[(n+1)\tilde{C} + \left(F(u^0) - F(u)\right)^{\frac{q-p}{q-1}}\right]^{\frac{q-1}{q-p}}, \tag{3.15}$$

where

$$\tilde{C} = \frac{p-q}{(p-1)\left(F(u^0) - F(u)\right)^{\frac{p-q}{q-1}} + (q-1)\hat{C}^{\frac{p-1}{q-1}}}. \tag{3.16}$$

Equation (3.15) is another form of the first estimate in (3.8), and the second one can be obtained using (3.15) and (3.13). The value of η in the the expression of $\hat{C}$ and $\bar{C}$ can be arbitrary in $(0, 1)$. On the other hand, we see that the constants in the error estimations of $F(u^n) - F(u)$ in (3.7) and (3.8) are some increasing functions of $\hat{C}$, and there is an $\eta_0 \in (0, 1)$ such that $\hat{C}(\eta_0) \leq \hat{C}(\eta)$ for any $\eta \in (0, 1)$. However, this value η_0 can be found by solving a nonlinear algebraic equation. □

4 The multiplicative Schwarz method as a subspace correction method

In the previous sections we have proved that the subspace method given in Algorithm 2.1 converges, for a general reflexive Banach space, provided that Assumption 2.1 holds. Also, under a little stronger Assumption 3.1, and some polynomial behaviors of the functions α_M and β_M in the neighborhood of zero, we have given error estimations. We shall prove in the following that for the problems in which we seek for the solution in a Sobolev space, Assumption 3.1 holds (and implicitly, Assumption 2.1, too) for any decomposition of the domain and any convex set described by some constraints on the function values at the points of the domain. In order to more precisely characterize the convex sets K for which our results hold, we assume that they satisfy the following

Property 4.1. *If $v, w \in K$, and if $\theta \in C^1(\Omega)$ with $0 \leq \theta \leq 1$, then $\theta v + (1-\theta)w \in K$.*

Let Ω be an open bounded domain in $\mathbf{R}^d$ with Lipschitz continuous boundary $\partial\Omega$. We take $V = W_0^{1,s}(\Omega)$, $1 < s < \infty$, and a convex closed set $K \subset V$ having the above property. We consider an overlapping decomposition of the domain Ω,

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{4.1}$$

in which Ω_i are open subdomains with Lipschitz continuous boundary. We associate to the domain decomposition (4.1) the subspaces $V_i = W_0^{1,s}(\Omega_i)$, $i = 1, \cdots, m$. In this case, Algorithm 2.1 represents the multiplicative Schwarz method.

Remark 4.1. *For the simplicity, we have chosen the above spaces V and V_i corresponding to Dirichlet boundary conditions. Similar results can be obtained if we consider mixed boundary conditions. We take $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ a partition of the boundary such that meas$(\Gamma_1) > 0$, and we consider the Sobolev space*

$$V = \{v \in W^{1,s}(\Omega) : v = 0 \text{ on } \Gamma_1\}$$

This space corresponds to Dirichlet boundary conditions on Γ_1 and Neumann boundary conditions on Γ_2. The subspaces V_i using domain decomposition (4.1) will be defined in this case as

$$V_i = \{v_i \in W^{1,s}(\Omega) : v_i = 0 \text{ in } \Omega - \bar{\Omega}_i,\ v_i = 0 \text{ in } \partial\Omega_i \cap \Gamma_1\},$$

$i = 1, \cdots, m.$

We shall denote in the following by $||\cdot||_{1,s}$ and $||\cdot||_{0,s}$ the norms in $W^{1,s}(\Omega)$ and $L^s(\Omega)$, respectively, and by $|\cdot|_{1,s}$ the semi-norm in $W^{1,s}(\Omega)$.

Concerning the decomposition (4.1), we assume that there are some functions $\theta_j^i \in C^1(\bar{\Omega})$, $i = 1, \cdots, m$, $j = i, \cdots, m$ such that for any $i = 1, \cdots, m$ we have

$$\operatorname{supp}(\theta_j^i) \subset (\bar{\Omega}_j),\ 0 \leq \theta_j^i \leq 1, \text{ for any } j = i, \cdots, m, \text{ and } \sum_{j=i}^{m} \theta_j^i \equiv 1 \text{ in } \bigcup_{j=i}^{m} \Omega_j. \tag{4.2}$$

This is a easy enough constraint on the domain decomposition (4.1). In [23] or [1], for instance, some conditions in which a domain decomposition satisfies (4.2) are given. The following result assures us that Assumption 3.1 holds for the above defined domain decomposition and convex sets. The proof is similar to that given in [2].

Proposition 4.1. *If the domain decomposition (4.1) satisfies (4.2), then Assumption 3.1 holds for any convex set K having Property 4.1.*

Proof. Let us consider $w \in K$, $w_i \in V_i$ such that $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \cdots, m$ and let v be another element in K. We recursively construct $v_i \in V_i$, $i = 1, \cdots, m$, satisfying (3.3)–(3.5) in Assumption 3.1.

We take

$$v_1 = \theta_1^1(v - w) + (1 - \theta_1^1)w_1. \tag{4.3}$$

Because supp$\theta_1^1 \subset \bar{\Omega}_1$ and $w + v_1 = \theta_1^1 v + (1 - \theta_1^1)(w + w_1)$ we have

$$v_1 \in V_1 \text{ and } w + v_1 \in K. \tag{4.4}$$

Since $v - v_1 + w_1 = (1-\theta_1^1)v + \theta_1^1(w + w_1)$ we have

$$v - v_1 + w_1 \in K. \tag{4.5}$$

Since $1 - \theta_1^1 = \theta_2^1 + \cdots + \theta_m^1$ we get $v - w - v_1 = (\theta_2^1 + \cdots + \theta_m^1)(v - w + w_1)$ and hence

$$\begin{aligned} & v - w - v_1 \in W_0^{1,s}(\bigcup_{i=2}^{m} \Omega_i), \\ & v - w - v_1 = 0 \text{ in } \Omega \backslash \overline{\cup_{i=2}^{m} \Omega_i}. \end{aligned} \tag{4.6}$$

In the following, for the domain $\cup_{i=2}^{m}\Omega_i$, we take $v - w - v_1$ in the place of $v - w$.

Assume now that up to an $i = 1, \cdots, m-2$ we have defined $v_1 \in V_1, \cdots, v_i \in V_i$ and we have the following relations corresponding to (4.4)–(4.6),

$$v_i \in V_i \text{ and } v_i + w + \sum_{j=1}^{i-1} w_j \in K, \tag{4.7}$$

$$v - \sum_{j=1}^{i} v_j + \sum_{j=1}^{i} w_j \in K, \tag{4.8}$$

$$\begin{aligned} & v - w - \sum_{j=1}^{i} v_j \in W_0^{1,s}(\bigcup_{j=i+1}^{m} \Omega_j), \\ & v - w - \sum_{j=1}^{i} v_j = 0 \text{ in } \Omega - \overline{\cup_{j=i+1}^{m} \Omega_j}. \end{aligned} \tag{4.9}$$

In the following we verify relations (4.7)–(4.9) which correspond to $i+1$. We define

$$v_{i+1} = \theta_{i+1}^{i+1}(v - w - \sum_{j=1}^{i} v_j) + (1 - \theta_{i+1}^{i+1}) w_{i+1}. \tag{4.10}$$

From (4.10) we get $v_{i+1} \in V_{i+1}$. Also, using (4.8) we get $v_{i+1} + w + \sum_{j=1}^{i} w_j = \theta_{i+1}^{i+1}(v - \sum_{j=1}^{i} v_j + \sum_{j=1}^{i} w_j) + (1 - \theta_{i+1}^{i+1})(w + \sum_{j=1}^{i+1} w_j) \in K$. Therefore, (4.7) corresponding to $i+1$ holds.

Using (4.8) corresponding to i, we get $v - \sum_{j=1}^{i+1} v_j + \sum_{j=1}^{i+1} w_j = (1 - \theta_{i+1}^{i+1})(v - \sum_{j=1}^{i} v_j + \sum_{j=1}^{i} w_j) + \theta_{i+1}^{i+1}(w + \sum_{j=1}^{i+1} w_j) \in K$. Therefore, (4.8) which corresponds to $i+1$ holds.

We have $v - w - \sum_{j=1}^{i+1} v_j = (1 - \theta_{i+1}^{i+1})(v - w - \sum_{j=1}^{i} v_j - w_{i+1})$. From the definition of θ_{i+1}^{i+1} and (4.9) we get that (4.9) corresponding to $i+1$ holds.

In this way we have proved that (4.7)–(4.9) hold for any $i = 1, \cdots, m-1$.

Now, for $i = m$, from (4.9), with $i = m-1$, we get $v - w - \sum_{j=1}^{m-1} v_j \in W_0^{1,s}(\Omega_m)$ and it vanishes in $\Omega - \bar{\Omega}_m$. We define

$$v_m = v - w - \sum_{j=1}^{m-1} v_j, \tag{4.11}$$

and we see that $v_m \in V_m$. Hence, (3.4) in Assumption 3.1 holds. Also, (4.4) and (4.7), for $i = 2, \cdots, m-1$, prove that (3.3) in Assumption 3.1 holds for $i = 1, \cdots, m-1$. Moreover, from (4.8) for $i = m-1$ we get $v_m + w + \sum_{j=1}^{m-1} w_j = v - \sum_{j=1}^{m-1} v_j + \sum_{j=1}^{m-1} w_j \in K$, and consequently, (3.3) in Assumption 3.1 holds for $i = m$, too. Finally, from (4.3), (4.10) and (4.11) we obtain that (3.5) in Assumption 3.1 holds, where C_0 depends on unity partitions (4.2), but it is independent of w, v, w_i and v_i. □

5 One and two level multiplicative Schwarz method

Since the finite element spaces have a finite dimension, the existence of the functions α_M and β_M is assured if we assume that the functional F is strictly convex and continuously differentiable. As we saw in Proposition 4.1, Assumption 3.1 holds for any closed convex K having Property 4.1, but the constant C_0 depends on the domain decomposition parameters. Consequently, since the constants $\hat{C}$ and $\bar{C}$ in the error estimations in Theorem 3.1 depend on C_0, then these estimations will depend on domain decomposition parameters, too. The goal of this section is to prove, for the one and two level multiplicative Schwarz methods, that Assumption 3.1 also holds for any closed convex K satisfying Property 4.1. In these cases we can explicitly write the dependence of C_0 on the domain decomposition and mesh parameters.

5.1 One-level multiplicative Schwarz method

Let us consider first that the domain $\Omega \subset \mathbf{R}^d$ has a non overlapping domain decomposition $\{O_i\}_{1 \le i \le M}$ and a simplicial mesh partition $\mathcal{T}_h$ of mesh sizes h. We assume that $\mathcal{T}_h$ is regular (ie. there exists a constant $C > 0$, independent of h, such that each τ in $\mathcal{T}_h$ contains a ball with the diameter of Ch, and evidently, it is contained in a ball with the diameter of h; see [8], pag. 124, for instance) and it supplies a mesh partition for each subdomain O_i, $i = 1, \cdots, M$, too. For each O_i, we consider an enlarged subdomain $O_i^\delta \subset \Omega$, consisting of the elements $\tau \in \mathcal{T}_h$ with $\mathrm{dist}(\tau, O_i) \le \delta$, where δ is a positive real number. In this way, $\{O_i^\delta\}_{1 \le i \le M}$ is an overlapping domain decomposition of Ω with overlaps of size δ. We assume that there exist m colors such that each subdomain O_i^δ can be marked with one color, and the subdomains with the same color do not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1$, $m \le 4$ if $d = 2$, and $m \le 8$ if $d = 3$. Let Ω_i be the union of the subdomains O_j^δ having the color i. In this way we have obtained an overlapping decomposition (4.1) with overlaps of size δ, and we can take the unity partitions $\{\theta_j^i\}_{j=i,\cdots,m}$, $i = 1, \cdots, m$, defined in (4.2), to satisfy

$$(5.1) \qquad |\partial_{x_k} \theta_j^i| \le C/\delta, \quad \text{for any } i = 1, \cdots, m, \ j = i, \cdots, m, \text{ and } k = 1, \cdots, d,$$

too. As in (5.1), we denote in the following by C a generic constant which does not depend on either the mesh or the domain decomposition parameters.

In this section we prove for the finite element spaces a similar result to that given in Proposition 4.1 for general Sobolev spaces. The proof is also similar to that given in [4] for the minimization of the quadratic forms. We consider the piecewise linear finite element space

$$(5.2) \qquad V^h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau), \ \tau \in \mathcal{T}_h, \ v = 0 \text{ on } \partial\Omega\},$$

and also, for $i = 1, \cdots, m$, we take

$$(5.3) \qquad V_i^h = \{v \in C^0(\bar{\Omega}) : v|_\tau \in P_1(\tau), \ \tau \in \mathcal{T}_h, \ v = 0 \text{ in } \Omega \backslash \Omega_i\}$$

as some subspaces of V^h corresponding to the domain decomposition $\Omega_1, \cdots, \Omega_m$. The spaces V^h and V_i^h, $i = 1, \cdots, m$, are considered as subspaces of $W^{1,s}$, for some fixed $1 \leq s \leq \infty$, and we use the usual norms $||\cdot||_{0,s}$, $||\cdot||_{1,s}$, and the seminorm $|\cdot|_{1,s}$. Contrary to the previous section, we may use the norm of $W^{1,s}$ for $s = 1$ and $s = \infty$ because the finite element spaces have a finite dimension, and consequently, they are reflexive Banach spaces.

In the following, I_h will always be the P_1–Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_h$. The convex set K^h will be defined as a subset of V^h by some constraints acting at the mesh nodes and satisfying

PROPERTY 5.1. *If $v, w \in K^h$, and if $\theta \in C^1(\Omega)$ with $0 \leq \theta \leq 1$, then $I_h(\theta v + (1-\theta)w) \in K^h$.*

In order to prove that Assumption 3.1 also holds in this case, we follow the same way as in the the proof of Proposition 4.1. It is easy to see that, taking into account the additivity of the Lagrangian interpolation I_h, (3.3) and (3.4) in Assumption 3.1 can be proved taking

$$v_1 = I_h\left(\theta_1^1(v-w) + (1-\theta_1^1)w_1\right) \tag{5.4}$$

and

$$v_{i+1} = I_h\left(\theta_{i+1}^{i+1}(v - w - \sum_{j=1}^{i} v_j) + (1-\theta_{i+1}^{i+1})w_{i+1}\right) \tag{5.5}$$

in the place of v_1 and v_{i+1} defined in (4.3) and (4.10), respectively. Evidently, we consider in the new proof the spaces V^h and V_i^h and the convex set K^h in the place of V, V_i and K, respectively. Also, we keep the same definition, (4.11), for v_m. To prove inequality (3.5) in Assumption 3.1, we first notice that, starting from v_1 given in (5.4), by the recurrent application of (5.5), and then taking v_m given in (4.11), we get that v_i, $i = 1, \cdots, m$, are of the form

$$v_i = I_h\left(\tau_0^i(v-w) + \sum_{j=1}^{i} \tau_j^i w_j\right), \ i = 1, \cdots, m. \tag{5.6}$$

By a simple calculus we get that

$$\begin{aligned}
&\tau_0^1 = \theta_1^1,\ \tau_1^1 = 1 - \theta_1^1,\\
&\tau_0^i = \theta_i^i(1-\theta_{i-1}^{i-1})\cdots(1-\theta_1^1),\ \tau_i^i = 1-\theta_i^i,\ \tau_j^i = -\theta_i^i(1-\theta_{i-1}^{i-1})\cdots(1-\theta_j^j),\\
&\text{for } i = 2, \cdots, m-1,\ j = 1, \cdots, i-1,\\
&\tau_0^m = (1-\theta_{m-1}^{m-1})\cdots(1-\theta_1^1),\ \tau_m^m = 0, \tau_{m-1}^m = -(1-\theta_{m-1}^{m-1}),\\
&\tau_j^m = \theta_{m-1}^{m-1}(1-\theta_{m-2}^{m-2})\cdots(1-\theta_j^j),\ \text{for } j = 1, \cdots, m-2.
\end{aligned}$$

Consequently, from (4.2) and (5.1), we have

$$|\tau_j^i| \leq 1 \text{ and } |\partial_{x_k}\tau_j^i| \leq C(m-1)/\delta,\ i = 1, \cdots, m,\ j = 0, \cdots, i,\ k = 1, \cdots, d. \tag{5.7}$$

For a $v \in V^h$, we can get (see Theorem 3.1.6, in [8], pag. 124, for instance) that

$$||\tau_j^i v - I_h(\tau_j^i v)||_{0,s} \leq Ch|\tau_j^i v|_{1,s},\ ||\tau_j^i v - I_h(\tau_j^i v)||_{1,s} \leq C|\tau_j^i v|_{1,s},$$

and therefore

$$||I_h(\tau_j^i v)||_{1,s} \leq C||\tau_j^i v||_{1,s},\ \text{with } v \in V^h, \tag{5.8}$$

for any $i = 1, \cdots, m$, $j = 0, \cdots, i$. On the other hand, from (5.7) we get

$$||\tau_j^i v||_{0,s} \le ||v||_{0,s}, \quad |\tau_j^i v|_{1,s} \le C(|v|_{1,s} + \frac{m-1}{\delta}||v||_{0,s}), \text{ for any } v \in W^{1,s}(\Omega),$$

and therefore, using (5.8), we get

$$||I_h(\tau_j^i v)||_{1,s} \le C(||v||_{1,s} + \frac{m-1}{\delta}||v||_{0,s}), \text{ for any } v \in V^h, \tag{5.9}$$

where τ_j^i, $i = 1, \cdots, m$, $j = 0, \cdots, i$, are given in (5.7). Now, by a application of (5.9) to (5.6) we get

$$||v_i||_{1,s} \le C(1 + \frac{m-1}{\delta}) \left(||v - w||_{1,s} + \sum_{j=1}^{i} ||w_j||_{1,s} \right), \text{ for any } i = 1, \cdots, m. \tag{5.10}$$

Using this equation we get (3.5) in Assumption 3.1, and we have

Proposition 5.1. *Let (4.1) be the overlapping domain decomposition of the domain Ω with overlaps of size δ which has been defined in this section starting from the non overlapping domain decomposition $\{O_i\}_{i=1,\cdots,M}$. Then, Assumption 3.1 holds in the piecewise linear finite element spaces, $V = V^h$ and $V_i = V_i^h$, $i = 1, \cdots, m$, for any convex set K^h defined by constraints acting on the function values at the mesh nodes of $\mathcal{T}_h$ and having Property 5.1. The constant in (3.5) of Assumption 3.1 can be taken of the form*

$$C_0 = C(m+1)^{\frac{p+1}{p}}(1 + \frac{m-1}{\delta}), \tag{5.11}$$

where C is independent of the mesh parameters and the domain decomposition.

Remark 5.1. *We notice that the number m of the subdomains Ω_i in the decomposition of Ω is in fact the number of the colors of the overlapping domain decomposition $\{O_i^\delta\}_{1\le i\le M}$, and it depends only on the dimension d of the space $\mathbf{R}^d$. Consequently, error estimations (3.7) and (3.8) in Theorem 3.1 depend only on the size δ of the overlaps through the intermediary of the constant C_0 given in (5.11).*

5.2 Two-level multiplicative Schwarz method

We consider a simplicial mesh partition $\mathcal{T}_h$ of the domain $\Omega \subset \mathbf{R}^d$ of a mesh size h, and a simplicial coarser mesh $\mathcal{T}_H$ with a mesh size H, $\mathcal{T}_h$ being a refinement of $\mathcal{T}_H$. The mesh size h is assumed to approach zero and we shall consider a family of mesh pairs (h, H). We assume that both the families, of fine meshes and coarse meshes are regular.

With h and H fixed, using the coarse mesh $\mathcal{T}_H$ we consider some non overlapping subsets $\{O_i\}_{1\le i\le M}$ of Ω, each subdomain O_i being an union of elements $\tau \in \mathcal{T}_H$, and we assume that there exists a constant C, which is independent of H, such that

$$\text{diameter}(O_i) \le CH, \ i = 1, \cdots, M. \tag{5.12}$$

Using the finer mesh $\mathcal{T}_h$ and the subsets $\{O_i\}_{1\le i\le M}$, we construct the overlapping subsets $\{O_i^\delta\}_{1\le i\le M}$ and $\{\Omega_i\}_{1\le i\le m}$ of Ω, with overlaps of size δ, as in the previous section. Here we assume that $\bar{O} = \cup_{i=1}^M \bar{O}_i$ might be different from $\bar{\Omega}$, but the overlapping subdomains $\{O_i^\delta\}_{1\le i\le M}$ and $\{\Omega_i\}_{1\le i\le m}$ cover Ω.

Now, we introduce the continuous, piecewise linear finite element space corresponding to the H-level

$$V_0^H = \left\{v \in C^0(\bar{O}) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial O\right\}, \tag{5.13}$$

and extending the functions of V_0^H with zero in $\Omega\backslash O$, it becomes a subspace of V^h. As in the previous section, the convex set $K^h \subset V^h = V$ is defined by constraints acting at the mesh nodes of V^h and having Property 5.1.

The two-level Schwarz method is also obtained from Algorithm 2.1 in which we take $V = V^h$, $K = K^h$ and the subspaces $V_0 = V_0^H$, $V_1 = V_1^h$, $V_2 = V_2^h, \cdots, V_m = V_m^h$. As in the previous subsection, the spaces V^h, V_0^H, V_1^h, $V_2^h, \cdots, V_m^h$, are considered as subspaces of $W^{1,s}$ for $1 \le s \le \infty$. We notice that this time the decomposition of the domain Ω contains m overlapping subdomains, but we use $m+1$ subspaces of V, $V_0, V_1, \cdots, V_m$, in Algorithm 2.1. Naturally, this algorithm will converge in this case if Assumption 2.1 or its stronger form, Assumption 3.1, written for $m+1$ subspaces, will be satisfied for the above choice of the convex set K and the subspaces $V_0, V_1, \cdots, V_m$, of V. As in the previous subsection we prove that Assumption 3.1 holds and find the constant C_0 depending on the mesh and domain decomposition parameters. First, we have the following lemma in which inequality (5.14) can be viewed as one of Friedrichs-Poincaré type for the finite element spaces.

Lemma 5.1. *Let $\omega \subset \mathbf{R}^d$ be a domain of diameter H, and ω_i, $i = 0, 1, \cdots, N$, be an overlapping decomposition of it, $\omega = \cup_{i=0}^N \omega_i$. We consider a simplicial regular mesh partition $\mathcal{T}_h$ of ω and assume that it supplies a mesh partition for ω_i, $i = 0, 1, \cdots, N$, too. Let $x^0 \in \bar{\omega}_0$ be a node of $\mathcal{T}_h$. We assume that the overlapping partition of ω satisfies:*

(i) for any $x \in \bar{\omega}_0$, the line segment $[x^0, x]$ lies in $\bar{\omega}_0$,

(ii) for $N > 0$, if $\omega_i \cap \omega_j \neq \emptyset$, $0 \le i \neq j \le N$, then for any $x \in \bar{\omega}_i$, $y \in \bar{\omega}_j$ and $z \in \bar{\omega}_i \cap \bar{\omega}_j$, the line segments $[x, z]$ and $[y, z]$ lie in $\bar{\omega}_i$ and $\bar{\omega}_j$, respectively.

On these conditions, if v is a continuous function which is linear on each $\tau \in \mathcal{T}_h$, and $v(x^0) = 0$, then

$$||v||_{0,s,\omega} \le C(N,s)C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega}, \tag{5.14}$$

where

$$C_{d,s}(H,h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or } 1 \le d < s \le \infty \\ \left(\ln\frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-s}{s}} & \text{if } 1 \le s < d < \infty, \end{cases} \tag{5.15}$$

$$C(d,s) = \begin{cases} C & \text{if } d = s = 1 \text{ or } 1 = s < d < \infty \\ C\left(d\frac{s-1}{s-d}\right)^{\frac{s-1}{s}} & \text{if } 1 \le d < s \le \infty \\ Cd^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ C(d\frac{s-1}{d-s})^{\frac{s-1}{s}} & \text{if } 1 < s < d < \infty. \end{cases} \tag{5.16}$$

and

$$C(N,s) = \begin{cases} 1 & \text{if } N = 0 \\ \text{if } (N+1)\frac{C_\omega^{(N+1)/s}-1}{C_\omega^{1/s}-1} & \text{if } N \neq 0 \end{cases} \tag{5.17}$$

with

$$C_\omega = \max_{\omega_i \cap \omega_j \neq \emptyset} \frac{|\omega_i|}{|\omega_i \cap \omega_j|} \tag{5.18}$$

In (5.18) we have denoted by $|\ \ |$ *the measure of a set, and we have marked in (5.14) that the norm and the semi-norm in* $W^{1,s}$, $1 \le s \le \infty$, *refer to the subdomain* ω. *The constant* C *in (5.16) is independent of* H, h, d, s *and the decomposition of* ω.

Proof. In this proof, we use the polar coordinates. The Jacobian determinant of the transformation from the rectangular coordinates to the polar coordinates can be written as

$$J(r, \varphi) = r^{d-1} E(\varphi),$$

where $E(\varphi)$ is an algebraic expression of cosines and sines of the component angles of φ.

We first consider that $N = 0$, ie. the decomposition of ω in the statement of the lemma has only one element, $\omega_0 = \omega$. Consequently, for any $x \in \bar{\omega}$, the line segment $[x^0, x]$ lies in $\bar{\omega}$. We take the origin of the system of coordinates at the point x^0, and, using the polar coordinates, a point $x = (x_1, \cdots, x_d)$, will be written as $x = (r, \varphi)$, φ being the system of $d-1$ angles giving the direction of the vector x. We denote by r_φ the maximum size of the radius in the direction φ of the points in $\bar{\omega}$, and consequently, the points on $\partial\omega$ will be written as (r_φ, φ). We denote by o the union of the $\tau \in \mathcal{T}_h$ having a vertex at x^0, let r_0 be the distance from x^0 to $\partial o \backslash \partial \omega$, and we consider the open ball with the center at x^0 of radius r_0, $B_{r_0}(x^0)$. For two points $x' = (r', \varphi) \in \omega \cap B_{r_0}(x^0)$ and $x = (r, \varphi) \in \omega \backslash \bar{B}_{r_0}(x^0)$, we have

(5.19)

$$\begin{aligned}
&|v(x)| = |v(r,\varphi)| \le |v(r',\varphi)| + |\textstyle\int_{r'}^{r} \frac{\partial v}{\partial r}(\rho,\varphi) d\rho| = |\frac{\partial v}{\partial r}(r',\varphi)| r' + |\int_{r'}^{r} \frac{\partial v}{\partial r}(\rho,\varphi) d\rho| \le \\
&|\nu_1 \tfrac{\partial v}{\partial x_1}(r',\varphi) + \cdots + \nu_d \tfrac{\partial v}{\partial x_d}(r',\varphi)| r' + |\textstyle\int_{r'}^{r_\varphi} \left(\nu_1 \frac{\partial v}{\partial x_1}(\rho,\varphi) + \cdots + \nu_d \frac{\partial v}{\partial x_d}(\rho,\varphi)\right) d\rho| \le \\
&(|\tfrac{\partial v}{\partial x_1}(r',\varphi)| + \cdots + |\tfrac{\partial v}{\partial x_d}(r',\varphi)|) r' + \textstyle\int_{r'}^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right) d\rho,
\end{aligned}$$

where $(\nu_1, \cdots, \nu_d)$ is the unity vector giving the direction of $x = (r, \varphi)$ in the rectangular system of coordinates $(x_1, \cdots, x_d)$. In the following we shall start from (5.19) for the various values of d and s.

For $d = s = 1$ or $1 \le d < s \le \infty$, we take $r' = 0$ in (5.19). If $d = s = 1$ we get

$$|v(x)| = |v(r,\varphi)| \le \int_0^{r_\varphi} |\frac{\partial v}{\partial x_1}(\rho,\varphi)| d\rho.$$

Here, we may have $\varphi = 0$ and $\varphi = \pi$ if x^0 is a inner point in ω, and only $\varphi = 0$ or $\varphi = \pi$ if $x^0 \in \partial\omega$. Integrating again from 0 to $r_\varphi \le H$, we get (5.14) for $N = 0$ and $d = s = 1$. If $1 \le d < s = \infty$, we have

$$|v(x)| \le r_\varphi d \max_{1 \le j \le d} \sup_{0 \le \rho \le r_\varphi} |\frac{\partial v}{\partial x_j}(\rho,\varphi)| \le CdH|v|_{1,\infty,\omega}.$$

If $1 \le d < s < \infty$ we have

$$|v(x)|^s \le d^{s-1} \left[\int_0^{r_\varphi} \rho^{\frac{1-d}{s-1}} d\rho\right]^{s-1} \int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho.$$

Multiplying the above inequality by r^{d-1} and integrating from 0 to $r_\varphi \leq H$ we get

$$\int_0^{r_\varphi} |v(r,\varphi)|^s r^{d-1} dr \leq$$
$$\left(d\frac{s-1}{s-d}\right)^{s-1} (CH)^s \int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho.$$

By multiplication of this equation with the Jacobian part depending on φ, $E(\varphi)$, and integrating over the $d-1$ dimensional domain of the angles φ, we get (5.14) for $N=0$ and $1 \leq d < s < \infty$.
Now, from (5.19) for an arbitrary $0 < r' < r_0$, we get

$$|v(x)| = |v(r,\varphi)| \leq$$
$$(|\frac{\partial v}{\partial x_1}(r',\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(r',\varphi)|) r_0 + \int_{r_0}^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right) d\rho. \tag{5.20}$$

Also, since for a fixed φ, $\frac{\partial v}{\partial r}(r',\varphi)$ is constant for $r' \in (0, r_0)$, we have

$$|v(x')|^s = |v(r',\varphi)|^s \leq \frac{(r')^{s-d}}{d} \int_0^{r_0} |\frac{\partial v}{\partial \rho}(\rho,\varphi)|^s \rho^{d-1} d\rho \leq$$
$$d^{s-2} (r')^{s-d} \int_0^{r_0} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho.$$

Multiplying the above inequality by $(r')^{d-1}$, and integrating from 0 to r_0, we get

$$\int_0^{r_0} |v(\rho,\varphi)|^s \rho^{d-1} d\rho \leq \frac{d^{s-2}}{s} r_0^s \int_0^{r_0} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho. \tag{5.21}$$

Now, if $1 = s < d < \infty$ we get from (5.20),

$$|v(x)| \leq \frac{1}{d} r_0^{1-d} \int_0^{r_0} (|\frac{\partial v}{\partial x_1}(\rho,\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|) \rho^{d-1} d\rho +$$
$$r_0^{1-d} \int_{r_0}^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right) \rho^{d-1} d\rho \leq$$
$$r_0^{1-d} \int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right) \rho^{d-1} d\rho.$$

Using the regularity of the mesh $\mathcal{T}_h$, we have $\frac{r_\varphi}{r} \leq C\frac{H}{h}$, and therefore,

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)| \rho^{d-1} d\rho \leq CH \left(\frac{H}{h}\right)^{d-1} \int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)| + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|\right) \rho^{d-1} d\rho.$$

From this last inequality and (5.21) we get (5.14) for $N = 0$ and $1 = s < d < \infty$ by a multiplication with $E(\varphi)$ and integrating over the domain of the angles φ.
Starting again from (5.20), for $1 < d = s < \infty$ or $1 < s < d < \infty$, we get

$$|v(x)|^s \leq (2d)^{s-1} (|\frac{\partial v}{\partial x_1}(r',\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(r',\varphi)|^s) r_0^s +$$
$$(2d)^{s-1} \left[\int_{r_0}^{r_\varphi} \rho^{\frac{1-d}{s-1}} d\rho\right]^{s-1} \int_{r_0}^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho =$$
$$2^{s-1} d^s r_0^{s-d} \int_0^{r_0} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho +$$
$$(2d)^{s-1} \left[\int_{r_0}^{r_\varphi} \rho^{\frac{1-d}{s-1}} d\rho\right]^{s-1} \int_{r_0}^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho.$$

Consequently,

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^s \rho^{d-1} d\rho \leq$$
$$(2d)^{s-1} r_\varphi^d r_0^{s-d} \int_0^{r_0} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho +$$
$$2^{s-1} d^{s-2} r_\varphi^d \left[\int_{r_0}^{r_\varphi} \rho^{\frac{1-d}{s-1}} d\rho\right]^{s-1} \int_{r_0}^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho. \tag{5.22}$$

Now, from (5.22), if $1 < d = s < \infty$ we get

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^d \rho^{d-1} d\rho \leq$$
$$2^{d-1} r_\varphi^d \max\left\{ d^{d-1}, d^{d-2}\left(\ln \frac{r_\varphi}{r_0}\right)^{d-1}\right\}$$
$$\int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^d + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^d\right) \rho^{d-1} d\rho.$$

Using regularity of the mesh $\mathcal{T}_h$, we get

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^d \rho^{d-1} d\rho \leq$$
$$d^{d-1} (CH)^d \left(\ln \frac{H}{h} + 1\right)^{d-1} \int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^d + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho.$$

This inequality together with (5.21) prove (5.14) for $N = 0$ and $1 < d = s < \infty$. Finally, if $1 < s < d < \infty$, we get from (5.22),

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^s \rho^{d-1} d\rho \leq 2^{s-1} \max\left\{ d^{s-1} r_\varphi^d r_0^{s-d}, d^{s-2} r_\varphi^s (\tfrac{s-1}{d-s})^{s-1}\left[\left(\frac{r_\varphi}{r_0}\right)^{\frac{d-s}{s-1}} - 1\right]^{s-1}\right\}$$
$$\int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho,$$

and consequently,

$$\int_{r_0}^{r_\varphi} |v(\rho,\varphi)|^s \rho^{d-1} d\rho \leq$$
$$(d\tfrac{s-1}{d-s})^{s-1} (CH)^s \left(\frac{H}{h}\right)^{d-s} \int_0^{r_\varphi} \left(|\frac{\partial v}{\partial x_1}(\rho,\varphi)|^s + \cdots + |\frac{\partial v}{\partial x_d}(\rho,\varphi)|^s\right) \rho^{d-1} d\rho.$$

Using again (5.21) and the last inequality we get (5.14) and for $N = 0$ and $1 < s < d < \infty$.

Assume now that $N > 0$, ie. we have more than one subdomain ω_i, $i = 0, 1, \cdots, N$ in the overlapping decomposition of ω. Such a decomposition is considered when there exist points $x \in \bar{\omega}$ for which the line segment $[x^0, x]$ do not wholly lie in $\bar{\omega}$. Let ω_i and ω_j, $i \neq j$, be two fixed subdomains such that $\omega_i \cap \omega_j \neq \emptyset$. We consider a fixed point $z \in \omega_i \cap \omega_j$, and denoting by z^k and ϕ_k, the nodes of $\mathcal{T}_h$ in $\bar{\omega}_i \cap \bar{\omega}_j$ and the corresponding functions in the nodal basis, respectively, for an $1 \leq s < \infty$, we have

$$||v||_{0,s,\omega_j} - |\sum_k v(z^k)\phi_k(z)||\omega_j|^{1/s} \leq ||v - \sum_k v(z^k)\phi_k(z)||_{0,s,\omega_j} =$$
$$||\sum_k \left(v - v(z^k)\right)\phi_k(z)||_{0,s,\omega_j} \leq \sum_k ||v - v(z^k)||_{0,s,\omega_j}\phi_k(z).$$

Since $v - v(z^k)$ vanishes at z^k, we get from the first part of the proof and the last equation that

$$||v||_{0,s,\omega_j} - |\sum_k v(z^k)\phi_k(z)||\omega_j|^{1/s} \leq \sum_k C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega_j}\phi_k(z) =$$
$$C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega_j},$$

and integrating over $\omega_i \cap \omega_j$, we get

$$|\omega_i \cap \omega_j| ||v||_{0,s,\omega_j} \leq |\omega_j|^{1/s} \int_{\omega_i \cap \omega_j} |v| + |\omega_i \cap \omega_j| C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega_j} \leq$$
$$|\omega_j|^{1/s} |\omega_i \cap \omega_j|^{(s-1)/s} ||v||_{0,s,\omega_i\cap\omega_j} + |\omega_i \cap \omega_j| C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega_j}.$$

Consequently, we have

$$(5.23) \qquad ||v||_{0,s,\omega_j} \le \left(\frac{|\omega_j|}{|\omega_i \cap \omega_j|}\right)^{1/s} ||v||_{0,s,\omega_i} + C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega_j}.$$

It is easy to see that equation (5.23) holds for $s = \infty$, too. Taking into account that

$$(5.24) \qquad ||v||_{0,s,\omega_0} \le C(d,s)HC_{d,s}(H,h)|v|_{1,s,\omega_0},$$

from (5.23) and (5.24), we get (5.14) for $N > 0$. □

Remark 5.2. *As we have said at the beginning of this subsection, we are interested in the error estimation when $H, h \to 0$. In general, since the mesh $\mathcal{T}_h$ is regular, the overlapping decomposition of ω in Lemma 5.1 can be taken such that the number N and the constant C_ω in (5.18) are bounded when $H, h \to 0$. In this point of view, the constants $C(d,s)$, $C(N,s)$ and C_ω, written in (5.16)-(5.18), can be considered as independent of H and h, and assimilated to the generic constant C. In the following we write (5.14) as*

$$(5.25) \qquad ||v||_{0,s,\omega} \le CHC_{d,s}(H,h)|v|_{1,s,\omega},$$

where $C = C(N,s)C(d,s)$ and $C_{d,s}(H,h)$ is given in (5.15).

The above lemma can be very useful in the various error estimations. The following result, for instance, generalizes that in Lemma 2.3 in [6].

Corollary 5.1. *Let ω be a subdomain of diameter H with a simplicial regular mesh partition $\mathcal{T}_h$. If v is a continuous function which is linear on each $\tau \in \mathcal{T}_h$, then for any $1 \le s \le \infty$ we have*

$$(5.26) \qquad ||v||_{0,\infty,\omega} \le CH^{\frac{s-d}{s}}C_{d,s}(H,h)||v||_{1,s,\omega},$$

where $C_{d,s}(H,h)$ is given in (5.15), and C is independent of H and h.

Proof. Let $x^0 \in \bar{\omega}$ be the point where $|v(x^0)| = ||v||_{0,\infty,\omega}$, and $x \in \omega$ a current point. We point out that x^0 is a node of $\mathcal{T}_h$. For $1 \le s < \infty$, we have

$$|v(x^0)|^s \le 2^{s-1}|v(x^0) - v(x)|^s + 2^{s-1}|v(x)|^s,$$

and integrating it over ω, using (5.26), we get

$$|\omega|||v||^s_{0,\infty,\omega} \le 2^{s-1}||v(x^0) - v(x)||^s_{0,s,\omega} + 2^{s-1}||v(x)||^s_{0,s,\omega} \le$$
$$2^{s-1}\left(CHC_{d,s}(H,h)\right)^s |v(x)|^s_{1,s,\omega} + 2^{s-1}||v(x)||^s_{0,s,\omega}.$$

If $s = \infty$, the proof is similar. □

Coming back to the two-level method, let us denote as above by x^i a node of $\mathcal{T}_H$, by ϕ_i the nodal basis functions in V_0^H associated with x^i, and by ω_i the support of ϕ_i. Given a $v \in V^h$, let $I_i^- v = \min_{x\in\bar{\omega}_i} v(x)^-$ and $I_i^+ v = \min_{x\in\bar{\omega}_i} v(x)^+$, where $v(x)^- = \max(0, -v(x))$ and $v(x)^+ = \max(0, v(x))$. Since v is piecewise linear, $I_i^- v$ and $I_i^+ v$ are attained at a node of $\mathcal{T}_h$. For a $v \in V^h$, we define

$$(5.27) \qquad I_H^- v := \sum_{x^i \text{ node of } \mathcal{T}_H} (I_i^- v)\phi_i(x) \text{ and } I_H^+ v := \sum_{x^i \text{ node of } \mathcal{T}_H} (I_i^+ v)\phi_i(x),$$

and we write

$$(5.28) \qquad I_H v = I_H^+ v - I_H^- v.$$

The following result extends that given in [34] where the operators I_i^- have been introduced.

Lemma 5.2. *For any $v \in V^h$, we have*

$$||I_H v - v||_{0,s} \le CHC_{d,s}(H,h)|v|_{1,s} \tag{5.29}$$

and

$$||I_H v||_{1,s} \le CC_{d,s}(H,h)||v||_{1,s}, \tag{5.30}$$

where $C_{d,s}(H,h)$ is defined in (5.15) and C is independent of H and h. Moreover, if $\mathcal{K}$ is a convex and closed set in V^h, described by some conditions at mesh nodes of $\mathcal{T}_h$, having Property 5.1, and $0 \in \mathcal{K}$, then for any $v \in \mathcal{K}$ we have $I_H v \in \mathcal{K} \cap V_0^H$.

Proof. Let us take an ω_i, the support of the basis function ϕ_i in V_0^H corresponding to the node x^i of $\mathcal{T}_H$, and a $v \in V^h$. If v vanishes at a point in $\bar{\omega}_i$, then $I_i^+ v = I_i^- v = 0$, and if $v \neq 0$ at any point of $\bar{\omega}_i$ then either $v^+ = I_i^+ v = 0$ or $v^- = I_i^- v = 0$. Consequently, there exits at least a node of $\mathcal{T}_h$ in $\bar{\omega}_i$ at which $v - I_i^+ v + I_i^- v = v^+ - v^- - I_i^+ v + I_i^- v$ vanishes. Applying Lemma 5.1, since $I_i^+ v - I_i^- v$ is a constant, we get

$$||v - I_i^+ v + I_i^- v||_{0,s,\omega_i} \le CHC_{d,s}(H,h)|v|_{1,s,\omega_i}. \tag{5.31}$$

We point out that since for any $x \in \bar{\omega}_i$ the line segment $[x^i, x]$ lies in $\bar{\omega}_i$, we can take a decomposition as in Lemma 5.1 of ω_i having $N \le 1$. Assuming that $N = 1$, let ω_{i0} and $\omega_{i1} = \omega$ this decomposition. Since ω_{i0} contains at least one $\tau \in \mathcal{T}_H$ and the mesh $\mathcal{T}_H$ is regular, then, according to (5.18), C_{ω_i} can be taken independent of H and h. Consequently, $C(N,s)$ in (5.17) is independent of H and h. Let $a_{\omega_i}(v) = \frac{1}{|\omega_i|}\int_{\omega_i} v$ be the average of v over ω_i. Since diameter$(\omega_i) \le CH$, we have

$$||v - a_{\omega_i}(v)||_{0,s,\omega_i} \le CH|v|_{1,s,\omega_i}. \tag{5.32}$$

Using (5.31) and (5.32), we get

$$\begin{aligned}
&||I_H v - a_{\omega_i}(v)||_{0,s,\omega_i} = ||\sum_{x^j \text{ node in } \mathcal{T}_H} (I_j^+ v - I_j^- v - a_{\omega_i}(v))\phi_j||_{0,s,\omega_i} \le \\
&\sum_{x^j \text{ node in } \mathcal{T}_H} ||I_j^+ v - I_j^- v - a_{\omega_i}(v)||_{0,s,\omega_i} \le \\
&\sum_{x^j \text{ node in } \mathcal{T}_H} \left(||I_j^+ v - I_j^- v - v||_{0,s,\omega_i} + ||v - a_{\omega_i}(v)||_{0,s,\omega_i}\right) \le \\
&CHC_{d,s}(H,h) \sum_{x^j \text{ node in } \mathcal{T}_H} |v|_{1,s,\omega_i} \le CHC_{d,s}(H,h)|v|_{1,s,\omega_i}.
\end{aligned}$$

We used in the last inequality the fact that, since the mesh is regular, the maximum number of simplexes having a node x^j in common is bounded, independent of H. From this equation, and using again (5.32), we get

$$\begin{aligned}
&||I_H v - v||_{0,s,\omega_i} \le ||I_H v - a_{\omega_i}(v)||_{0,s,\omega_i} + ||a_{\omega_i}(v) - v||_{0,s,\omega_i} \le \\
&CHC_{d,s}(H,h)|v|_{1,s,\omega_i}.
\end{aligned}$$

As we said at the beginning of this subsection, $O = \cup_{i=1,M}\omega_i$ might be different from Ω, but dist$(O, \partial\Omega) \le \delta \le H$. Consequently, since $I_H v = 0$ on $\Omega \backslash O$ and $v = 0$ on $\partial\Omega$, the above inequality holds on $\Omega \backslash O$, too. Therefore, we get (5.29) from this remark and the last

equation, since the regularity of the mesh implies that the number of ω_j which intersect a fixed ω_i is independent oh H.
It follows from (5.29) that

$$||I_H v||_{0,s} \leq CHC_{d,s}(H,h)||v||_{1,s}, \tag{5.33}$$

From the definition of $I_i^+ v$ and $I_i^- v$ we have for any $x \in \omega_i$,

$$0 \leq I_i^+ v - I_i^- v \leq v(x) \text{ if } v(x) \geq 0, \text{ and } 0 \geq I_i^+ v - I_i^- v \geq v(x) \text{ if } v(x) \leq 0. \tag{5.34}$$

Therefore,

$$|I_i^+ v - I_i^- v| \leq |v(x)| \text{ for any } x \in \omega_i,$$

and, from Corollary 5.1, we get

$$|I_H v|_{1,s,\omega_i} = |\sum_{x^j \text{ node in } \mathcal{T}_H} (I_j^+ v - I_j^- v)\phi_j|_{1,s,\omega_i} \leq \sum_{x^j \text{ node in } \mathcal{T}_H} |(I_j^+ v - I_j^- v)\phi_j|_{1,s,\omega_i} \leq$$
$$Cd^{\frac{1}{s}} H^{-1+\frac{d}{s}} ||v||_{0,\infty,\omega_i} \leq d^{\frac{1}{s}} CC_{d,s}(H,h)||v||_{1,s,\omega_i}.$$

We have again used above that $\mathcal{T}_H$ is regular. The last inequality together with (5.33) prove (5.30).
From (5.34), (5.27) and (5.28), we get that for any $x \in \Omega$ we have

$$0 \leq I_H v(x) \leq v(x) \text{ if } v(x) \geq 0, \text{ and } 0 \geq I_H v(x) \geq v(x) \text{ if } v(x) \leq 0$$

Finally, if $0, v \in \mathcal{K}$, and $\mathcal{K}$ is described by some conditions at mesh nodes of $\mathcal{T}_h$ and having Property 5.1, then we get from the above equation that $I_H v \in \mathcal{K}$. □

Now, we can prove the following proposition which shows that the constant C_0 in Assumption 3.1 is independent of the mesh and domain decomposition parameters if H/δ and H/h are constant when $h \to 0$. This result is similar to that given in [4] for the inequalities coming from minimization of the quadratic forms. In the first part of the proof, the construction of v_i, $i = 1, \cdots, m$, is similar to that given for one-level method. In the second part we define an appropriate v_0 using the previous lemma.

Proposition 5.2. *Let (4.1) be the overlapping domain decomposition of the domain Ω with overlaps of size δ defined in this section. Then Assumption 3.1 is verified for the piecewise linear finite element spaces, $V = V^h$ and $V_0 = V_0^H$, $V_i = V_i^h$, $i = 1, \cdots, m$, defined in (5.2), (5.3) and (5.13), respectively, and any convex set $K = K^h$ defined by constraints on the function values at the nodes of $\mathcal{T}_h$ and having Property 5.1. The constant in (3.5) of Assumption 3.1 can be taken of the form*

$$C_0 = C(m+2)^{\frac{p+1}{p}} \left(1 + (m-1)\frac{H}{\delta} C_{d,s}(H,h)\right), \tag{5.35}$$

where C is independent of the mesh and domain decomposition parameters, and $C_{d,s}(H,h)$ is given in (5.15).

Proof. Let us consider $w \in K$, $w_i \in V_i$ such that $w + \sum_{j=0}^{i} w_i \in K$, $i = 0, \cdots, m$, and let v be another element in K. In the following we use the unity partitions $(\theta_j^i)_{j=i,\cdots,m}$, of the domains $\cup_{j=i,m}^{m} \Omega_j$, $i = 1, \cdots, m$, with the properties in (4.2) and (5.1).

Step 1. We assume that we have $v_0 \in V_0$ satisfying

$$w + v_0,\ v + w_0 - v_0 \in K, \tag{5.36}$$

and we shall construct recursively $v_i \in V_i$, $i = 1, \cdots, m$, which will satisfy (3.3) and (3.4) in Assumption 3.1. We define

$$v_1 = I_h\left(\theta_1^1(v - w - v_0) + (1 - \theta_1^1)w_1\right), \tag{5.37}$$

and, as in the previous section, we get

$$\begin{aligned} &v_1 \in V_1 \text{ and } w + w_0 + v_1 \in K, \\ &v - v_0 - v_1 + w_0 + w_1 \in K, \\ &v - w - v_0 - v_1 \in W_0^{1,s}(\bigcup_{j=2}^{m} \Omega_j) \text{ and} \\ &v - w - v_0 - v_1 = 0 \text{ in } \Omega - \overline{\cup_{j=2}^{m}\Omega_j}. \end{aligned}$$

Also, for $i = 2, \cdots, m-1$ we take

$$v_i = I_h\left(\theta_i^i(v - w - \sum_{j=0}^{i-1} v_j) + (1 - \theta_i^i)w_i\right), \tag{5.38}$$

and we prove

$$\begin{aligned} &v_i \in V_i \text{ and } v_i + w + \sum_{j=0}^{i-1} w_j \in K, \\ &v - \sum_{j=0}^{i} v_j + \sum_{j=0}^{i} w_j \in K, \\ &v - w - \sum_{j=0}^{i} v_j \in W_0^{1,s}(\bigcup_{j=i+1}^{m} \Omega_j) \text{ and} \\ &v - w - \sum_{j=0}^{i} v_j = 0 \text{ in } \Omega - \overline{\cup_{j=i+1}^{m}\Omega_j}. \end{aligned}$$

Finally, we take

$$v_m = v - w - \sum_{j=0}^{m-1} v_j \tag{5.39}$$

and we get that (3.3) and (3.4) in Assumption 3.1 hold.

Step 2. We shall define a $v_0 \in V_0$ satisfying (5.36) and prove that condition (3.5) in Assumption 3.1 is satisfied with the constant C_0 given in (5.35). It is easy to see that (5.36) is equivalent with

$$v_0 - w_0 \in (K - (w + w_0)) \cap (v - K), \tag{5.40}$$

and also, since $w, w + w_0 \in K$, we get

$$v - w - w_0 \in (K - (w + w_0)) \cap (v - K). \tag{5.41}$$

We write $\mathcal{K} = (K - (w + w_0)) \cap (v - K)$, and from the above equation and Lemma 5.2 we get that $I_H(v - w - w_0) \in \mathcal{K}$. From (5.29) and (5.30) we have

$$||v - w - w_0 - I_H(v - w - w_0)||_{0,s} \le CHC_{d,s}(H,h)|v - w - w_0|_{1,s} \tag{5.42}$$

and

$$||I_H(v - w - w_0)||_{1,s} \le CC_{d,s}(H,h)||v - w - w_0||_{1,s}, \tag{5.43}$$

where $C_{d,s}(H,h)$ is defined in (5.15). Now, we take

$$v_0 = w_0 + I_H(v - w - w_0), \tag{5.44}$$

and we see that it satisfies condition (5.36). To prove condition (3.5) in Assumption 3.1, we first notice that, starting from v_1 given in (5.37), by the recurrent application of (5.38), as in the proof of Proposition 5.1, we get v_i, $i = 1, \cdots, m$, of the form

$$v_i = I_h(\tau_0^i(v - w - v_0) + \sum_{j=1}^{i} \tau_j^i w_j),\ i = 1, \cdots, m, \tag{5.45}$$

where τ_j^i, $i = 1, \cdots, m$, $j = 0, \cdots, i$, satisfy (5.7). Using (5.7) and (5.8), we get

$$||I_h(\tau_j^i w_j)||_{1,s} \le C(||\tau_j^i w_j||_{0,s} + |\tau_j^i w_j|_{1,s}) \le C(||w_j||_{0,s} + \frac{m-1}{\delta}||w_j||_{0,s} + |w_j|_{1,s}).$$

Since $w_i \in V_i$ and the diameters of the connected component of Ω_i are less than $C(H + \delta)$, we get from the classical Friedrichs-Poincaré inequality,

$$||I_h(\tau_j^i w_j)||_{1,s} \le C[1 + (m-1)\frac{H}{\delta}]|w_j|_{1,s},\ i = 1, \cdots, m,\ j = 1, \cdots, i. \tag{5.46}$$

On the other hand, taking into account the definition of v_0, (5.42), (5.7) and (5.8), we get

$$\begin{aligned}
&||I_h(\tau_0^i(v - w - v_0))||_{1,s} \le C[|v - w - v_0|_{1,s} + (1 + \tfrac{m-1}{\delta})||v - w - v_0||_{0,s}] = \\
&C[|v - w - v_0|_{1,s} + (1 + \tfrac{m-1}{\delta})||v - w - w_0 - I_H(v - w - w_0)||_{0,s}] \le \\
&C\left[|v - w - v_0|_{1,s} + (m-1)C_{d,s}(H,h)\tfrac{H}{\delta}|v - w - w_0|_{1,s}\right] \le \\
&C(|v - w|_{1,s} + |v_0|_{1,s}) + C(m-1)C_{d,s}(H,h)\tfrac{H}{\delta}(|v - w|_{1,s} + |w_0|_{1,s}).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&||I_h(\tau_0^i(v - w - v_0))||_{1,s} \le \\
&C\left[1 + (m-1)C_{d,s}(H,h)\tfrac{H}{\delta}\right](|v - w|_{1,s} + |w_0|_{1,s}) + C|v_0|_{1,s},\ i = 1, \cdots, m.
\end{aligned} \tag{5.47}$$

Also, from the definition of v_0 and (5.43) we get

$$\begin{aligned}
&|v_0|_{1,s} = |w_0 + I_H(v - w - w_0)|_{1,s} \le |w_0|_{1,s} + |I_H(v - w - w_0)|_{1,s} \le \\
&|w_0|_{1,s} + CC_{d,s}(H,h)||v - w - w_0||_{1,s},
\end{aligned}$$

and therefore,

$$|v_0|_{1,s} \le C[1 + C_{d,s}(H,h)](||v - w||_{1,s} + ||w_0||_{1,s}). \tag{5.48}$$

Now, taking into account that $\delta < H$, from (5.47) and (5.48), we get

$$(5.49) \qquad \begin{array}{l} ||I_h(\tau_0^i(v-w-v_0))||_{1,s} \leq \\ C\left[1+(m-1)C_{d,s}(H,h)\frac{H}{\delta}\right](||v-w||_{1,s}+||w_0||_{1,s}),\ i=1,\cdots,m. \end{array}$$

Finally, from (5.45), (5.46), (5.48), and (5.49) we obtain that condition (3.5) in Assumption 3.1 holds with C_0 given in (5.35). □

Remark 5.3. *As in Remark 5.1, we notice that, since the number m of the subdomains Ω_i is the number of colors of the overlapping domain decomposition $\{O_i^\delta\}_{1\leq i\leq M}$, the error estimates in Theorem 3.1 depends only on C_0 given in (5.35). Therefore, if the overlapping size δ and coarse mesh sizes H and h are chosen such that H/h and H/δ are constant, then the convergence rate of the two-level multiplicative Schwarz method is mesh and domain decomposition independent when H, $h \to 0$.*

6 Numerical example

For a domain $\Omega \subset \mathbf{R}^d$ and an $1 < s < \infty$, let $K \subset V \equiv W_0^{1,s}(\Omega)$ be a closed and convex set. Given an $f \in V' \equiv W^{-1,s'}(\Omega)$, $1/s + 1/s' = 1$, we consider the problem

$$(6.1) \qquad u \in K : \int_\Omega |\nabla u|^{s-2}\nabla u \nabla(v-u) \geq f(v-u), \text{ for any } v \in K,$$

which is equivalent with

$$(6.2) \qquad u \in K : F(u) = \min_{v\in K} F(v),$$

where

$$F(v) = \left[\frac{1}{s}\int_\Omega |\nabla v|^s - f(v)\right].$$

We know (see [15]) that if $1 < s \leq 2$, then there exist two positive constants α and β such that

$$\begin{array}{l} < F'(v)-F'(u), v-u > \geq \alpha\frac{||v-u||^2_{1,s}}{(||v||_{1,s}+||u||_{1,s})^{2-s}} \\ \beta||v-u||^{s-1}_{1,s} \geq ||F'(v)-F'(u)||_{V'}, \end{array}$$

for any $v, u \in W_0^{1,s}(\Omega)$. Consequently, the functions introduced in (3.1) can be written as

$$\alpha_M(\tau) = \frac{\alpha}{(2M)^{2-s}}\tau^2,\ \beta_M(\tau) = \beta\tau^{s-1},$$

and therefore, $A_M = \frac{\alpha}{(2M)^{2-s}}$, $B_M = \beta$, $p = 2$ and $q = s$ in (3.1). If $s \geq 2$, then there exist two positive constants α and β such that (see [8])

$$\begin{array}{l} < F'(v)-F'(u), v-u > \geq \alpha||v-u||^s_{1,s} \\ \beta(||v||_{1,s}+||u||_{1,s})^{s-2}||v-u||_{1,s} \geq ||F'(v)-F'(u)||_{V'}, \end{array}$$

for any $v, u \in W_0^{1,s}(\Omega)$. Therefore, for given $M > 0$, we have

$$\alpha_M(\tau) = \alpha\tau^s,\ \beta_M(\tau) = \beta(2M)^{s-2}\tau,$$

and therefore, $A_M = \alpha$, $B_M = \beta(2M)^{s-2}$, $p = s$ and $q = 2$ in (3.1).

We can conclude from the above comments that Algorithm 2.1 can be applied for the solving of problem (6.1) if the convex set K has Property 4.1. Naturally, all the error estimations in the previous sections hold.

We have tested the correctness of the error estimations given in the previous sections, by an numerical example concerning the two-obstacle problem of a nonlinear elastic membrane. This is a problem of type (6.1) in which $\Omega \subset \mathbf{R}^2$ and the convex set is of the form $K = [a, b]$, where $a, b \in W_0^{1,s}(\Omega)$, $a \leq b$. In our numerical tests, the exterior forces f are zero, ie. we have considered the two-obstacle nonlinear problem,

$$u \in [a, b] : \int_\Omega |\nabla u|^{s-2} \nabla u \nabla (v - u) \geq 0, \text{ for any } v \in [a, b].$$

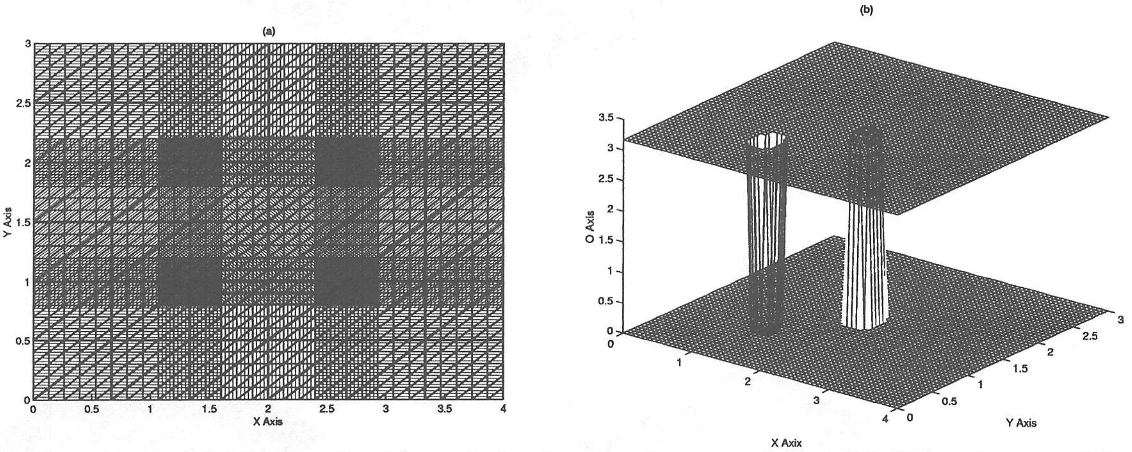

Figure 6.1. (a) Meshes $\mathcal{T}_H$, $\mathcal{T}_h$, and the domain decomposition, (b) Obstacles a and b.

In the numerical experiments, the domain Ω is the rectangle $(0, 4) \times (0, 3)$. The meshes $\mathcal{T}_H$ and $\mathcal{T}_h$ contains right-angled triangles, which are obtained by partitioning the sides of the rectangular domain in the same number of segments. We show these meshes in Figure 6.1.a, where we have considered 30 segments for $\mathcal{T}_h$ and 6 segments for $\mathcal{T}_H$, on each side. In the same figure we have shown the domain decomposition, the number of the non overlapping subdomains O_i being 9, and evidently, the number of the subdomains Ω_i is 4. The width of the overlaps in this figure is of 2 triangles in $\mathcal{T}_h$. The obstacles a and b are shown in Figure 6.1.b for a mesh $\mathcal{T}_h$ having 60 nodes on a side of the rectangular domain. The obstacle a is given by the plane $z = 0$ with a circular cylinder having a basis on this plane and the other one, in the plane $z = 3.0$, is ended with a semisphere. The cylinder has the radius of $1/6$, and the center of its first basis is at the center of the rectangle Ω. The obstacle b is the plane $z = 3 + 1/6$ with a circular cylinder having a basis on this plane and the other one, in the plane $z = 1/6$, is ended with a semisphere. The cylinder has the radius of $1/6$, and his axis passes through the point $(4/3, 3/4, 0)$.

The computed solutions for $s = 2.0$, $s = 1.5$ and $s = 3.0$ are plotted in Figure 6.2 for a mesh $\mathcal{T}_h$ having 60 nodes on a side of the rectangular domain Ω.

We have seen in the previous sections that the constant C_0 depends on $1/\delta$ in equation (5.11), in the case of the one-level method, and on H/h and H/δ in equation (5.35), for the two-level method. We have tried to verify it by numerical tests for the nonlinear membrane problem taking various values of H, h and δ. In all the numerical tests the calculus has been stopped at a relative error of 1.E-03 at the nodes of $\mathcal{T}_h$ between two consecutive computed solutions. The solution on the subdomains have been calculated by the relaxation method, which is a particular case of the Schwarz domain decomposition method. The computing of the solutions on subdomains has been stopped at a relative error of 1.E-05 at the nodes of $\mathcal{T}_h$ between two consecutive computed subsolutions.

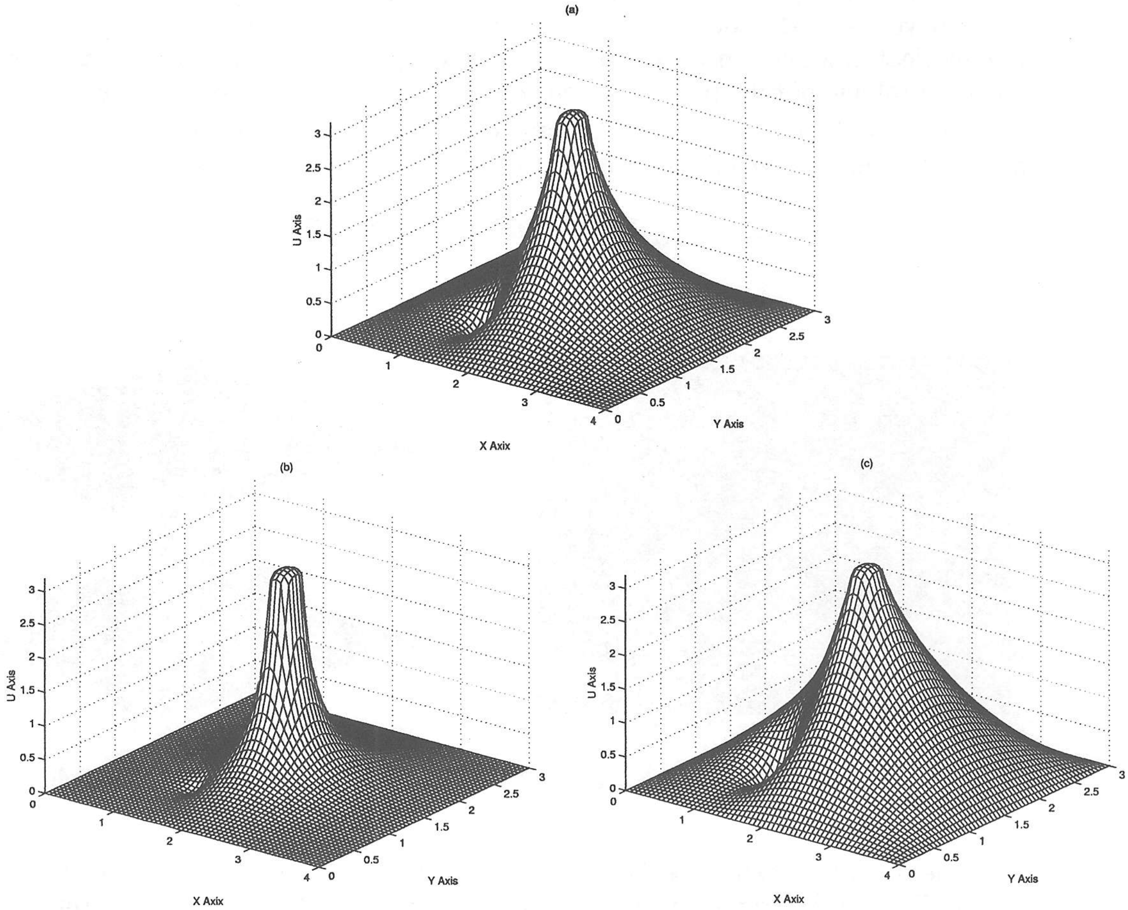

Figure 6.2. Solution for: (a) s=2, (b) s=1.5, (c) s=3.

The tests in Figure 6.3 have been made for $H/h = 6$ and $H/\delta = 2$, and the points on the two curves are obtained for various coarse meshes $\mathcal{T}_H$ corresponding to 20, 18, 16, ... , 2 segments on a side of the rectangular domain Ω. We see that the number of the iterations is bounded for the two-level method, and it is in concordance with the fact that C_0 in (5.35) is constant. Also, the number of iterations is an decreasing function of H for the one-level method. Since H/δ is constant, it follows that the number of iterations is an increasing function of $1/\delta$, and it is in concordance with C_0 in (5.11).

In the tests in Figures 6.4, 6.5 and 6.6, two of the parameters H, h or δ are constant and the third is variable.

For the tests in Figure 6.4 we have taken $H = 5.0/12$, $h = 5.0/120$ and $\delta = 1h, 2h, \cdots, 10h$. We see that, in both cases, the number of iterations is a decreasing function of δ, and it is concordance with the expressions of C_0 in (5.11) and (5.35).

The tests in Figure 6.5 have been made for $H = 5.0/6$, $\delta = 5.0/12$, and h corresponds to partitions $\mathcal{T}_h$ with $2 \cdot 6, 4 \cdot 6, 6 \cdot 6, \cdots, 20 \cdot 6$ segments on each side of the rectangular domain Ω. We see that the number of iterations is constant for $h \leq 5/24$ in the case of the one-level method, and it is in concordance with C_0 in (5.11). In the case of the two-level method, the number of iterations is a decreasing function of h for $s = 1.5$ and $s = 2$, and it is also in concordance with C_0 in (5.35). For $s = 3 > d = 2$, the number of iterations should be constant, $C_{d,s}(H, h)$ in (5.15) being equal to 1 in this case. In Figure 6.5.b, we see that the three curves are similar; however it is not excluded the fact that, as $h \to 0$, the number of iterations slowly increases for $s = 1.5$ and $s = 2$, and it remains bounded for $s = 3$.

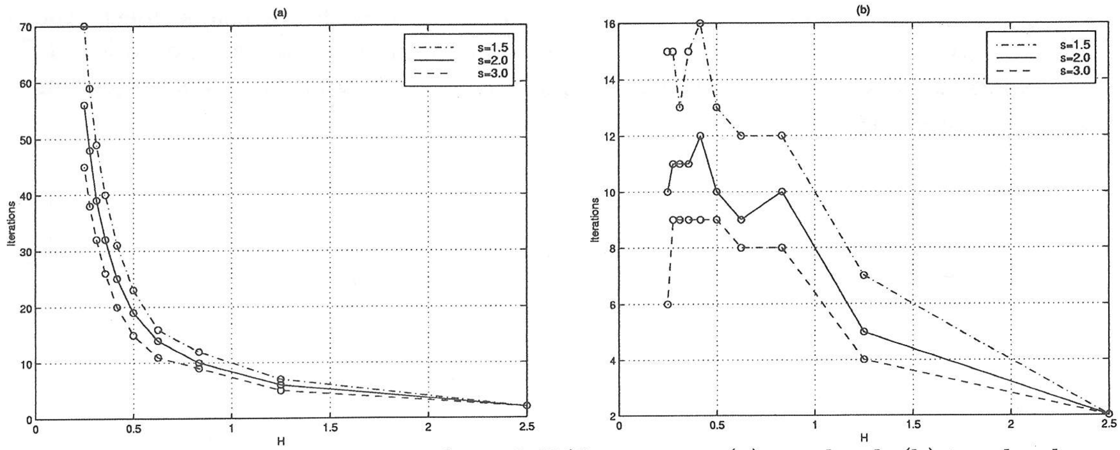

Figure 6.3. Iterations for H/h and H/δ constant: (a) one level, (b) two levels.

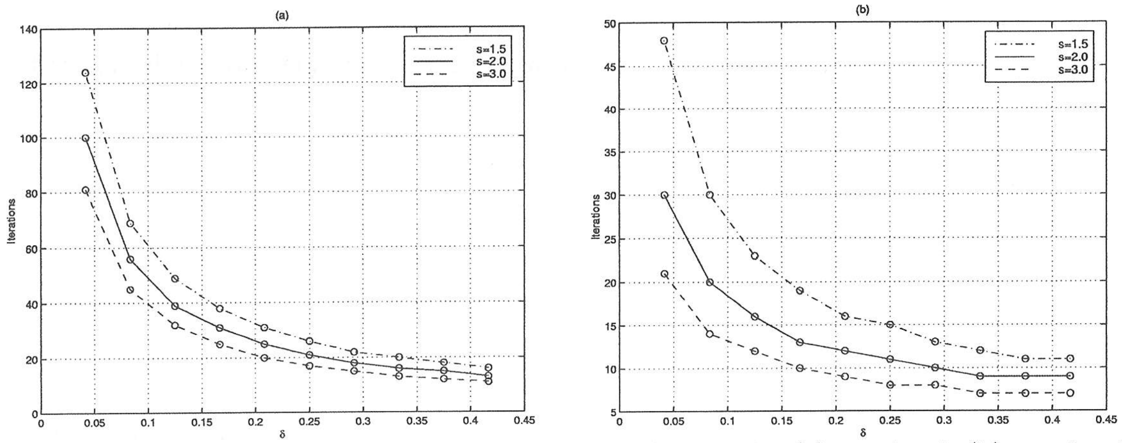

Figure 6.4. Iterations for H and h constant, and δ variable: (a) one level, (b) two levels.

In the tests in Figure 6.6 we have taken $h = 5.0/120$, $\delta = 5.0/20$ and $H = 5.0/20$, $5.0/12$, $5.0/10$, $5.0/8$ and $5.0/6$. Since the number of the subdomains O_i^δ depends on H, even if inside an iteration we have found the solution first for the subdomains O_i^δ of the first color, then for those of the second color, and so on, from Figure 6.6.a we see that the number of iterations for the one-level method depends on the number of the subdomains O_i^δ. In the case of the two-level method, the number of iterations is an increasing function of H which is in concordance with our constant C_0 in (5.35).

Finally, we see from our numerical tests that the number of iterations for the two-level method is always less than that for the one-level method.

In the end of this section, we give some details concerning the method we have used in the computing code to solve the problems on subdomains. As we have already said, the subproblems, the subproblem corresponding to the coarse mesh included, have been solved by the relaxation method. Consequently, for a fixed subdomain, we have to solve iteratively one-dimensional problems. Since the functional is strictly convex and the convex set is a one-dimensional segment, the solutions of these nonlinear problems can be found by the same method as for the quadratic functionals: we solve first the one-dimensional nonlinear equation corresponding to the inequality, and then we project it on the convex set. This projection is very simple for the subdomains covered with the mesh $\mathcal{T}_h$ because the convex is an interval and the constraints of the convex set operate on the function values at the nodes of the fine mesh. In the case of the domain with the coarse mesh, the projection is a little more complicated because the constraints operates at the nodes of $\mathcal{T}_h$ and the functions belong to V_0^H. We shall explain in the following how this projection is made in our computing code.

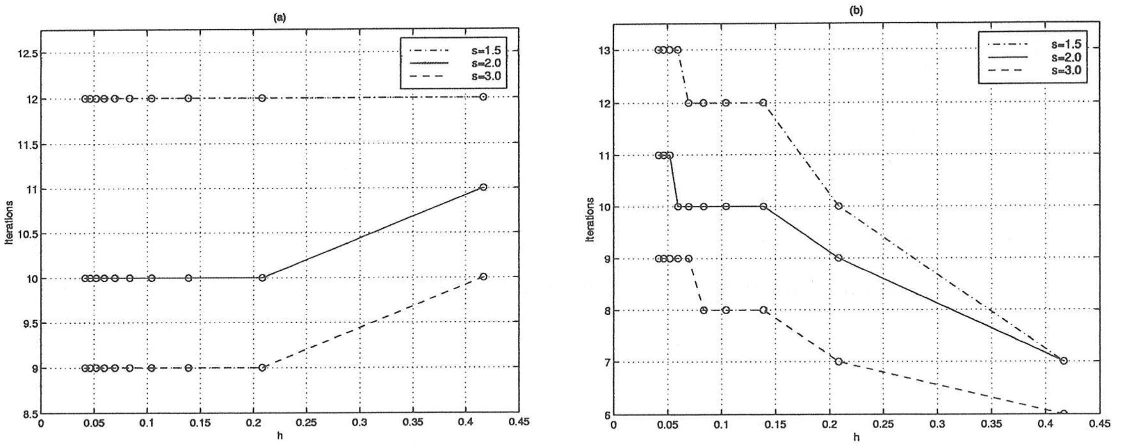

Figure 6.5. Iterations for H and δ constant, and h variable: (a) one level, (b) two levels.

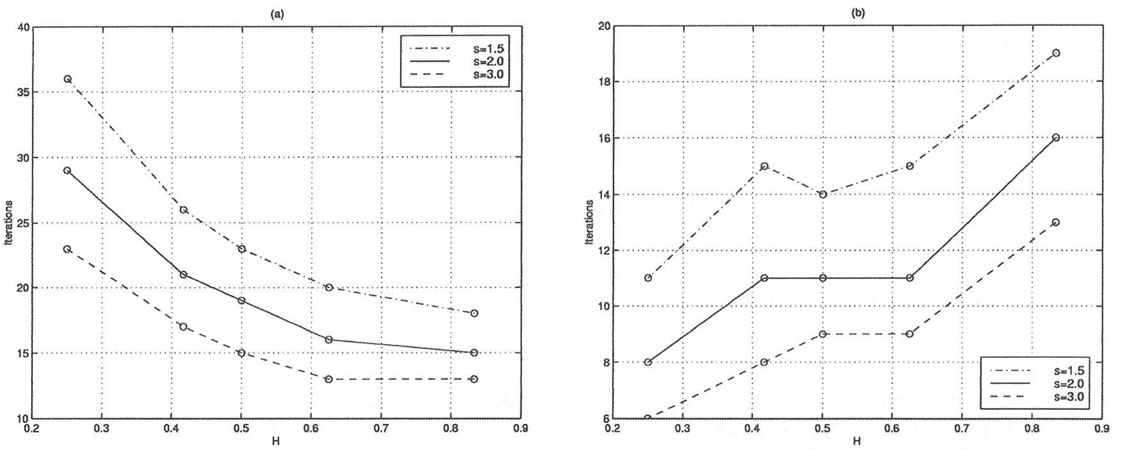

Figure 6.6. Iterations for h and δ constant, and H variable: (a) one level, (b) two levels.

We have two vectors $u(k)$ and $w(k)$, k runs from 1 to the number of nodes in $\mathcal{T}_h$, containing the values of $u^{n+\frac{i-1}{m}}$ and w_i^{n+1} obtained from Algorithm 2.1. At the iteration n, for a given subdomain i, the values of $w(k)$ are obtained by the relaxation method, and we update $u(k)$ with $w(k)$, in order to obtain $u^{n+\frac{i}{m}}$, after the obtaining of the values w_i^{n+1} with the wanted error. Naturally, we have two vectors $a(k)$ and $b(k)$, containing the values of the two obstacles at the mesh nodes in $\mathcal{T}_h$.

Assume now that we are computing the solution on a subdomain Ω_i and we seek for the value $w(k)$ of the correction at the node k of $\mathcal{T}_h$. As we have already said, we first find the solution v of the corresponding nonlinear equation, and then we project it as usually: if $a(k) - u(k) \leq v \leq b(k) - u(k)$, then we take $w(k) = v$; if $v \leq a(k) - u(k)$, we take $w(k) = a(k) - u(k)$; and if $b(k) - u(k) \leq v$, we take $w(k) = b(k) - u(k)$.

In order to compute the corrections at the nodes of the coarse mesh $\mathcal{T}_H$, we have introduced the matrix $c(l,k)$, where l runs from 1 to the number of the nodes of $\mathcal{T}_H$, and k takes the values from 1 to the number of nodes of $\mathcal{T}_h$ contained in the support of ϕ_l, the function in the nodal basis in V_0^H. The coefficients $c(l,k)$ depend on the relative position of the mesh nodes l and k, and they give the correction introduced by ϕ_l at the node k when the correction at the node l is 1, ie. $c(l,k) = \phi_l(x^k)$. Consequently, with a new correction w at the node l in the place of the old one, $w(l)$, the new correction at the node k will be $w(k) + c(l,k)(w - w(l))$. Therefore, the new value satisfies the constraint of the convex at the node k if

$$a(k) \leq u(k) + w(k) + c(l,k)(w - w(l)) \leq b(k), \tag{6.3}$$

or

$$\frac{1}{c(l,k)}\left[a(k) - u(k) - w(k) + c(l,k)w(l)\right] \leq w \leq \frac{1}{c(l,k)}\left[b(k) - u(k) - w(k) + c(l,k)w(l)\right].$$

Now, if v is the solution of the nonlinear equation corresponding to the inequality in the relaxation method at the node l, in order to obtain the real correction satisfying also the constraints at the nodes k, we have to project it on the interval $[a_l, b_l]$, where

$$a_l = \max\{a(l) - u(l), \max_k \frac{1}{c(l,k)}\left[a(k) - u(k) - w(k) + c(l,k)w(l)\right]\}$$
$$b_l = \min\{a(l) - u(l), \min_k \frac{1}{c(l,k)}\left[b(k) - u(k) - w(k) + c(l,k)w(l)\right]\},$$

where $\max_k$ and $\min_k$ are taken over the nodes k of $\mathcal{T}_h$ contained in suppϕ_l. We point out that since before the new correction we had $u(k) + w(k) \in K^h$, from (6.3), we get that $w(l) \in [a_l, b_l]$. The projection of v on $[a_l, b_l]$ is made as in the relaxation method for the subdomains Ω_i, and if w_p is this projection, the new updated corrections will be: $w(l) := w_p$ and $w(k) := w(k) + c(l,k)(w_p - w(l))$, k being the nodes of $\mathcal{T}_h$ in suppϕ_l.

We notice that the projection for the two-level method is a little more complicated than that in the one-level method, but since the number of iterations is less in the two-level method than that in the one-level method, the two-level method is more efficient in point of view of the computing time. For instance, we see in Figure 6.3 that for $H = 5.0/10$, $h = 5.0/60$ and $\delta = 5.0/20$, the number of iteration is: 23 for $s = 1.5$, 19 for $s = 2.0$, and 15 for $s = 3.0$, in the case of the one-level method, and 13 for $s = 1.5$, 10 for $s = 2.0$, and 9 for $s = 3.0$, in the case of the two-level method. The computing time obtained on a PC with one processor Intel Pentium III of 600MHz was: 18min45sec for $s = 1.5$, 6min16sec for $s = 2.0$, and 17min8sec for $s = 3.0$, in the case of the one-level method, and 13min54sec for $s = 1.5$, 4min43sec for $s = 2.0$, and 14min27sec for $s = 3.0$, in the case of the two-level method. Naturally, the computing time for $s = 2.0$ is less than that for $s = 1.5$ or $s = 3.0$ since in this case we solve linear equations in the relaxation method. This case corresponds to the minimization of a quadratic functional. The finite element problem in these computing time tests has 3481 unknowns.

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