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by

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Elementary Diagrams in Institutions

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Abstract. We generalise the method of diagrams from conventional model theory to a simple institution-independent (i.e. independent of the details of the actual logic or institution) framework based on a novel categorical concept of elementary diagram of a model. We illustrate the power of our institution-independent method of elementary diagrams by developing several applications to institution liberality, institution-independent quasi-varieties, and limits and colimits of theory models. The results obtained are illustrated systematically with examples from four different algebraic specification logics. In the introduction we also discuss the relevance of our institution-independent approach to the model theory of algebraic specification and computing science, but also to conventional and abstract model theory.

1. Introduction

The theory of institutions (Goguen and Burstall, 1992) is a categorical abstract model theory which formalises the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them. Institutions constitute the modern level of algebraic specification theory and can be considered its most fundamental mathematical structure. It is already an algebraic specification tradition to have an institution underlying each language or system, in which all language/system constructs and features can be rigorously explained as mathematical entities. Most modern algebraic specification languages follow this tradition, including CASL (Mossakowski, 2001), Maude (Meseguer, 1993), or CafeOBJ (Diaconescu and Futatsugi, 2002). There is an increasing multitude of logics in use as institutions in algebraic specification and computing science. Some of them, such as first order predicate (in many variants), second order, higher order, Horn, type theoretic, equational, modal (in many variants), infinitary logics, etc., are well known or at least familiar to the ordinary logicians, while others such as behavioural or rewriting logics are known and used mostly in computing science.

The original goals of institution theory are to do as much computing science and model theory as possible, independent of what the actual logic may be (Goguen and Burstall, 1992). This mathematical paradigm is often called 'institution-independent' computing science or model theory. While the former goal has been greatly accomplished in the algebraic specification literature, there were only very few and rather isolated attempts towards the latter (Tarlecki, 1986a; Tarlecki, 1986b; Salibra and Scollo, 1996). This situation contrasts with the feeling shared by some researchers that deep concepts and results in model theory can be reached in significantly via institution theory. This paper can be regarded as a new step towards this goal, part of a recent series of works in institution-independent model theory starting with (Diaconescu, 2002).

The significance of institution-independent model theory is manifold:

- It provides model-theoretic results and analysis for various logics in a generic way. Only a limited number of model-theoretic properties are usually studied for the logics in use in computing science and algebraic specification, however it is important to have as deep as possible understanding of the model-theoretic properties of the underlying logic because the specification or software engineering properties of the logic depend intimately on the model theoretic ones ((Diaconescu et al., 1993) is only one of the works that support this argument). We sometimes notice that the failure of some specification properties of a logic is due to the rather subtle wrong definition of some details of the logic. We also notice that often the right definition of a logic can be checked through its model-theoretic properties, otherwise said good model-theoretic properties lead to good specification properties.
- It exports model-theoretic methods from classical logic to other logics. Classical first-order predicate logic has developed very rich a powerful model-theoretic methods, which exported

to an institution-independent framework can become available for the multitude of computing science or algebraic specification logics.

– It provides a new way of doing model theory. While the points we made above have a more application oriented significance, this point has a pure mathematics methodological significance. The institution-independent way of obtaining a model theoretic result, or just viewing a concept, leads to a deeper understanding of *why* a certain model theoretic phenomenon holds. Such top-down understanding is not suffocated by the details of the actual logic, it decomposes the model-theoretic phenomenon (into various layers of abstract conditions), and provides a clear picture of its limits.

Although these points are largely valid for any form of abstract model theory, they are especially relevant for the institution-independent abstract model theory. One of the reasons for this is that up to our knowledge, the theory of institutions provide the most complete definition of abstract model theory, the only one including signature morphisms, model reducts, and even mappings (morphisms) between logical systems, as primary concepts. Also, as mentioned above, the current algebraic specification logics and an increasing number of computing science logics are formalised as institutions.

This work exports one of the most important and powerful conventional model theory methods, namely the method of diagrams (C.C.Chang and H.J.Keisler, 1973), to an institution-independent framework. This framework is based on a simple novel categorical formulation of the concept of *elementary diagram* (of a model). Notice that a different institution-independent formulation of the method of diagrams has been used by Tarlecki as part of the definition of the so-called "abstract algebraic institutions" (Tarlecki, 1986a; Tarlecki, 1986b).[1]

1.1. Summary and Contributions of this Work

The preliminary section gives a very brief overview of concepts, terminology, and notations from category theory (including inclusion systems) and institution theory.

The next section is devoted to the institution-independent definition of the concept of elementary diagram of a model. We illustrate this definition with rather detailed examples from four of the most important logics in algebraic specification and model theory (formalized as institutions).

In the last section we illustrate the power of our institution-independent method of elementary diagrams by exploring several applications. Some of the results obtained by instantiating the results of this section to actual institutions might be already known in the theory of algebraic specification, however our goal here is to illustrate the power of the concept and to show ways of using it. The reader may easily compare the more conventional way of obtaining some of these results, involving rather complex proofs, to the simplicity of the results based on elementary diagrams. This simplicity has various aspects: on one hand elementary diagrams are a natural property of institutions (provided the details of the institution are correctly defined) easy to check, on the other hand the existence of elementary diagrams is very rich in mathematical consequences especially when combined with other basic properties of the institutions.

The first application concerns institution liberality, which is one of the most important properties of algebraic specification institutions. We show that under the existence of elementary diagrams, full liberality follows almost directly from the existence of initial models of theories (which is a special restricted case of liberality). A related result has been obtained in (Tarlecki, 1986a) but under a much more complex set of conditions providing a lot of additional structure to the concept of institution.

The second application develops an institution-independent approach to quasi-varieties based on elementary diagrams and inclusion systems. Due to both elementary diagrams and the use of inclusion systems instead of (the more conventional) factorisation systems, our institution-independent approach to quasi-varieties is much simpler than "abstract algebraic institutions" of (Tarlecki, 1986a) and (Tarlecki, 1986b).

[1] This definition can be formulated only in institutions with additional structure such as regular categories of models, etc.

The last application is perhaps the most surprising. We show how under the existence of elementary diagrams, limits/colimits of models in theories with initial model can be obtained from limits/colimits of signatures. It is well known that in actual institutions, the categories of signatures have easy limits and colimits[2], while limits and especially colimits of theory models are rather difficult.

The applications developed are illustrated with examples from four different institutions: first-order predicate logic (with or without equality), rewriting logic, partial algebra, and hidden algebra for behavioural logic. These logics are very briefly presented in the Appendix, mainly for setting up some notation and terminology.

2. Preliminaries

2.1. Categories

This work assumes some familiarity with category theory, and generally uses the same notations and terminology as Mac Lane (MacLane, 1998), except that composition is denoted by ";" and written in the diagrammatic order. The application of functions (functors) to arguments may be written either normally using parentheses, or else in diagrammatic order without parentheses, or, more rarely, by using sub-scripts or super-scripts. The category of sets is denoted as $\mathbb{S}et$, and the category of categories[3] as $\mathbb{C}at$. The opposite of a category $\mathbb{C}$ is denoted by $\mathbb{C}^{op}$. The class of objects of a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$; also the set of arrows in $\mathbb{C}$ having the object a as source and the object b as target is denoted as $\mathbb{C}(a,b)$. The isomorphism of objects in categories is denoted by $\simeq$. We use $\Rightarrow$ to denote natural transformations.

For any object a, the comma category $a/\mathbb{C}$ has

- arrows $f \in \mathbb{C}(a,b)$, as objects, and
- arrows $h \in \mathbb{C}(b,b')$ such that $f;h = f'$, as arrows between $f \in \mathbb{C}(a,b)$ and $f' \in \mathbb{C}(a,b')$.

2.1.1. *Inclusion Systems*

Inclusion systems where introduced in (Diaconescu et al., 1993) for the institution-independent study of structuring specifications. In (Diaconescu et al., 1993) they provide the underlying mathematical concept for module imports, which are the most fundamental structuring construct. Mathematically, inclusion systems capture categorically the concept of set-theoretic 'inclusion' in a way reminiscent of factorization systems (Borceux, 1994); however in many applications the former are more convenient than the latter. *Weak inclusion systems* were introduced in (Căzănescu and Roşu, 1997) as a weakening of the original definition of inclusion systems of (Diaconescu et al., 1993).

DEFINITION 1. $\langle \mathcal{I}, \mathcal{E} \rangle$ is a *weak inclusion system* for a category $\mathbb{C}$ if $\mathcal{I}$ and $\mathcal{E}$ are two subcategories with $|\mathcal{I}| = |\mathcal{E}| = |\mathbb{C}|$ such that

1. $\mathcal{I}$ is a partial order, and
2. every arrow f in $\mathbb{C}$ can be factored uniquely as $f = e;i$ with $e \in \mathcal{E}$ and $i \in \mathcal{I}$.

The arrows of $\mathcal{I}$ are called *inclusions*, and the arrows of $\mathcal{E}$ are called *surjections*.[4] The domain (source) of the inclusion i in the factorization of f is called called the *image of* f and denoted as $\mathrm{Im}(f)$.

A weak inclusion system $\langle \mathcal{I}, \mathcal{E} \rangle$ is a *epic weak inclusion system* if and only if all surjections are epics and it is an *inclusion system* if in addition $\mathcal{I}$ has finite least upper bounds (denoted $+$) (see (Diaconescu et al., 1993)). □

[2] For example, by using arguments from indexed category theory (Tarlecki et al., 1991).

[3] We steer clear of any foundational problem related to the "category of all categories"; several solutions can be found in the literature, see, for example (MacLane, 1998).

[4] Surjections of some weak inclusion systems need not necessarily be surjective in the ordinary set-theoretic sense.

EXAMPLE 1. Let (S,Σ,Π) be a signature in first-order predicate logic (see Appendix A). The category of (Σ,Π)-models has an epic weak inclusion system such that a (Σ,Π)-model homomorphism $h\colon M \to M'$

- is inclusion if for each sort $s \in S$, $h_s\colon M_s \hookrightarrow M'_s$ are set-theoretic inclusions, and
- is surjection if for each sort $s \in S$, $h_s\colon M_s \to M'_s$ are set-theoretic surjections and $M'_\pi = \{h_w(\underline{a}) \mid \underline{a} \in M_\pi\}$ for each arity w and each predicate symbol $\pi \in \Pi_w$.

In the particular case of the inclusion system of many-sorted algebra (obtained by the absence of the predicate symbols), algebra homomorphisms are algebra inclusions/surjections, if and only if they are component-wise set-theoretic inclusions/surjections, respectively. □

EXAMPLE 2. Given an algebraic signature Σ, we can define an epic weak inclusion system for the category of Σ-preorder models (see Appendix B) such that a Σ-preorder homomorphism $h\colon M \to M'$

- is inclusion if for each sort s of the signature Σ, $h_s\colon M_s \hookrightarrow M'_s$ are preorder inclusions, and
- is surjection if for each sort s of the signature Σ, $h_s\colon M_s \to M'_s$ are preorder surjections and for each $m'_1, m'_2 \in M'$, $m'_1 \le m'_2$ if and only if there exists $m_1, m_2 \in M$ such that $m_1 \le m_2$ and $m'_1 = h(m_1)$ and $m'_2 = h(m_2)$.

□

EXAMPLE 3. Given a partial algebra signature (Σ,Δ), we can define an epic weak inclusion system for the category of partial (Σ,Δ)-algebras (see Appendix C) such that a partial (Σ,Δ)-homomorphism $h\colon A \to A'$

- is inclusion if for each sort s, $h_s\colon A_s \hookrightarrow A'_s$ are total inclusions, and
- is surjection if for each sort s, $h_s\colon A_s \to A'_s$ are partial surjections.

□

2.2. INSTITUTIONS

In this section we briefly review some of the basic concepts on institutions. Besides the seminal paper (Goguen and Burstall, 1992), (Diaconescu et al., 1993) contains many results about institutions with direct application to modularisation in algebraic specification languages.

DEFINITION 2. An *institution* $\mathfrak{I} = (\mathbb{S}ign, Sen, \textsc{Mod}, \models)$ consists of

1. a category $\mathbb{S}ign$, whose objects are called *signatures*,
2. a functor $Sen\colon \mathbb{S}ign \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\textsc{Mod}\colon \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *morphisms*, and
4. a relation $\models_\Sigma \subseteq |\textsc{Mod}(\Sigma)| \times Sen(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign|$, called Σ-*satisfaction*,

such that for each morphism $\varphi\colon \Sigma \to \Sigma'$ in $\mathbb{S}ign$, the *satisfaction condition*

$$m' \models_{\Sigma'} Sen(\varphi)(e) \ \text{ iff } \ \textsc{Mod}(\varphi)(m') \models_\Sigma e$$

holds for each $m' \in |\textsc{Mod}(\Sigma')|$ and $e \in Sen(\Sigma)$. We may denote the reduct functor $\textsc{Mod}(\varphi)$ by $\_\restriction_\varphi$ and the sentence translation $Sen(\varphi)$ simply by $\varphi(\_)$. When $M = M'\restriction_\varphi$ we will say that M' is an *expansion of* M *along* φ. □

DEFINITION 3. An institution is *closed under isomorphisms* if and only if each two isomorphic models satisfy the same sentences. □

DEFINITION 4. Let $\mathfrak{I} = (\mathbb{S}ign, Sen, \text{MOD}, \models)$ be an institution. For any signature Σ the closure of a set E of Σ-sentences is $E^\bullet = \{e \mid E \models_\Sigma e\}$[5]. (Σ, E) is a *theory* if and only if E is closed, i.e., $E = E^\bullet$.

A *theory morphism* $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi\colon \Sigma \to \Sigma'$ such that $\varphi(E) \subseteq E'$. Let $\mathbb{T}h_\mathfrak{I}$ denote the category of all theories in $\mathfrak{I}$. □

REMARK 1. For any institution $\mathfrak{I}$, the model functor MOD extends from the category of its signatures $\mathbb{S}ign$ to the category of its theories $\mathbb{T}h_\mathfrak{I}$, by mapping a theory (Σ, E) to the full subcategory $\text{MOD}(\Sigma, E)$ of $\text{MOD}(\Sigma)$ formed by the Σ-models which satisfy E. □

DEFINITION 5. A theory morphism $\varphi\colon (\Sigma, E) \to (\Sigma', E')$ is *liberal* if and only if the reduct functor $\_\restriction_\varphi\colon \text{MOD}(\Sigma', E') \to \text{MOD}(\Sigma, E)$ has a left-adjoint $(\_)^\varphi$.

The institution $\mathfrak{I}$ is *liberal* if and only if each theory morphism is liberal. □

Exactness properties for institutions formalise the possibility of amalgamating models of different signatures when they are consistent on some kind of 'intersection' of the signatures (formalised as a pushout square):

DEFINITION 6. An institution $\mathfrak{I} = (\mathbb{S}ign, Sen, \text{MOD}, \models)$ is *exact* if and only if the model functor $\text{MOD}\colon \mathbb{S}ign^{\text{op}} \to \mathbb{C}at$ preserves finite limits. $\mathfrak{I}$ is *semi-exact* if and only if MOD preserves only pullbacks. □

FACT 1. In a semi-exact institution $\mathfrak{I}$ consider a pushout of signatures

$$\begin{tikzcd} \Sigma \arrow[r, "\phi_1"] \arrow[d, "\phi_2"'] & \Sigma_1 \arrow[d, "\phi'_1"] \\ \Sigma_2 \arrow[r, "\phi'_2"'] & \Sigma' \end{tikzcd}$$

and two models, a Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1\restriction_{\phi_1} = M_2\restriction_{\phi_2}$. Then by the semi-exactness, there exists an unique Σ'-model M' such that $M'\restriction_{\phi'_1} = M_1$ and $M'\restriction_{\phi'_2} = M_2$. We call this model the *amalgamation* of M_1 and M_2 and denote it by $M_1 \otimes M_2$.

This amalgamation concept is also extended to model homomorphisms □

3. Institutions with Elementary Diagrams

In this section we introduce the main concept of this paper.

DEFINITION 7. An institution $\mathfrak{I} = (\mathbb{S}ign, \text{MOD}, Sen, \models)$ has *elementary diagrams* if and only if there exists a natural transformation $\iota\colon \text{MOD} \Rightarrow (-/\mathbb{T}h_\mathfrak{I})$, such that $\text{MOD}(\Sigma_M, E_M)$ and the comma category $M/\text{MOD}(\Sigma)$ are naturally isomorphic, i.e. the following diagram commutes by the isomorphism $i_{\Sigma,M}$ natural in M

$$\begin{tikzcd} \text{MOD}(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] \arrow[dr, "\text{MOD}(\iota_\Sigma(M))"'] & (M/\text{MOD}(\Sigma)) \arrow[d, "forgetful"] \\ & \text{MOD}(\Sigma) \end{tikzcd}$$

where $\iota_\Sigma(M)\colon (\Sigma, \emptyset) \to (\Sigma_M, E_M)$ for each signature Σ and each Σ-model M.

The signature morphism $\iota_\Sigma(M)\colon \Sigma \to \Sigma_M$ is called the *elementary extension of* Σ *via* M and the set E_M of Σ_M-sentences is called the *elementary diagram* of the model M. □

[5] $E \models_\Sigma e$ means that $M \models_\Sigma e$ for any Σ-model M that satisfies all sentences in E.

REMARK 2. The naturality of ι means that for each signature morphism $\phi\colon \Sigma \to \Sigma'$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{MOD}(\Sigma) \arrow[r, "\iota_\Sigma"] & \Sigma/\mathbb{T}h_{\mathfrak{I}} \\
\mathrm{MOD}(\Sigma') \arrow[u, "\mathrm{MOD}(\phi)"] \arrow[r, "\iota_{\Sigma'}"'] & \Sigma'/\mathbb{T}h_{\mathfrak{I}} \arrow[u, "\phi/\mathbb{T}h_{\mathfrak{I}}"']
\end{tikzcd}$$

Given a Σ-model homomorphism $h\colon M \to M'$ for a signature Σ, by the the functoriality of ι_Σ, we have that $\iota_\Sigma(h)$ is a theory morphism $\iota_\Sigma(h)\colon (\Sigma_M, E_M) \to (\Sigma_{M'}, E_{M'})$ such that

$$\begin{tikzcd}
\Sigma \arrow[r, "\iota_\Sigma(M)"] \arrow[dr, "\iota_\Sigma(M')"'] & \Sigma_M \arrow[d, "\iota_\Sigma(h)"] \\
& \Sigma_{M'}
\end{tikzcd}$$

The naturality of $i_{\Sigma,M}$ in M means that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{MOD}(\Sigma_M, E_M) \arrow[r, "i_{\Sigma,M}"] & M/\mathrm{MOD}(\Sigma) \\
\mathrm{MOD}(\Sigma_{M'}, E_{M'}) \arrow[u, "\mathrm{MOD}(\iota_\Sigma(h))"] \arrow[r, "i_{\Sigma,M'}"'] & M/\mathrm{MOD}(\Sigma) \arrow[u, "h/\mathrm{MOD}(\Sigma)"']
\end{tikzcd}$$

□

Informally, Definition 7 says that for each model M, the class of model homomorphisms from M can be axiomatised as a class of models of a theory. In other words, the elementary diagram of M is the theory encoding the class of model homomorphisms from M. In the particular concrete cases this categorical abstract model-theoretic concept of elementary diagram coincides with the ordinary concepts of elementary diagram, as know from conventional model theory (C.C.Chang and H.J.Keisler, 1973) (see Example 4 below).

The rest of this section presents a serie of concrete examples for the concept of elementary diagram.

EXAMPLE 4. *Elementary diagrams in many-sorted first-order predicate logic with equality.* Consider a first-order predicate logic signature (Σ,Π) and a (Σ,Π)-model M. Then the the elementary extension of (Σ,Π) via M is $\iota_{\Sigma,\Pi}(M)\colon (\Sigma,\Pi) \hookrightarrow (\Sigma_M,\Pi)$ where

– $(\Sigma_M)_{w\to s} = \Sigma_{w\to s}$ for any non-empty arity w and any sort s, and
– $(\Sigma_M)_{\to s} = \Sigma_{\to s} \cup M_s$ for any sort s.

Notice that the (Σ,Π)-model M can be canonically expanded to a (Σ_M,Π)-model M_M by interpreting the new constants of $(\Sigma_M)_{\to s}$ by the corresponding elements of M_s, i.e. $(M_M)_a = a$ for each $a \in M$. Then we define the elementary diagram E_M by

$$E_M = \{(\forall\emptyset)\ t\ =\ t' \text{ ground equation} \mid M_M \models_{\Sigma_M,\Pi} (\forall\emptyset)\ t\ =\ t'\} \cup \{(\forall\emptyset)\pi(\underline{t}) \text{ ground atom} \mid M_M \models_{\Sigma_M,\Pi} (\forall\emptyset)\pi(\underline{t}),\ \pi\in\Pi,\ \underline{t} \text{ list of ground } \Sigma_M\text{-terms corresponding to the arity of } \pi\}$$

The isomorphism $i_{\Sigma,\Pi,M}$ maps any (Σ_M,Π)-model N satisfying E_M to the (Σ,Π)-model homomorphism $h_N\colon M \to N\restriction_{\iota_{\Sigma,\Pi}(M)}$ such that $h_N(a) = N_a$ for each element $a \in M$. For each operation $\sigma \in \Sigma_{w\to s}$ and for each list of elements $\underline{a} \in M_w$, we have that $h_N(M_\sigma(\underline{a})) = N_{M_\sigma(\underline{a})} = N_{\sigma(\underline{a})}$ (because $N \models_{\Sigma_M,\Pi} (\forall\emptyset)\ \sigma(\underline{a})\ =\ M_{\sigma(\underline{a})}) = N_\sigma(N_{\underline{a}}) = N_\sigma(h_N(\underline{a}))$, therefore h_N is a Σ-homomorphism. For each predicate $\pi \in \Pi_w$ and for each list of elements $\underline{a} \in M_w$, we prove that $\underline{a} \in M_\pi$ implies $h_N(\underline{a}) \in N_\pi$. But $\underline{a} \in M_\pi$ implies $M_M \models_{\Sigma_M,\Pi} (\forall\emptyset)\pi(\underline{a})$ which means that $(\forall\emptyset)\pi(\underline{a}) \in E_M$. This implies $N \models_{\Sigma_M,\Pi} (\forall\emptyset)\pi(\underline{a})$ which further implies $N_{\underline{a}} \in N_\pi$ which means $h_N(\underline{a}) \in N_\pi$. Therefore h_N is also a Π-homomorphism.

The reverse isomorphism $i^{-1}_{\Sigma,\Pi,M}$ maps any (Σ,Π)-model homomorphism $h\colon M \to N$ to the (Σ_M,Π)-model $i^{-1}_{\Sigma,\Pi,M}(h) = N_h$ where $(N_h)\restriction_{\iota_{\Sigma,\Pi}(M)} = N$ and $(N_h)_a = h(a)$ for each $a \in M$. For each ground Σ_M-equation $(\forall\emptyset)\ t\ =\ t'$, we have that $N_h \models (\forall\emptyset)\ t\ =\ t'$ if and only if $(N_h)_t = (N_h)_{t'}$ if and only if $h(M_t) = h(M_{t'})$ which holds when $M_t = M_{t'}$ which is the same with $M \models (\forall\emptyset)\ t\ =\ t'$. Also, for each ground Π-atom $(\forall\emptyset)\pi(\underline{t})$ where $\underline{t}$ is a list of Σ_M-terms, we have that $N_h \models (\forall\emptyset)\pi(\underline{t})$ if and only if $N_{\underline{t}} \in (N_h)_\pi$ if $h(M_{\underline{t}}) \in (N_h)_\pi$ which holds whenever $M_{\underline{t}} \in M_\pi$ which is the same with $M \models (\forall\emptyset)\pi(\underline{t})$. This argument resumes the proof that $N_h \models E_M$.

Notice that in the unsorted case without equality, E_M gives exactly the elementary diagram of M in the sense of conventional model theory (C.C.Chang and H.J.Keisler, 1973). □

EXAMPLE 5. *Elementary diagrams in rewriting logic*
Consider an algebraic signature Σ and a preorder Σ-model M. Then the the elementary extension of Σ via M is $\iota_\Sigma(M)\colon \Sigma \hookrightarrow \Sigma_M$ where

- $(\Sigma_M)_{w\to s} = \Sigma_{w\to s}$ for any non-empty arity w and any sort s, and
- $(\Sigma_M)_{\to s} = \Sigma_{\to s} \cup |M_s|$ for any sort s.

As in the previous example, the Σ-model M can be canonically expanded to a Σ_M-model M_M by interpreting the new constants of $(\Sigma_M)_{\to s}$ by the corresponding elements of M_s, i.e. $(M_M)_a = a$ for each $a \in M$. Then the elementary diagram E_M is defined by

$$E_M = \{(\forall\emptyset)\ t\ =\ t' \text{ ground equation} \mid M_M \models_{\Sigma_M} (\forall\emptyset)\ t\ =\ t'\} \cup$$
$$\{(\forall\emptyset)\ t\ \Rightarrow\ t' \text{ ground transition} \mid M_M \models_{\Sigma_M} (\forall\emptyset)\ t\ \Rightarrow\ t'\}$$

The isomorphism $i_{\Sigma,M}$ maps any preorder Σ_M-model N satisfying E_M to the preorder Σ-homomorphism $i_{\Sigma,M}(N) = h_N\colon M \to N\restriction_{\iota_\Sigma(M)}$ such that $h_N(a) = N_a$ for each element $a \in M$. Because N satisfies the equations of E_M, we can prove that h_N is a Σ-homomorphism in the same way as in Example 4. We still have to prove that h_N is a preorder functor. Consider two elements $a, a' \in M$ of the same sort such that $a \le a'$. This implies that $M_M \models_{\Sigma_M} (\forall\emptyset)\ a\ \Rightarrow\ a'$, which implies that $(\forall\emptyset)\ a\ \Rightarrow\ a' \in E_M$, which implies that $N \models_{\Sigma_M} (\forall\emptyset)\ a\ \Rightarrow\ a'$, which implies that $N_a \le N_{a'}$, which means $h_N(a) \le h_N(a')$.

The reverse isomorphism $i^{-1}_{\Sigma,M}$ maps any preorder Σ-homomorphism $h\colon M \to N$ to the preorder Σ_M-model N_h where $(N_h)\restriction_{\iota_\Sigma(M)} = N$ and $(N_h)_a = h(a)$ for each element $a \in M$. We can prove that N_h satisfies all equations of E_M in the same way as in Example 4. Consider a transition $(\forall\emptyset)\ t\ \Rightarrow\ t' \in E_M$. Then $M_M \models_{\Sigma_M} (\forall\emptyset)\ t\ \Rightarrow\ t'$, which means that $M_t \le M_{t'}$. Because h is a preorder functor, this implies that $h(M_t) \le h(M_{t'})$, which means that $(N_h)_t \le (N_h)_{t'}$, which finally means $N_h \models_{\Sigma_M} (\forall\emptyset)\ t\ \Rightarrow\ t'$. This resumes the proof that N_h also satisfies all transitions from E_M. □

EXAMPLE 6. *Elementary diagrams in partial algebra.*
Consider a partial algebra signature (Σ,Δ) and a partial (Σ,Δ)-algebra A. The elementary extension of (Σ,Δ) via A is $\iota_{\Sigma,\Delta}(A)\colon (\Sigma,\Delta) \hookrightarrow (\Sigma,\Delta_A)$ where

- $(\Delta_A)_{w\to s} = \Delta_{w\to s}$ for any non-empty arity w and any sort s, and
- $(\Delta_A)_{\to s} = \Delta_{\to s} \cup A_s$ for any sort s.

The partial (Σ,Δ)-algebra A can be canonically expanded to a partial (Σ,Δ_A)-algebra A_A by interpreting the new constants of $(\Sigma_A)_{\to s}$, for each sort s by the corresponding elements of A_s, i.e. $(A_A)_a = a$ for each $a \in A$. Then the elementary diagram E_A is defined by

$$E_A = \{(\forall\emptyset)\ t \stackrel{s}{=} t' \text{ ground strong equation} \mid A_A \models_{\Sigma,\Delta_A} (\forall\emptyset)\ t \stackrel{s}{=} t'\} \cup$$
$$\{(\forall\emptyset)\ t\downarrow \text{ ground undefinedness predicate} \mid A_A \models_{\Sigma,\Delta_A} (\forall\emptyset)\ t\downarrow\}$$

The isomorphism $i_{\Sigma,\Delta,A}$ maps any partial (Σ,Δ_A)-algebra B satisfying E_A to the partial (Σ,Δ)-homomorphism $i_{\Sigma,\Delta,A}(B) = h_B\colon A \twoheadrightarrow B\restriction_{\iota_{\Sigma,\Delta}(A)}$ such that $h_B(a) \stackrel{s}{=} B_a$ for each element $a \in A$, i.e. either both of them are undefined or both of them are defined and equal. For each operation symbol $\sigma \in (\Sigma\cup\Delta)_{w\to s}$ and for each list of elements $\underline{a} \in A_w$, we have to prove that $h_B(A_\sigma(\underline{a})) \stackrel{s}{=} B_\sigma(h_B(\underline{a}))$. By definition we have that $h_B(A_\sigma(\underline{a})) \stackrel{s}{=} B_{A_\sigma(\underline{a})}$ and $B_\sigma(h_B(\underline{a})) \stackrel{s}{=} B_\sigma(B_{\underline{a}})$. But $B_{A_\sigma(\underline{a})} \stackrel{s}{=} B_\sigma(B_{\underline{a}})$ because

- when $A_\sigma(\underline{a})$ is undefined then $B_\sigma(B_{\underline{a}})$ is also undefined because $B \models_{\Sigma,\Delta_A} (\forall\emptyset)\ \sigma(\underline{a}) \downarrow$ as $(\forall\emptyset)\ \sigma(\underline{a}) \downarrow\in E_A$, and
- when $A_\sigma(\underline{a})$ is defined then $B_{A_\sigma(\underline{a})} \stackrel{s}{=} B_\sigma(B_{\underline{a}})$ because $B \models_{\Sigma,\Delta_A} (\forall\emptyset)\ \sigma(\underline{a}) \stackrel{s}{=} A_\sigma(\underline{a})$ as $(\forall\emptyset)\ \sigma(\underline{a}) \stackrel{s}{=} A_\sigma(\underline{a}) \in E_A$.

Therefore, by the transitivity of the strong partial equality $\stackrel{s}{=}$ relation, h_B is a (Σ,Δ)-homomorphism.

The reverse isomorphism $i^{-1}_{\Sigma,\Delta,A}$ maps any (Σ,Δ)-homomorphism $h\colon A \twoheadrightarrow B$ to the partial (Σ,Δ_A)-algebra B_h where $(B_h)\restriction_{\iota_{\Sigma,\Delta}(A)} = B$ and $(B_h)_a \stackrel{s}{=} h(a)$ for each element $a \in A$. The fact that B_h satisfies all strong equations and undefinedness predicates of E_A follows from the fact that for each ground term t, $h(A_t) \stackrel{s}{=} (B_h)_t$ (easily shown by structural induction on t). □

EXAMPLE 7. *Elementary diagrams in hidden algebra.*
Consider a CHA signature $(H,V,\Sigma,\Sigma^{\mathrm{b}})$ and a Σ-algebra A. Then the elementary extension of $(H,V,\Sigma,\Sigma^{\mathrm{b}})$ via A is $\iota_\Sigma(A)\colon (H,V,\Sigma,\Sigma^{\mathrm{b}}) \hookrightarrow (H,V,\Sigma_A,\Sigma^{\mathrm{b}})$, where

- $(\Sigma_A)_{w\to s} = \Sigma_{w\to s}$ for any non-empty arity w and any sort s, and
- $(\Sigma_A)_{\to s} = \Sigma_{\to s} \cup A_s$ for any sort s.

The Σ-algebra A can be canonically expanded to a Σ_A-algebra A_A by interpreting the new constants of $(\Sigma_A)_{\to s}$ by the corresponding elements of A_s, i.e. $(A_A)_a = a$ for each $a \in A$. Then the elementary diagram E_A is defined by

$$E_A = \{(\forall\emptyset)\ t\ =\ t' \text{ ground strict equation} \mid A_A \models_{\Sigma_A} (\forall\emptyset)\ t\ =\ t'\} \cup$$
$$\{(\forall\emptyset)\ t\ \sim\ t' \text{ ground behavioural equation} \mid A_A \models_{\Sigma_A} (\forall\emptyset)\ t\ \sim\ t'\}$$

The isomorphism $i_{\Sigma,A}$ maps any Σ_A-algebra B satisfying E_A to the CHA Σ-algebra homomorphism $h_B\colon A \to B\restriction_{\iota_\Sigma(A)}$ such that $h_B(a) = B_a$ for each element $a \in A$. For each operation $\sigma \in \Sigma_{w\to s}$ and for each list of elements $\underline{a} \in A_w$, we have that $h_B(A_\sigma(\underline{a})) = B_{A_\sigma(\underline{a})} = B_{\sigma(\underline{a})}$ (because $B \models_{\Sigma_A} (\forall\emptyset)\ \sigma(\underline{a})\ =\ A_{\sigma(\underline{a})}) = B_\sigma(B_{\underline{a}}) = B_\sigma(h_B(\underline{a}))$, therefore h_B is a Σ-homomorphism. Now let us consider two behavioural equivalent elements $a, a' \in A$ such that $a \sim a'$. This implies $(\forall\emptyset)\ a\ \sim\ a' \in E_A$ which implies $B \models_{\Sigma_A} (\forall\emptyset)\ a\ \sim\ a'$ which implies $B_a \sim B_{a'}$ which means that $h_B(a) \sim h_B(a')$. This shows that h_B is a CHA Σ-homomorphism.

The reverse isomorphism $i^{-1}_{\Sigma,A}$ maps any CHA Σ-algebra homomorphism $h\colon A \to B$ to the Σ_A-algebra B_h where $(B_h)\restriction_{\iota_\Sigma(A)} = B$ and $(B_h)_a = h(a)$ for each $a \in A$. Each ground strict Σ_A-equation $(\forall\emptyset)\ t\ =\ t'$ in E_A is satisfied by B_h by the virtue of the same argument as in Example 4. Now consider a ground behavioural Σ_A-equation $(\forall\emptyset)\ t\ \sim\ t'$ in E_A. This means that $A_t \sim A_{t'}$, which implies $h(A_t) \sim h(A_{t'})$ because h preserves behavioural equivalence, which means $(B_h)_t \sim (B_h)_{t'}$, which means that $B_h \models_{\Sigma_A} (\forall\emptyset)\ t\ \sim\ t'$. This resumes the proof that $B_h \models E_A$. □

The examples of this section point out to a certain pattern for the elementary diagrams. In the particular cases they are the *basic* ground sentences satisfied by the model over the signature extended with the elements of the model as constants. Informally, the basic sentences are the simplest sentences of the institution that match the fundamental model-theoretic structure of the institution, not involving logical connectives or quantifications.

4. Some Applications

This section is devoted to several applications illustrating the power of the institution-independent method of elementary diagrams.

4.1. Institution Liberality

Intuitively, liberality can be regarded as a generalised form of initial semantics. The following result makes this intuition precise for the institutions with elementary diagrams.

PROPOSITION 1. *Let $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ be an institution with elementary diagrams such that each theory has an initial model. Then*

1. *for each theory (Σ, E), the forgetful functor $\mathrm{MOD}(\Sigma, E) \to \mathrm{MOD}(\Sigma)$ has a left adjoint, and*
2. *if the institution $\mathfrak{I}$ has pushouts of signatures and is semi-exact, then for each signature morphism Φ the reduct functor $\mathrm{MOD}(\Phi)$ has a left adjoint.*

Proof. For each theory (Σ, E), we denote its initial model by $0_{\Sigma,E}$.

1. Consider a theory (Σ, E) and let M be a Σ-model. Let $\iota_\Sigma(M)\colon \Sigma \to \Sigma_M$ be the elementary extension of Σ via M and let $E' = \iota_\Sigma(M)(E)$. We show that $M' = 0_{\Sigma_M, E_M \cup E'} \upharpoonright_{\iota_\Sigma(M)}$ is the free (Σ, E)-model over M with the universal arrow $\eta_M = (0_{\Sigma_M,E_M} \to 0_{\Sigma_M,E_M\cup E'}) \upharpoonright_{\iota_\Sigma(M)}$.

We have to prove that for each model homomorphism $h\colon M \to N$ for $N \models_\Sigma E$, there exists an unique $h'\colon M' \to N$ such that $\eta_M; h' = h$. Let $N_h = i^{-1}_{\Sigma,M}(h)$. Then $N_h \models_{\Sigma_M} E'$ because $N_h \upharpoonright_{\iota_\Sigma(M)} = N$ and $N \models_\Sigma E$. Let h'' be the unique model homomorphism $h''\colon 0_{\Sigma_M,E_M\cup E'} \to N_h$. Let h' be $h'' \upharpoonright_{\iota_\Sigma(M)}$. Then $\eta_M; h' = (0_{\Sigma_M,E_M} \to N_h) \upharpoonright_{\iota_\Sigma(M)} = h$.

The uniqueness of h' follows by the bijection between $(M/\mathrm{MOD}(\Sigma))(\eta_M, h)$ and $\mathrm{MOD}(\Sigma_M, E_M)(0_{\Sigma_M,E_M\cup E'}, N_h)$.

2. Let $\Phi\colon \Sigma \to \Sigma'$ be a signature morphism and let M be a Σ-model. Consider the pushout of signatures

$$\begin{tikzcd}
\Sigma_M \arrow[r, "\Phi'"] & \Sigma'' \\
\Sigma \arrow[u, "\iota_\Sigma(M)"] \arrow[r, "\Phi"'] & \Sigma' \arrow[u, "\iota'"']
\end{tikzcd}$$

We define M^Φ to be $(0_{\Sigma'',\Phi'(E_M)}) \upharpoonright_{\iota'}$ and the universal arrow $\eta_M\colon M \to (M^\Phi)\upharpoonright_\Phi$ to be $(0_{\Sigma_M,E_M} \to (0_{\Sigma'',\Phi'(E_M)})\upharpoonright_{\Phi'}) \upharpoonright_{\iota_\Sigma(M)}$.

For proving the universal property of η_M, consider $h\colon M \to N\upharpoonright_\Phi$ with N any Σ'-model. Let $M_h = i^{-1}_{\Sigma,M}(h)$. Notice that $M_h \upharpoonright_{\iota_\Sigma(M)} = N\upharpoonright_\Phi$. Because $\mathfrak{I}$ is semi-exact, let $N \otimes M_h$ be the amalgamation of N and M_h. Notice that $N \otimes M_h \models \Phi'(E_M)$ because $(N \otimes M_h)\upharpoonright_{\Phi'} = M_h \models E_M$. Therefore there exists an unique model homomorphism $h''\colon 0_{\Sigma'',\Phi'(E_M)} \to N \otimes M_h$. Let $h' = h''\upharpoonright_{\iota'}$. We have that $h'\colon M^\Phi \to N$ and $\eta_M; h'\upharpoonright_\Phi = \eta_M; h''\upharpoonright_{\iota'}\upharpoonright_\Phi = \eta_M; h''\upharpoonright_{\Phi'}\upharpoonright_{\iota_\Sigma(M)} = (0_{\Sigma_M,E_M} \to M_h)\upharpoonright_{\iota_\Sigma(M)} = h$.

The uniqueness of h' follows from the uniqueness of h'' and of the amalgamation of model homomorphisms.

The following is a corollary of Proposition 1:

THEOREM 1. *A semi-exact institution with elementary diagrams and finite colimits of signatures is liberal if and only if each theory has an initial model.*

Proof. Let $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ be an institution with elementary diagrams and finite colimits of signatures.

In each institution with initial signatures, the existence of initial models for a theory is the same with the liberality of the unique theory morphism from the initial (empty) theory to that theory. Therefore, we have to prove only that the existence of initial models for theories implies the liberality of the institution.

Consider a theory morphism $\Phi\colon (\Sigma, E) \to (\Sigma', E')$.

$$\begin{tikzcd}
\mathrm{MOD}(\Sigma) & \mathrm{MOD}(\Sigma') \arrow[l, "\mathrm{MOD}(\Phi)"'] \\
\mathrm{MOD}(\Sigma, E) \arrow[u, "\textit{forgetful subcategory}"] & \mathrm{MOD}(\Sigma', E') \arrow[l, "\mathrm{MOD}(\Phi)"] \arrow[u, "\textit{forgetful subcategory}"']
\end{tikzcd}$$

By Proposition 1, both $\mathrm{MOD}(\Phi)\colon \mathrm{MOD}(\Sigma') \to \mathrm{MOD}(\Sigma)$ and the forgetful functor $\mathrm{MOD}(\Sigma', E') \to \mathrm{MOD}(\Sigma')$ have left-adjoints. By composition of adjunctions (see (MacLane, 1998)), the composite functor $\text{MOD}(\Sigma', E') \to \text{MOD}(\Sigma)$ has a left-adjoint. Now, by the following simple categorical lemma (we omit its proof):

LEMMA 1. *Let $C' \hookrightarrow C$ be a full subcategory and consider a functor $D \to C'$. If the composite functor $D \to C$ has a left-adjoint F, then the restriction of F to C' is a left-adjoint to $D \to C'$.*

$$\begin{tikzcd} C & \\ C' \arrow[u] & D \arrow[l] \arrow[ul] \end{tikzcd}$$

we resume the proof of this theorem by substituting the category D by $\text{MOD}(\Sigma', E')$, the category C by $\text{MOD}(\Sigma)$, and the category C' by $\text{MOD}(\Sigma, E)$.

4.2. QUASI-VARIETIES IN INSTITUTIONS

We develop an institution-independent approach on quasi-varieties, based on the method of elementary diagrams and inclusion systems.

4.2.1. *Quasi-varieties in inclusion systems*

In this section we rephrase abstractly some classical model theoretic concepts within the framework of inclusion systems. Similar concepts have been formulated and results obtained within the framework of factorisation systems (see (Tarlecki, 1986a; Tarlecki, 1986b) or (Andréka and Németi, 1981) for a very general approach), however the inclusion systems framework leads to greater simplicity.

Firstly, we may use the concept of inclusion system for rephrasing the category theoretic concepts of subobjects and quotients (that are traditionally (MacLane, 1998) defined in terms of monics and epics).

DEFINITION 8. Consider a weak inclusion system $\langle I, \mathcal{E} \rangle$ for a category $\mathbb{C}$. Then

- a is a *subobject* of b if there exists an inclusion $a \hookrightarrow b$, and
- an object b is a *quotient representation* of a if there exists a surjection $a \twoheadrightarrow b$. A *quotient* of a is an isomorphism class of quotient representations.

The weak inclusion systems $\langle I, \mathcal{E} \rangle$ is *well-powered*, respectively *co-well-powered*, if the class of subobjects, respectively quotients, of each object is a *set*. □

DEFINITION 9. Consider a category $\mathbb{C}$ with a weak inclusion system. Then an object of $\mathbb{C}$ is *reachable* if and only if it has no proper subobjects. □

FACT 2. Consider a category $\mathbb{C}$ with a weak inclusion system and with an initial object $0_{\mathbb{C}}$. Then

1. for each reachable object a, the unique arrow $0_{\mathbb{C}} \twoheadrightarrow a$ is a surjection, and
2. each object has exactly one reachable subobject.

□

DEFINITION 10. Consider a category $\mathbb{C}$ with finite products and with a weak inclusion system. Then a class of objects of $\mathbb{C}$ closed under isomorphisms

- is a *quasi-variety* if it is closed under finite products and subobjects, and
- is a *variety* if it is a quasi-variety closed under quotients.

□

PROPOSITION 2. Consider a category $\mathbb{C}$ with an initial object $0_{\mathbb{C}}$ and with a co-well-powered epic weak inclusion system. Then each quasi-variety Q of $\mathbb{C}$ has a reachable initial object. □

Proof. Let $\{A_i \mid i \in I\}$ be the class of of all reachable subobjects of all objects of Q. Then we consider a subclass of indices $I' \subseteq I$ such that there are no isomorphic objects in $\{A_i \mid i \in I'\}$ and for each $i \in I$ there exists $j \in I'$ such that $A_i \simeq A_j$. I' is a *set* because the weak inclusion system of $\mathbb{C}$ is co-well-powered and by Fact 2. Let 0_Q be the reachable subobject of the product $\Pi_{i \in I'} A_i$. We prove that 0_Q is initial in Q.

For each object A of Q, there exists $j \in I$ such that A_j is a reachable subobject of A. Then there exists $i \in I'$ such that A_i is isomorphic to A_j, therefore there exists and arrow $\Pi_{i \in I'} A_i \to A$. Because 0_Q is a subobject of $\Pi_{i \in I'} A_i$, there exists an arrow $0_Q \to A$. Because 0_Q is reachable, the unique arrow $0_{\mathbb{C}} \to 0_Q$ is a surjection, hence it is an epic, which implies the uniqueness of the arrow $0_Q \to A$.

4.2.2. *Quasi-varieties in institutions with elementary diagrams*

The following result extends the conclusion of Proposition 2 with its opposite implication, thus obtaining an 'if and only if' characterization of quasi-varieties. This generalizes a classical result from universal algebra (Grätzer, 1979) or conventional first-order model theory (Malcev, 1971). A similar institution-independent result has been obtained by Tarlecki (Tarlecki, 1986a) within the framework of the so-called "abstract algebraic institutions". However, the concept of abstract algebraic institution provides a set of conditions much more complex than the conditions of Theorem 2, the greater simplicity of our approach leading also to simpler and somehow different proofs.

THEOREM 2. *Consider an institution* $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ *closed under isomorphisms and with elementary diagrams such that*

1. *the category of* Σ*-models has an initial object* 0_Σ*, finite products, and a co-well-powered epic weak inclusion system for each signature* Σ*,*
2. *all model reduct functors preserve the inclusions and the surjections,*
3. *the model reduct functors corresponding to the elementary extensions reflect identities.*

Then each theory has a reachable initial model if and only if the class of models of each theory is a quasi-variety.

Proof. By Proposition 2 we have to prove only one implication. Let (Σ, E) be a theory and consider $B \hookrightarrow A$ a submodel of $A \in |\mathrm{MOD}(\Sigma, E)|$. We prove that $B \models_\Sigma E$. Let $i_{\Sigma,B}^{-1}(B \hookrightarrow A) = h\colon 0_{\Sigma_B, E_B} \to A_B$. Let us factor $h = e; j$ in the inclusion system of Σ_B with e surjection and j inclusion. Because the reduct functor $\mathrm{MOD}(\iota_\Sigma(B))$ preserves both the inclusions and the surjections, $B \hookrightarrow A$ gets factored as $B \hookrightarrow A = e\restriction_{\iota_\Sigma(B)}; j\restriction_{\iota_\Sigma(B)}$ in the inclusion system of Σ. Because $B \hookrightarrow A$ is an inclusion, we deduce that $e\restriction_\iota = 1_B$, which means that e is identity because $\mathrm{MOD}(\iota_\Sigma(B))$ reflects identities. Therefore h is inclusion.

$A \models_\Sigma E$ implies that $A_B \models_{\Sigma_B} E'$, where $E' = \iota_\Sigma(B)(E)$, which means that there exists an unique arrow $f\colon 0_{\Sigma_B, E_B \cup E'} \to A_B$. Because $h\colon 0_{\Sigma_B, E_B} \hookrightarrow A_B$ is inclusion and $0_{\Sigma_B, E_B} \to 0_{\Sigma_B, E_B \cup E'}$ is surjection, by factoring f in the inclusion system of Σ_B and by using the initiality properties, it follows that $0_{\Sigma_B, E_B}$ and $0_{\Sigma_B, E_B \cup E'}$ are isomorphic. Therefore, $0_{\Sigma_B, E_B} \models_{\Sigma_B} E'$, which by the satisfaction condition implies $B \models_\Sigma E$.

For the second part of this proof, consider $(\pi_i\colon B \to A_i)_{i \in I}$ a product of models in a signature Σ and such that $A_i \models E$ for each $i \in I$. We prove that $B \models E$. Because of the canonical isomorphism $i_{\Sigma,B}\colon \mathrm{MOD}(\Sigma_B, E_B) \to B/\mathrm{MOD}(\Sigma)$ and because the forgetful functor $B/\mathrm{MOD}(\Sigma) \to \mathrm{MOD}(\Sigma)$ reflects the products, we deduce that $((\pi_i)_B\colon 0_{\Sigma_B, E_B} \to (A_i)_B)_{i \in I}$ is a product in $\mathrm{MOD}(\Sigma_B, E_B)$, where $(\pi_i)_B = i_{\Sigma,B}^{-1}(\pi_i)$ for each $i \in I$.

By the satisfaction condition $(A_i)_B \models E'$ for each $i \in I$, where $E' = \iota_\Sigma(B)(E)$. Therefore we get an unique arrow $0_{\Sigma_B, E_B \cup E'} \to (A_i)_B$ for each $i \in I$. By the universal property of products, we thus get an arrow $0_{\Sigma_B,E_B\cup E'} \to 0_{\Sigma_B,E_B}$. Because we already have an arrow $0_{\Sigma_B,E_B} \to 0_{\Sigma_B,E_B\cup E'}$, by the universal property of the initial objects $0_{\Sigma_B,E_B}$ and $0_{\Sigma_B,E_B\cup E'}$, we have that $0_{\Sigma_B,E_B}$ and $0_{\Sigma_B,E_B\cup E'}$ are isomorphic. This implies that $0_{\Sigma_B,E_B} \models_{\Sigma_B} E'$, which by the satisfaction condition implies $B \models_\Sigma E$.

COROLLARY 1. Consider a semi-exact institution with finite colimits of signatures and satisfying the conditions of Theorem 2. If the class of models of each theory is a quasi-variety, then the institution is liberal. □

EXAMPLE 8. The institutions of first-order predicate logic, rewriting logic, partial algebra, have finite colimits of signatures and are semi-exact[6] and admit elementary diagrams (by Examples 4, 5, and 6). One may easily notice that the epic weak inclusion systems for the categories of models of these three institutions, given in Examples 1, 2, and 3, satisfy the specific conditions of Theorem 2.

In first-order predicate logic, each universal Horn sentence theory (see (C.C.Chang and H.J.Keisler, 1973)) is preserves by products and submodels, therefore each morphism between universal Horn theories is liberal (by applying Corollary 1 for the subinstitution of universal Horn sentences). For the particular case of the signatures without predicate symbols, we get the well know result that the institution of many-sorted conditional equational logic is liberal (Goguen and Burstall, 1992).

The rewriting logic institution is liberal because each transition is preserved under products and preorder submodels.[7]

From Proposition 3, we can easily deduce that the strong equations and the undefinedness predicates are preserved under products and (partial) subalgebras. Thus, by Corollary 1, the morphisms between universal theories of strong equations and undefinedness predicates are liberal. □

4.3. Model Limits and Colimits

Existence of limits and colimits of models are important properties of theories, both in institution-independent model theory or institution-independent computing science. For example, the institution-independent ultraproduct method (Diaconescu, 2002) requires both products and filtered colimits of models, while in the semantics of constraint logic programming (Diaconescu, 2000) finite colimits of models play an important rôle.

4.3.1. *Small Limits*

By the technology of indexed categories (Tarlecki et al., 1991), in most institutions in use in logic and computing science small limits of signatures are easy. In this section we show that by the method of elementary diagrams, limits of theory models can be obtained from limits of signatures.

THEOREM 3. *Consider an institution with elementary diagrams and initial models for theories. If its category of signatures has limits, then the category of models of each theory has limits too.*

Proof. Let $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ be an institution with elementary diagrams and initial models for theories. Let $\mathbb{J}$ be a category such that $\mathbb{S}ign$ has $\mathbb{J}$-limits. Let $M\colon \mathbb{J} \to \mathrm{MOD}(\Sigma, E)$ be a $\mathbb{J}$-diagram of models for a theory (Σ, E).

$$\begin{tikzcd}
 & & \Sigma_{M^i} \arrow[dd, "\iota_\Sigma(M^u)"] \\
\Sigma \arrow[r, "\varphi"] \arrow[urr, "\iota_\Sigma(M^i)"] \arrow[drr, "\iota_\Sigma(M^j)"'] & \Sigma' \arrow[ur, "\phi^i"'] \arrow[dr, "\phi^j"] & \\
 & & \Sigma_{M^j}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M^i \arrow[dd, "M^u"] \\
N \arrow[ur, "\mu^i"] \arrow[dr, "\mu^j"'] & \\
 & M^j
\end{tikzcd}
\qquad
\begin{tikzcd}
i \arrow[dd, "u"] \\
\\
j
\end{tikzcd}$$

Let $\phi\colon \Sigma' \Rightarrow \Sigma_M$ be the limit cone where $\Sigma_M\colon \mathbb{J} \to \mathbb{S}ign$ with

[6] These properties of these institutions are well known in the theory of algebraic specification.

[7] We leave this as exercise to the reader.

- $(\Sigma_M)^i = \Sigma_{M^i}$ for each index $i \in |\mathbb{J}|$, and
- $(\Sigma_M)^u = \iota_\Sigma(M^u)$ for each index morphism $u \in \mathbb{J}$.

Let $E' = \{e \in Sen(\Sigma') \mid \phi^i(e) \in E_{M_i}$ for each $i \in |\mathbb{J}|\}$. By the universal property of the limit cone ϕ, let $\varphi\colon \Sigma \to \Sigma'$ be the unique signature morphism such that $\varphi;\phi^i = \iota_\Sigma(M^i)$ for each $i \in |\mathbb{J}|$. Let $N = 0_{\Sigma',E'\cup\varphi(E)}\restriction_\varphi$, where $0_{\Sigma',E'\cup\varphi(E)}$ is the initial model of the theory $(\Sigma', E' \cup \varphi(E))$. Notice that for each $i \in |\mathbb{J}|$, $0_{\Sigma_{M^i},E_{M^i}}\restriction_{\phi^i} \models E' \cup \varphi(E)$ (by the definition of E', by the fact that $M^i \models E$, by the fact that $0_{\Sigma_{M^i},E_{M^i}}\restriction_{\iota_\Sigma(M^i)} = M^i$, and by the satisfaction condition), where $0_{\Sigma_{M^i},E_{M^i}}$ is the initial model of the elementary diagram (Σ_{M^i}, E_{M^i}). Therefore let ν^i be the unique model homomorphism $0_{\Sigma',E'\cup\varphi(E)} \to 0_{\Sigma_{M^i},E_{M^i}}\restriction_{\phi^i}$. Let $\mu^i = \nu^i\restriction_\varphi$ for each $i \in |\mathbb{J}|$.

We prove that $\mu\colon N \Rightarrow M$ is a limit cone. For each $u \in \mathbb{J}(i,j)$, we have that $(0_{\Sigma_{M^i},E_{M^i}} \to 0_{\Sigma_{M^j},E_{M^j}}\restriction_{\iota_\Sigma(M^u)})\restriction_{\iota_\Sigma(M^i)} = M^u$. Since $\iota_\Sigma(M^i) = \varphi;\phi^i$ and because $\nu^i;(0_{\Sigma_{M^i},E_{M^i}} \to 0_{\Sigma_{M^j},E_{M^j}}\restriction_{\iota_\Sigma(M^u)})\restriction_{\phi^i} = \nu^j$, we have that $\nu^i\restriction_\varphi;M^u = \nu^j\restriction_\varphi$, which means that $\mu^i;M^u = \mu^j$. Therefore $\mu\colon N \Rightarrow M$ is a cone.

Now consider another cone $\mu'\colon N' \Rightarrow M$. Let $\iota_\Sigma(N')\colon \Sigma \to \Sigma_{N'}$ be the elementary extension of Σ via N'. Notice that $\{\iota_\Sigma(\mu'^i)\}_{i\in|\mathbb{J}|}\colon \Sigma_{N'} \Rightarrow \Sigma_M$ is a cone. Therefore let $\varphi'\colon \Sigma_{N'} \to \Sigma'$ be the unique signature morphism such that $\varphi';\phi^i = \iota_\Sigma(\mu'^i)$ for each $i \in |\mathbb{J}|$. For each $i \in |\mathbb{J}|$, $\iota_\Sigma(\mu'^i)$ is a theory morphism $(\Sigma_{N'}, E_{N'}) \to (\Sigma_{M^i}, E_{M^i})$, which implies that $\phi^i(\varphi'(E_{N'})) \subseteq E_{M^i}$ for each $i \in |\mathbb{J}|$, which implies $\varphi'(E_{N'}) \subseteq E'$ which implies $0_{\Sigma',E'\cup\varphi(E)} \models \varphi'(E_{N'})$. By the satisfaction condition this is equivalent to $0_{\Sigma',E'\cup\varphi(E)}\restriction_{\varphi'} \models E_{N'}$. Therefore let h be the unique model homomorphism $0_{\Sigma_{N'},E_{N'}} \to 0_{\Sigma',E'\cup\varphi(E)}\restriction_{\varphi'}$.

We show that $h\restriction_{\iota_\Sigma(N')}$ is the unique model homomorphism such that $\mu'^i = h\restriction_{\iota_\Sigma(N')};\mu^i$ for each $i \in |\mathbb{J}|$. Indeed, for each $i \in |\mathbb{J}|$, $h\restriction_{\iota_\Sigma(N')};\mu^i = h\restriction_{\iota_\Sigma(N')};\nu^i\restriction_\varphi = h\restriction_{\iota_\Sigma(N')};\nu^i\restriction_{\varphi'}\restriction_{\iota_\Sigma(N')} = (h;(0_{\Sigma',E'\cup\varphi(E)}\restriction_{\varphi'} \to 0_{\Sigma_{M^i},E_{M^i}}\restriction_{\phi^i}\restriction_{\varphi'}))\restriction_{\iota_\Sigma(N')} = (0_{\Sigma_{N'},E_{N'}} \to 0_{\Sigma_{M^i},E_{M^i}}\restriction_{\phi^i}\restriction_{\varphi'})\restriction_{\iota_\Sigma(N')} = (0_{\Sigma_{N'},E_{N'}} \to 0_{\Sigma_{M^i},E_{M^i}}\restriction_{\iota_\Sigma(\mu'^i)})\restriction_{\iota_\Sigma(N')} = \mu'^i$. The uniqueness of $h\restriction_{\iota_\Sigma(N')}$ follows from the isomorphism $i_{\Sigma,N'}$ and from the initiality property of $0_{\Sigma_{N'},E_{N'}}$.

EXAMPLE 9. By applying the methods of (Tarlecki et al., 1991), the institutions of first-order predicate logic, rewriting logic, partial algebra, have small limits of signatures. By Theorem 3, cf. Example 8, the categories of models of universal Horn theories in the in first-order predicate logic, of equational theories in general (many-sorted) algebra, of rewriting logic theories, and of strong equational and universal undefinedness predicate theories in partial algebra, have small limits. □

4.3.2. *Finite Colimits*

The following result gives a simple sufficient condition for the existence finite colimits of theory models.

THEOREM 4. *In any liberal institution with elementary diagrams the category of models of any theory has finite colimits.*

Proof. Let $\mathfrak{I} = (\mathbb{S}ign, \mathrm{MOD}, Sen, \models)$ be a liberal institution with elementary diagrams.

Let us first notice that by the Satisfaction Condition, for each model M of a theory (Σ, E), the restriction of the natural isomorphism $i_{\Sigma,M}\colon \mathrm{MOD}(\Sigma_M, E_M) \to M/\mathrm{MOD}(\Sigma)$ to the subcategory $\mathrm{MOD}(\Sigma_M, E_M \cup E') \subseteq \mathrm{MOD}(\Sigma_M, E_M)$ is an isomorphism $\mathrm{MOD}(\Sigma_M, E_M \cup E') \to M/\mathrm{MOD}(\Sigma, E)$, where $E' = \iota_\Sigma(M)(E)$.

Now consider $h_i\colon M \to M_i$ two model homomorphisms in $\mathrm{MOD}(\Sigma, E)$. Notice that the pushout of h_1 and h_2 is the same with an universal arrow from h_2 to the functor $h_1/\mathrm{MOD}(\Sigma)\colon (M_1/\mathrm{MOD}(\Sigma,E)) \to (M/\mathrm{MOD}(\Sigma,E))$. This universal arrow exists because $h_1/\mathrm{MOD}(\Sigma)$ has a left adjoint since

- the reduct functor $\mathrm{MOD}(\iota_\Sigma(h_1))\colon \mathrm{MOD}(\Sigma_{M_1}, E_{M_1} \cup E'_1) \to \mathrm{MOD}(\Sigma_M, E_M \cup E')$ has left adjoint by the liberality of the institution, where $E'_1 = \iota_\Sigma(M_1)(E)$,

- the diagram

$$\begin{tikzcd}
(M_1/\textsc{Mod}(\Sigma,E)) \arrow[r, "h_1/\textsc{Mod}(\Sigma)"] & (M/\textsc{Mod}(\Sigma,E)) \\
\textsc{Mod}(\Sigma_{M_1},E_{M_1}) \arrow[u, "i_{\Sigma,M_1}"] \arrow[r, "\textsc{Mod}(\iota_\Sigma(h_1))"'] & \textsc{Mod}(\Sigma_M,E_M) \arrow[u, "i_{\Sigma,M}"']
\end{tikzcd}$$

  commutes by the naturality of i (cf. Remark 2),
- as isomorphisms, i^{-1}_{Σ,M_1} and $i_{\Sigma,M}$ have left adjoints, and
- the composition of right adjoint functors is a right adjoint functor (MacLane, 1998).

Therefore $\textsc{Mod}(\Sigma,E)$ has pushouts. By the liberality of the institution it also has an initial model, thus, by the construction of any finite colimit from initial objects and pushouts (see (MacLane, 1998)), it has *all* finite colimits.

EXAMPLE 10. Cf. Example 8, the categories of models of universal Horn theories in the in first-order predicate logic, of equational theories in general (many-sorted) algebra, of rewriting logic theories, and of strong equational and universal undefinedness predicate theories in partial algebra, have finite colimits. □

4.3.3. *Small Colimits*

Here we give an alternative way of obtaining colimits of theory models which is based on colimits of signatures in the style of Section 4.3.1 rather than liberality (as in Theorem 4) and which has the advantage of going beyond the finiteness restriction. Notice that, as in the case of limits, indexed categories (Tarlecki et al., 1991) provides an easy method for proving the existence of small colimits of signatures in most institutions in use in logic and computing science.

THEOREM 5. *Consider an institution with elementary diagrams and initial models for theories. If its category of signatures has colimits, then the category of models of each theory has colimits too.*

Proof. Because the proof of this result uses the same technique and follows the same steps as the proof of Theorem 3 we will omit the details.

Let $\mathfrak{I} = (\mathbb{S}ign, \textsc{Mod}, Sen, \models)$ be an institution with elementary diagrams and initial models for theories. Let $\mathbb{J}$ be a category such that $\mathbb{S}ign$ has $\mathbb{J}$-colimits. Let $M\colon \mathbb{J} \to \textsc{Mod}(\Sigma,E)$ be a $\mathbb{J}$-diagram of models for a theory (Σ,E).

$$\begin{tikzcd}
i \arrow[dd, "u"'] & M^i \arrow[dd, "M^u"'] \arrow[dr, "\mu^i"] & & & & \Sigma_{M^i} \arrow[dd, "\iota_\Sigma(M^u)"'] \arrow[dr, "\phi^i"] \arrow[drr, "\iota_\Sigma(\mu'^i)"] & & \\
 & & N & & \Sigma \arrow[ur, "\iota_\Sigma(M^i)"] \arrow[dr, "\iota_\Sigma(M^j)"'] & & \Sigma' \arrow[r, "\varphi'"] & \Sigma_{N'} \\
j & M^j \arrow[ur, "\mu^j"'] & & & & \Sigma_{M^j} \arrow[ur, "\phi^j"'] \arrow[urr, "\iota_\Sigma(\mu'^j)"'] & &
\end{tikzcd}$$

Let $\phi\colon \Sigma_M \Rightarrow \Sigma'$ be the colimit cone where $\Sigma_M\colon \mathbb{J} \to \mathbb{S}ign$ with

- $(\Sigma_M)^i = \Sigma_{M^i}$ for each index $i \in |\mathbb{J}|$, and
- $(\Sigma_M)^u = \iota_\Sigma(M^u)$ for each index morphism $u \in \mathbb{J}$.

Let $E' = \bigcup_{i\in|\mathbb{J}|} \phi^i(E_{M_i})$. Let $\varphi = \iota_\Sigma(M^i);\phi^i$. Let $N = 0_{\Sigma',E'\cup\varphi(E)}\restriction_\varphi$, where $0_{\Sigma',E'\cup\varphi(E)}$ is the initial model of the theory $(\Sigma', E'\cup\varphi(E))$. Let ν^i be the unique model homomorphism $0_{\Sigma_{M^i},E_{M^i}} \to 0_{\Sigma',E'\cup\varphi(E)}\restriction_{\phi^i}$. Let $\mu^i = \nu^i\restriction_\varphi$ for each $i \in |\mathbb{J}|$.

We prove that $\mu\colon M \Rightarrow N$ is a colimit cocone. By calculations similar to those of Theorem 3, we get that μ is cocone. Now consider another cocone $\mu'\colon M \Rightarrow N'$. Let $\iota_\Sigma(N')\colon \Sigma \to \Sigma_{N'}$ be the elementary extension of Σ via N'. Notice that $\{\iota_\Sigma(\mu'^i)\}_{i\in|\mathbb{J}|}\colon \Sigma_M \Rightarrow \Sigma_{N'}$ is a cocone. Therefore let $\varphi'\colon \Sigma' \to \Sigma_{N'}$ be the unique signature morphism such that $\phi^i;\varphi' = \iota_\Sigma(\mu'^i)$ for each $i \in |\mathbb{J}|$. Let h be the unique model homomorphism $0_{\Sigma',E'\cup\varphi(E)} \to 0_{\Sigma_{N'},E_{N'}}\upharpoonright_{\varphi'}$. Then $h\upharpoonright_\varphi$ is the unique model homomorphism such that $\mu'^i = h\upharpoonright_{\iota_\Sigma(N')};\mu^i$ for each $i \in |\mathbb{J}|$.

EXAMPLE 11. By applying the methods of (Tarlecki et al., 1991), the institutions of first-order predicate logic, rewriting logic, partial algebra, have small colimits of signatures. By Theorem 5, cf. Example 8, the categories of models of universal Horn theories in the in first-order predicate logic, of equational theories in general (many-sorted) algebra, of rewriting logic theories, and of strong equational and universal undefinedness predicate theories in partial algebra, have small colimits. □

5. Conclusions and Future Research Work

We generalised the method of diagrams from conventional model theory to an institution-independent framework based on a novel categorical definition of elementary diagram of a model. We showed that this is a natural easy to check property of actual institutions, and illustrated the power of our institution-independent method of diagrams with some applications such as institution liberality, institution-independent quasi-varieties, and limits and colimits of theory models. A side contribution of this paper is the new institution-independent approach on quasi-varieties based on inclusion systems. We also illustrated the concepts and results of this work with examples from four different logics or institutions.

The applications developed in this paper suggest a great application potential for the institution-independent method of elementary diagrams. We plan to further explore this potential.

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Appendix

In the Appendix we give very brief presentations of a number of institutions which are used in this paper as examples for illustrating some of the concepts introduced by this work and some of the applications of the main results. Although we assume some familiarity with these institutions, the reader is encouraged to consult the recommended references for more details. Also, some notations and terminology used in some sections of the Appendix rely on notations and terminology from previous sections.

A. (Many-sorted) First-order Predicate Logic with Equality

The role of this very brief presentation of many-sorted first-order predicate logic with equality is mainly for fix some notations and conventions. A detailed definition of the first-order predicate logic institution can be found in (Goguen and Burstall, 1992).

Recall that a (many-sorted) signature in first-order predicate logic is a tuple (S,Σ,Π) (often denoted just by (Σ,Π)) where S is the set of sorts, Σ is the set of (S-sorted) operation symbols, and Π is the set of (S-sorted) relation symbols. By $\Sigma_{w\to s}$ we denote the set of operations with arity w and sort s (in particular, when the arity w is empty, $\Sigma_{\to s}$ denotes the set of *constants* of sort s), and by Π_w we denote the set of relations with arity w.

Given a signature (Σ,Π), a *model* M of first-order predicate logic interprets:

- each sort s as a set M_s,
- each operation symbol $\sigma \in \Sigma_{w\to s}$ as a function $M_\sigma\colon M_w \to M_s$, where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ for $w = s_1 \ldots s_n$, and
- each relation symbol $\pi \in \Pi_w$ as a relation $M_\pi \subseteq M_w$.

Any *ground* (i.e., without variables) Σ-term $t = \sigma(t_1 \ldots t_n)$, where σ is an operation symbol and $t_1,\ldots,t_n$ are subterms, gets interpreted as an element M_t in a Σ-model M by $M_t = M_\sigma(M_{t_1} \ldots M_{t_n})$.

A (Σ,Π)-*model homomorphism* $h\colon M \to M'$ is an indexed family of functions $\{h_s\colon M_s \to M'_s\}_{s\in S}$ such that

- h is a Σ-algebra homomorphism $M \to M'$, i.e., $h(M_\sigma(m)) = M'_\sigma(h(m))$ for each $\sigma \in \Sigma_{w\to s}$ and each $m \in M_w$,[8] and
- $h(m) \in M'_\pi$ if $m \in M_\pi$ for each relation $\pi \in \Pi_w$ and each $m \in M_w$.

[8] By $h(m)$ we mean in fact $h_w(m)$, where $h_w\colon M_w \to M'_w$ is the canonical component-wise extension of h.

The sentences are the well-known first-order closed formulæ (including equations), and their satisfaction by the models is the well-known Tarskian satisfaction (see (Goguen and Burstall, 1992; C.C.Chang and H.J.Keisler, 1973) for details).

A *signature morphism* $\phi = (\phi^{\mathrm{sort}}, \phi^{\mathrm{op}}, \phi^{\mathrm{rel}}) \colon (S, \Sigma, \Pi) \to (S', \Sigma', \Pi')$ consists of a function between the sets of sorts $\phi^{\mathrm{sort}} \colon S \to S'$, a function between the sets of operation symbols $\phi^{\mathrm{op}} \colon \Sigma \to \Sigma'$, and a function between the sets of relation symbols $\phi^{\mathrm{rel}} \colon \Pi \to \Pi'$ such that $\phi^{\mathrm{op}}(\Sigma_{w \to s}) \subseteq \Sigma'_{\phi^{\mathrm{sort}}(w) \to \phi^{\mathrm{sort}}(s)}$ and $\phi^{\mathrm{rel}}(\Pi_w) \subseteq \Pi'_{\phi^{\mathrm{sort}}(w)}$ for any string of sorts $w \in S^*$ and each sort $s \in S$.[9]

Given a signature morphism $\phi \colon (S, \Sigma, \Pi) \to (S', \Sigma', \Pi')$, the *reduct* $M' \restriction_\phi$ of a (S', Σ', Π')-model M' is defined by $(M' \restriction_\phi)_s = M'_{\phi^{\mathrm{sort}}(s)}$ for each sort $s \in S$, $(M' \restriction_\phi)_\sigma = M'_{\phi^{\mathrm{op}}(\sigma)}$ for each operation symbol $\sigma \in \Sigma$, and $(M' \restriction_\phi)_\pi = M'_{\phi^{\mathrm{rel}}(\pi)}$ for each relation symbol $\pi \in \Pi$.

The *sentence translation* along ϕ of any sentence is defined inductively on the structure of the sentences by replacing the symbols from (S, Σ, Π) with symbols from (S', Σ', Π') as defined by ϕ.

Notice that by discarding the relational part, we get the many-sorted algebra institution with full first-order equational sentences.

B. Rewriting Logic

Rewriting logic (Meseguer, 1992) is emerging as one of the most important new algebraic specification logics. Here we refer to a simplified variant of rewriting logic which is used for defining the CafeOBJ institution (Diaconescu and Futatsugi, 2002), however this example can be extended to the original definition of rewriting logic without any difficulty.

Recall (from (Diaconescu and Futatsugi, 2002)) that our rewriting logic signatures are just ordinary (many-sorted) algebraic signatures. The models are *preorder models* which are (algebraic) interpretations of the signatures into $\mathbb{P}re$ (the category of preorders) rather than in $\mathbb{S}et$ (the category of sets) as in the case of ordinary algebras. More precisely, given a signature Σ, a model M interprets:

- each sort s as a preorder M_s, and
- each operation $\sigma \in \Sigma_{w \to s}$ as a preorder functor $M_\sigma \colon M_w \to M_s$, where M_w stands for $M_{s_1} \times \ldots \times M_{s_n}$ for $w = s_1 \ldots s_n$.

The *sentences* are either ordinary equations or *transitions*, both in their unconditional or conditional form. For example, the unconditional Σ-transitions for a signature Σ, are sentences of the form

$$(\forall X)\ t \ \texttt{=>}\ t'$$

where X is a many-sorted set of variables for Σ and t, t' are Σ-terms with variables X. Conditional sentences in rewriting logic are universally quantified implications where the hypotheses are finite conjunctions of transitions or equations and the conclusion is a transition or an equation.

The signature morphisms, the model reducts, and the sentence translations along signature morphisms are defined in the same way with ordinary (many-sorted) algebra (Appendix A).

A preorder model M *satisfies* a transition $M \models (\forall X)\ t \ \texttt{=>}\ t'$, if and only if $M'_t \le M'_{t'}$ for each expansion M' of M along the signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$. The satisfaction of conditional sentences extends the satisfaction of equations and transitions to the conditional case; we leave this as exercise to the reader.

More details of this institution of rewriting logic can be found in (Diaconescu and Futatsugi, 2002), while (Meseguer, 1992) has the details of the institution of full rewriting logic.

9 For any string of sorts $w = s_1 \ldots s_n$, by $\phi^{\mathrm{sort}}(w)$ we mean the string of sorts $\phi^{\mathrm{sort}}(s_1) \ldots \phi^{\mathrm{sort}}(s_n)$.

C. Partial Algebra

There are many approaches to partial algebra, two classical references being (Burmeister, 1986; Reichel, 1984). Our formalisation of the partial algebra institution is tailored to the needs of this paper but without affecting the logic and model theory of partial algebra.

A *partial algebraic signature* is a pair (Σ,Δ), where Σ is the set of the total operations and Δ is the set of the partial operations.[10] A *partial* (Σ,Δ)*-algebra* A is just like a $\Sigma\cup\Delta$-algebra but interpreting the operations of Δ as partial functions rather than total functions. A *homomorphism* $h\colon A \to B$ between partial algebras, is a family of partial functions $\{h_s\colon A_s \to\!\!\!\!\!\circ\; B_s\}_{s\in S}$ indexed by the set of sorts S of (Σ,Δ) such that $h(A_\sigma(a)) \stackrel{s}{=} B_\sigma(h(a))$, i.e. either both $h(A_\sigma(a))$ and $B_\sigma(h(a))$ are undefined or they are defined and equal, for each operation $\sigma \in (\Sigma\cup\Delta)_{w\to s}$ and each argument $a \in A_w$.[11]

The *interpretation* A_t of a $\Sigma\cup\Delta$-ground term t in a partial (Σ,Δ)-algebra is defined inductively by

- A_t is undefined if A_{t_k} is undefined for some $k \in \{1,\dots,n\}$ or $(A_{t_1},\dots,A_{t_n})$ does not belong to the definition domain of A_σ, otherwise
- $A_t = A_\sigma(A_{t_1},\dots,A_{t_n})$.

where $t = \sigma(t_1\dots t_n)$ is a term with σ any (Σ,Δ)-operation and $t_1,\dots,t_n$ subterms.

Signature morphisms, model reducts, and sentence translations are defined similarly to the case of the total algebra (see Appendix A).

The sentences are either *undefinedness* predicates or *strong* or *existential* equations, the equations both in their conditional or unconditional form.

For each undefinedness $(\Sigma\cup\Delta)$-predicate $(\forall X)\, t\downarrow$, where X is a many-sorted set of variables for (Σ,Δ) and t is a $\Sigma\cup\Delta$-term over X, a partial (Σ,Δ)-algebra A satisfies it if and only if A'_t is undefined for each expansion A' of the partial algebra A along the signature inclusion $(\Sigma,\Delta) \hookrightarrow (\Sigma\cup X,\Delta)$.

For any unconditional strong $(\Sigma\cup\Delta)$-equation $(\forall X)\, t \stackrel{s}{=} t'$, where X is a many-sorted set of variables for (Σ,Δ) and t,t' are $\Sigma\cup\Delta$-terms over X, a partial (Σ,Δ)-algebra A satisfies it if and only if

- A'_t and $A'_{t'}$ are both undefined, or
- A'_t and $A'_{t'}$ are both defined and $A'_t = A'_{t'}$.

for each expansion A' of the partial algebra A along the signature inclusion $(\Sigma,\Delta) \hookrightarrow (\Sigma\cup X,\Delta)$.

For any unconditional existential $(\Sigma\cup\Delta)$-equation $(\forall X)\, t \stackrel{e}{=} t'$, where X is a many-sorted set of variables for (Σ,Δ) and t,t' are $\Sigma\cup\Delta$-terms over X, a partial (Σ,Δ)-algebra A satisfies it if and only if

- A'_t and $A'_{t'}$ are both defined and $A'_t = A'_{t'}$.

for each expansion A' of the partial algebra A along the signature inclusion $(\Sigma,\Delta) \hookrightarrow (\Sigma\cup X,\Delta)$. These definitions extend without any problems to the conditional case. We leave it as exercise to the reader.

The following result show how this version of partial algebra is equivalent to an equationally defined class (i.e. variety) of total algebras, which is very useful for establishing some properties of partial algebras. We omit its straightforward proof.

PROPOSITION 3. For any partial algebra signature (Σ,Δ) with S the set of sorts, let $\downarrow = \{\downarrow_s\}_{s\in S}$ be an indexed set of new constant symbols and let Γ be set of the equations

$$(\forall x_1 \dots \forall x_n)\sigma(x_1 \dots \downarrow_s \dots x_n) = \downarrow_{s'}$$

[10] In this notation we ignore the set of sorts, which are of course common to the total and the partial operations.

[11] Notice that by convention $h(a)$ is defined if and only if is defined on all components of a.

for all operations $\sigma \in \Sigma \cup \Delta$.

Then the functor mapping each partial (Σ,Δ)-algebra A to the total $(\Sigma \cup \Delta \cup \downarrow, \Gamma)$-algebra $\overline{A}$ such that

- $\overline{A}_s = A_s \cup \{\downarrow_s\}$ for each sort $s \in S$,
- for each operation $\sigma \in \Sigma \cup \Delta, \overline{A}_\sigma(a) = A_\sigma(a)$ if a belongs to the definition domain of A_σ, and
- $\overline{A}_\sigma(a) = \downarrow_s$ otherwise, where s is the sort of σ,

and mapping each partial algebra homomorphism $h\colon A \to B$ to the total algebra homomorphism $\overline{h}\colon \overline{A} \to \overline{B}$ such that for each sort s,

- $\overline{h}_s(a) = h_s(a)$ if a belongs to the definition domain of h_s, and
- $\overline{h}_s(a) = \downarrow_s$ otherwise.

is an isomorphism between the category of partial (Σ,Δ)-algebras and the category of total $(\Sigma \cup \Delta \cup \downarrow, \Gamma)$-algebras.

Moreover,

$$A \models_{\Sigma,\Delta} (\forall X)\, t \stackrel{s}{=} t' \text{ iff } \overline{A} \models_{\Sigma\cup\Delta\cup\downarrow} (\forall X)\, t \ = \ t'$$

for each strong equation $(\forall X)\, t \stackrel{s}{=} t'$,

$$A \models_{\Sigma,\Delta} (\forall X)\, t \stackrel{e}{=} t' \text{ iff } \overline{A} \models_{\Sigma\cup\Delta\cup\downarrow} ((\forall X)\, t \ = \ t' \text{ and } \neg(\exists X) t = \downarrow)$$

for each existential equation $(\forall X)\, t \stackrel{e}{=} t'$, and

$$A \models_{\Sigma,\Delta} (\forall X)\, t \downarrow \text{ iff } \overline{A} \models_{\Sigma\cup\Delta\cup\downarrow} (\forall X)\, t \ = \ \downarrow$$

for each universal undefinedness predicate $(\forall X)\, t \downarrow$. $\square$

D. Hidden Algebra

Hidden algebra is the institution underlying behavioural specification, which is one of the most important new algebraic specification formalisms. In the literature there are several versions of hidden algebra, with only slight technical differences between them (Diaconescu and Futatsugi, 2000; Hennicker and Bidoit, 1999; Goguen and Roşu, 1999). Here we adopt a slightly modified version of *coherent hidden algebra* (abbreviated *CHA*) of (Diaconescu and Futatsugi, 2000).

A *CHA signature* is a tuple $(H, V, \Sigma, \Sigma^{\mathrm{b}})$, where

- H and V are disjoint sets of *hidden* sorts and *visible* sorts, respectively,
- Σ is a $H \cup V$-sorted signature,
- $\Sigma^{\mathrm{b}} \subseteq \Sigma$ is a subset of *behavioural operations* such that $\sigma \in \Sigma^{\mathrm{b}}_{w \to s}$ has *exactly* one hidden sort in w.

A CHA model M for a signature $(H, V, \Sigma, \Sigma^{\mathrm{b}})$ is just an ordinary Σ-algebra.

CHA sentences can be ordinary (strict) equations, *behavioural equations* (both in conditional or unconditional format), or *coherence declarations* (see (Diaconescu and Futatsugi, 2000; Diaconescu and Futatsugi, 2002) for details). Recall ((Diaconescu and Futatsugi, 2000; Diaconescu and Futatsugi, 2002)) that coherence declarations are semantically equivalent to conditional behavioural equations and that the strict equations are treated in the same way as in the case of the ordinary algebra. An unconditional *behavioural equation* is a sentence of the form

$$(\forall X)\, t \sim t'$$

where X is a set of variables and t, t' are Σ-terms over X.

Recall that a Σ-*context* $c[z]$ is any Σ-term c with a marked variable z occurring only once in c. A context $c[z]$ is *behavioural* iff all operations above[12] z are behavioural.

Given a Σ-algebra A, two elements (of the same sort s) a and a' are called *behaviourally equivalent*, denoted $a \sim_s a'$ (or just $a \sim a'$), iff $A'^a_c = A'^{a'}_c$ for each *visible* behavioural context c, where A'^a and $A'^{a'}$ are any expansions of A along the signature inclusion $\Sigma \hookrightarrow \Sigma \cup Y$, where Y is the set of variables of c, and such that $A'^a_y = A'^{a'}_y$ for each $y \in Y \setminus \{z\}$, $A'^a_z = a$, and $A'^{a'}_z = a'$.

Then, a Σ-algebra A satisfies an (unconditional) behavioural equation $A \models (\forall X)\ t \sim t'$, iff $A'_t \sim A'_{t'}$ for each A' expansion of the algebra A along the signature inclusion $\Sigma \hookrightarrow \Sigma \cup X$. This definition extends without any problems to the conditional case. We leave it as exercise to the reader.

Given a CHA signature $(H, V, \Sigma, \Sigma^{\mathrm{b}})$, a CHA *algebra homomorphism* $h\colon A \to B$ between Σ-algebras is a Σ-algebra homomorphism which preserves the behavioural equivalence, i.e., $h(a) \sim h(a')$ if $a \sim a'$ for each elements $a, a' \in A$.

Recall also that a *CHA signature morphism* $\phi\colon (H, V, \Sigma, \Sigma^{\mathrm{b}}) \to (H', V', \Sigma', \Sigma'^{\mathrm{b}})$ is an many-sorted signature morphism $(H \cup V, \Sigma) \to (H' \cup V', \Sigma')$ such that

(M1) $\phi(V) \subseteq V'$ and $\phi(H) \subseteq H'$,

(M2) $\phi(\Sigma^{\mathrm{b}}) = \Sigma'^{\mathrm{b}}$ and $\phi^{-1}(\Sigma'^{\mathrm{b}}) \subseteq \Sigma^{\mathrm{b}}$,

Finally, model reducts and sentence translations along CHA signature morphisms are the same with those from ordinary many-sorted algebra (Appendix A).

[12] Meaning that z is in the subterm determined by the operation.