INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

BOUNDARY CONTROL OF A NON STATIONARY MAGNETOHYDRODYNAMIC FLOW

by

RUXANDRA STAVRE

Preprint nr. 10/2003

BUCUREŞTI

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY

BOUNDARY CONTROL OF A NON STATIONARY MAGNETOHYDRODYNAMIC FLOW

by

RUXANDRA STAVRE

Preprint nr. 10/2003

BOUNDARY CONTROL OF A NON STATIONARY MAGNETOHYDRODYNAMIC FLOW

by

RUXANDRA STAVRE[•]

August, 2003

• Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1–764, 70700 Bucharest, Romania.
E–mail: Ruxandra.Stavre@imar.ro

Boundary control of a non stationary magnetohydrodynamic flow

Ruxandra Stavre, Institute of Mathematics "Simion Stoilow", Romanian Academy, P.O.Box 1-764, RO-014700 Bucharest, Romania; e-mail: Ruxandra.Stavre@imar.ro

Abstract

The purpose of this paper is to study a boundary control problem associated with the non stationary motion of an incompressible, viscous, magnetic fluid, describing the liquid-lithium flow inside a Tokamak cooling system. The existence, the uniqueness and the regularity of the unknowns of the state system are obtained by transforming it into a new system, with homogeneous boundary conditions. Then we introduce the control problem and we look for an exterior magnetic field which realises a magnetohydrodynamic flow without recirculation. The existence of an optimal control is proved and the necessary conditions of optimality are derived.

1 Introduction

Magnetohydrodynamics is concerned with the interactions of magnetic fields with fluid matter (liquids and gases). It finds practical use in many areas of engineering and pure science. Some areas of applications are: pumping and levitation of liquid metals, orientation and confinement of extremely hot ionized gases or plasmas as in thermonuclear fusion experiments, space propulsion resulting from the electromagnetic acceleration of ionized gases, etc. A complete approach to the electromagnetic theory and the continuum mechanics presented from a unified point of view can be found in [1].

The present paper is concerned with an optimal control problem associated with a non stationary magnetohydrodynamic viscous flow. This subject is motivated by the great number of technological applications of electromagnetic fluids (see e. g. [1], [2], [3]).

In the last ten years, there was an increasing interest in the study of optimal control problems associated with viscous flows: [4], [5], [6], [7], [8], [9], [10], [11] are only a few examples of works dealing with theoretical and numerical approaches to control of Navier-Stokes equations.

Since many physical phenomena which possess important technological applications lie at the intersection of continuum mechanics and electromagnetic theory, we considered interesting to extend this type of problems to the class of magnetic fluids. We study a non stationary flow of a liquid metal in the presence of a magnetic field. This motion describes, for instance, the liquid-lithium flow inside a Tokamak cooling system. The fluid motion across the magnetic fields drives an electric current in the liquid metal, and the electric current produces the electromagnetic body force opposing the motion. Thus, stagnant regions or even recirculation regions may appear. In [12] a qualitative discussion of the flow in an elbow in the plane of an uniform magnetic field is presented. The authors conclude that the flow becomes concentrated into jets with a large region of stagnant liquid. In [5] it was proved that, even without a force opposing the flow, for a Navier-Stokes fluid, there exists a recirculation region near the corner of the flow domain whose size increases with the Reynolds' number.

When we formulate an optimal control problem, finding a cost functional which is relevant to the physics of the flow is a very important step. Since we are interested in obtaining flows without recirculation regions, we introduce a suitable optimal control problem associated with the considered magnetohydrodynamic motion. We look for an exterior magnetic field which realises a magnetohydrodynamic flow without recirculation.

The paper is organized as follows: in Section 2 we introduce the coupled, nonlinear system which describes the non stationary flow of an incompressible, magnetic, viscous fluid, with non homogeneous boundary conditions and initial data. The existence, the uniqueness and the regularity of the unknowns of the problem are obtained by transforming the system into a new one, with homogeneous boundary conditions. The next section deals with the optimal control problem. We consider a cost functional given by

$$\frac{1}{2}\int_{\Omega_T}(|\min(v_1,0)|^2+|\min(v_2,0)|^2)dxdt, \tag{1.1}$$

where $\Omega \subset I\!R^2$ is the bounded domain of motion, $\Omega_T = \Omega\times]0,T[$ and $\vec{v} =$

(v_1, v_2) is the velocity of the magnetic fluid.

This choice is motivated by the fact that a flow without recirculation is characterized by $v_1 \geq 0$ and $v_2 \geq 0$ (i. e. the fluid is moving upward and to the right).

From the physical point of view, the most relevant control variable is the exterior magnetic field; hence, we are dealing with a boundary control problem: *we look for an exterior magnetic field which realises a magnetohydrodynamic flow without recirculation.* Next, we prove the existence of an optimal control. In the last section we derive the necessary conditions of optimality.

2 Analysis of the state system

We consider an incompressible, viscous fluid with electromagnetic properties occupying an open, bounded, connected set $\Omega \subset I\!R^2$. Let T be a positive given constant. The non stationary motion of the fluid in the presence of an exterior magnetic field, with non homogeneous boundary conditions and non homogeneous initial data is described by the following coupled system, obtained by using the assumptions of the magnetohydrodynamic approximation (see e. g. [1], p. 507):

$$(2.1) \quad \begin{cases} \vec{v}' + (\vec{v} \cdot \nabla)\vec{v} - \mu \bigtriangleup \vec{v} + \nabla(p + \frac{r}{2}\vec{h} \cdot \vec{h}) - r(\vec{h} \cdot \nabla)\vec{h} = \vec{f} \text{ in } \Omega_T, \\ \vec{h}' + \alpha(\vec{v} \cdot \nabla)\vec{h} - \alpha(\vec{h} \cdot \nabla)\vec{v} - \nu \bigtriangleup \vec{h} = 0 \text{ in } \Omega_T, \\ \text{div } \vec{v} = 0 \text{ in } \Omega_T, \\ \text{div } \vec{h} = 0 \text{ in } \Omega_T, \\ \vec{v} = \vec{u} \text{ on } \partial\Omega \times]0, T[, \\ \vec{h} = \vec{g} \text{ on } \partial\Omega \times]0, T[, \\ \vec{v}(x, 0) = \vec{v}_i(x) \text{ in } \Omega, \\ \vec{h}(x, 0) = \vec{h}_i(x) \text{ in } \Omega, \end{cases}$$

where $\Omega_T = \Omega \times]0, T[$, μ, r, α, ν are positive given constants associated with the properties of the material, $\vec{f}$ is an exterior given field, $\vec{u}$ is the velocity of the fluid on the boundary of the flow domain, $\vec{g}$ is the exterior magnetic field, $\vec{v}_i$, $\vec{h}_i$ are the initial velocity and the initial magnetic field, respectively and $\vec{v}$, $\vec{h}$, p are the unknowns of the system: the velocity, the magnetic field and the pressure of the fluid, respectively.

In the sequel we introduce the hypotheses of regularity for the data, which will allow us to obtain the existence, the uniqueness and the regularity of the unknowns by replacing the system (2.1) with another system with homogeneous boundary conditions.

(2.2) The boundary of Ω is at least Lipschitz continuous;

(2.3) $\vec{f} \in L^2(0,T;(H^{-1}(\Omega))^2)$,

(2.4) $\vec{u},\ \vec{u}' \in L^2(0,T;(H^{1/2}(\partial\Omega))^2)$,

(2.5) $\int_{\partial\Omega} \vec{u}(x,t)\cdot\vec{n}_x ds_x = 0$ a.e. in $]0,T[$,

(2.6) $\vec{g},\ \vec{g}' \in L^2(0,T;(H^{1/2}(\partial\Omega))^2)$,

(2.7) $\int_{\partial\Omega} \vec{g}(x,t)\cdot\vec{n}_x ds_x = 0$ a.e. in $]0,T[$,

(2.8) $\vec{v}_i \in (L^2(\Omega))^2$, div $\vec{v}_i = 0$ in Ω, $\vec{v}_i\cdot\vec{n} = \vec{u}(0)\cdot\vec{n}$ on $\partial\Omega$,

(2.9) $\vec{h}_i \in (L^2(\Omega))^2$, div $\vec{h}_i = 0$ in Ω, $\vec{h}_i\cdot\vec{n} = \vec{g}(0)\cdot\vec{n}$ on $\partial\Omega$,

In the sequel we shall need the following spaces and notation:

$V = \{\vec{v} \in (H_0^1(\Omega))^2 \,/\, \text{div}\ \vec{v} = 0\}$,

$H = \{\vec{v} \in (L^2(\Omega))^2 \,/\, \text{div}\ \vec{v} = 0,\ \vec{v}\cdot\vec{n} = 0 \text{ on } \partial\Omega\}$,

$W(0,T;X,X') = \{u \in L^2(0,T;X)/u' \in L^2(0,T;X')\}$, with X a Hilbert space,

$\tilde{H}^{1/2} = \{\vec{v} \in (H^{1/2}(\partial\Omega))^2 \,/\, \int_{\partial\Omega} \vec{v}\cdot\vec{n} ds = 0\}$,

$b(\vec{u},\vec{v},\vec{w}) = \int_\Omega (\vec{u}\cdot\nabla)\vec{v}\cdot\vec{w} dx,\ B(\vec{v},\vec{h}) = (\vec{v}\cdot\nabla)\vec{h}$,

γ_0 is the trace operator,

$\langle\cdot,\cdot\rangle_{V',V}$ is the duality pairing between a space V and its dual,

$((\cdot,\cdot))_0$, $\|\cdot\|_0$ are the scalar product and the norm in $(H_0^1(\Omega))^2$,

$(\cdot,\cdot)$, $|\cdot|$ are the scalar product and the norm in $(L^2(\Omega))^2$.

The properties of the spaces V, H and of the function b can be found in [13]. In order to obtain the system with homogeneous boundary conditions we use the following result:

Theorem 2.1. If $\Omega \subset I\!R^2$ is an open, bounded, connected set satisfying (2.2) and $\vec{u}$ verifies the hypotheses (2.4)-(2.5), then for any $\delta > 0$ there exists an unique function $\vec{U}_\delta \in L^2(0,T;(H^1(\Omega))^2)$ with $\vec{U}'_\delta \in L^2(0,T;(H^1(\Omega))^2)$ such that:

$$
(2.10) \qquad \begin{cases} \text{div } \vec{U}_\delta = 0 \text{ in } \Omega_T, \\ \gamma_0 \vec{U}_\delta = \vec{u} \text{ on } \partial\Omega \times]0,T[, \\ |b(\vec{h}, \vec{U}_\delta(t), \vec{v})| \leq \delta \|\vec{h}\|_0 \|\vec{v}\|_0 \ \forall \ \vec{h},\ \vec{v} \in V, \text{ a. e. in }]0,T[. \end{cases}
$$

Proof. For functions not depending on t, the proof of this theorem can be found for Lipschitz continuous domains in [14] and for domains of class $\mathcal{C}^2$ in [13]. Since we need to apply it for domains with corners, we use the result of [14]. It is known that for any $\delta > 0$ there exists a linear and continuous operator $\Lambda_\delta : \tilde{H}^{1/2} \mapsto (H^1(\Omega))^2$ with the properties:

$$
(2.11) \qquad \begin{cases} \text{div } \Lambda_\delta \vec{u} = 0, \\ \gamma_0 \Lambda_\delta \vec{u} = \vec{u}, \\ |b(\vec{h}, \Lambda_\delta \vec{u}, \vec{v})| \leq \delta \|\vec{h}\|_0 \|\vec{v}\|_0 \ \forall \ \vec{h},\ \vec{v}, \in V,\ \forall \vec{u} \in \tilde{H}^{1/2}. \end{cases}
$$

Since from (2.4)-(2.5) it follows that $\vec{u}(t)$, $\vec{u}'(t) \in \tilde{H}^{1/2}$, we put:

$$
(2.12) \qquad \begin{cases} \vec{U}_\delta(t) = \Lambda_\delta(\vec{u}(t)) \text{ a. e. in }]0,T[, \\ \vec{V}_\delta(t) = \Lambda_\delta(\vec{u}'(t)) \text{ a. e. in }]0,T[. \end{cases}
$$

Using Proposition 2, p. 566, [15] and Lemma 1.1., p. 169, [13] we obtain $\vec{V}_\delta = \vec{U}'_\delta$ and the desired regularity for the function $\vec{U}_\delta$ and the proof is achieved.

Corrolary 2.2. If Ω is as in Theorem 2.1. and $\vec{g}$ verifies the hypotheses (2.6)-(2.7), then for any $\delta > 0$ there exists an unique function $\vec{\psi}_\delta \in L^2(0,T;(H^1(\Omega))^2)$ with $\vec{\psi}'_\delta \in L^2(0,T;(H^1(\Omega))^2)$ such that:

$$
(2.13) \qquad \begin{cases} \text{div } \vec{\psi}_\delta = 0 \text{ in } \Omega_T, \\ \gamma_0 \vec{\psi}_\delta = \vec{g} \text{ on } \partial\Omega \times]0,T[, \\ |b(\vec{h}, \vec{\psi}_\delta(t), \vec{v})| \leq \delta \|\vec{h}\|_0 \|\vec{v}\|_0 \ \forall \ \vec{h},\ \vec{v} \in V, \text{ a. e. in }]0,T[. \end{cases}
$$

In order to study the existence, the uniqueness and the regularity of the unknowns of the system (2.1), we define the new functions:

$$
(2.14) \qquad \begin{cases} P = p + \frac{r}{2} \vec{h} \cdot \vec{h}, \\ \vec{V} = \vec{v} - \vec{U}_\delta, \\ \vec{H} = \vec{h} - \vec{\psi}_\delta, \end{cases}
$$

where $\vec{U}_\delta$ and $\vec{\psi}_\delta$ are the functions introduced in Theorem 2.1. and in Corolary 2.2., respectively. The system (2.1), written for the new functions is:

$$(2.15)\quad \begin{cases} \vec{V}' + B(\vec{V},\vec{V}) + B(\vec{U}_\delta,\vec{V}) + B(\vec{V},\vec{U}_\delta) - \mu\,\triangle\,\vec{V} + \nabla P \\ -rB(\vec{H},\vec{H}) - rB(\vec{\psi}_\delta,\vec{H}) - rB(\vec{H},\vec{\psi}_\delta) = \vec{F} \text{ in } \Omega_T, \\ \vec{H}' + \alpha B(\vec{V},\vec{H}) + \alpha B(\vec{U}_\delta,\vec{H}) + \alpha B(\vec{V},\vec{\psi}_\delta) - \alpha B(\vec{H},\vec{V}) \\ -\alpha B(\vec{\psi}_\delta,\vec{V}) - \alpha B(\vec{H},\vec{U}_\delta) - \nu\,\triangle\,\vec{H} = \vec{G} \text{ in } \Omega_T, \\ \text{div } \vec{V} = 0 \text{ in } \Omega_T,\ \text{div } \vec{H} = 0 \text{ in } \Omega_T, \\ \vec{V} = 0 \text{ on } \partial\Omega\times]0,T[,\ \vec{H} = 0 \text{ on } \partial\Omega\times]0,T[, \\ \vec{V}(x,0) = \vec{V}_i(x) \text{ in } \Omega,\ \vec{H}(x,0) = \vec{H}_i(x) \text{ in } \Omega, \end{cases}$$

where the functions $\vec{F}$, $\vec{G}$, $\vec{V}_i$, $\vec{H}_i$ are defined as follows:

$$(2.16)\quad \begin{cases} \vec{F} = \vec{f} - \vec{U}_\delta' - B(\vec{U}_\delta,\vec{U}_\delta) + rB(\vec{\psi}_\delta,\vec{\psi}_\delta) + \mu\,\triangle\,\vec{U}_\delta, \\ \vec{G} = -\vec{\psi}_\delta' - \alpha B(\vec{U}_\delta,\vec{\psi}_\delta) + \alpha B(\vec{\psi}_\delta,\vec{U}_\delta) + \nu\,\triangle\,\vec{\psi}_\delta, \\ \vec{V}_i(x) = \vec{v}_i(x) - \vec{U}_\delta(x,0), \\ \vec{H}_i(x) = \vec{h}_i(x) - \vec{\psi}_\delta(x,0). \end{cases}$$

Remark 2.3. For obtaining the regularity of the unknowns we need at least $\vec{F}$, $\vec{G} \in L^2(0,T;V')$. This regularity is a consequence of hypothesis (2.3), of the regularities for $\vec{U}_\delta$, $\vec{\psi}_\delta$ and of the properties of B, which can be found in [13]. Moreover, the initial conditions $\vec{V}_i$, $\vec{H}_i$ must be elements of the space H. This regularity follows from the hypotheses (2.8)-(2.9) and from the properties of the $\vec{U}_\delta$, $\vec{\psi}_\delta$.

As usual, by taking test functions from V, we obtain the next variational formulation of the system (2.15):

$$(2.17)\quad \begin{cases} (\vec{V},\vec{H}) \in (W(0,T;V,V'))^2, \\ \langle \vec{V}'(t),\vec{z}\rangle_{V',V} + \mu((\vec{V}(t),\vec{z}))_0 + \langle B(\vec{V}(t),\vec{V}(t)),\vec{z}\rangle_{V',V} \\ +\langle B(\vec{U}_\delta(t),\vec{V}(t)),\vec{z}\rangle_{V',V} + \langle B(\vec{V}(t),\vec{U}_\delta(t)),\vec{z}\rangle_{V',V} \\ -r\langle B(\vec{H}(t),\vec{H}(t)),\vec{z}\rangle_{V',V} - r\langle B(\vec{\psi}_\delta(t),\vec{H}(t)),\vec{z}\rangle_{V',V} \\ -r\langle B(\vec{H}(t),\vec{\psi}_\delta(t)),\vec{z}\rangle_{V',V} = \langle \vec{F}(t),\vec{z}\rangle_{V',V}\,\forall\,\vec{z} \in V, \text{ a. e. in }]0,T[, \\ \langle \vec{H}'(t),\vec{w}\rangle_{V',V} + \nu((\vec{H}(t),\vec{w}))_0 + \alpha\langle B(\vec{V}(t),\vec{H}(t)),\vec{w}\rangle_{V',V} \\ +\alpha\langle B(\vec{U}_\delta(t),\vec{H}(t)),\vec{w}\rangle_{V',V} + \alpha\langle B(\vec{V}(t),\vec{\psi}_\delta(t)),\vec{w}\rangle_{V',V} \\ -\alpha\langle B(\vec{H}(t),\vec{V}(t)),\vec{w}\rangle_{V',V} - \alpha\langle B(\vec{\psi}_\delta(t),\vec{V}(t)),\vec{w}\rangle_{V',V} \\ -\alpha\langle B(\vec{H}(t),\vec{U}_\delta(t)),\vec{w}\rangle_{V',V} = \langle \vec{G}(t),\vec{w}\rangle_{V',V}\,\forall\,\vec{w} \in V, \text{ a. e. in }]0,T[, \\ \vec{V}(0) = \vec{V}_i \text{ in } \Omega,\ \vec{H}(0) = \vec{H}_i \text{ in } \Omega, \end{cases}$$

The main result of this section is the next theorem, in which we establish existence, regularity and uniqueness properties and some *a priori* estimates.

Theorem 2.4. a) There exists at least a pair $(\vec{V}, \vec{H})$ satisfying the variational problem (2.17); the pair $(\vec{v}, \vec{h})$, obtained from $(\vec{V}, \vec{H})$ with (2.14) is unique. Moreover, there exists a distribution $P \in \mathcal{D}'(\Omega_T)$, unique up to the addition of a distribution of t, which satisfies, together with $(\vec{v}, \vec{h})$, the system (2.1).

b) There exists a positive constant δ_0 such that for any $\delta \leq \delta_0$ the following estimate holds:

$$\max\{\|\vec{V}\|_{L^2(0,T;V)}, \|\vec{V}\|_{L^\infty(0,T;H)}, \|\vec{H}\|_{L^2(0,T;V)}, \|\vec{H}\|_{L^\infty(0,T;H)}\} \leq C, \tag{2.18}$$

where C is a constant depending on the data and on $\vec{U}_\delta, \vec{\psi}_\delta$ and it will be defined in the proof.

Proof. a) The existence of the functions $\vec{V}$ and $\vec{H}$ is obtained using the Galerkin's method, as in [13], hence we shall skip the proof. The regularity of the functions defined as the Galerkin's approximation gives the regularity of $\vec{V}$ and $\vec{H}$, stated in $(2.17)_1$.

Let $(\vec{v}_j, \vec{h}_j, P_j)$, $j=1,2$ be two solutions of the problem (2.1) and $(\vec{v}, \vec{h}, P)= (\vec{v}_1, \vec{h}_1, P_1)-(\vec{v}_2, \vec{h}_2, P_2)$. Subtracting the two systems, corresponding to $j=1$ and $j=2$, respectively, taking as test function the pair $(\vec{v}(t), \vec{h}(t))$ and using the regularity of these functions, obtained from $(2.17)_1$, and the properties of B (see Lemma 3.4, p. 198, [13]) the uniqueness of the velocity and of the magnetic field follows in a classical way. Next, the assertion for the pressure is obtained as in [13].

b) We take as test function in (2.17) $(\vec{z}, \vec{w}) = (\vec{V}(t), \vec{H}(t))$ and we compute $(2.17)_1+\frac{r}{\alpha}(2.17)_2$. It follows:

$$\begin{aligned}
&\frac{1}{2}(|\vec{V}(t)|^2 + \frac{r}{\alpha}|\vec{H}(t)|^2)' + \mu\|\vec{V}(t)\|_0^2 + \frac{\nu r}{\alpha}\|\vec{H}(t)\|_0^2 = \langle \vec{f}(t), \vec{V}(t)\rangle_{V',V} - \\
&(\vec{U}_\delta'(t), \vec{V}(t)) - \mu(\nabla\vec{U}_\delta(t), \nabla\vec{V}(t)) - \frac{r}{\alpha}(\vec{\psi}_\delta'(t), \vec{H}(t)) - \frac{\nu r}{\alpha}(\nabla\vec{\psi}_\delta(t), \nabla\vec{H}(t)) \\
&-b(\vec{V}(t), \vec{U}_\delta(t), \vec{V}(t)) + rb(\vec{H}(t), \vec{\psi}_\delta(t), \vec{V}(t)) - rb(\vec{V}(t), \vec{\psi}_\delta(t), \vec{H}(t)) \\
&+rb(\vec{H}(t), \vec{U}_\delta(t), \vec{H}(t)) - b(\vec{U}_\delta(t), \vec{U}_\delta(t), \vec{V}(t)) + rb(\vec{\psi}_\delta(t), \vec{\psi}_\delta(t), \vec{V}(t)) \\
&-rb(\vec{U}_\delta(t), \vec{\psi}_\delta(t), \vec{H}(t)) + rb(\vec{\psi}_\delta(t), \vec{U}_\delta(t), \vec{H}(t)).
\end{aligned} \tag{2.19}$$

Using $(2.11)_3$ and $(2.13)_3$ and choosing

$$\delta_0 = \max\left\{\frac{\mu}{4(1+r)}, \frac{\nu}{8\alpha}\right\} \tag{2.20}$$

it follows that for any $\delta \leq \delta_0$ the corresponding pair $(\vec{V}, \vec{H})$ satisfies:

$$(2.21) \quad \frac{1}{2}(|\vec{V}(t)|^2 + \frac{r}{\alpha}|\vec{H}(t)|^2)' + \frac{\mu}{4}\|\vec{V}(t)\|_0^2 + \frac{\nu r}{4\alpha}\|\vec{H}(t)\|_0^2 \leq C^2,$$

where C is a constant defined by:

$$C^2 = \frac{8}{\mu}\|\vec{f}(t)\|_{V'}^2 + \frac{8}{\mu}|\vec{U}_\delta'(t)|^2 + 8|\nabla\vec{U}_\delta(t)|^2 + \frac{6}{\nu}|\vec{\psi}_\delta'(t)|^2 + 6|\nabla\vec{\psi}_\delta(t)|^2$$
$$+\frac{8}{\mu}(\|\vec{U}_\delta(t)\|_{(L^4(\Omega))^2}^2 + r\|\vec{\psi}_\delta(t)\|_{(L^4(\Omega))^2}^2)^2 + \frac{12\alpha}{\nu}\|\vec{U}_\delta(t)\|_{(L^4(\Omega))^2}^2\|\vec{\psi}_\delta(t)\|_{(L^4(\Omega))^2}^2.$$

Hence, the proof is complete.

Remark 2.5. The functions $\vec{V}$, $\vec{H}$ are not unique, since for different $\delta > 0$ we obtain different functions. However, for a fixed value of δ, their uniqueness holds. In the sequel, the functions $\vec{V}$, $\vec{H}$ will be those corresponding to $\delta = \delta_0$, with δ_0 defined in the previous theorem.

3 The optimization problem: boundary control with constraints

We begin by giving a precise statement of the optimal control problem we consider. We introduce the space

$$(3.1) \qquad \tilde{W} = \{\vec{g} \in L^2(0,T;\tilde{H}^{1/2}) / \vec{g}' \in L^2(0,T;\tilde{H}^{1/2})\}.$$

Let $\vec{g} \in \tilde{W}$ be the boundary control (the exterior magnetic field which acts on the fluid) and $(\vec{v}, \vec{h}, P)$ the state variable. The state and control variables are constrained to satisfy the system (2.1). The cost functional is $J : \tilde{W} \mapsto I\!R_+$,

$$(3.2) \qquad J(\vec{g}) = \frac{1}{2}\int_{\Omega_T}(|\min(v_1,0)|^2 + |\min(v_2,0)|^2)dxdt.$$

Since we have proved the uniqueness of the pair $(\vec{v}, \vec{h})$, the correspondence $\vec{g} \mapsto \vec{v}$ is univalued; hence the cost functional is well defined.

We denote by $\mathcal{U}$ a bounded, closed, convex subset of the space $\tilde{W}$ and we consider the following optimization problem:

$$(3.3) \qquad \begin{cases} \text{Find } \vec{g}^* \in \mathcal{U} \text{ such that} \\ J(\vec{g}^*) = \min\{J(\vec{g}) \,/\, \vec{g} \in \mathcal{U}\}. \end{cases}$$

Our choice of the cost functional and of the optimization problem is motivated by the following physical consideration: when a viscous fluid moves in a magnetic field, recirculation regions may appear, from different reasons (the intensity of the exterior magnetic field, the viscosity of the fluid, the structure of the flow domain, etc). The purpose of the considered control problem is to find a magnetic field (in a set of admissible functions from the physical viewpoint) so that, without changing the other characteristics of the problem, recirculation regions do not appear.

Theorem 3.1. There exists an optimal solution to the control problem (3.3).

Proof. Let $(\vec{g}_n)_n$ be a sequence in $\mathcal{U}$ such that $\lim_{n\to\infty} J(\vec{g}_n) = \inf\{J(\vec{g}) \,/\, \vec{g} \in \mathcal{U}\}$. $\mathcal{U}$ being a bounded set, on a subsequence, denoted also by $(\vec{g}_n)_n$, we have $\vec{g}_n \rightharpoonup \vec{g}^*$ weakly in $\tilde{W}$; as $\mathcal{U}$ is convex and closed, it is closed in the weak topology, thus $\vec{g}^* \in \mathcal{U}$.

The positive constant δ being fixed at the value δ_0, we denote by $\vec{\psi}_n$ the function given by Corrolary 2.2. which corresponds to $\vec{g}_n$. Since the operator Λ is bounded, we get the boundedness of the sequences $(\vec{\psi}_n)_n$, $(\vec{\psi}_n')_n$ in $L^2(0,T;(H^1(\Omega))^2)$ and (2.18) yields:

$$(3.4)\max\{\|\vec{V}_n\|_{L^2(0,T;V)}, \|\vec{V}_n\|_{L^\infty(0,T;H)}, \|\vec{H}_n\|_{L^2(0,T;V)}, \|\vec{H}_n\|_{L^\infty(0,T;H)}\} \le C(\mathcal{U}),$$

where $C(\mathcal{U})$ is the constant introduced in the proof of Theorem 2.4., which corresponds to the bounded set $\mathcal{U}$. We establish next the boundedness of the sequences $(\vec{V}_n')_n, (\vec{H}_n')_n$ in $L^2(0,T;V')$. $(2.17)_2$ and $(2.17)_3$ corresponding to $\vec{g}_n$ can be written as follows:

$$(3.5)\quad \begin{cases} \vec{V}_n' = \mu \triangle \vec{V}_n - B(\vec{V}_n,\vec{V}_n) - B(\vec{U},\vec{V}_n) - B(\vec{V}_n,\vec{U}) \\ +rB(\vec{H}_n,\vec{H}_n) + rB(\vec{\psi}_n,\vec{H}_n) + rB(\vec{H}_n,\vec{\psi}_n) + \vec{F}_n \text{ in } L^2(0,T;V'), \\ \vec{H}_n' = \nu \triangle \vec{H}_n - \alpha B(\vec{V}_n,\vec{H}_n) - \alpha B(\vec{U},\vec{H}_n) - \alpha B(\vec{V}_n,\vec{\psi}_n) \\ +\alpha B(\vec{H}_n,\vec{V}_n) + \alpha B(\vec{\psi}_n,\vec{V}_n) + \alpha B(\vec{H}_n,\vec{U}) + \vec{G}_n \text{ in } L^2(0,T;V'), \end{cases}$$

where $\vec{U}$ is the function given by Theorem 2.1., corresponding to δ_0 and $\vec{F}_n$, $\vec{G}_n$ are the functions defined in (2.16) written for $\vec{\psi}_n$. We establish next the boundedness in $L^2(0,T;V')$ of the right hand sides of (3.5). The boundedness of the first term is obvious since $\triangle$ is a bounded operator from $L^2(0,T;V)$ to $L^2(0,T;V')$. The estimates for the terms with B in the case when its two arguments belong to $L^2(0,T;V) \cap L^\infty(0,T;H)$ are obtained as a consequence of Lemma 3.4, p. 198, [13]. For the other terms, we proceed as follows:

$$\|B(\vec{\psi}_n,\vec{V}_n)\|^2_{L^2(0,T;V')} \leq \int_0^T \|\vec{\psi}_n(t)\|^2_{(L^4(\Omega))^2}\|\vec{V}_n(t)\|^2_{(L^4(\Omega))^2}dt$$

$$\leq \sqrt{2}\int_0^T \|\vec{\psi}_n(t)\|^2_{(L^4(\Omega))^2}|\vec{V}_n(t)|\|\vec{V}_n(t)\|_0 dt$$

$$\leq \sqrt{2}\|\vec{V}_n\|_{L^\infty(0,T;H)}\|\vec{V}_n\|_{L^2(0,T;V)}\|\vec{\psi}_n\|^2_{L^4(0,T;(L^4(\Omega))^2)} \leq C(\mathcal{U}).$$

We obtained (on subsequences):

$$(3.6)\qquad \begin{cases} \vec{V}_n \rightharpoonup \vec{V}^* \text{ weakly in } W(0,T;V,V') \\ \vec{V}_n \rightharpoonup \vec{V}^* \text{ weakly-* in } L^\infty(0,T;H) \\ \vec{H}_n \rightharpoonup \vec{H}^* \text{ weakly in } W(0,T;V,V') \\ \vec{H}_n \rightharpoonup \vec{H}^* \text{ weakly-* in } L^\infty(0,T;H) \\ \vec{\psi}_n \rightharpoonup \vec{\psi}^* \text{ weakly in } W(0,T;(H^1((\Omega))^2,(H^1((\Omega))^2) \end{cases}$$

The above convergences are sufficient for passing to the limit for all the terms except those of the type $B(\vec{\psi}_n,\vec{V}_n)$, etc. For them, we need a strong convergence. Since $(\vec{V}_n)_n$, $(\vec{H}_n)_n$ are bounded in $W(0,T;V,V')$ it follows, from Theorem 2.1, p. 184, [13], that on a subsequence, also denoted by $(\vec{V}_n)_n$, $(\vec{H}_n)_n$ we have $\vec{V}_n \to \vec{V}^*$ $\vec{H}_n \to \vec{H}^*$ strongly in $L^2(0,T;H)$. Moreover, $\vec{\psi}_n \to \vec{\psi}^*$ strongly in $L^2(0,T;(L^2(\Omega))^2)$. Using the technique of [13], we can now pass to the limit in (2.17), written for $\vec{g}_n$ and we obtain that $\vec{V}^*$, $\vec{H}^*$, $\vec{\psi}^*$ satisfy (2.17). Hence we proved the $(\vec{V}^*, \vec{H}^*)$ is the unique solution corresponding to $\vec{\psi}^*$. Now we have to answer to the following question: is $\vec{\psi}^*$ the only weak limit point corresponding to $\vec{g}^*$? The answer is yes, and we shall explain it. We denote $\gamma_0(\vec{\psi}^*)$ by $\vec{g}_0$. We have to prove that the subsequence $(\vec{g}_n)_n$, converging to $\vec{g}^*$, has $\vec{g}_0$ as weak limit point in $L^2(0,T;\tilde{H}^{1/2})$. Using Theorem 7.9, p. 119, [16] we obtain the compactness of the embedding $H^1(\Omega) \subset H^{1/2+\epsilon}(\Omega)$ for $\epsilon \in]0,1/2[$. From Theorem 2.1, p. 184, [13] it follows that $\vec{\psi}_n \to \vec{\psi}^*$ strongly in $L^2(0,T;(H^{1/2+\epsilon}(\Omega))^2)$. Since the operator $\gamma_0 : L^2(0,T;(H^{1/2+\epsilon}(\Omega))^2) \mapsto L^2(0,T;(H^\epsilon(\partial\Omega))^2)$ is linear and continuous (see e.g. Theorem 8.7, p. 126, [16]), it follows that $\gamma_0(\vec{\psi}_n) \to \gamma_0(\vec{\psi}^*) = \vec{g}_0$ strongly in $L^2(0,T;(L^2(\partial\Omega))^2)$. The last step of the proof is to show that $J(\vec{g}^*) \leq \liminf_{n\to\infty} J(\vec{g}_n)$, i.e. that J is a lower semicontinuous functional with respect to the weak topology. In fact, we shall prove a stronger result. The convergence $\phi_n \to \phi$ strongly in $L^2(\Omega)$ yields

$\int_\Omega |\min(0,\phi_n)|^2 dx \to \int_\Omega |\min(0,\phi)|^2 dx$; hence the functional J is weakly continuous. Using a Weierstrass theorem, the proof is achieved.

4 The first order necessary conditions of optimality

The first result we obtain, in order to derive the optimality conditions, is:

Theorem 4.1. *J is G-differentiable on $\tilde{W}$ and for any $\vec{g},\ \vec{g}_0 \in \tilde{W}$*

$$(4.1)\quad \langle DJ(\vec{g}_0), \vec{g}-\vec{g}_0\rangle = \int_{\Omega_T} ((v^*-v_0)_1 \min(0,(v_0)_1) + (v^*-v_0)_2 \min(0,(v_0)_2))dxdt,$$

where $(\vec{v}_0, \vec{h}_0, P_0)$ is the unique solution of the system (2.1) corresponding to $\vec{g}_0$ and $(\vec{v}^*, \vec{h}^*, P^*)$ is the unique solution of the following auxiliary system:

$$(4.2)\quad \begin{cases} \vec{v}^{*\prime} + B(\vec{v}^*,\vec{v}_0) + B(\vec{v}_0,\vec{v}^*) - \mu \triangle \vec{v}^* + \nabla P^* \\ -r(B(\vec{h}^*,\vec{h}_0) + B(\vec{h}_0,\vec{h}^*)) = \vec{f} + B(\vec{v}_0,\vec{v}_0) - rB(\vec{h}_0,\vec{h}_0) \text{ in } \Omega_T, \\ \vec{h}^{*\prime} + \alpha(B(\vec{v}^*,\vec{h}_0) + B(\vec{v}_0,\vec{h}^*) - B(\vec{h}^*,\vec{v}_0) - B(\vec{h}_0,\vec{v}^*)) \\ -\nu \triangle \vec{h}^* = \alpha(B(\vec{v}_0,\vec{h}_0) - B(\vec{h}_0,\vec{v}_0)) \text{ in } \Omega_T, \\ \operatorname{div} \vec{v}^* = 0 \text{ in } \Omega_T, \\ \operatorname{div} \vec{h}^* = 0 \text{ in } \Omega_T, \\ \vec{v}^* = \vec{u} \text{ on } \partial\Omega\times]0,T[,\ \ \vec{h}^* = \vec{g} \text{ on } \partial\Omega\times]0,T[, \\ \vec{v}^*(x,0) = \vec{v}_i(x) \text{ in } \Omega,\ \ \vec{h}^*(x,0) = \vec{h}_i(x) \text{ in } \Omega, \end{cases}$$

Proof. For any $\epsilon \in]0,1[$, we denote by $(\vec{v}_{\epsilon g}, \vec{h}_{\epsilon g}, P_{\epsilon g})$ the solution of (2.1) corresponding to the boundary condition $\vec{h}_{\epsilon g} = \vec{g}_0 + \epsilon(\vec{g}-\vec{g}_0)$ on $\partial\Omega\times]0,T[$. We introduce the new functions:

$$(4.3)\qquad \vec{v}_\epsilon = \frac{\vec{v}_{\epsilon g} - \vec{v}_0}{\epsilon} + \vec{v}_0,\ \ \vec{h}_\epsilon = \frac{\vec{h}_{\epsilon g} - \vec{h}_0}{\epsilon} + \vec{h}_0.$$

We define now

$$\vec{V}_\epsilon = \vec{v}_\epsilon - \vec{U},\ \ \vec{H}_\epsilon = \vec{h}_\epsilon - \vec{\psi}$$

and we write the variational problem satisfied by $(\vec{V}_\epsilon, \vec{H}_\epsilon)$. Following the same steps as those from Theorem 3.1., we obtain convergences of the type (3.6) for the sequences $(\vec{V}_\epsilon)_\epsilon$ and $(\vec{H}_\epsilon)_\epsilon$. If we denote by $\vec{V}^*$ and $\vec{H}^*$ their limits, we define

$$(4.4)\qquad \begin{cases} \vec{v}^* = \vec{V}^* + \vec{U}, \\ \vec{h}^* = \vec{H}^* + \vec{\psi} \end{cases}$$

and we introduce the pressure P^* in a usual way, it follows that $(\vec{v}^*, \vec{h}^*, P^*)$ satisfies the system (4.2).

For obtaining the expression (4.1) of the G-differential of J, we define:

$$J_1 = \frac{1}{2\epsilon}\int_{\Omega_T} (|\min(0, v_{\epsilon g})|^2 - |\min(0, v_0 + \epsilon(v^* - v_0))|^2)dxdt,$$
$$J_2 = \frac{1}{2\epsilon}\int_{\Omega_T} (|\min(0, v_0 + \epsilon(v^* - v_0))|^2 - |\min(0, v_0)|^2)dxdt$$

For computing $\lim_{\epsilon\to 0} J_1$ we use the inequality $|\min(0, a) - \min(0, b)| \leq |a - b|\ \forall a, b \in I\!R$. It follows $J_1 \leq \frac{1}{2}||v_\epsilon - v^*||_{L^2(\Omega_T)} a_\epsilon$, with $(a_\epsilon)_\epsilon$ a bounded sequence in $I\!R$ which obviously yields (taking into account the convergences previously established) $\lim_{\epsilon\to 0} J_1 = 0$. For computing the limit of J_2 as $\epsilon \to 0$ we denote $w = v^* - v_0$ and we write $J_2 = I_1 + I_2 + I_3$, where

$$I_1 = \frac{1}{2\epsilon}\int_{\Omega_T\cap\{w<0\}\cap\{v_0<0\}} ((v_0 + \epsilon w)^2 - v_0^2)dxdt,$$
$$I_2 = \frac{1}{2\epsilon}\int_{\Omega_T\cap\{w<0\}\cap\{v_0\geq 0\}} |\min(0, v_0 + \epsilon w)|^2 dxdt,$$
$$I_3 = \frac{1}{2\epsilon}\int_{\Omega_T\cap\{w\geq 0\}\cap\{v_0<0\}} (|\min(0, v_0 + \epsilon w)|^2 - v_0^2)dxdt.$$

Let compute now the limit of each integral, as $\epsilon \to 0$.

$$\lim_{\epsilon\to 0} I_1 = \int_{\Omega_T\cap\{w<0\}\cap\{v_0<0\}} v_0 w dxdt = \int_{\Omega_T\cap\{w<0\}} w\min(0, v_0)dxdt.$$

The second integral satisfies the inequality $I_2 \leq a_\epsilon b_\epsilon$ with

$$a_\epsilon = (\int_{\Omega_T\cap\{w<0\}\cap\{v_0\geq 0\}} |\min(0, v_0/\epsilon + w)|^2 dxdt)^{1/2} \text{ and } b_\epsilon = \frac{1}{2}(\int_{\Omega_T\cap\{w<0\}\cap\{v_0\geq 0\}} |\min(0, v_0 + \epsilon w)|^2 dxdt)^{1/2}.$$

Since $\lim_{\epsilon\to 0} b_\epsilon = 0$, if we prove that $(a_\epsilon)_\epsilon$ is a bounded sequence, we shall obtain $\lim_{\epsilon\to 0} I_2 = 0$. The boundedness of $(a_\epsilon)_\epsilon$ is a consequence of the fact that this sequence is monotone increasing and, hence, for any $\epsilon \in]0,1[$, it follows that $0 \leq a_\epsilon \leq a_1$. Finally, for the last integral we have:

$$I_3 = \frac{1}{2\epsilon}\int_{\Omega_T \cap \{w \geq 0\} \cap \{v_0 < 0\} \cap \{v_0 + \epsilon w < 0\}} (2\epsilon v_0 w + \epsilon^2 w^2) dxdt - \frac{1}{2\epsilon}\int_{\Omega_T \cap \{w \geq 0\} \cap \{v_0 < 0\} \cap \{v_0 + \epsilon w \geq 0\}} v_0^2 dxdt.$$

Since the limit of the second term vanishes as $\epsilon \to 0$ it follows that

$$\lim_{\epsilon \to 0} I_3 = \int_{\Omega_T \cap \{w \geq 0\} \cap \{v_0 < 0\}} v_0 w dxdt = \int_{\Omega_T \cap \{w \geq 0\}} w \min(0, v_0) dxdt.$$

Adding the above limits, the expression (4.1) is obtained.

As a consequence of the expression (4.1) we get

Corrolary 4.2. If $\vec{g}_0$ is an optimal control, then

$$\int_{\Omega_T} ((v^* - v_0)_1 \min(0, (v_0)_1) + (v^* - v_0)_2 \min(0, (v_0)_2)) dxdt \geq 0. \tag{4.5}$$

The constrained inequality (4.5) will be replaced by an unconstrained one, by introducing the optimality system.

Theorem 4.3. Let $\vec{g}_0$ be a solution to the control problem (3.3) and $\vec{\psi}_0$ the corresponding function given by Corrolary 2.2. Then there exist the unique pairs $(\vec{V}_0, \vec{H}_0)$, $(\vec{W}_0, \vec{K}_0) \in (W(0,T;V,V'))^2$ satisfying the following optimality system:

–system (2.17) written for δ_0 and corresponding to $\vec{\psi}_0$,

$$\begin{cases} -\langle \vec{W}_0'(t), \vec{z}\rangle_{V',V} + \mu((\vec{W}_0(t), \vec{z}))_0 + b(\vec{z}, \vec{V}_0(t) + \vec{U}(t), \vec{W}_0(t)) \\ -b(\vec{V}_0(t) + \vec{U}(t), \vec{W}_0(t), \vec{z}) + \alpha b(\vec{z}, \vec{H}_0(t) + \vec{\psi}_0(t), \vec{K}_0(t)) \\ +\alpha b(\vec{H}_0(t) + \vec{\psi}_0(t), \vec{K}_0(t), \vec{z}) = (\min(0, \vec{V}_0(t) + \vec{U}(t)), \vec{z}) \,\forall \vec{z} \in V, \\ -\langle \vec{K}_0'(t), \vec{w}\rangle_{V',V} + \nu((\vec{K}_0(t), \vec{w}))_0 - rb(\vec{w}, \vec{H}_0(t) + \vec{\psi}_0(t), \vec{W}_0(t)) \\ +rb(\vec{H}_0(t) + \vec{\psi}_0(t), \vec{W}_0(t), \vec{w}) - \alpha b(\vec{w}, \vec{V}_0(t) + \vec{U}(t), \vec{K}_0(t)) \\ -\alpha b(\vec{V}_0(t) + \vec{U}(t), \vec{K}_0(t), \vec{w}) = 0 \,\forall \vec{w} \in V, \\ \vec{W}_0(T) = 0 \text{ in } \Omega,\ \vec{K}_0(T) = 0 \text{ in } \Omega, \end{cases} \tag{4.6}$$

$$\begin{cases} (\Lambda(\vec{g}_0(0) - \vec{g}(0)), \vec{K}_0(0)) + \int_0^T ((\Lambda(\vec{g}_0(t) - \vec{g}(t)))', \vec{K}_0(t)) dt \\ -r\int_0^T (b(\Lambda(\vec{g}_0(t) - \vec{g}(t)), \vec{h}_0(t), \vec{W}_0(t)) + b(\vec{h}_0(t), \Lambda(\vec{g}_0(t) - \vec{g}(t)), \vec{W}_0(t))) dt \\ -\alpha\int_0^T (b(\Lambda(\vec{g}_0(t) - \vec{g}(t)), \vec{v}_0(t), \vec{K}_0(t)) - b(\vec{v}_0(t), \Lambda(\vec{g}_0(t) - \vec{g}(t)), \vec{K}_0(t))) dt \\ +\nu \int_0^T (\nabla(\Lambda(\vec{g}_0(t) - \vec{g}(t))), \nabla \vec{K}_0(t)) dt \geq 0 \ \forall \vec{g} \in \mathcal{U}. \end{cases} \tag{4.7}$$

Proof. The first step is to explain the assertion concerning the pair $(\vec{W}_0, \vec{K}_0)$. We obtain it with the same techniques as those used in Theorem 2.4., but with easier computations since the system (4.6) is linear. We prove next the unconstrained inequality (4.7). We write the variational problem satisfied by $(\vec{V}_0 - \vec{V}^*, \vec{H}_0 - \vec{H}^*)$ and we take as test function $(\vec{z}, \vec{w}) = (\vec{W}_0(t), \vec{K}_0(t))$. Then we take as test function $(\vec{z}, \vec{w}) = (\vec{V}_0 - \vec{V}^*, \vec{H}_0 - \vec{H}^*)$ in (4.6). Subtracting these two systems and applying the Green's formula, we get as the expression of $\langle DJ(\vec{g}_0), \vec{g} - \vec{g}_0 \rangle$ the left hand side of (4.7); the inequality is now an obvious consequence of (4.5).

References

[1] Eringen, A. C. and Maugin, G. A., **Electrodynamics of continua**, Springer-Verlag 1990.

[2] Perry, M. P.,*A survey of ferromagnetic liquid applications*, Thermomechanics of magnetic fluids, ed. B. Berkovsky, pp. 219-230, Hemisphere, Washington, 1978.

[3] Rosenweig, R. E., **Ferrohydrodynamics**, Cambridge University Press, 1985.

[4] Abergel, F. and Casas, E., *Some optimal control problems of multistate equations appearing in fluid mechanics*, M^2AN, **27**, 2, pp. 223-247, 1993.

[5] Desai, M. and Ito, K., *Optimal controls of Navier-Stokes equations*, SIAM J. Control Optim., **32**, 5, pp. 1428-1446, 1994.

[6] Capatina, A. and Stavre, R., *Optimal control of a non-isothermal Navier-Stokes flow*, Int. J. Engng. Sci., **34**, 1, pp. 59-66, 1996.

[7] Capatina, A. and Stavre, R., *Numerical analysis of a control problem in heat conducting Navier-Stokes fluid*, Int. J. Engng. Sci., **34**, 13, pp. 1467-1476, 1996.

[8] Casas, E., *An optimal control problem governed by the evolution Navier-Stokes equations*, Optimal control of viscous flow, SIAM, Philadelphia, pp. 79-95, 1998.

[9] Fursikov, A. V., **Optimal control of distributed systems. Theory and applications**, AMS, Providence, 2000.

[10] Stavre, R., *Distributed control of a heat-conducting, time-dependent, Navier-Stokes fluid*, Glasgow Math. J., **44**, 2, pp. 191-200, 2002.

[11] Roubiček, T., *Optimization of steady-state flow of incompressible fluids*, in Analysis and optimization of differential systems, V. Barbu, I. Lasiecka, D. Tiba, C. Varsan editors, Kluwer Academic Publishers, Boston/Dordrecht/London, pp. 357-368, 2003.

[12] Hunt, J. C. R. and Holroyd, R. J., *Applications of the laboratory and theoretical MHD duct flow studies in fusion reactor technology*, Culham Laboratory Report CLM-R169, Abingdon, Oxfordshire, 1977.

[13] Temam, R., **Navier-Stokes equations**, AMS, Providence, 1984.

[14] Girault, V. and Raviart, P. A., **Finite element methods for Navier-Stokes equations**, Springer-Verlag, Berlin, 1984.

[15] Dautray, R. and Lions, J. L., **Analyse mathématique et calcule numérique pour les sciences et les techniques**, Masson, Paris, 1988.

[16] Wloka, J., **Partial differential equations**, Cambridge University Press, 1987.