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EXPONENTIAL STABILITY FOR DISCRETE TIME LINEAR EQUATIONS DEFINED BY POSITIVE OPERATORS

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Abstract

In this paper the problem of exponential stability of the zero state equilibrium of a discrete -time time-varying linear equation described by a sequence of linear positive operators acting on an ordered finite dimensional Hilbert space is investigated.

The class of linear equations considered in this paper contains as particular cases linear equations described by Liapunov operators or symmetric Stein operators as well as nonsymmetric Stein operators. Such equations occurs in connection with the problem of mean square exponential stability for a class of difference stochastic equations affected by independent random perturbations and Markovian jumping as well as in connection with some iterative procedures which allow as to compute global solutions of discrete time generalized symmetric or nonsymmetric Riccati equations.

The exponential stability is characterized in terms of the existence of some globaly defined and bounded solutions of some suitable backward affine equations (inequations) or forward affine equations (inequations).

1 Introduction

The stabilization problem together with various control problems for linear stochastic systems was intensively investigated in the last four decades. For the readers convenience we refer to ones of the most popular monographies in the field: [1, 6, 9, 19, 23, 32, 33] and references therein.

It is well known that the mean square exponential stability or equivalently the second moments exponential stability of the zero solution of a linear stochastic differential equation or linear stochastic difference equation is equivalent with the exponential stability of the zero state equilibrium of a suitable deterministic linear differential equation or a deterministic linear difference equation. Such deterministic differential (difference) equations are defined by the so called Liapunov type operators associated to the given stochastic linear differential (difference) equations.

The exponential stability in the case of differential equations or difference equations described by Liapunov operators have been investigated as a problem with interest in itself in a lot of works. In the time invariant case results concerning the exponential stability of the linear differential equations defined by Liapunov type operators were derived based on spectral properties of linear positive operators on an ordered Banach space obtained by Krein and Rutman [22] and Schneider [31]. A significant extension of the results of [22] and [31] to the class of positive rezolvent operators was provided by Damm and Hinrichsen in [7, 8]. Similar results were derived also for discrete-time time-invariant case see [16, 30].

In the case of continuous-time time-varying systems in [11] a class of linear differential equations on the space of $n \times n$ symmetric matrices $\mathcal{S}_n$ is studied. Such equations have the property that the corresponding linear evolution operator is a positive operator on $\mathcal{S}_n$. They contain as particular cases linear differential equations of Liapunov type arising in connection with the problem of investigation of mean square exponential stability.

In this paper the discrete-time time-varying counterpart of [11] is provided. While in [11] the considered linear differential equations are defined by operator valued functions acting on the space $\mathcal{S}_n$, in this paper we consider discrete-time time-varying linear equations described by sequences of linear positive operators acting on a suitable ordered finite dimensional Hilbert space.

The ordered spaces considered in this paper contain as special casses the spaces $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ ordered by the component wise order relation and the space $\mathcal{S}_n$ of the $n \times n$ symmetric matrices ordered by the order induced by the cone of the positive semidefinite matrices.

Tha main results of this paper provide necessary and sufficient conditions which guarantee the exponential stability of the zero state equilibrium of a discrete-time time-varying linear equation described by a sequence of positive operators.

To characterize the exponential stability a crucial role is played by the unique bounded solution of some suitable backward affine equations as well as of some forward affine equations. We show that if the considered equations are described by periodic sequences of operators then the bounded solution if it exists is also a periodic sequence. Moreover, in the time-invariant case the bounded solutions to both backward affine equation and forward affine equation are constant. Thus, the results concerning the exponential stability for the time-invariant case are recovered as special casses of the results proved in this paper.

The outline of the paper is as follows: Section 2 collects some definitions, some auxiliary results in order to display the framework where the main results are proved. Section 3 contains results which characterize the exponential stability of the zero state equilibrium of a discrete-time time-varying linear equation described by a sequence of linear positive operators on a ordered finite dimensional Hilbert space. Section 4 deals with a class of linear positive operators acting on a space of symmetric matrices. Such operators contain as a special case the Liapunov type operators arising in connection with discrete-time linear stochastic equations affected to both independent random perturbations and Markovian jumping.

2 Discrete time linear equations defined by positive operators

2.1 Preliminary considerations

Let $\mathcal{X}$ be a finite dimensional real Hilbert space. We assume that $\mathcal{X}$ is ordered by a order relation " $\leq$ " induced by a regular solid closed pointed selfdual convex cone $\mathcal{X}^+$. For detailed definitions and other properties of convex cones we refer to [2, 8, 16, 20].

Here we recall only that if $\mathcal{C} \subset \mathcal{X}$ is a convex cone then the corresponding dual cone $\mathcal{C}^* \subset \mathcal{X}^*$ consists of the set of all functionals $y^* \in \mathcal{X}^*$ such that $y^*(x) \geq 0$ for all $x \in \mathcal{C}$. A cone $\mathcal{C}$ is called selfdual if $\mathcal{C}^* = \mathcal{C}$. For the last equality we take into account that based on Ritz theorem the dual $\mathcal{X}^*$ is identified with $\mathcal{X}$.

Therefore the cone $\mathcal{C}$ is selfdual is equivalent with the fact that $x \in \mathcal{C}$ if and only if $\langle x, y \rangle \geq 0$ for all $y \in \mathcal{C}$, $\langle \cdot , \cdot \rangle$ being the inner product on $\mathcal{X}$.

We also recall that a cone is said to be regular if for arbitrary bounded bellow sequence $x_1 \geq x_2 \geq ... \geq x_n \geq ... \geq \hat{x}$ there exists $\lim_{k\to\infty} x_k \in \mathcal{X}$.

By $|\cdot|_2$ we denote the norm on $\mathcal{X}$ induced by the inner product on $\mathcal{X}$ i.e.

$$|x|_2 = [< x, x >]^{1/2}.$$

Throughout this paper we suppose that together with $|\cdot|_2$ on $\mathcal{X}$ there exists also another norm denoted by $|\cdot|_1$ with the following properties:

$\mathbf{P_1}$) There exists $\xi_{\mathcal{X}} \in Int\mathcal{X}^+$ such that $|\xi_{\mathcal{X}}|_1 = 1$ and

$$-\xi_{\mathcal{X}} \leq x \leq \xi_{\mathcal{X}} \tag{2.1}$$

for arbitrary $x \in \mathcal{X}$ with $|x|_1 \leq 1$.

$\mathbf{P_2}$) If $x, y, z \in \mathcal{X}$ are such that $y \leq x \leq z$ then

$$|x|_1 \leq \max(|y|_1, |z|_1). \tag{2.2}$$

If $T : \mathcal{X} \to \mathcal{X}$ is a linear operator then $||T||_k$ is the norm of T induced by $|\cdot|_k$, $k = 1, 2$, that is

$$||T||_k = \sup_{|x|_k \leq 1} \{|Tx|_k\}. \tag{2.3}$$

Remark 2.1 a) Since $\mathcal{X}$ is a finite dimensional space then $|\cdot|_1$ and $|\cdot|_2$ are equivalent. From (2.3) it follows that $||\cdot||_1$ and $||\cdot||_2$ are also equivalent. This means that there are two positive constants c_1 and c_2 such that

$$c_1||T||_1 \leq ||T||_2 \leq c_2||T||_1$$

for all linear operators $T : \mathcal{X} \to \mathcal{X}$.

b) If $T^* : \mathcal{X} \to \mathcal{X}$ is the adjoint operator of T with respect to the inner product on $\mathcal{X}$, then $||T||_2 = ||T^*||_2$. In general the equality $||T||_1 = ||T^*||_1$ is not true.

However, based on a) it follows that there are two positive constants $\tilde{c}_1, \tilde{c}_2$ such that

$$\tilde{c}_1||T||_1 \leq ||T^*||_1 \leq \tilde{c}_2||T||_1. \tag{2.4}$$

Let $(\mathcal{X}, \mathcal{X}^+)$ and $(\mathcal{Y}, \mathcal{Y}^+)$ be two ordered Hilbert spaces. An operator $T : \mathcal{X} \to \mathcal{Y}$ is called positive operator if $T(\mathcal{X}^+) \subset \mathcal{Y}^+$. In this case we shall write $T \geq 0$.

If $T(Int\mathcal{X}^+) \subset Int\mathcal{Y}^+$ we shall write $T > 0$.

Proposition 2.2 *If $T : \mathcal{X} \to \mathcal{X}$ is a linear operator then the following hold:*

(i) $T \geq 0$ if and only if $T^* \geq 0$.

(ii) If $T \geq 0$ then $||T||_1 = |T\xi_{\mathcal{X}}|_1$.

Proof: (i) is a direct consequence of the fact that $\mathcal{X}^+$ is a selfdual cone.

(ii) If $T \geq 0$ then from (2.1) we have

$$-T\xi_{\mathcal{X}} \leq Tx \leq T\xi_{\mathcal{X}}.$$

From (2.2) it follows that $|Tx|_1 \leq |T\xi_{\mathcal{X}}|_1$ for all $x \in \mathcal{X}$ with $|x|_1 \leq 1$ which leads to

$$\sup_{|x|_1 \leq 1} |Tx|_1 \leq |T\xi_{\mathcal{X}}|_1 \leq \sup_{|x|_1 \leq 1} |Tx|_1$$

hence $||T||_1 = |T\xi_{\mathcal{X}}|_1$ and thus the proof is complete.

Example 2.3 (i) Consider $\mathcal{X} = \mathbf{R}^n$ ordered by the order relation induced by the cone $\mathbf{R}^n_+$. Recall that $\mathbf{R}^n_+ = \{x = (x_1, ..., x_n)^T \in \mathbf{R}^n | x_i \geq 0, 1 \leq i \leq n\}$. It is not difficult to see that $\mathbf{R}^n_+$ is a regular solid closed selfdual pointed convex cone. If $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear operator then $T \geq 0$ iff its corresponding matrix A with respect to the cannonical basis on $\mathbf{R}^n$ has nonnegative entries. Together with the Euclidian norm $|\cdot|_2$ on $\mathbf{R}^n$ we consider the norm $|\cdot|_1$ defined by

$$|x|_1 = \max_{1 \leq i \leq n} |x_i|. \tag{2.5}$$

The properties $\mathbf{P_1}$ and $\mathbf{P_2}$ are fulfilled for the norm defined by (2.5). The element $\xi_{\mathcal{X}}$ is now $\xi_{\mathcal{X}} = (1, 1, 1..., 1)^T \in Int(\mathbf{R}^n_+)$. The ordered space $(\mathbf{R}^n, \mathbf{R}^n_+)$ is considered in connection with Perron-Frobenius Theorem.

(ii) Let $\mathcal{X} = \mathbf{R}^{m\times n}$ be the space of $m \times n$ real matrices, endowed with the inner product

$$\langle A, B\rangle = Tr(B^T A) \tag{2.6}$$

$\forall\, A, B \in \mathbf{R}^{m\times n}, Tr(M)$ denoting as usually the trace of a matrix M.

On $\mathbf{R}^{m\times n}$ we consider the order relation induced by the cone $\mathcal{X}^+ = \mathbf{R}_+^{m\times n}$ where

$$\mathbf{R}_+^{m\times n} = \{A \in \mathbf{R}^{m\times n} | A = \{a_{ij}\}, a_{ij} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n\}. \tag{2.7}$$

The interior of the cone $\mathbf{R}_+^{m\times n}$ is not empty. Let A be an element of the dual cone $(\mathbf{R}_+^{m\times n})^*$. This means that $\langle A, B\rangle \geq 0$ for arbitrary $B \in \mathbf{R}_+^{m\times n}$. Let $E^{ij} \in \mathbf{R}_+^{m\times n}$ be such that $E^{ij} = \{e_{lk}^{ij}\}_{l.k}$, with $e_{lk}^{ij} = 0$ if $(l, k) \neq (i, j)$, $e_{lk}^{ij} = 1$ if $(l, k) = (i, j)$. We have $0 \leq \langle A, E^{ij}\rangle = a_{ij}$ which show that $A \in \mathbf{R}_+^{m\times n}$ and it follows that the cone (2.7) is selfdual. On $\mathbf{R}^{m\times n}$ we consider also the norm $|\cdot|_1$ defined by

$$|A|_1 = \max_{i,j} |a_{ij}|. \tag{2.8}$$

Properties $\mathbf{P_1}$ and $\mathbf{P_2}$ are fulfilled for norm (2.8) with

$$\xi_{\mathcal{X}} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \in Int\mathbf{R}_+^{m\times n}.$$

An important class of linear operators on $\mathbf{R}^{m\times n}$ is that of the form $\mathcal{L}_{A,B} : \mathbf{R}^{m\times n} \to \mathbf{R}^{m\times n}$ by $\mathcal{L}_{A,B}Y = AYB$, for all $Y \in \mathbf{R}^{m\times n}$ where $A \in \mathbf{R}^{m\times m}, B \in \mathbf{R}^{n\times n}$ are given fixed matrices. These operators are often called "nonsymmetric Stein operators". It can be checked that $\mathcal{L}_{A,B} \geq 0$ iff $a_{ij}b_{lk} \geq 0, \ \ \forall\ i, j \in \{1, ..., m\}, l, k \in \{1, ..., n\}$. Hence $\mathcal{L}_{A,B} \geq 0$ iff the matrix $A \otimes B$ defines a positive operator on the ordered space $(\mathbf{R}^{mn}, \mathbf{R}_+^{mn})$ where $\otimes$ is the Kronecker product.

(iii) Let $\mathcal{S}_n \subset \mathbf{R}^{n\times n}$ be the subspace of $n \times n$ symmetric matrices. Let $\mathcal{X} = \mathcal{S}_n \oplus \mathcal{S}_n \oplus ... \oplus \mathcal{S}_n = \mathcal{S}_n^N$ with $N \geq 1$ fixed. On $\mathcal{S}_n^N$ we consider the inner product

$$\langle X, Y\rangle = \sum_{i=1}^{N} Tr(Y_i X_i) \tag{2.9}$$

for arbitrary $X = (X_1\ X_2\ ...\ X_N)$ and $Y = (Y_1\ Y_2\ ...\ Y_N)$ in $\mathcal{S}_n^N$. The space $\mathcal{S}_n^N$is ordered by the convex cone

$$\mathcal{S}_n^{N,+} = \{X = (X_1\ X_2\ ...\ X_N) | X_i \geq 0, 1 \leq i \leq N\}. \tag{2.10}$$

The cone $\mathcal{S}_n^{N,+}$ has the interior nonempty.

$$Int\mathcal{S}_n^{N,+} = \{X \in \mathcal{S}_n^N | X_i > 0, 1 \leq i \leq N\}.$$

Here $X_i \geq 0, (X_i > 0)$ respectively, means that X_i is a positive semidefinite matrix, positive definite matrix. With a similar reasoning as in [16] for $N = 1$ one may show that $\mathcal{S}_n^{N,+}$ is a selfdual cone.

Together with the norm $|\cdot|_2$ induced by the inner product (2.9), on $\mathcal{S}_n^N$ we consider the norm $|\cdot|_1$ defined by

$$|X|_1 = \max_{1 \leq i \leq N} |X_i|, \qquad (\forall) X = (X_1 \ldots X_N) \in \mathcal{S}_n^N \tag{2.11}$$

where $|X_i| = \max_{\lambda \in \sigma(X_i)} |\lambda|$, $\sigma(X_i)$ being the set of eigenvalues of the matrix X_i. For the norm defined by (2.11) the properties $\mathbf{P_1}$ and $\mathbf{P_2}$ are fulfilled with $\xi_{\mathcal{X}} = (I_n\ I_n\ \ldots\ I_n) = J \in \mathcal{S}_n^N$.

An important class of positive linear operators on $\mathcal{S}_n^N$ will be widely investigated in Section 4. The operators considered in Section 4 contain as a particular case the symmetric Stein operators.

2.2 Discrete time affine equations

Let $\mathbf{L} = \{\mathcal{L}_k\}_{k \geq k_0}$ be a sequence of linear operators $\mathcal{L}_k : \mathcal{X} \to \mathcal{X}$ and $f = \{f_k\}_{k \geq k_0}$ be a sequence of elements $f_k \in \mathcal{X}$. These two sequences define two affine equations on $\mathcal{X}$:

$$x_{k+1} = \mathcal{L}_k x_k + f_k \tag{2.12}$$

which will be called "the forward" affine equation or "causal affine equation" defined by $(\mathbf{L}, f)$ and

$$x_k = \mathcal{L}_k x_{k+1} + f_k \tag{2.13}$$

which will be called "the backward affine equation" or "anticausal affine equation" defined by $(\mathbf{L}, f)$. For each $k \geq l \geq k_0$ let $T^c_{kl} : \mathcal{X} \to \mathcal{X}$ be the causal evolution operator defined by the sequence $\mathbf{L}$, $T^c_{kl} = \mathcal{L}_{k-1}\mathcal{L}_{k-2} \ldots \mathcal{L}_l$ if $k > l$ and $T^c_{kl} = I_{\mathcal{X}}$ if $k = l$, $I_{\mathcal{X}}$ being the identity operator on $\mathcal{X}$.

For all $k_0 \leq k \leq l, T^a_{kl} : \mathcal{X} \to \mathcal{X}$ stands for the anticausal evolution operator on $\mathcal{X}$ defined by the sequence $\mathbf{L}$, that is

$$T^a_{kl} = \mathcal{L}_k \mathcal{L}_{k+1} \ldots \mathcal{L}_{l-1}$$

if $k < l$ and $T^a_{kl} = I_{\mathcal{X}}$ if $k = l$.

Often the superscripts a and c will be omitted if any confusion is not possible.

Let $\tilde{x}_k = T^c_{kl} x, k \geq l, l \geq k_0$ be fixed. One obtains that $\{\tilde{x}_k\}_{k \geq l}$ verifies the forward linear equation

$$x_{k+1} = \mathcal{L}_k x_k \tag{2.14}$$

with initial value $x_l = x$. Also, if $y_k = T^a_{kl} y, k_0 \leq k \leq l$ then from definition of T^a_{kl} one obtains that $\{y_k\}_{k_0 \leq k \leq l}$ is the solution of the backward linear equation

$$y_k = \mathcal{L}_k y_{k+1} \tag{2.15}$$

with given terminal value $y_l = y$.

It must be remarked that, in contrast with the continuous time case, a solution $\{x_k\}_k$ of the forward linear equation (2.14) with given initial values $x_l = x$ is well defined for $k \geq l$ while a solution $\{y_k\}_k$ of the backward linear equation (2.15) with given terminal condition $y_l = y$ is well defined for $k_0 \leq k \leq l$.

If for each k, the operators $\mathcal{L}_k$ are invertible, then all solutions of the equations (2.14), (2.15) are well defined for all $k \geq k_0$.

If $(T^c_{kl})^*$ is the adjoint operator of the causal evolution operator T^c_{kl} we define

$$z_l = (T^c_{kl})^* z, \ (\forall)\ k_0 \leq l \leq k.$$

By direct calculation one obtains that $z_l = \mathcal{L}^*_l z_{l+1}$ which shows that the adjoint of the causal evolution operator associated with the sequence $\mathbf{L}$ generates anticausal evolution.

Definition 2.4 We say that the sequence $\mathbf{L} = \{\mathcal{L}_k\}_{k \geq k_0}$ defines a positive evolution if for all $k \geq l \geq k_0$ the causal linear evolution operator $T^c_{kl} \geq 0$.

Since $T^c_{l+1l} = \mathcal{L}_l$ it follows that the sequence $\{\mathcal{L}_k\}_{k \geq k_0}$ generates a positive evolution if and only if for each $k \geq k_0, \mathcal{L}_k$ is a positive operator. Hence, in contrast with the continuous time case, in the discrete time case only sequences of positive operators define equations which generate positive evolutions (see [11].)

At the end of this subsection we recall the representation formulae of the solutions of affine equations (2.12), (2.13).

Each solution of the forward affine equation (2.12) has the representation:

$$x_k = T^c_{kl} x_l + \sum_{i=l}^{k-1} T^c_{ki+1} f_i \tag{2.16}$$

for all $k \geq l+1$. Also, any solution of the backward affine equation (2.13) has a representation

$$y_k = T^a_{kl} y_l + \sum_{i=k}^{l-1} T^a_{ki} f_i, \quad k_0 \leq k \leq l-1.$$

3 Exponential stability

In this section we deal with the exponential stability of the zero solution of a discrete time linear equation defined by a sequence of linear positive operators.

Definition 3.1 We say that the zero solution of the equation

$$x_{k+1} = \mathcal{L}_k x_k \tag{3.1}$$

or equivalently that the sequence $\mathbf{L} = \{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution if there are $\beta > 0, q \in (0,1)$ such that

$$||T_{kl}||_1 \leq \beta q^{k-l}, \quad (\forall)\ k \geq l \geq k_0 \tag{3.2}$$

T_{kl} being the causal linear evolution operator defined by the sequence $\mathbf{L}$.

In the case when $\mathcal{L}_k = \mathcal{L}$ for all k, if (3.2) is satisfied we shall say that the operator $\mathcal{L}$ generates a discrete-time exponentially stable evolution.

It is well known that $\mathcal{L}$ generates a discrete-time exponentially stable evolution if and only if the eigenvalues of $\mathcal{L}$ are located in the inside of the disk $|\lambda| < 1$ or equivalently, $\rho[\mathcal{L}] < 1, \rho[\dot{]}$ being the spectral radius.

It must be remarked that if the sequence $\{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution then it is a bounded sequence.

In this section we shall derive several conditions which are equivalent with the exponnetial stability of the zero solution of the equation (3.1) in the case $\mathcal{L}_k \geq 0, k \geq k_0$. Such results can be viewed as an alternative characterization of exponential stability to the one in terms of Liapunov functions. We remark that since $\mathcal{X}$ is a finite dimensional space in (3.2) we may consider any norm on $\mathcal{X}$.

Firstly, we prove:

Theorem 3.2 *Let $\{\mathcal{L}_k\}_{k\geq 0}$ be a sequence of linear positive operators $\mathcal{L}_k : \mathcal{X} \to \mathcal{X}$. Then the following are equivalent:*

(i) The sequence $\{\mathcal{L}_k\}_{k\geq 0}$ generates an exponentially stable evolution.

(ii) There exists $\delta > 0$ such that

$$\sum_{l=k_0}^{k} ||T_{kl}||_1 \leq \delta$$

for arbitrary $k \geq k_0 \geq 0$.

(iii) There exists $\delta > 0$, such that $\sum_{l=k_1}^{k} T_{kl}\xi_{\mathcal{X}} \leq \delta\xi_{\mathcal{X}}$ for arbitrary $k \geq k_1 \geq 0, \delta > 0$ being independent of k, k_1.

(iv) For arbitrary bounded sequence $\{f_k\}_{k\geq 0} \subset \mathcal{X}$ the solution with zero initial value of the forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + f_k, \quad k \geq 0$$

is bounded.

Proof: The implication $(iv) \to (i)$ is the discrete-time counter part of the Perron's Theorem (see [17].) It remains to prove the implications $(i) \to (ii) \to (iii) \to (iv)$.

If (i) is true then (ii) follows immediately from (3.2) with $\delta = \frac{\beta}{1-q}$.

Let us prove that:

$$0 \leq T_{kl}\xi_{\mathcal{X}} \leq ||T_{kl}||_1 \xi_{\mathcal{X}} \tag{3.3}$$

for arbitrary $k \geq l \geq 0$. If $T_{kl}\xi_{\mathcal{X}} = 0$ then from Proposition 2.2 (ii) it follows that $||T_{kl}||_1 = 0$ and (3.3) is fulfilled. If $T_{kl}\xi_{\mathcal{X}} \neq 0$ then from (2.1) applied to $x = \frac{1}{|T_{kl}\xi_{\mathcal{X}}|_1}T_{kl}\xi_{\mathcal{X}}$ one gets $0 \leq T_{kl}\xi_{\mathcal{X}} \leq |T_{kl}\xi_{\mathcal{X}}|_1\xi_{\mathcal{X}}$ and (3.3) follows based on Proposition 2.2 (ii).

If (ii) holds then (iii) follows from (3.3). We have to prove that $(iii) \to (iv)$. Let $\{f_k\}_{k\geq 0} \subset \mathcal{X}$ be a bounded sequence, that is $|f_k|_1 \leq \mu$, $k \geq 0$. Based on (2.1) we obtain that $-|f_l|_1\xi_{\mathcal{X}} \leq f_l \leq |f_l|_1\xi_{\mathcal{X}}$ which leads to $-\mu\xi_{\mathcal{X}} \leq f_l \leq \mu\xi_{\mathcal{X}}$ for all $l \geq 0$.

Since for each $k \geq l+1 \geq 0$, T_{kl+1} is a positive operator we have:

$$-\mu T_{kl+1}\xi_{\mathcal{X}} \leq T_{kl+1}f_l \leq \mu T_{kl+1}\xi_{\mathcal{X}}$$

and

$$-\mu\sum_{l=0}^{k-1} T_{kl+1}\xi_{\mathcal{X}} \leq \sum_{l=0}^{k-1} T_{kl+1}f_l \leq \mu\sum_{l=0}^{k-1} T_{kl+1}\xi_{\mathcal{X}}.$$

Applying (2.2) we deduce that:

$$|\sum_{l=0}^{k-1} T_{kl+1}f_l|_1 \leq \mu|\sum_{l=0}^{k-1} T_{kl+1}\xi_{\mathcal{X}}|_1.$$

If (iii) is valid we conclude by using again (2.2) that

$$|\sum_{l=0}^{k-1} T_{kl+1}f_l|_1 \leq \mu\delta, \quad (\forall)\ k \geq 1$$

which shows that (iv) is fulfilled using (2.16) and thus the proof ends.

We note that the proof of the above theorem shows that in the case of a discrete time linear equation (3.1) defined by a sequence of linear positive operators the exponential stability is equivalent with the boundedness of the solution with the zero initial value of the forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + \xi_{\mathcal{X}}.$$

We recall that in the general case of a discrete time linear equation if we want to use the Perron's Theorem to characterize the exponential stability we have to check the boundedness of the solution with zero initial value of the forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + f_k$$

for arbitrary bounded sequence $\{f_k\}_{k\geq 0} \subset \mathcal{X}$.

Definition 3.3 We say that a sequence $\{f_k\}_{k\geq k_0} \subset \mathcal{X}^+$ is uniformly positive if there exists $c > 0$ such that $f_k > c\xi_{\mathcal{X}}$ for all $k \geq k_0$. If $\{f_k\}_{k\geq k_0} \subset \mathcal{X}^+$ is uniformly positive we shall write $f_k \gg 0, k \geq k_0$. If $-f_k \gg 0, k \geq k_0$ then we shall write $f_k \ll 0, k \geq k_0$.

The next result provides a characterization of the exponential stability, using solutions of some suitable backward affine equations.

Theorem 3.4 *Let $\{\mathcal{L}_k\}_{k\geq k_0}$ be a sequence of linear and positive operators $\mathcal{L}_k : \mathcal{X} \to \mathcal{X}$. Then the following are equivalent:*

(i) The sequence $\{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution.

(ii) There exist $\beta_1 > 0, q \in (0,1)$ such that $||T^*_{kl}||_1 \leq \beta_1 q^{k-l}$, $(\forall)\ k \geq l \geq k_0$.

(iii) There exists $\delta > 0$, independent of k, such that $\sum_{l=k}^{\infty} T^*_{lk}\xi_{\mathcal{X}} \leq \delta\xi_{\mathcal{X}}$.

(iv) The backward affine discrete time equation

$$x_k = \mathcal{L}^*_k x_{k+1} + \xi_{\mathcal{X}} \tag{3.4}$$

has a bounded and uniformly positive solution.

(v) For arbitrary bounded and uniformly positive sequence $\{f_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ the backward affine equation

$$x_k = \mathcal{L}^*_k x_{k+1} + f_k, \quad k \geq k_0 \tag{3.5}$$

has a bounded and uniformly positive solution.

(vi) There exists a bounded and uniformly positive sequence $\{f_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ such that the corresponding backward affine equation (3.5) has a bounded solution $\{\tilde{x}_k\}_{k\geq k_0} \subset \mathcal{X}^+$.

(vii) There exists a bounded and uniformly positive sequence $\{y_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ which verifies

$$\mathcal{L}^*_k y_{k+1} - y_k \ll 0, \quad k \geq k_0. \tag{3.6}$$

Proof: The equivalence $(i) \leftrightarrow (ii)$ follows immediatelly from (2.4). In a similar way as in the proof of inequality (3.3) one obtains:

$$0 \leq T^*_{lk}\xi_{\mathcal{X}} \leq ||T^*_{lk}||_1\xi_{\mathcal{X}} \tag{3.7}$$

for all $l \geq k \geq k_0$.

If (ii) holds, then (iii) follows immediatelly from (3.7) together with the property that $\mathcal{X}^+$ is a regular cone. To show that $(iii) \to (iv)$ we define $y_k = \sum_{l=k}^{\infty} T^*_{lk}\xi_{\mathcal{X}},\ k \geq k_0$. If (iii) holds it follows that $\{y_k\}_{k\geq 0}$ is well defined. Since $y_k = \xi_{\mathcal{X}} + \mathcal{L}^*_k \sum_{l=k+1}^{\infty} T^*_{lk+1}\xi_{\mathcal{X}}$ one obtains that $y_k \gg 0, k \geq k_0$ and $\{y_k\}_{k\geq k_0}$ solves (3.4) and thus (iv) is true.

Let us prove now that $(iv) \to (iii)$. Let $\{x_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ be a bounded and uniform positive solution of (3.4) that is

$$0 < \mu_1\xi_{\mathcal{X}} \leq x_k \leq \mu_2\xi_{\mathcal{X}} \tag{3.8}$$

for some positive constants μ_i independent of k. The solution $\{x_k\}_{k\geq k_0}$ has the representation formula

$$x_k = T^*_{jk}x_j + \sum_{l=k}^{j-1} T^*_{lk}\xi_{\mathcal{X}}$$

for all $j \geq k+1 \geq k_0$. Since $T^*_{lk} \geq 0$ we obtain

$$\sum_{l=k}^{j-1} T^*_{lk}\xi_{\mathcal{X}} \leq x_k. \tag{3.9}$$

For each fixed $k \geq k_0$ we define

$$z_j = \sum_{l=k}^{j-1} T^*_{lk}\xi_{\mathcal{X}}$$

for all $j \geq k+1$. The sequence $\{z_j\}_{j\geq k+1}$ is monotone increasing. From (3.8) and (3.9) we obtain that

$$\xi_{\mathcal{X}} \leq z_j \leq \mu_2\xi_{\mathcal{X}}.$$

Since $\mathcal{X}^+$ is a regular cone we may conclude that there exists

$$\lim_{j\to\infty} z_j = \sum_{l=k}^{\infty} T^*_{lk}\xi_{\mathcal{X}} \leq \mu_2\xi_{\mathcal{X}}$$

and thus (iii) is valid.

Now we prove $(iii) \to (v)$. Let $\{f_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ be a bounded and uniformly positive sequence. This means that there exists $\nu_i > 0$ such that

$$\nu_1\xi_{\mathcal{X}} \leq f_l \leq \nu_2\xi_{\mathcal{X}}, \quad \forall l \geq k_0.$$

Since $T^*_{lk} \geq 0$ one obtains $\nu_1 T^*_{lk}\xi_{\mathcal{X}} \leq T^*_{lk}f_l \leq \nu_2 T^*_{lk}\xi_{\mathcal{X}}, \ \forall\, l \geq k \geq k_0$.

Further we may write the inequalities: $\nu_1\xi_{\mathcal{X}} \leq \nu_1 \sum_{l=k}^{j} T^*_{lk}\xi_{\mathcal{X}} \leq \sum_{l=k}^{j} T^*_{lk}f_l \leq \nu_2 \sum_{l=k}^{j} T^*_{lk}\xi_{\mathcal{X}} \leq \nu_2 \sum_{l=k}^{\infty} T^*_{lk}\xi_{\mathcal{X}} \leq \nu_2\delta\xi_{\mathcal{X}}, \quad (\forall)\ j \geq k \geq k_0$.

Since $\mathcal{X}^+$ is a regular cone one concludes that the sequence $\{\sum_{l=k}^{j} T^*_{lk}f_l\}_{j\geq k}$ is convergent.

We define $\tilde{x}_k = \sum_{l=k}^{\infty} T^*_{lk}f_l, \ \ k \geq k_0$. One obtains that $\tilde{x}_k = f_k + \mathcal{L}^*_k \sum_{l=k+1}^{\infty} T^*_{lk+1}f_l$ which shows that $\{\tilde{x}_k\}_{k\geq k_0}$ is a solution with desired properties of the equation (3.5) and thus (v) holds.

$(v) \to (vi)$ is obvious. We prove now $(vi) \to (ii)$. Let us assume that there exists a bounded and uniformly positive sequence $\{f_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ such that the discrete time backward affine equation (3.5) has a bounded solution $\{\hat{x}_k\}_{k\geq k_0} \subset \mathcal{X}^+$.

Therefore there exist positive constants γ_i such that

$$\begin{aligned} 0 < \gamma_1\xi_{\mathcal{X}} \leq f_l \leq \gamma_2\xi_{\mathcal{X}} \\ 0 < \gamma_1\xi_{\mathcal{X}} \leq \hat{x}_l \leq \gamma_3\xi_{\mathcal{X}} \end{aligned} \tag{3.10}$$

for all $l \geq k_0$. Writing the representation formula

$$\hat{x}_k = T^*_{jk}\hat{x}_j + \sum_{l=k}^{j-1} T^*_{lk}f_l$$

and taking into account that $T_{jk}^* \geq 0$ if $j \geq k$ one obtains

$$f_k \leq \sum_{l=k}^{j-1} T_{lk}^* f_l \leq \hat{x}_k, \quad (\forall)\ j-1 \geq k \geq k_0. \tag{3.11}$$

Set $y_k = \sum_{l=k}^{\infty} T_{lk}^* f_l, \quad k \geq k_0$, $\mathcal{X}^+$ being a regular cone together with (3.10), (3.11) guarantee that y_k is well defined and

$$\gamma_1 \xi_{\mathcal{X}} \leq y_k \leq \gamma_3 \xi_{\mathcal{X}} \tag{3.12}$$

for all $k \geq k_0$. Let $k_1 \geq k_0$ be fixed. We define $\tilde{y}_k = T_{kk_1}^* y_k$, $k \geq k_1$. Since $T_{kk_1}^* \geq 0$ one obtains that

$$\gamma_1 T_{kk_1}^* \xi_{\mathcal{X}} \leq \tilde{y}_k \leq \gamma_3 T_{kk_1}^* \xi_{\mathcal{X}} \tag{3.13}$$

for all $k \geq k_1$.

On the other hand we have $\tilde{y}_k = \sum_{l=k}^{\infty} T_{lk_1}^* f_l$. This allows us to write

$$\tilde{y}_{k+1} - \tilde{y}_k = -T_{kk_1}^* f_k.$$

From (3.10) we get

$$\tilde{y}_{k+1} - \tilde{y}_k \leq -\gamma_1 T_{kk_1}^* \xi_{\mathcal{X}}.$$

Further, (3.13) leads to:

$$\tilde{y}_{k+1} \leq (1 - \frac{\gamma_1}{\gamma_3})\tilde{y}_k, \quad (\forall)\ k \geq k_1.$$

Inductively we deduce

$$\tilde{y}_k \leq q^{k-k_1} \tilde{y}_{k_1}, \quad \forall\ k \geq k_1 \tag{3.14}$$

where $q = 1 - \frac{\gamma_1}{\gamma_3}, q \in (0,1)$ (in (3.13) γ_3 may be choosen large enough so that $\gamma_3 > \gamma_1$). Invoking again (3.13) we may write

$$T_{kk_1}^* \xi_{\mathcal{X}} \leq \frac{\gamma_3}{\gamma_1} q^{k-k_1} \xi_{\mathcal{X}}$$

which by (2.2) leads to $|T_{kk_1}^* \xi_{\mathcal{X}}|_1 \leq \frac{\gamma_3}{\gamma_1} q^{k-k_1}$, $\forall\ k \geq k_1$. Based on Proposition 2.2 (ii) we have

$$||T_{kk_1}^*||_1 \leq \frac{\gamma_3}{\gamma_1} q^{k-k_1}$$

that means that (ii) is fulfilled.

The implication $(iv) \to (vii)$ follows immediatelly since a bounded and uniformly positive solution of (3.4) is a solution with desired properties of (3.6). To end the proof we show that $(vii) \to (vi)$. Let $\{z_k\}_{k \geq k_0} \subset Int\mathcal{X}^+$ be a bounded and uniform positive solution of (3.6). Define $\hat{f}_k = z_k - \mathcal{L}_k^* z_{k+1}$. It follows that $\{\hat{f}_k\}_{k \geq k_0}$ is bounded and uniform positive, therefore $\{z_k\}_{k \geq 0}$ will be a bounded and positive solution of (3.5) corresponding to $\{\hat{f}_k\}_{k \geq k_0}$ and thus the proof ends.

We remark that in the proof of Theorem 3.4 the fact that $\mathcal{X}^+$ is assumed to be a regular cone, was used in order to guarantee the convergence of several series in $\mathcal{X}$.

The result proved in Theorem 3.2 holds even if $\mathcal{X}^+$ is not a regular cone.

The next result provides more information concerning the bounded solution of the discrete time backward affine equations.

Theorem 3.5 *Let $\{\mathcal{L}_k\}_{k\geq k_0}$ be a sequence of linear operators which generates an exponentially stable evolution on $\mathcal{X}$. Then the following hold:*

(i) for each bounded sequence $\{f_k\}_{k\geq k_0} \subset \mathcal{X}$ the discrete-time backward affine equation

$$x_k = \mathcal{L}_k^* x_{k+1} + f_k \tag{3.15}$$

has an unique bounded solution which is given by

$$\tilde{x}_k = \sum_{l=k}^{\infty} T_{lk}^* f_l, \quad k \geq k_0. \tag{3.16}$$

(ii) If there exists an integer $\theta > 1$ such that $\mathcal{L}_{k+\theta} = \mathcal{L}_k$, $f_{k+\theta} = f_k$ for all k then the unique bounded solution of equation (3.15) is also a periodic sequence with period θ.

(iii) If $\mathcal{L}_k = \mathcal{L}$, $f_k = f$ for all k then the unique bounded solution of the equation (3.15) is constant and it is given by

$$\tilde{x} = (I_{\mathcal{X}} - \mathcal{L}^*)^{-1} f \tag{3.17}$$

with $I_{\mathcal{X}}$ the indentity operator on $\mathcal{X}$.

(iv) If $\mathcal{L}_k$ are positive operators and $\{f_k\}_{k\geq k_0} \subset \mathcal{X}^+$ is a bounded sequence then the unique bounded solution of the equation (3.15) satisfies $\tilde{x}_k \geq 0$ for all $k \geq k_0$.

Moreover if $\{f_k\}_{k\geq k_0} \subset Int\mathcal{X}^+$ is a bounded and uniformly positive sequence then the unique bounded solution $\{\tilde{x}_k\}_{k\geq k_0}$ of the equation (3.15) is also uniformly positive.

Proof: (i) Based on $(i) \to (ii)$ of Theorem 3.4 we deduce that for all $k \geq k_0$ the series $\{\sum_{l=k}^{j} T_{lk}^* f_l\}_{j\geq k}$ is absolutely convergent and there exists $\delta > 0$ independent of k and j such that

$$|\sum_{l=k}^{j} T_{lk}^* f_l|_1 \leq \delta. \tag{3.18}$$

Set $\tilde{x}_k = \lim_{j\to\infty} \sum_{l=k}^{j} T_{lk}^* f_l = \sum_{l=k}^{\infty} T_{lk}^* f_l$. Taking into account the definition of T_{lk}^* we obtain $\tilde{x}_k = f_k + \mathcal{L}_k^* \sum_{l=k+1}^{\infty} T_{lk+1}^* f_l = f_k + \mathcal{L}_k^* \tilde{x}_{k+1}$ which shows that $\{\tilde{x}_k\}_{k\geq k_0}$ solves (3.15).

From (3.18) it follows that $\{\tilde{x}_k\}$ is a bounded solution of (3.15). Let $\{\hat{x}_k\}_{k\geq k_0}$ be another bounded solution of the equation (3.15). For each $0 \leq k < j$ we may write

$$\hat{x}_k = T_{j+1k}^* \hat{x}_{j+1} + \sum_{l=k}^{j} T_{lk}^* f_l. \tag{3.19}$$

Since $\{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution and $\{\hat{x}_k\}_{k\geq k_0}$ is a bounded sequence we have $\lim_{j\to\infty} T^*_{j+1k}\hat{x}_{j+1} = 0$. Taking the limit for $j \to \infty$ in (3.19) we conclude that $\hat{x}_k = \sum_{l=k}^{\infty} T^*_{lk} f_l = \tilde{x}_k$ which proved the uniqueness of the bounded solution of the equation (3.15).

(ii) If $\{\mathcal{L}_k\}_{k\geq k_0}, \{f_k\}_{k\geq k_0}$ are periodic sequences with period θ then in a standard way using the representation formula (3.16) one shows that the unique bounded solution of the equation (3.15) is also periodic with period θ.

In this case we may take that $k_0 = -\infty$.

(iii) If $\mathcal{L}_k = \mathcal{L}, f_k = f$ for all k, then they may be viewed as periodic sequences with period $\theta = 1$. Based on the above result of (ii) one obtains that the unique bounded solution of the equation (3.15) is also periodic with period $\theta = 1$, so it is constant. In this case $\tilde{x}$ will verify the equation $\tilde{x} = \mathcal{L}^*\tilde{x} + f$.

Since the operator $\mathcal{L}$ generates an exponentially stable evolution it follows that its eigenvalues are located in the inside of unit disk $|\lambda| < 1$. Hence, the operator $I_{\mathcal{X}} - \mathcal{L}^*$ is invertible and one obtains that $\tilde{x}$ is given by (3.17). Finally, if $\mathcal{L}_k$ are positive operators the assertions of (iv) follow immediatelly from the representation formula (3.16) and thus the proof ends.

Remark 3.6 From the representation formula (2.16) one obtains that if the sequence $\{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution and $\{f_k\}_{k\geq k_0}$ is a bounded sequence, then all solutions of the discrete time forward affine equation (2.12) with given initial values at time $k = k_0$ are bounded on the interval $[k_0, \infty)$. On the other hand from Theorem 3.5 (i) it follows that the discrete time backward equation (2.13) has a unique bounded solution on the interval $[k_0, \infty)$ which is the solution provided by the formula (3.16).

In the case of $k_0 = -\infty$ with the same techniques as in the proof of Theorem 3.5 we may obtain a result concerning the existence and uniqueness of the bounded solution of a forward affine equation similar to the one proved for the case of backward affine equation.

Theorem 3.7 *Assume that $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ is a sequence of linear operators which generates an exponentially stable evolution on $\mathcal{X}$. Then the following assertions hold:*

(i) For each bounded sequence $\{f_k\}_{k\in Z}$ the discrete time forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + f_k \tag{3.20}$$

has a unique bounded solution $\{\hat{x}_k\}_{k\in\mathbf{Z}}$. Moreover this solution has a representation formula:

$$\hat{x}_k = \sum_{l=-\infty}^{k-1} T_{k,l+1} f_l, \quad \forall\, k \in \mathbf{Z}. \tag{3.21}$$

(ii) If $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}, \{f_k\}_{k\in\mathbf{Z}}$ are periodic sequences with period θ then the unique bounded solution of the equation (3.20) is periodic with period θ.

(iii) If $\mathcal{L}_k = \mathcal{L}, f_k = f, k \in \mathbf{Z}$ then the unique bounded solution of the equation (3.20) is constant and it is given by $\hat{x} = (I_{\mathcal{X}} - \mathcal{L})^{-1} f$.

(iv) If $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ are positive operators and if $\{f_k\}_{k\in\mathbf{Z}} \subset \mathcal{X}^+$, then the unique bounded solution of the equation (3.20) satisfies $\hat{x}_k \geq 0$ for all $k \in \mathbf{Z}$. Moreover, if $f_k \gg 0, k \in \mathbf{Z}$ then $\hat{x}_k \gg 0, k \in \mathbf{Z}$.

If $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ is a sequence of linear operators on $\mathcal{X}$ we may associate a new sequence of linear operators $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ defined as follows:

$$\mathcal{L}^{\#}{}_k = \mathcal{L}^*_{-k}.$$

Lemma 3.8 *Let $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ be a sequence of linear operators on $\mathcal{X}$. The following assertions hold:*

(i) If $T^{\#}_{kl}$ is the causal linear evolution operator on $\mathcal{X}$ defined by the sequence $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ we have

$$T^{\#}_{kl} = T^*_{-l+1,-k+1}$$

where T_{ij} is the causal linear evolution operator defined on $\mathcal{X}$ by the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$.

(ii) $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ is a sequence of positive linear operators if and only if $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ is a sequence of positive linear operators.

(iii) The sequence $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution if and only if the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution.

(iv) The sequence $\{x_k\}_{k\in\mathbf{Z}}$ is a solution of the discrete time backward affine equation (3.15) if and only if the sequence $\{y_k\}_{k\in\mathbf{Z}}$ defined by $y_k = x_{-k+1}$ is a solution of the discrete time forward equation $y_{k+1} = \mathcal{L}^{\#}_k y_k + f_{-k},\ \ k \in \mathbf{Z}$.

The proof is omitted for shortness.

The next result provide a characterization of exponential stability in terms of the existence of the bounded solution of some suitable forward affine equation.

Theorem 3.9 *Let $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ be a sequence of positive linear operators on $\mathcal{X}$. Then the following are equivalent:*

(i) The sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution.

(ii) There exists $\delta > 0$, independent of k such that

$$\sum_{l=-\infty}^{k} T_{kl}\xi_{\mathcal{X}} \leq \delta \xi_{\mathcal{X}}, \quad \forall\, k \in \mathbf{Z}.$$

(iii) The forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + \xi_{\mathcal{X}} \tag{3.22}$$

has a bounded and uniformly positive solution.

(iv) For any bounded and uniformly positive sequence $\{f_k\}_{k\in\mathbf{Z}} \subset Int\mathcal{X}^+$ the corresponding forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + f_k \tag{3.23}$$

has a bounded and uniformly positive solution.

(v) There exists a bounded and uniformly positive sequence $\{f_k\}_{k\in\mathbf{Z}} \subset Int\mathcal{X}^+$ such that the corresponding forward affine equation (3.23) has a bounded solution $\tilde{x}_k, k \in \mathbf{Z} \subset \mathcal{X}^+$.

(vi) There exists a bounded and uniformly positive sequence $\{y_k\}_{k\in\mathbf{Z}}$ which verifies $y_{k+1} - \mathcal{L}_k y_k \gg 0$.

The proof follows immediatelly combining the result proved in Theorem 3.4 and Lemma 3.8.

At the end of this section we prove some results which provide a "measure" of the robustness of the exponential stability in the case of positive linear operators. To state and prove this result some preliminary remarks are needed.

So, $\ell^\infty(\mathbf{Z}, \mathcal{X})$ stands for the real Banach space of bounded sequences of elements of $\mathcal{X}$. If $x \in \ell^\infty(\mathbf{Z}, \mathcal{X})$ we denote $|x| = \sup_{k\in\mathbf{Z}} |x_k|_1$.

Let $\ell^\infty(\mathbf{Z}, \mathcal{X}^+) \subset \ell^\infty(\mathbf{Z}, \mathcal{X})$ be the subset of bounded sequences $\{x_k\}_{k\in\mathbf{Z}} \subset \mathcal{X}^+$. It can be checked that $\ell^\infty(\mathbf{Z}, \mathcal{X}^+)$ is a solid closed normal convex cone. Therefore, $\ell^\infty(\mathbf{Z}, \mathcal{X})$ is an ordered real Banach space for which the assumptions of Theorem 2.11 in [8] are fulfilled.

Now we are in position to prove:

Theorem 3.10 *Let $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$, $\{\mathcal{G}_k\}_{k\in\mathbf{Z}}$ be sequences of positive linear operators such that $\{\mathcal{G}_k\}_{k\in\mathbf{Z}}$ is a bounded sequence. Under these conditions the following are equivalent:*

(i) The sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution and $\rho[\mathcal{T}] < 1$ where $\rho[\mathcal{T}]$ is the spectral radius of the operator $\mathcal{T} : \ell^\infty(\mathbf{Z}, \mathcal{X}) \to \ell^\infty(\mathbf{Z}, \mathcal{X})$ defined by

$$y = \mathcal{T}x, \quad y_k = \sum_{l=-\infty}^{k-1} T_{kl+1}\mathcal{G}_l x_l. \tag{3.24}$$

T_{kl} being the linear evolution operator on $\mathcal{X}$ defined by the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$.

(ii) The sequence $\{\mathcal{L}_k + \mathcal{G}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution on $\mathcal{X}$.

Proof: $(i) \to (ii)$ If the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ defines an exponentially stable evolution, then we define the sequence $\{f_k\}_{k\in\mathbf{Z}}$ by $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$

$$f_k = \sum_{l=-\infty}^{k-1} T_{kl+1}\xi_{\mathcal{X}}. \tag{3.25}$$

We have $f_k = \xi_{\mathcal{X}} + \sum_{l=-\infty}^{k-2} T_{kl+1}\xi_{\mathcal{X}}$ which leads to $f_k \geq \xi_{\mathcal{X}}$ thus $f_k \in Int\mathcal{X}^+$ for all $k \in \mathbf{Z}$. This allows us to conclude that $f = \{f_k\}_{k\in\mathbf{Z}} \in Int\ell^\infty(\mathbf{Z}, \mathcal{X}^+)$.

Applying Theorem 2.11 [8] with $R = -I_{\ell^\infty}$ and $P = \mathcal{T}$ we deduce that there exists $x = \{x_k\}_{k\in\mathbf{Z}} \in Int\ell^\infty(\mathbf{Z}, \mathcal{X}^+)$ which verifies the equation:

$$(I_{\ell^\infty} - \mathcal{T})(x) = f. \tag{3.26}$$

Here I_{ℓ^∞} stands for the identity operator on $\ell^\infty(\mathbf{Z}, \mathcal{X})$. Partitioning (3.26) and taking into account (3.24)-(3.25) we obtain that for each $k \in \mathbf{Z}$ we have :

$$x_{k+1} = \sum_{l=-\infty}^{k} T_{k+1,l+1}\mathcal{G}_l x_l + \sum_{l=-\infty}^{k} T_{k+1,l+1}\xi_{\mathcal{X}}.$$

Further we may write:

$$x_{k+1} = \mathcal{G}_k x_k + \xi_{\mathcal{X}} + \mathcal{L}_k \sum_{l=-\infty}^{k-1} T_{kl+1}\mathcal{G}_l x_l + \mathcal{L}_k \sum_{l=-\infty}^{k-1} T_{kl+1}\xi_{\mathcal{X}} = \mathcal{G}_k x_k + \xi_{\mathcal{X}} + \mathcal{L}_k x_k.$$

This shows that $\{x_k\}_{k\in\mathbf{Z}}$ verifies the equation

$$x_{k+1} = (\mathcal{L}_k + \mathcal{G}_k)x_k + \xi_{\mathcal{X}}. \tag{3.27}$$

Since $\mathcal{L}_k$ and $\mathcal{G}_k$ are positive operators and $x \geq 0$, (3.27) shows that $x_k \geq \xi_{\mathcal{X}}$. Thus we get that the equation (3.22) associated to the sum operator $\mathcal{L}_k + \mathcal{G}_k$ has a bounded and uniform positive solution. Applying implication $(iii) \to (i)$ of Theorem 3.9 we conclude that the sequence $\{\mathcal{L}_k + \mathcal{G}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution.

Now we prove the converse implication.

If (ii) holds then based on the implication $(i) \to (iii)$ of Theorem 3.9 we deduce that the equation (3.27) has a bounded and uniform positive solution $\{\tilde{x}_k\}_{k\in\mathbf{Z}} \subset Int\mathcal{X}^+$. The equation (3.27) verified by $\tilde{x}_k$ may be rewritten as:

$$\tilde{x}_{k+1} = \mathcal{L}_k\tilde{x}_k + \tilde{f}_k \tag{3.28}$$

where $\tilde{f}_k = \mathcal{G}_k\tilde{x}_k + \xi_{\mathcal{X}}, \quad k \in \mathbf{Z}, \tilde{f}_k \geq \xi_{\mathcal{X}}, \quad k \in \mathbf{Z}$. Using the implication $(v) \to (i)$ of Theorem 3.9 we deduce that the sequence $\mathcal{L}_k$ generates an exponentially stable evolution. Since the equation (3.28) has an unique bounded solution which is given by the representation formula (3.21), we have: $\tilde{x}_k = \sum_{l=-\infty}^{k-1} T_{kl+1}\tilde{f}_l$, $\forall\, k \in \mathbf{Z}$,

$$\tilde{x}_k = \sum_{l=-\infty}^{k-1} T_{kl+1}\mathcal{G}_l\tilde{x}_l + \sum_{l=-\infty}^{k-1} T_{kl+1}\xi_{\mathcal{X}}. \tag{3.29}$$

Invoking (3.24) the equality (3.29) may be written:

$$\tilde{x} = \mathcal{T}\tilde{x} + \tilde{g} \tag{3.30}$$

where $\tilde{g} = \{\tilde{g}_k\}_{k\in\mathbf{Z}}, \tilde{g}_k = \sum_{l=-\infty}^{k-1} T_{kl+1}\xi_{\mathcal{X}}$. It is obvious that $\tilde{g}_k \geq \xi_{\mathcal{X}}$ for all $k \in \mathbf{Z}$. Hence $\tilde{g} \in Int\ell^\infty(\mathbf{Z}, \mathcal{X}^+)$.

Applying implication $(v) \to (vi)$ of Theorem 2.11 in [8] for $R = -I_{\ell^\infty}$ and $P = \mathcal{T}$ one obtains that $\rho[\mathcal{T}] < 1$ and thus the proof is complete.

In the time invariant case one obtains the following version of Theorem 3.10:

Theorem 3.11 *Let $\mathcal{L} : \mathcal{X} \to \mathcal{X}, \mathcal{G} : \mathcal{X} \to \mathcal{X}$ be linear and positive operators.*

Then the following are equivalent:

(i) The eigenvalues of the operators $\mathcal{L}$ *and* $(I_{\mathcal{X}} - \mathcal{L})^{-1}\mathcal{G}$ *are located in the inside of the disk* $|\lambda| < 1$.

(ii) The eigenvalues of the sum operator $\mathcal{L}+\mathcal{G}$ *are located in the inside of the disk* $|\lambda| < 1$.

Proof: If (i) holds, then based on (iii), (iv) of Theorem 3.7 we deduce that $(I_{\mathcal{X}}-\mathcal{L})^{-1}\xi_{\mathcal{X}} \in Int\mathcal{X}^{+}$.

Applying $(vi) \to (iii)$ in Theorem 2.11 [8] for $R = -I_{\mathcal{X}}$ and $P = (I_{\mathcal{X}} - \mathcal{L})^{-1}\mathcal{G}$ one obtains that there exists $\tilde{x} \in Int\mathcal{X}^{+}$ which verifies

$$\tilde{x} = (I_{\mathcal{X}} - \mathcal{L})^{-1}\mathcal{G}\tilde{x} + (I_{\mathcal{X}} - \mathcal{L})^{-1}\xi_{\mathcal{X}},$$

which leads to $(I_{\mathcal{X}} - \mathcal{L})\tilde{x} = \mathcal{G}\tilde{x} + \xi_{\mathcal{X}}$.

Therefore we obtain that the equation

$$x_{k+1} = [\mathcal{L} + \mathcal{G}]x_k + \xi_{\mathcal{X}} \tag{3.31}$$

has a bounded and uniform positive solution $\{\tilde{x}_k\}_{k\in\mathbf{Z}}$ namely $\tilde{x}_k = \tilde{x}$ for all $k \in \mathbf{Z}$.

Applying $(iii) \to (i)$ of Theorem 3.9 one obtains that the operator $\mathcal{L} + \mathcal{G}$ generates a discrete-time exponentially stable evolution which shows that the implication $(i) \to (ii)$ is valid. Let us prove the converse implication. If (ii) holds then based on the implication $(i) \to (iii)$ of Theorem 3.9 we obtain that the equation (3.31) has a bounded and uniform positive solution, $\tilde{x}_k, k \in \mathbf{Z}$. Further, from (iii), (iv) of Theorem 3.7 we conclude that $\tilde{x}_k = \tilde{x} \in Int\mathcal{X}^{+}$, for all $k \in \mathbf{Z}$. Hence $\tilde{x} = \mathcal{L}\tilde{x} + \tilde{f}$ where $\tilde{f} = \mathcal{G}\tilde{x} + \xi_{\mathcal{X}} \in Int\mathcal{X}^{+}$.

Invoking again $(iii) \to (i)$ of Theorem 3.9 one gets that $\mathcal{L}$ generates a discrete-time exponentially stable evolution. We may write $\tilde{x} = (I_{\mathcal{X}} - \mathcal{L})^{-1}\tilde{f}$ which leads to

$$\tilde{x} = (I_{\mathcal{X}} - \mathcal{L})^{-1}\mathcal{G}\tilde{x} + (I_{\mathcal{X}} - \mathcal{L})^{-1}\xi_{\mathcal{X}}.$$

Since $(I_{\mathcal{X}} - \mathcal{L})^{-1}\xi_{\mathcal{X}} \in Int\mathcal{X}^{+}$ then from $(iv) \to (vi)$ of Theorem 2.11 in [8] we obtain that $\rho[(I_{\mathcal{X}} - \mathcal{L})^{-1}\mathcal{G}] < 1$ which ends the proof of the implication $(ii) \to (i)$ and the proof is complete.

An infinite dimensional counter part of the result proved in the Theorem 3.11 may be also obtained based on Theorem 2.11 in [8].

In a similar way with the proof of Theorem 3.10 we may prove the following result:

Theorem 3.12 *Let* $\{\mathcal{L}_k\}_{k\geq k_0}, \{\mathcal{G}_k\}_{k\geq k_0}$ *be two sequences of linear and positive operators on* $\mathcal{X}$ *such that* $\{\mathcal{G}_k\}_{k\geq k_0}$ *is a bounded sequence.*

Then the following are equivalent:

(i) The sequence $\{\mathcal{L}_k\}_{k\geq k_0}$ *generates an exponentially stable evolution and* $\rho[\mathcal{T}^a] < 1$ *where* $\mathcal{T}^a : \ell^\infty[Z_{k_0}, \mathcal{X}] \to \ell^\infty[Z_{k_0}, \mathcal{X}]$ *is defined by* $y = \mathcal{T}^a x$

$$y_k = \sum_{l=k}^{\infty} T_{lk}^{*}\mathcal{G}_l^{*}x_l, \quad k \geq k_0 \tag{3.32}$$

T_{lk} being the causal linear evolution operator defined by the sequence $\{\mathcal{L}_k\}_{k\geq k_0}, Z_{k_0}] \subset Z, Z_{k_0} = \{k \in \mathbf{Z} | k \geq k_0\}$.

(ii) The sequence $\{\mathcal{L}_k + \mathcal{G}_k\}_{k\in Z_{k_0}}$ generates an exponentially stable evolution on $\mathcal{X}$.

The proof is made combining the results of the above theorems 3.4 and 3.5 and Theorem 2.11 in [8]. It is omitted for shortness.

4 Application to the problem of mean square exponential stability

In this section we consider discrete time linear equations defined by some linear positive operators arising in connection with the problem of mean square exponential stability for a class of discrete-time linear stochastic systems. To be more specific let us consider the space $\mathcal{S}_n^N$ introduced in Example 2.3 (iii). On $\mathcal{S}_n^N$ we consider the linear operators $\mathcal{L}_k$ defined as follows: $\mathcal{L}_k X = ((\mathcal{L}_k X)(1)(\mathcal{L}_k X)(2)...(\mathcal{L}_k X)(N))$ where

$$(\mathcal{L}_k X)(i) = \sum_{j=1}^{N} p_k(j,i) A_{0k}(j) X(j) A_{0k}^T(j) + \sum_{r=1}^{N_1} \sum_{j=1}^{N} \mu_k(r) p_k(j,i) A_{rk}(j) X(j) A_{rk}^T(j) \quad (4.1)$$

for all $X = (X(1)...X(N)) \in \mathcal{S}_n^N$, where $A_{rk}(j) \in \mathbf{R}^{n\times n}$, $0 \leq r \leq N_1, 1 \leq j \leq N, k \geq 0$ and $\mu_k(r)$ and $p_k(j,i)$ are nonnegative scalars. It is clear that $\mathcal{L}_k$ is a positive operator.

If the scalars $p_k(j,i)$ have the additional property:

$$\sum_{i=1}^{N} p_k(j,i) = 1, \quad 1 \leq j \leq N, \quad k \geq 0 \quad (4.2)$$

then the operators (4.1) are associated to the discrete-time linear stochastic equations of the form:

$$x_{k+1} = [A_{0k}(\eta_k) + \sum_{r=1}^{N_1} A_{rk}(\eta_k) w_k(r)] x_k, \quad k \geq 0 \quad (4.3)$$

where $\{w_k(r)\}_{k\geq 0}, 1 \leq r \leq N_1$ are sequences of zero mean square integrable random variables on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ having the additional properties:

1. if $w_k = (w_k(1) w_k(2) ... w_k(N_1))^T$ then $\{w_k\}_{k\geq 0}$ is a sequence of independent random vectors,

2. $$E[w_k w_k^T] = diag[\mu_k(1)\mu_k(2)...\mu_k(N_1)].$$

The sequence $\{\eta_k\}_{k\geq 0}$ is a Markov chain with the state space the finite set $\mathcal{N} = \{1, 2, ..., N\}$ and $\{P_k\}_{k\geq 0}, P_k = \{p_k(i,j)\}$ $i, j \in \mathcal{N}$ the sequence of transition probability matrices, that is $\mathcal{P}\{\eta_{k+1} = i | \eta_0, \eta_1, ..., \eta_k\} = p_k(\eta_k, i)$ a.s. For details see [10].

Assume that the stochastic processes $\{w_k\}_{k\geq 0}$ and $\{\eta_k\}_{k\geq 0}$ are independent.

If $P_k = P$ then $\{\eta_k\}_{k\geq 0}$ is called homogeneous Markov chain.

Two important cases of linear stochastic systems of type (4.3) were intensively investigated in the literature, namely for $N = 1$ or for $A_{rk}(i) = 0, 1 \leq r \leq N_1, k \geq 0, i \in \mathcal{N}$.

In the case $N = 1$, (4.3) becomes:

$$x_{k+1} = [A_{0k} + \sum_{r=1}^{N_1} A_{rk} w_k(r)] x_k. \tag{4.4}$$

The exponential stability in mean square for the systems of type (4.4) was investigated in [3, 27, 28, 29, 34].

In the case $A_{rk}(i) = 0,\ 1 \leq r \leq N_1,\ k \geq 0,\ i \in \mathcal{N}$ the system (4.3) reduces to

$$x_{k+1} = A_{0k}(\eta_k) x_k \tag{4.5}$$

which was studied in [4, 5, 12, 13, 14, 18, 21, 23, 24, 25, 26] and references therein.

Setting $A_k = A_{0k}(\eta_k) + \sum_{r=1}^{N_1} A_{rk}(\eta_k) w_k(r)$ the system (4.3) may be written in a compact form as:

$$x_{k+1} = A_k x_k.$$

For each $k \geq l \geq 0$ define Φ_{kl} by $\Phi_{kl} = A_{k-1} A_{k-2} \ldots A_l$ if $k > l$ and $\Phi_{kl} = I_n$ if $k = l$, Φ_{kl} is the fundamental matrix solution of the system (4.3).

If $\{x_k\}_{k\geq 0}$ is a solution of the equation (4.3) then we have $x_k = \Phi_{kl} x_l,\quad k \geq l \geq 0$. The next result provided the relationship between the evolution defined on $\mathcal{S}_n^N$ by the operator $\mathcal{L}_k$ introduced by (4.1) and the evolution defined by the equation (4.3).

Theorem 4.1 *If $T_{kl}, k \geq l \geq 0$ is the causal evolution operator on $\mathcal{S}_n^N$ defined by the sequence $\{\mathcal{L}_k\}_{k\geq 0}$ defined by (4.1) and (4.2), then :*

$$[T_{kl}^* X](i) = E[\Phi_{kl}^T X(\eta_k) \Phi_{kl} | \eta_l = i] \tag{4.6}$$

for all $X = (X(1)X(2)\ldots X(N)) \in \mathcal{S}_n^N, k \geq l \geq 0, i \in \mathcal{N}$, such that $\mathcal{P}\{\eta_l = i\} > 0, E[\cdot|\eta_l = i]$ stands for the conditional expectation with respect to the event $\{\eta_l = i\}$.

A proof of the result stated in the above theorem in particular case when system (4.3) reduces to system (4.4) may be found in [28], while if the system (4.3) reduces to the system (4.5) the equality (4.6) was proved in [26].

A complete proof of the Theorem 4.1 in the general case of systems (4.3) will be given in an acompany paper which deals with the problem of the exponential stability in mean square.

To avoid some inconvenience due to the presence of Markov chain in the matrix coefficients of the system we assume that the following property holds.

$\mathbf{P_3}$) The Markov chain $\{\eta_k\}_{k\geq 0}$ has the property

$$\mathcal{P}\{\eta_k = i\} > 0 \tag{4.7}$$

for all $i \in \mathcal{N}$ and $k \geq 0$.

It can be checked inductively that (4.7) is fulfilled if for each $k \geq 0$ and $1 \leq j \leq N$ there exists $i \in \mathcal{N}$ such that $p_k(i,j) > 0$ and $\mathcal{P}\{\eta_0 = l\} > 0, l \in \mathcal{N}$.

Definition 4.2 We say that the zero solution of the system 4.3 is exponentially stable in mean square (ESMS) if there exist $\beta > 0, q \in (0,1)$ such that

$$E[|\Phi_{kl}x|^2|\eta_l = i] \leq \beta q^{k-l}|x|^2 \tag{4.8}$$

for all $k \geq l \geq 0,\ i \in \mathcal{N},\ x \in \mathbf{R}^n$.

Applying (4.6) for $X = J = (I_n...I_n)$ one obtains:

$$x^T[T_{kl}J](i)x = E[|\Phi_{kl}x|^2|\eta_l = i] \tag{4.9}$$

for all $k \geq l \geq 0,\ i \in \mathcal{N}$ and $x \in \mathbf{R}^n$.

Thus we obtain:

Corollary 4.3 *Under the considered assumptions the following are equivalent:*

(i) The zero solution of the equation (4.3) is (ESMS).

(ii) The sequence $\{\mathcal{L}_k\}_{k\geq 0}$ defined by (4.1) and (4.2) generates an exponentially stable evolution on $\mathcal{S}_n^N$.

It must be remarked that the discrete time linear equations defined by the operators $\mathcal{L}_k$ introduced by (4.1) offer a deterministic framework which allow us to obtain informations concerning the exponential stability for the equations (4.3) which are probabilistic objects. The theorems proved in the previous section for the linear positive operators provide necessary and sufficient conditions for the mean square exponential stability of the zero solution of the equation (4.3). If the stochastic system is in one of the particular form (4.4) or (4.5) respectively, the results proved in Section 3 recover some results proved in [4, 5, 12, 13, 14, 18, 21, 23, 24, 25, 26, 27, 28, 29]. If the system (4.3) reduces to (4.4) then the corresponding operator (4.1) becomes:

$$\hat{\mathcal{L}}_k Y = A_{0k} Y A_{0k}^T + \sum_{r=1}^{N_1} \mu_k(r) A_{rk} Y A_{rk}^T \tag{4.10}$$

for all $Y \in \mathcal{S}_n$. If the system (4.3) reduces to (4.5) then (4.1) reduces to

$$(\check{\mathcal{L}}_k X)(i) = \sum_{j=1}^{N} p_k(j,i) A_{0k}(j) X(j) A_{0k}^T(j) \tag{4.11}$$

for all $X = (X(1)...X(N)) \in \mathcal{S}_n^N, i \in \mathcal{N}$.

For the readers convenience we provide the formulae of the adjoint operators corresponding to (4.1), (4.10) and (4.11).

These formulae may be deduced in a standard way taking into account the definition of the inner product on $\mathcal{S}_n^N$ and $\mathcal{S}_n$, respectively.

We have $\mathcal{L}_k^* X = (\mathcal{L}_k^* X(1) \ \dots \ \mathcal{L}_k^* X(N))$ where

$$\mathcal{L}_k^* X(i) = A_{0k}^T(i)(\sum_{j=1}^{N} p_k(i,j)X(j))A_{0k}(i) + \sum_{r=1}^{N_1} \mu_k(r) A_{rk}^T(i)(\sum_{j=1}^{N} p_k(i,j)X(j))A_{rk}(i), \quad (4.12)$$

$1 \leq i \leq N$

$$\hat{\mathcal{L}}_k^* Y = A_{0k}^T Y A_{0k} + \sum_{r=1}^{N_1} \mu_k(r) A_{rk}^T Y A_{rk} \quad (4.13)$$

for all $Y \in \mathcal{S}_n$, $\check{\mathcal{L}}_k^* X = (\check{\mathcal{L}}_k^* X(1) \ \dots \ \check{\mathcal{L}}_k^* X(N))$ where

$$\check{\mathcal{L}}_k^* X(i) = A_{0k}^T(i)(\sum_{j=1}^{N} p_k(i,j)X(j))A_{0k}(i), \ 1 \leq i \leq N, \ X = (X(1) \dots X(N)) \in \mathcal{S}_n^N. \quad (4.14)$$

Using the formulae (4.1), (4.10), (4.11) or (4.12)-(4.14), respectively, we may rewrite the equations arising in Theorem 3.2, Theorem 3.4 in order to provide necessary and sufficient conditions for mean square exponential stability of the zero solution of the systems (4.3), (4.4) or (4.5), respectively.

These results show that the mean square exponential stability of the zero solution of the systems (4.3) and (4.5), respectively, does not depend upon the initial distribution of the Markov chain, it depending only by the sequences $\{A_{rk}(i)\}_{k\geq 0}, \{p_k(i,j)\}_{k\geq 0}, \{\mu_k(r)\}_{k\geq 0}$.

The equation (4.3) may be view as a perturbation of the equation (4.5). In the same time equation (4.4) may be view as a perturbation of the deterministic equation

$$x_{k+1} = A_{0k} x_k, \quad k \geq 0. \quad (4.15)$$

The results of Theorem 3.11 and Theorem 3.12 allow us to obtain conditions which guarantee the preservation of the exponential stability of the zero solution of the perturbed equations (4.3) and (4.4) if the zero solution of the unperturbed equations (4.5) and (4.15), respectively are exponentially stable.

We recall that if $M \in \mathbf{R}^{n\times n}$ is a given matrix then the corresponding discrete-time Liapunov operator or Stein operator is defined as: $\mathcal{L}_M : \mathcal{S}_n \to \mathcal{S}_n, \ \mathcal{L}_M Y = MYM^T$.

From Theorem 3.11 one obtains:

Corollary 4.4 *a) Assume that the system (4.4) is time invariant that is* $A_{rk} = A_r, \mu_k(r) = \mu(r)$ *for all* $k \geq 0, 0 \leq r \leq N_1$.

Then the following are equivalent:

(i) The eigenvalues of the matrix A_0 *and the eigenvalues of the operator* $(I_{\mathcal{S}_n} - \mathcal{L}_{A_0})^{-1}(\hat{\mathcal{L}} - \mathcal{L}_{A_0})$ *are located in the inside of the disk* $|\lambda| < 1$.

(ii) The zero solution of the discrete-time stochastic equation (4.4) is ESMS.

b) Assume that the system (4.3) is time invariant, that is $A_{rk} = A_r, \mu_k(r) = \mu(r), p_k(i,j) = p(i,j)$ *for all* $k \geq 0, 0 \leq r \leq N_1, i,j \in \{1,2,\dots,N\}$. *Let* $\mathcal{L}, \check{\mathcal{L}}$ *be the corresponding operators defined by (4.1) and (4.11), respectively.*

The following are equivalent:

(j) The zero solution of the equation (4.5) is ESMS and the eigenvalues of the operator $(I_{\mathcal{S}_n^N} - \check{\mathcal{L}})^{-1}(\mathcal{L} - \check{\mathcal{L}})$ *are located in the inside of the disk* $|\lambda| < 1$.

(jj) The zero solution of the equation (4.3) is ESMS.

Similar results may be obtained in the time varying case based on Theorem 3.12, but they are omitted for shortness.

It must be remarked that the operators which are involved in Corollary 4.4 are working on finite dimensional linear spaces. Therefore their eigenvalues may be computed since they are the eigenvalues of the corresponding matrices, with respect to the canonical basis of the considered linear space.

In the last part of this section we shall prove a necessary and sufficient condition for the exponential stability of the evolution generated by the operators (4.1).

Such condition may not be directly derived from the result proved in Section 3. In order to state that result we need to introduce the concept of detectability.

Definition 4.5 Let $\{\mathcal{L}_k\}_{k\geq 0}$ be a sequence of operators of type (4.1) and $\{C_k\}_{k\geq 0}$ be such that $C_k = (C_k(1)...C_k(N)), C_k(i) \in \mathbf{R}^{p\times n}$. We say that the pair $(C_k, \mathcal{L}_k)$ is detectable if there exist a bounded sequence $\{H_k\}_{k\geq 0}$ where $H_k = (H_k(1)...H_k(N)), H_k(i) \in \mathbf{R}^{n\times p}$ such that the sequence $\{\mathcal{L}_k^H\}_{k\geq 0}$ generates an exponentially stable evolution, where $\mathcal{L}_k^H$ is defined by $\mathcal{L}_k^H X = (\mathcal{L}_k^H X(1) \ \ldots \ \mathcal{L}_k^H X(N))$ with

$$\mathcal{L}_k^H X(i) = \sum_{j=1}^{N} p_k(j,i)[A_{0k}(j) + H_k(j)C_k(j)]X(j)[A_{0k}(j) + H_k(j)C_k(j)]^T + \sum_{r=1}^{N_1} \mu_k(r) \sum_{j=1}^{N} p_k(j,i) A_{rk}(j) X(j) A_{rk}^T(j) \qquad (4.16)$$

for all $X = (X(1) \ \ldots \ X(N)) \in \mathcal{S}_n^N$.

The sequence $\{H_k\}_{k\geq 0}$ involved in the above definition will be called stabilizing injection. If the sequences $\{\mathcal{L}_k\}_{k\geq 0}$, $\{C_k\}_{k\geq 0}$ are periodic with period θ then the definition of the detectability is restricted to the stabilizing injections which are periodic sequences with the same period θ. Moreover if $\mathcal{L}_k = \mathcal{L}, C_k = C$ then the definition of detectability is restricted to the constant stabilizing injection.

A possible motivation of the above definition of detectability is given by its relation with the concept of stochastic detectability.

We recall:

Definition 4.6 We say that the system (4.3) together with the output $y_k = C_k(\eta_k)x_k$ is stochastically detectable if there exists a bounded sequence $\{H_k\}_{k\geq 0}$, $H_k = (H_k(1) \ \ldots \ H_k(N))$,

$H_k(i) \in \mathbf{R}^{n\times p}$ such that the zero solution of the discrete-time stochastic equation:

$$x_{k+1} = [A_{0k}(\eta_k) + H_k(\eta_k)C_k(\eta_k) + \sum_{r=1}^{N_1} A_{rk}(\eta_k)w_k(r)]x_k \tag{4.17}$$

is ESMS.

Fron Definition 4.5, Definition 4.6 and Corollary 4.3 we obtain:

Corollary 4.7 *Assume that the scalars $p_k(i,j)$ satisfy the additional condition (4.2). Then the following are equivalent:*

(i) The system (4.3) together with the output $y_k = C_k(\eta_k)x_k$ is stochastically detectable.

(ii) The pair $(C_k, \mathcal{L}_k)$ is detectable, where $C_k = (C_k(1) \ldots C_k(N))$.

We mention that Definition 4.5 is done without condition (4.2). This condition is needed when we want to specify that the operators $\mathcal{L}_k$ correspond to a discrete-time linear stochastic system.

The result proved below hold without condition (4.2).

Theorem 4.8 *Let $\{\mathcal{L}_k\}_{k\geq 0}$ be a sequence defined by (4.1) with additional property that $\{p_k(i,j)\}_{k\geq 0}$ and $\{A_{0k}\}_{k\geq 0}$ are bounded sequences.*

Consider the discrete-time backward affine equation

$$Y_k = \mathcal{L}_k^* Y_{k+1} + \tilde{C}_k, \quad k \geq 0 \tag{4.18}$$

where $\tilde{C}_k = (\tilde{C}_k(1) \quad \tilde{C}_k(2) \quad \ldots \quad \tilde{C}_k(N)), \tilde{C}_k(i) = C_k^T C_k$. Assume that $\{C_k(i)\}_{k\geq 0}$ are bounded sequences and the pair $(C_k, \mathcal{L}_k)$ is detectable.

Under these conditions the following are equivalent:

(i) the sequence $\{\mathcal{L}_k\}_{k\geq 0}$ generates an exponentially stable evolution,

(ii) the equation (4.18) has a bounded solution $\{\tilde{Y}_k\}_{k\geq 0} \subset \mathcal{S}_n^{N,+}$.

Proof. The implication $(i) \to (ii)$ follows imediately from Theorem 3.5 (iv).

It remains to prove the converse implication.

Let $\{X_k\}_{k\geq k_0}$ be a solution of the problem with given initial values:

$$X_{k+1} = \mathcal{L}_k X_k, \quad k \geq k_0 \tag{4.19}$$

$$X_{k_0} = H, \quad H \in \mathcal{S}_n^{N,+}. \tag{4.20}$$

We show that there exists $\gamma > 0$ not depending upon k_0 and H such that

$$\sum_{k=k_0}^{\infty} |X_k|_1 \leq \gamma |H|_1 \tag{4.21}$$

for all $k_0 \geq 0, H \in \mathcal{S}_n^{N,+}$.

Let $\{H_k\}_{k\geq 0}$ be a stabilizing injection. This means that there exist $\beta_1 > 0, q_1 \in (0,1)$ such that

$$||T_{kl}^H||_1 \leq \beta_1 q_1^{k-l}$$

for all $k \geq l \geq 0$, T_{kl}^H being the causal linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $\{\mathcal{L}_k^H\}_{k\geq 0}$ where $\mathcal{L}_k^H$ is defined as in (4.16).

The equation (4.19) may be rewritten as:

$$X_{k+1} = \mathcal{L}_k^H X_k + \mathcal{G}_k X_k \tag{4.22}$$

where $\mathcal{G}_k X_k = (\mathcal{G}_k X_k(1) \ \ldots \ \mathcal{G}_k X_k(N))$,

$$\mathcal{G}_k X_k(i) = -\sum_{j=1}^{N} p_k(j,i)[H_k(j)C_k(j)X_k(j)A_{0k}^T(j) +$$

$$A_{0k}(j)X_k(j)C_k^T(j)H_k^T(j) + H_k(j)C_k(j)X_k(j)C_k^T(j)H_k^T(j)].$$

Further we define the perturbed operators

$$\mathcal{L}_k^\varepsilon = \mathcal{L}_k^H + \varepsilon^2 \hat{\mathcal{G}}_k \tag{4.23}$$

where $\hat{\mathcal{G}}_k X = (\hat{\mathcal{G}}_k X(1) \ \ldots \hat{\mathcal{G}}_k(N))$ with

$$\hat{\mathcal{G}}_k X(i) = \sum_{j=1}^{N} p_k(j,i)A_{0k}(j)X(j)A_{0k}^T(j)$$

for all $X = (X(1) \ \ldots \ X(N)) \in \mathcal{S}_n^N$.

If $q \in (q_1, 1)$ one shows in a standard way using discrete-time version of Belman-Gronwall Lemma that there exists $\varepsilon_0 > 0$ such that

$$||T_{kl}^\varepsilon||_1 \leq \beta q^{k-l}, \tag{4.24}$$

for all $k \geq l \geq 0, 0 < \varepsilon \leq \varepsilon_0$, T_{kl}^ε being the causal linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $(\mathcal{L}_k^\varepsilon)_{k\geq 0}$.

Let $\varepsilon \in (0, \varepsilon_0)$ be fixed and $\{Z_k\}_{k\geq k_0}$ be the solution of the problem with given initial condition:

$$Z_{k+1} = \mathcal{L}_k^\varepsilon Z_k + \frac{1}{\varepsilon^2}\Psi_k, \qquad Z_{k_0} = H \tag{4.25}$$

where $\Psi_k = (\Psi_k(1) \ \ldots \ \Psi_k(N))$,

$$\Psi_k(i) = \sum_{j=1}^{N} p_k(j,i)H_k(j)C_k(j)X_k(j)C_k^T(j)H_k^T(j). \tag{4.26}$$

If we set $\tilde{Z}_k = Z_k - X_k$ then by direct calculations based on (4.22) and (4.25) one obtains that $\tilde{Z}_k$ solves:

$$\tilde{Z}_{k+1} = \mathcal{L}_k^\varepsilon \tilde{Z}_k + \tilde{\Psi}_k, \qquad \tilde{Z}_{k_0} = 0 \tag{4.27}$$

where $\tilde{\Psi}_k = (\tilde{\Psi}_k(1) \ ... \ \tilde{\Psi}_k(N))$,

$$\tilde{\Psi}_k(i) = \sum_{j=1}^{N} p_k(j,i)(\varepsilon A_{0k}(j) + \frac{1}{\varepsilon} H_k(j) C_k(j)) X_k(j) (\varepsilon A_{0k}(j)$$

$$+\frac{1}{\varepsilon} H_k(j) C_k(j))^T + \sum_{j=1}^{N} p_k(j,i) H_k(j) C_k(j) X_k(j) C_k^T(j) H_k^T(j).$$

Since the solution of (4.19) is in $\mathcal{S}_n^{N,+}$ it follows that $\tilde{\Psi}_k(i) \geq 0$ for all $k \geq k_0$ and $i \in \mathcal{N}$, that is $\tilde{\Psi}_k \in \mathcal{S}_n^{N,+}$.

Since $\mathcal{L}_k^\varepsilon$ are positive operators, then one obtains inductively based on (4.27) that $\tilde{Z}_k \geq 0$ for all $k \geq k_0$, which is equivalent to $X_k \leq Z_k$ for all $k \geq k_0$.

The last inequality allows us to write

$$|X_k|_1 \leq |Z_k|_1, \quad k \geq k_0. \tag{4.28}$$

From (4.25) we obtain the representation formula

$$Z_k = T_{kk_0}^\varepsilon H + \frac{1}{\varepsilon^2} \sum_{l=k_0}^{k-1} T_{k,l+1}^\varepsilon \Psi_l, \qquad k \geq k_0 + 1.$$

Based on (4.24) we get:

$$|Z_k|_1 \leq \beta q^{k-k_0} |H|_1 + \frac{\beta}{\varepsilon^2} \sum_{l=k_0}^{k-1} q^{k-l-1} |\Psi_l|_1. \tag{4.29}$$

Taking into account the definition of the norm $|\dot{|}_1$ on $\mathcal{S}_n^N$ one obtains (see also (4.26)):

$$|\Psi_l|_1 = \max_{i \in \mathcal{N}} |\Psi_l(i)| \leq \max_{i \in \mathcal{N}} \sum_{j=1}^{N} p_k(j,i) |H_l(j) C_l(j) X_l(j) C_l^T(j) H_l^T(j)|$$

which leads to

$$|\Psi_l|_1 \leq \rho_1 \rho_2 \sum_{j=1}^{N} |C_l(j) X_l(j) C_l(j)^T| \tag{4.30}$$

where $\rho_1 \geq p_l(j,i), \rho_2 \geq |H_l(j)|^2$ for all $l \geq 0, i, j \in \mathcal{N}$.

Since $|C_l(j) X_l(j) C_l^T(j)| = \lambda_{max}[C_l(j) X_l(j) C_l^T(j)]$ we may write

$$|\Psi_l|_1 \leq \rho_1 \rho_2 \sum_{j=1}^{N} Tr(C_l(j) X_l(j) C_l^T(j)) = \rho_1 \rho_2 \sum_{j=1}^{N} Tr(C_l^T(j) C_l(j) X_l(j)).$$

In view of definition of inner product on $\mathcal{S}_n^N$ we get:

$$|\Psi_l|_1 \leq \rho_1 \rho_2 \langle \tilde{C}_l, X_l \rangle. \tag{4.31}$$

Based on equation (4.18) verified by $\{\tilde{Y}_l\}_{l\geq 0}$ we may write

$$\langle \tilde{C}_l, X_l\rangle = \langle \tilde{Y}_l, X_l\rangle - \langle \mathcal{L}_l^* \tilde{Y}_{l+1}, X_l\rangle = \langle \tilde{Y}_l, X_l\rangle - \langle \tilde{Y}_{l+1}, X_{l+1}\rangle. \tag{4.32}$$

Since $(\tilde{Y}_l)_{l\geq 0}$ is a bounded sequence and $\langle \tilde{Y}_i, X_i\rangle \geq 0$ for arbitrary $i \geq 0$, we obtain from (4.31) and (4.32) that

$$\sum_{l=k_0}^{k_1} |\Psi_l|_1 \leq \rho_3 |H|_1, \qquad (\forall)\ k_1 > k_0 \tag{4.33}$$

with $\rho_3 > 0$ independent of l and H.

Using (4.29) we may write:

$$\sum_{k=k_0}^{k_1} |Z_k|_1 = |H|_1 + \sum_{k=k_0+1}^{k_2} |Z_k|_1 \leq$$

$$(1+\beta \sum_{k=k_0+1}^{k_2} q^{k-k_0})|H|_1 + \frac{\beta}{\varepsilon^2} \sum_{k=k_0+1}^{k_2} \sum_{l=k_0}^{k_1} q^{k-l-1} |\Psi_l|_1.$$

Changing the order of summation and taking into account (4.33) we obtain finally

$$\sum_{k=k_0}^{k_2} |Z_k|_1 \leq \gamma |H|_1, \qquad \forall\ k_2 > k_0$$

and $\gamma = 1 + \frac{\beta q}{1-q} + \frac{\beta\varepsilon^{-2}\rho_3}{1-q}$ does not depend upon k_0, k_2, H.

Taking the limit for $k_2 \to \infty$ one gets:

$$\sum_{k=k_0}^{\infty} |Z_k|_1 \leq \gamma |H|_1.$$

Invoking (4.28) we conclude that (4.21) is valid. Taking $H = J = (I_n\ I_n\ ...\ I_n)$, (4.21) becomes $\sum_{k=k_0}^{\infty} |T_{kk_0} J|_1 \leq \gamma$ for all $k_0 \geq 0$, or equivalently

$$\sum_{k=k_0}^{\infty} \|T_{kk_0}\|_1 \leq \gamma. \tag{4.34}$$

Based on (2.4), (4.34) leads to $\sum_{k=k_0}^{\infty} \|T_{kk_0}^*\|_1 \leq \gamma_1$ for all $k \geq k_0$, $\gamma_1 > 0$ being independent of k_0.

Since $T_{kk_0}^* J \leq \|T_{kk_0}^*\|_1 J$ one obtains $0 \leq \sum_{k=k_0}^{\infty} T_{kk_0}^* J \leq \delta J$.

Aplying now the implication $(iii) \to (i)$ of Theorem 3.4 we conclude that the sequence $\{\mathcal{L}_k\}_{k\geq 0}$ generates an exponentially stable evolution and thus the proof ends.

The result proved in the above theorem may be view as an alternative of the equivalence $(i) \leftrightarrow (vi)$ of Theorem 3.4 for the case when the forced term of the corresponding equation (3.5) is not uniform positive.

The lost of the uniform positivity is supplied by the detectability property. The continuous-time time-invariant version of the result proved in Theorem 4.8 may be found in [15] Lemma 3.2, while the continuous-time time-varying counterpart of this result may be found in [11].

Such result is sometimes useful to derive the existence of stabilizing solutions for generalized Riccati equations.

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