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by

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RANK ONE MAXIMAL COHEN-MACAULAY MODULES OVER SINGULARITIES OF TYPE $Y_1^3+Y_2^3+Y_3^3+Y_4^3$

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Abstract. We describe, by matrix factorizations, the rank one graded maximal Cohen-Macaulay modules over the hypersurface $Y_1^3+Y_2^3+Y_3^3+Y_4^3$.

Key words: hypersurface ring, maximal Cohen-Macaulay modules, Ulrich modules

2000 Mathematics Subject Classification: 13C14,13H10, 13P10, 14J60

1. Introduction

Let R be a hypersurface ring, that is $R=S/(f)$ for a regular local ring (S,m) and $0\neq f\in m$. After Eisenbud [10], any maximal Cohen-Macaulay module has a minimal free resolution of periodicity 2 which is completely given by a matrix factorization (ϕ,ψ), ϕ,ψ being square matrices over S such that $\phi\psi=\psi\phi=fI_n$, for a certain positive integer n. So, in order to describe the maximal Cohen-Macaulay R-modules, it is enough to describe their matrix factorizations (this we did for instance in [11] in order to describe the maximal Cohen-Macaulay modules over singularities of type X^t+Y^3). A different approach was used by Cipu, Herzog and Popescu in [7] to describe generalized Cohen-Macaulay modules (see also [5] or [8]). A powerful method seems to be also the lifting theory in the sense of Auslander-Ding-Solberg [2], which was used in [18] in order to complete Knörrer Periodicity Theorem [15] in char $p>0$ (see also [19], [6]).

Let $R_n:=K[Y_1,\ldots,Y_n]/(f_n)$, where $f_n=Y_1^3+Y_2^3+\ldots+Y_n^3$ and K is an algebraically closed field of characteristic 0. Using the classification of vector bundles over elliptic curves obtained by Atiyah [1], C. Kahn gives a "geometrically" de-

scription of the graded maximal Cohen-Macaulay (briefly MCM) modules over R_3 and also describe the Auslander-Reiten quivers of MCM over R_3 [14]. His method does not give the matrix factorizations of the indecomposable MCM R_3-modules. In a recent paper [17], Laza, Pfister and Popescu use Atiyah classification to describe the matrix factorizations of the graded, indecomposable, reflexive modules over R_3. They give canonical normal forms for the matrix factorizations of these modules of rank one and show how one may obtain the modules of rank ≥ 2 using SINGULAR. Since over the completion $K[[Y_1,Y_2,Y_3]]/(f_3)$ of R_3, every reflexive module is gradable (see [20]), the authors obtain a description of MCM-modules over $K[[Y_1,Y_2,Y_3]]/(f_3)$.
Now we consider $n=4$. In this case we do not have the support of Atiyah classification used in the previous one, but we may give the matrix factorizations for the rank one indecomposable MCM modules over R_4.
Let M be a MCM module over R_4 and let $\mu(M)$ be the minimal number of generators of M. By Corollary 1.3 of [13], we obtain that $\mu(M) \in \{2,3\}$. We shall prove that there exists a finite number of indecomposable MCM modules of rank one over R_4. We note that, by [17], there exists infinitely many indecomposable MCM modules of rank one over R_3. In [13], Bruns showed that if M is a MCM module over a hypersurface ring, then rank $M \geq (\dim R-1)/2$. This implies that there are no rank one MCM modules over R_n, for $n \geq 5$.

2. Rank one MCM modules over R_4 with two generators

For every $a,b \in K$ with $a^3=b^3=-1$ and for every permutation $(i\ j\ s)$ of the set $\{2,3,4\}$ with $i<j$ we denote:

$$\varphi_{ij}(a,b) = \begin{pmatrix} Y_1 - aY_s & -(Y_i^2+bY_iY_j+b^2Y_j^2) \\ Y_i - bY_j & Y_1^2+aY_1Y_s+a^2Y_s^2 \end{pmatrix}$$

and

$$\psi_{ij}(a,b) = \begin{pmatrix} Y_1^2+aY_1Y_s+a^2Y_s^2 & (Y_i^2+bY_iY_j+b^2Y_j^2) \\ -(Y_i-bY_j) & Y_1-aY_s \end{pmatrix}$$

THEOREM 2.1.

$(\varphi_{ij}(a,b),\psi_{ij}(a,b))$ is a matrix factorization for all $a,b \in K$ with $a^3=b^3=-1$ and $i,j \in \{2,3,4\}$ with $i<j$. The sets of graded MCM modules

$$\mathcal{M} = \{Coker\ \varphi_{ij}(a,b) | a,b,i,j\}$$

and

$$\mathcal{N} = \{Coker\ \psi_{ij}(a,b) | a,b,i,j\}$$

have the following properties:

(i) *Every two generated, non free, graded MCM module is isomorphic with one of the modules of $\mathcal{M} \cup \mathcal{N}$.*

(ii) *Every two different graded MCM modules from $\mathcal{M} \cup \mathcal{N}$ are not isomorphic.*

(iii) *The modules of $\mathcal{M}$ are the syzygies and also the duals of the modules from $\mathcal{N}$.*

(iv) *The modules of $\mathcal{M} \cup \mathcal{N}$ are all of rank one.*

Proof. (i) Obviously $(\varphi_{ij}(a,b), \psi_{ij}(a,b))$ is a matrix factorization. Now let (φ, ψ) be a reduced 2×2-matrix factorization of f_4 over $K[Y_1, Y_2, Y_3, Y_4]$ with homogeneous entries. Then $\det\varphi \det\psi = f_4^2$ and, since f_4 is irreducible, we have $\det\varphi = \det\psi = f_4$, after multiplication of a row of φ and ψ with some elements from K^*. The matrix ψ is the adjoint of φ, so it suffices to find φ such that $\det\varphi = f_4$. After elementary transformations we may suppose that the entries of the first column of φ are linear forms which must be linear independent since f_4 is irreducible. So, applying some elementary transformations on the matrix φ, we may suppose that the entries of the first column of φ are of the form:

$$\varphi_{11} = Y_1 - a_{i_1} Y_{i_1} - a_{i_2} Y_{i_2}$$

and

$$\varphi_{21} = Y_i - b_{i_1} Y_{i_1} - b_{i_2} Y_{i_2}$$

for some $a_{i_1}, a_{i_2}, b_{i_1}, b_{i_2} \in K$, $\{i, i_1, i_2\} = \{2,3,4\}$ and that the second column of φ has the entries homogeneous forms of degree 2. Since $\det\varphi = f_4$ we have that

$$f(a_{i_1} Y_{i_1} + a_{i_2} Y_{i_2}, b_{i_1} Y_{i_1} + b_{i_2} Y_{i_2}, Y_{i_1}, Y_{i_2}) = 0.$$

This implies that $a_{i_1}, a_{i_2}, b_{i_1}, b_{i_2}$ satisfy the following identities:

$$\begin{aligned} a_{i_1}^3 + b_{i_1}^3 + 1 &= 0, \\ a_{i_2}^3 + b_{i_2}^3 + 1 &= 0, \\ a_{i_1}^2 a_{i_2} + b_{i_1}^2 b_{i_2} &= 0, \\ a_{i_1} a_{i_2}^2 + b_{i_1} b_{i_2}^2 &= 0. \end{aligned}$$

If $b_{i_1} b_{i_2} \neq 0$, then $a_{i_1} a_{i_2} \neq 0$ and, from the last two equations, it results

$$(\frac{a_{i_1}}{b_{i_1}})^2 = -\frac{b_{i_2}}{a_{i_2}}$$

and

$$(\frac{a_{i_2}}{b_{i_2}})^2 = -\frac{b_{i_1}}{a_{i_1}},$$

so

$$(\frac{a_{i_1}}{b_{i_1}})^3 = -1,$$

which contradicts the first identity. Thus $b_{i_1}b_{i_2}=0$ and $a_{i_1}a_{i_2}=0$. We may suppose $b_{i_1}=0$. It results:

$$a_{i_1}^3=-1,\ a_{i_2}=0,\ b_{i_2}^3=-1.$$

We have obtained that

$$\varphi_{11}=Y_1-aY_s$$

and

$$\varphi_{21}=Y_i-bY_j,$$

where $a,b\in K$, $a^3=b^3=-1$, and $(i\ j\ s)$ is a permutation of the set $\{2,3,4\}$. It is clear that we may transform the matrix such that $i<j$. Let

$$\varphi=\begin{pmatrix} Y_1-aY_s & \gamma' \\ Y_i-bY_j & \delta' \end{pmatrix}$$

where γ',δ' are homogeneous forms of degree 2. Then we obtain that φ and $\varphi_{ij}(a,b)$ define the same MCM module as in ([16], Prop. 1.1).

(ii) It is clear that no module of $\mathcal{M}$ is isomorphic with one of $\mathcal{N}$. The first Fitting ideal of $\varphi_{ij}(a,b)$ is $\mathrm{Fitt}_1(\varphi_{ij}(a,b))=(Y_1-aY_s,Y_i-bY_j,Y_s^2,Y_j^2)$. Suppose that $\varphi_{ij}(a,b)$ and $\varphi_{uv}(a',b')$ define the same MCM module of $\mathcal{M}$. Then

$$\mathrm{Fitt}_1(\varphi_{\mathrm{ij}}(\mathrm{a},\mathrm{b}))=\mathrm{Fitt}_1(\varphi_{\mathrm{uv}}(\mathrm{a}',\mathrm{b}'))$$

which implies

$$\varphi_{ij}(a,b)=\varphi_{uv}(a',b'),$$

as we can easy check. Since the modules of $\mathcal{N}$ are the syzygies of those of $\mathcal{M}$ it results that any two different modules of $\mathcal{N}$ are not isomorphic. (3) and (4) follows as in ([17], Theorem 3.1).

REMARK 2.2. We note that every matrix factorization of a two generated, non free, graded MCM module over R_4 is the tensor product of the matrix factorizations of $Y_1^3+Y_s^3$ and $Y_i^3+Y_j^3$ (see [21]).

3. Rank one MCM modules over R_4 with three generators

Let M be a rank one MCM module over R_4 with three generators and let (φ,ψ) be a matrix factorization of M. We may suppose $\det\varphi=f_4$ (if necessary replacing M by its first syzygy). Thus the entries of φ are linear forms.

LEMMA 3.1. *Let $\alpha,\beta,\gamma,\delta$ be independent linear forms in $K[Y_1,Y_2,Y_3,Y_4]$ such that $f_4\in(\alpha,\beta)\cap(\gamma,\delta)$. Then there exists some linear forms m,n,w,t such that*

$$\det\begin{pmatrix} 0 & \alpha & \beta \\ \gamma & m & n \\ \delta & w & t \end{pmatrix}=f_4.$$

Proof. Since $f_4 \in (\alpha, \beta)$ there exist non unique $2-$forms η_1, η_2 such that

$$f_4 = \alpha\eta_1 + \beta\eta_2. \tag{1}$$

η_1, η_2 can be expressed as:

$$\eta_1 = \eta_{11}\alpha + \eta_{12}\beta + \eta_{13}\gamma + \eta_{14}\delta \tag{2}$$

$$\eta_2 = \eta_{21}\alpha + \eta_{22}\beta + \eta_{23}\gamma + \eta_{24}\delta, \tag{3}$$

where η_{ij} are linear forms, since $\alpha, \beta, \gamma, \delta$ are independent and so generate the linear form space. By hypothesis $f_4 \in (\gamma, \delta)$ so

$$\alpha\eta_1 + \beta\eta_2 \equiv 0 \pmod{(\gamma, \delta)},$$

which implies $\alpha\eta_1 \equiv 0 \pmod{(\beta, \gamma, \delta)}$. But α is not contained in the prime ideal (β, γ, δ). It results that

$$\eta_1 \equiv 0 \pmod{(\beta, \gamma, \delta)},$$

thus we may take $\eta_{11} = 0$. Replacing the expressions of η_1 and η_2 in the equality (1) we get

$$(\eta_{12} + \eta_{21})\alpha\beta + \eta_{22}\beta^2 \in (\gamma, \delta).$$

Since $\beta \notin (\gamma, \delta)$ we deduce that

$$(\eta_{12} + \eta_{21})\alpha + \eta_{22}\beta \in (\gamma, \delta). \tag{4}$$

This implies that

$$\eta_{22}\beta \equiv 0 \pmod{(\alpha, \gamma, \delta)}.$$

Moreover, we have $\eta_{22} \equiv 0 \pmod{(\alpha, \gamma, \delta)}$. It follows that there exists $\lambda_1, \lambda_2, \lambda_3 \in K$ such that

$$\eta_{22} = \lambda_1\alpha + \lambda_2\gamma + \lambda_3\delta.$$

By the relation (4) we have that

$$\eta_{12} + \eta_{21} + \lambda_1\beta \equiv 0 \pmod{(\gamma, \delta)}$$

so

$$\eta_{21} \equiv -\eta_{12} - \lambda_1\beta \pmod{(\gamma, \delta)}.$$

Therefore we may write η_2 in the following form:

$$\eta_2 = -\eta_{12}\alpha + \eta'_{23}\gamma + \eta'_{24}\delta.$$

Denote $\eta'_1 = \eta_1 - \eta_{12}\beta$ and $\eta'_2 = \eta_2 + \eta_{12}\alpha$. Then

$$f_4 = \alpha\eta'_1 + \beta\eta'_2$$

and

$$\eta_1', \eta_2' \equiv 0 \pmod{(\gamma, \delta)}.$$

Thus we may find some linear forms with the required property.

For $1 \leq i \leq 4$, let $\mathcal{L}_i$ be the set of the linear forms $Y_i - aY_j$, where $a \in K$, $a^3 = -1$ and $j \in \{1,2,3,4\}$, $j > i$.

PROPOSITION 3.2. *Let M be a three generated, rank one, graded MCM module over R_4. Then there exist some independent linear forms $\alpha, \beta, \gamma, \delta$ with $\alpha, \gamma \in \mathcal{L}_1$, $\beta \in \mathcal{L}_j$ and $\delta \in \mathcal{L}_i$ for some $i, j \geq 2$ and there exist m, n, w, t linear forms such that*

$$\varphi = \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & m & n \\ \delta & w & t \end{pmatrix}$$

and its adjoint matrix, ψ, form a matrix factorization of M.

Proof. As rank $M = 1$, every matrix factorization (φ, ψ) of M has $\det \varphi = f_4$. Since $f_4 \in (Y_1 + Y_2, Y_3 + Y_4)$, we obtain that φ has a generalized zero (see [9]). By elementary transformations φ can be arranged in the form

$$\varphi = \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & m & n \\ \delta & w & t \end{pmatrix}.$$

As in the two generated case we obtain

$$\alpha = Y_1 - aY_{j_1}, \beta = Y_j - bY_{j_2}, \gamma = Y_1 - cY_{i_1}, \delta = Y_i - dY_{i_2},$$

where (j, j_1, j_2) and (i, i_1, i_2) are permutations of the set $\{2,3,4\}$ such that $j < j_2$ and $i < i_2$, that is $\alpha, \gamma \in \mathcal{L}_1$, $\beta \in \mathcal{L}_j$ and $\delta \in \mathcal{L}_i$. We shall prove that since $\det \varphi = f_4$ we must have $\alpha, \beta, \gamma, \delta$ linear independent. We have the following possibilities to choose φ :

(i) $A = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_2 & \star & \star \\ Y_3 - dY_4 & \star & \star \end{pmatrix}$

(ii) A^t

(iii) $B = \begin{pmatrix} 0 & Y_1 - aY_3 & Y_2 - bY_4 \\ Y_1 - cY_2 & \star & \star \\ Y_3 - dY_4 & \star & \star \end{pmatrix}$

(iv) B^t

(v) $C = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_3 & \star & \star \\ Y_2 - dY_4 & \star & \star \end{pmatrix}$

(vi) C^t

(vii) $D = \begin{pmatrix} 0 & Y_1 - aY_2 & Y_3 - bY_4 \\ Y_1 - cY_2 & \star & \star \\ Y_3 - dY_4 & \star & \star \end{pmatrix}$

(viii) $E = \begin{pmatrix} 0 & Y_1 - aY_3 & Y_2 - bY_4 \\ Y_1 - cY_3 & \star & \star \\ Y_2 - dY_4 & \star & \star \end{pmatrix}$

(ix) $F = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_4 & \star & \star \\ Y_2 - dY_3 & \star & \star \end{pmatrix}$

We shall give the proof only for the first case. The others are similar.

Let $\varphi = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_2 & m & n \\ Y_3 - dY_4 & w & t \end{pmatrix}$. Since $\det\varphi = f_4$ we obtain:

$$-(Y_1 - aY_4)((Y_1 - cY_2)t - (Y_3 - dY_4)n) + (Y_2 - bY_3)((Y_1 - cY_2)w - (Y_3 - dY_4)m) =$$
$$(Y_1 - aY_4)(Y_1^2 + aY_1Y_4 + a^2Y_4^2) + (Y_2 - bY_3)(Y_2^2 + bY_2Y_3 + b^2Y_3^2).$$

This equality is equivalent with

$$(Y_1 - aY_4)((Y_1^2 + aY_1Y_4 + a^2Y_4^2) + (Y_1 - cY_2)t - (Y_3 - dY_4)n) =$$
$$(Y_2 - bY_3)((Y_1 - cY_2)w - (Y_3 - dY_4)m - (Y_2^2 + bY_2Y_3 + b^2Y_3^2)).$$

It results that there exists a linear form η such that

$$(Y_1^2 + aY_1Y_4 + a^2Y_4^2) + (Y_1 - cY_2)t - (Y_3 - dY_4)n = \eta(Y_2 - bY_3).$$

Put

$$Y_1 = bcd, Y_2 = bd, Y_3 = d \text{ and } Y_4 = 1$$

in the above equality. It follows

$$(bcd)^2 + abcd + a^2 = 0,$$

which gives $a \neq bcd$. This condition means exactly that $\alpha, \beta, \gamma, \delta$ are linearly independent. Thus $bcd = \varepsilon a$,where ε is in K, $\varepsilon^3 = 1$ and $\varepsilon \neq 1$. An example of such A is given by:

$$A = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_2 & -b^2Y_3 - abc^2\varepsilon^2Y_4 & b^2c^2Y_3 - abc\varepsilon^2Y_4 \\ Y_3 - dY_4 & c^2Y_2 + bc^2Y_3 + acY_4 & -Y_1 - cY_2 - aY_4 \end{pmatrix}.$$

Then A and its adjoint, A^*, form a matrix factorization of f_4 (by the following lemma we see that always A can be supposed of the above form after some elementary transformations).
The condition of linear independence of $\alpha,\beta,\gamma,\delta$, in the case **(iii)** is $ad \neq bc$, that is $ad = \varepsilon bc$. Then

$$B = \begin{pmatrix} 0 & Y_1 - aY_3 & Y_2 - bY_4 \\ Y_1 - cY_2 & a^2cY_3 + (abc^2 + a^2cd)Y_4 & a^2Y_3 - a^2d\varepsilon Y_4 \\ Y_3 - dY_4 & c^2Y_2 + acY_3 + bc^2Y_4 & -Y_1 - cY_2 - aY_3 \end{pmatrix}$$

and its adjoint matrix, B^*, form a matrix factorization of f_4.
The condition of linear independence of $\alpha,\beta,\gamma,\delta$, in the case **(v)** is $ab \neq cd$, that is $ab = \varepsilon cd$. Then

$$C = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_3 & -Y_2 - bY_3 - dY_4 & -b^2c^2Y_3 + bc^2d\varepsilon^2Y_4 \\ Y_2 - dY_4 & b^2c^2Y_3 - bc^2d\varepsilon^2Y_4 & -Y_1 - cY_3 - aY_4 \end{pmatrix}$$

and its adjoint matrix, C^*, form a matrix factorization of f_4.
For the last three cases we obtain that $\alpha,\beta,\gamma,\delta$ are linear independent if and only if $a \neq c$ and $b \neq d$. Then the pairs (D,D^*), (E,E^*) and (F,F^*) are matrix factorizations, where

$$D = \begin{pmatrix} 0 & Y_1 - aY_2 & Y_3 - bY_4 \\ Y_1 - cY_2 & -Y_3 - (b+d)Y_4 & 0 \\ Y_3 - dY_4 & 0 & -Y_1 - (a+c)Y_2 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & Y_1 - aY_3 & Y_2 - bY_4 \\ Y_1 - cY_3 & -Y_2 - (b+d)Y_4 & 0 \\ Y_2 - dY_4 & 0 & -Y_1 - (a+c)Y_3 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_4 & -Y_2 - (b+d)Y_3 & 0 \\ Y_2 - dY_3 & 0 & -Y_1 - (a+c)Y_4 \end{pmatrix}.$$

The next Lemma will show that every three generated, rank one, non-free graded MCM module over R_4 is isomorphic with a module given by one of the above matrix factorizations.

LEMMA 3.3. *If $\alpha,\beta,\gamma,\delta$ are independent linear forms as in the above Proposition and*

$$\varphi = \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & m & n \\ \delta & w & t \end{pmatrix}, \ \varphi' = \begin{pmatrix} 0 & \alpha & \beta \\ \gamma & m' & n' \\ \delta & w' & t' \end{pmatrix}$$

then Coker $\varphi \cong$ Coker φ'.

Proof. Let η and ν be two homogeneous forms of degree 2 such that $f_4 = \alpha\eta + \beta\nu$. It results that

$$\alpha n\delta + \beta\gamma w - \delta\beta m - \alpha\gamma t = \alpha\eta + \beta\nu,$$

that is

$$\alpha(n\delta - \gamma t - \eta) = \beta(\nu - \gamma w + \delta m).$$

Therefore we obtain the following equalities:

$$n\delta - \gamma t - \eta = \theta\beta \tag{5}$$

and

$$\delta m - \gamma w + \nu = \theta\alpha, \tag{6}$$

for some linear form θ. In the same way we obtain that there exists a linear form θ' such that

$$n'\delta - \gamma t' - \eta = \theta'\beta \tag{7}$$

and

$$\delta m' - \gamma w' + \nu = \theta'\alpha, \tag{8}$$

Subtracting the identities (5) and (7) we obtain:

$$(n - n')\delta - (t - t')\gamma = (\theta - \theta')\beta. \tag{9}$$

Since $\beta \notin (\gamma, \delta)$ it follows that there exist $a, b \in K$ such that $\theta - \theta' = a\delta + b\gamma$. Replacing in the equation (9) we get:

$$(n - n' - a\beta)\delta = (t - t' + b\beta)\gamma.$$

Thus there exists $c \in K$ such that

$$t' = t + b\beta - c\delta \tag{10}$$

and

$$n' = n - a\beta - c\gamma. \tag{11}$$

Starting with the equations (6) and (8) we obtain analogously that there exists $c' \in K$ such that

$$m' = m - a\alpha - c'\gamma \tag{12}$$

and

$$w' = w + b\alpha - c'\delta. \tag{13}$$

The last four equalities show that φ' is obtained from φ after some elementary transformations and so prove our Lemma.

From now on, the most difficult task is to decide which of the modules given by the matrix factorizations defined in the proof of Proposition 3.2 are isomorphic.

We recall that two matrices, φ and φ', define the same module over R_4 (i.e. Coker $\varphi \simeq$ Coker φ'), if and only if they are equivalent, that is there exist U and V two square matrices with entries in $K[Y_1,\dots,Y_n]$ such that $\varphi' = U\varphi V$ and $\det(U) = \det(V) = 1$ (see [10]). In this case we denote $\varphi \sim \varphi'$.
The proof of the main theorem of this section will be done with the help of the computer algebra system SINGULAR [12].
For $a,b,c,d,\varepsilon \in K$ such that $a^3 = b^3 = c^3 = d^3 = -1, \varepsilon^3 = 1, \varepsilon \neq 1$ and $bcd = \varepsilon a$, we set

$$\alpha(b,c,d,\varepsilon) = \begin{pmatrix} 0 & Y_1 - aY_4 & Y_2 - bY_3 \\ Y_1 - cY_2 & -b^2Y_3 - abc^2\varepsilon^2Y_4 & b^2c^2Y_3 - abc\varepsilon^2Y_4 \\ Y_3 - dY_4 & c^2Y_2 + bc^2Y_3 + acY_4 & -Y_1 - cY_2 - aY_4 \end{pmatrix}$$

and

$$\beta(b,c,d,\varepsilon) = \alpha(b,c,d,\varepsilon)^t,$$

that is the transpose of $\alpha(a,b,c,d)$. We know from the proof of the Proposition 3.2 that $(\alpha(b,c,d,\varepsilon), \alpha(b,c,d,\varepsilon)^*)$ and $(\beta(b,c,d,\varepsilon), \beta(b,c,d,\varepsilon)^*)$ are matrix factorizations of f_4.
For $a,b,c \in K$, distinct roots of -1, and ε as above, we set

$$\eta(a,b,c,\varepsilon) = \begin{pmatrix} 0 & Y_1 + Y_2 & Y_3 - aY_4 \\ Y_1 + \varepsilon Y_2 & -Y_3 + cY_4 & 0 \\ Y_3 - bY_4 & 0 & -Y_1 - \varepsilon^2 Y_2 \end{pmatrix},$$

$$\vartheta(a,b,c) = \begin{pmatrix} 0 & Y_1 + Y_3 & Y_2 - aY_4 \\ Y_1 - a^2bY_3 & -Y_2 + cY_4 & 0 \\ Y_2 - bY_4 & 0 & -Y_1 + ab^2Y_3 \end{pmatrix}.$$

These matrices are of the type D and E. Thus, every matrix forms with its adjoint a matrix factorization of f_4.

THEOREM 3.4. *Let*

$$\mathcal{M} = \{Coker\ \alpha(b,c,d,\varepsilon),\ Coker\ \beta(b,c,d,\varepsilon) \mid b,c,d,\varepsilon \in K,$$
$$b^3 = c^3 = d^3 = -1,\ bcd = \varepsilon a,\ \varepsilon^3 = 1, \varepsilon \neq 1\}$$

and

$$\mathcal{N} = \{Coker\ \eta(a,b,c,\varepsilon),\ Coker\ \vartheta(a,b,c),\ \mid \varepsilon^3 = 1,\ \varepsilon \neq 1$$
$$and\ (a,b,c)\ is\ a\ permutation\ of\ the\ roots\ of\ -1\}.$$

Then the sets $\mathcal{M}, \mathcal{N}$ of rank one, three generated, MCM graded R_4-modules have the following properties:

(i) *every three generated, rank one, non-free, graded MCM R_4−module is isomorphic with one module from $\mathcal{M} \cup \mathcal{N}$.*

(ii) *if* $M = Coker\ \alpha(b,c,d,\varepsilon)$ *(or* $M = Coker\ \beta(b,c,d,\varepsilon)$*) belongs to* $\mathcal{M}$ *and* $N \in \mathcal{M}$*, then* $N \simeq M$ *if and only if* $N = Coker\ \alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$ *(or* $N = \beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$*).*

(iii) *any two different modules from* $\mathcal{N}$ *are not isomorphic.*

(iv) *any module of* $\mathcal{N}$ *is not isomorphic with some module of* $\mathcal{M}$*.*

Proof. For the beginning we shall prove that any module of the type B, B^t, C and C^t of the proof of Proposition 3.2 is isomorphic with one of type A or A^t. This can be done using SINGULAR. For instance, to establish that the modules of type B are isomorphic with modules of type A^t, we use the following procedure (see [17], Lemma 5.1):

```
LIB"matrix.lib";
option(redSB);
proc isomorf(matrix X, matrix Y)
{
matrix U[3][3]=u(1..9);
matrix V[3][3]=v(1..9);
matrix C=U*X-Y*V;
ideal I=flatten(C);
ideal I1=transpose(coeffs(I,y(1)))[2];
ideal I2=transpose(coeffs(I,y(2)))[2];
ideal I3=transpose(coeffs(I,y(3)))[2];
ideal I4=transpose(coeffs(I,y(4)))[2];
ideal J=I1+I2+I3+I4+ideal(det(U)-1,det(V)-1);
ideal L=std(J);
L;
}
```

We apply this procedure for the matrices A^t and B:

```
ring R=0,(u(1..9),v(1..9),y(1..4),x,a,b,c,d,m,n,p,q,y),lp;
ideal F=a3+1,b3+1,c3+1,d3+1,x*a-b*c*d,x2+x+1,m3+1,n3+1,p3+1,q3+1,
m*q-y*n*p,y2+y+1;
qring Q=std(F);
matrix A[3][3]=0,y(1)-a*y(4),y(2)-b*y(3),y(1)-c*y(2),-b2*y(3)
-a*b*c2*x2*y(4), b2*c2*y(3)-a*b*c*x2*y(4),y(3)-d*y(4),c2*y(2)
+b*c2*y(3)+a*c*y(4),-y(1)-c*y(2)-a*y(4);
matrix B[3][3]=0,y(1)-m*y(3),y(2)-n*y(4),y(1)-p*y(2),m2*p*y(3)
+m*n*p2*y(4)+m2*p*q*y(4),m2*y(3)-m2*q*y*y(4),y(3)-q*y(4),p2*y(2)
+m*p*y(3)+n*p2*y(4), -y(1)-p*y(2)-m*y(3);
// Now we test the equivalence between the matrices transpose(A)
```

```
// and B
isomorf(transpose(A),B);
```

We obtain that A^t and B are equivalent if and only if

$$d^2 - dqy - dq + q^2y = 0 \tag{14}$$

$$c - dpq^2y - dpq^2 = 0 \tag{15}$$

$$b - dnq = 0 \tag{16}$$

$$a + dnpq^2 + npy + np = 0 \tag{17}$$

and

$$ab^2c^2d^2 + dq^2y - 1 = 0. \tag{18}$$

If m, n, p, q and y are fixed such that $m^3 = n^3 = p^3 = q^3 = -1$, $y^2 + y + 1 = 0$ and $mq = npy$, then we may obtain a, b, c, d and x such that the above equations are satisfied and $a^3 = b^3 = c^3 = d^3 = -1$, $x^2 + x + 1 = 0$, $bcd = ax$.
For instance, we may take

$$a = -np,\ b = ynq^2,\ c = p \text{ and } d = qy.$$

In this case, if $\lambda \in K$ verifies the equation $9\lambda^3 = 8mp^2$, one finds the matrices

$$U = \begin{pmatrix} 0 & -\frac{1}{2}p\lambda & \frac{1}{2}p^2\lambda \\ \frac{3}{2}m^2\lambda & 0 & 0 \\ 0 & -\frac{1}{2}p^2\lambda & \lambda \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & \frac{3}{2}m^2\lambda & 0 \\ -\frac{1}{2}p\lambda & 0 & -\frac{1}{2}p^2\lambda \\ \frac{1}{2}p^2\lambda & 0 & \lambda \end{pmatrix},$$

such that $UA^t = BV$.
With the same procedure, we obtain that every matrix of type C wich depends on m, n, p, q and y is equivalent with the transpose of a matrix of type A depending on a, b, c, d and x, where

$$a = -n^2pqy^2,\ b = n,\ c = n^2p,\ d = n^2q \text{ and } \mathrm{x} = \mathrm{y}.$$

Now we study the equivalence of the matrices of type D, E, F. Let (a, b, c) and (p, q, r) be two permutations of the third roots of -1 and

$$D((a,b,c),(p,q,r)) = \begin{pmatrix} 0 & Y_1 - aY_2 & Y_3 - pY_4 \\ Y_1 - cY_2 & -Y_3 + rY_4 & 0 \\ Y_3 - qY_4 & 0 & -Y_1 + bY_2 \end{pmatrix}.$$

One can apply elementary transformations on the columns and on the rows of $D((a,b,c),(p,q,r))$ to obtain

$$D((a,b,c),(p,q,r)) \sim D((c,a,b),(q,r,p)) \sim D((b,c,a),(r,p,q)).$$

We deduce similar equivalences for the matrices of type E and F. This means that we may restrict our study to the matrices

$$\eta(p,q,r,\varepsilon) = D((-1,-\varepsilon,-\varepsilon^2),(p,q,r)),$$

$$E((-1,-\varepsilon,-\varepsilon^2),(p,q,r)),\ F((-1,-\varepsilon,-\varepsilon^2),(p,q,r)),$$

where ε is a root $\neq 1$ of 1 in K.
Using the procedure `isomorf(matrix X, matrix Y)`, we get that any two different matrices $D((-1,-x,-x^2),(a,b,c))$ and $D((-1,-y,-y^2),(p,q,r))$ are not equivalent.
Now, let us consider the matrices $D((-1,-x,-x^2),(a,b,c))$ and $E((-1,-y,-y^2),(p,q,r))$. Applying our procedure, it results that they are equivalent if and only if

$$y+pq^2=0,\ b-q=0,\ a+p+q=0 \text{ and } x^2-pq^2+1=0.$$

Since $-1, pq^2, p^2q$ are the solutions of the equation $x^3+1=0$, we get $x^2 = -1+pq^2 = -p^2q$. Thus, if $y=-pq^2$, then $E((-1,-y,-y^2),(p,q,r))$ is equivalent with $D((-1,pq^2,p^2q),(-p-q,q,p))$. If $y=-p^2q$ (the only left case!) then $E((-1,-y,-y^2),(p,q,r))$ is $\vartheta(p,q,r)$. Until now we have obtained that the matrices which define the modules of the set $\mathcal{N}$ are pairwise non-equivalent. Finally, we find that the matrices of the form $F((-1,-y,-y^2),(p,q,r))$, where $y=-pq^2$, are equivalent with some matrices of type D, and those $F((-1,-y,-y^2),(p,q,r))$, where $y=-p^2q$, are equivalent with some matrices of the type $\vartheta(a,b,c)$, with $b=q$ and $a+p+q=0$.
So we have proved the parts **(i)** and **(iii)** of the theorem. For the rest, one can use the procedure `isomorf(matrix X, matrix Y)`, as in the previous part. For instance, to prove **(iv)**:

```
ring R=0,(u(1..9),v(1..9),y(1..4),x,a,b,c,d,w,p,q),lp;
ideal F=a3+1,b3+1,c3+1,d3+1,x*a-b*c*d,x2+x+1,p3+1,q3+1,w2-w+1;
qring Q=std(F);
matrix A[3][3]=0,y(1)-a*y(4),y(2)-b*y(3),y(1)-c*y(2),-b2*y(3)
-a*b*c2*x2*y(4),b2*c2*y(3)-a*b*c*x2*y(4),y(3)-d*y(4),c2*y(2)
+b*c2*y(3)+a*c*y(4),-y(1)-c*y(2)-a*y(4);
matrix D[3][3]=0,y(1)+y(2),y(3)-p*y(4),y(1)-w*y(2),-y(3)-p*y(4)
-q*y(4),0,y(3)-q*y(4),0,-y(1)-w2*y(2);
// Now we test the equivalence between the matrices A and D
isomorf(A,D);
// Now we test the equivalence between the matrices transpose(A)
// and D
isomorf(transpose(A),D);
```

In both cases we obtain:

```
L[1]=1
```

This proves that there is no module of type D which is isomorphic with a module of the set $\mathcal{M}$. Analogously we may check that there is no module of type E which is isomorphic with a module of the set $\mathcal{M}$. This shows **(iv)**.
Finally, for the part **(ii)**, we apply the procedure `isomorf(matrix X, matrix Y)` for the matrices $\alpha(b,c,d,x)$ and $\alpha(n,p,q,y)$. We obtain that these two matrices are equivalent if and only if the following equations are satisfied:

$$d^2 - dqy - dq + q^2y = 0 \tag{19}$$

$$c + dpq^2 = 0 \tag{20}$$

$$b + dnq^2 = 0 \tag{21}$$

$$a + dnpy + npq = 0 \tag{22}$$

$$ab^2c^2d^2 + dq^2y - 1 = 0. \tag{23}$$

From the equation (19) we obtain: (i) $\frac{d}{qy^2} = y$ or (ii) $\frac{d}{qy^2} = y^2$. In the case (i), it follows $d = q$ and, from the above equations, we obtain $c = p$, $b = n$ and $a = y^2npq = my^3 = m$. The equation (23) is obviously verified. Thus in the first case we get that $\alpha(b,c,d,x) = \alpha(n,p,q,y)$. In the second one, using the equations (19)-(23), we obtain

$$d = qy,\ c = py,\ b = ny \text{ and } x = y^2.$$

To finish the proof of **(ii)** we apply the procedure `isomorf(matrix X, matrix Y)` for the matrices $\beta(b,c,d,x)$ and $\alpha(n,p,q,y)$:

```
ring R=0,(u(1..9),v(1..9),y(1..4),x,a,b,c,d,m,n,p,q,y),lp;
ideal F=a3+1,b3+1,c3+1,d3+1,x*a-b*c*d,x2+x+1,m3+1,n3+1,p3+1,
q3+1,y*m-n*p*q,y2+y+1;
qring Q=std(F);
matrix A[3][3]=0,y(1)-a*y(4),y(2)-b*y(3),y(1)-c*y(2),-b2*y(3)
-a*b*c2*x2*y(4),b2*c2*y(3)-a*b*c*x2*y(4),y(3)-d*y(4),c2*y(2)
+b*c2*y(3)+a*c*y(4),-y(1)-c*y(2)-a*y(4);
matrix AA[3][3]=0,y(1)-m*y(4),y(2)-n*y(3),y(1)-p*y(2),-n2*y(3)
-m*n*p2*y2*y(4),n2*p2*y(3)-m*n*p*y2*y(4),y(3)-q*y(4),p2*y(2)
+n*p2*y(3)+m*p*y(4),-y(1)-p*y(2)-m*y(4);
isomorf(transpose(A),AA);
```

We obtain:

```
L[1]=1
```

This shows that no matrix of type A is equivalent with one of type A^t.

The three generated, rank one, MCM modules over R_4 are linear MCM or Ulrich modules (see [3], [4]). Thus, from the above theorem we obtain:

COROLLARY 3.5. *There are* 72 *isomorphism classes of Ulrich modules of rank one over the ring* R_4.

Proof. The modules of the set $\mathcal{M}$ depends on b, c, d and ε, thus there are $3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 108$ elements in this set. Since these modules are isomorphic in couples, we obtain 54 isomorphism classes which have the representatives in the set $\mathcal{M}$. The modules Coker $\eta(a, b, c, \varepsilon)$ depends on ε and on the permutation $(a\ b\ c)$ of the cubic roots of -1 and the modules Coker $\vartheta(a, b, c)$ are determined by the permutation $(a\ b\ c)$ of the roots of -1, thus we get $6 \cdot 2 + 6 = 18$ isomorphism classes which have the representatives in the set $\mathcal{N}$.

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