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ELEMENTS OF LIE THEORY
IN FINITE AND INFINITE DIMENSIONS
(APPENDICES)

by

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(APPENDICES)

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ELEMENTS OF LIE THEORY IN FINITE AND INFINITE DIMENSIONS (APPENDICES)

DANIEL BELTIŢĂ

ABSTRACT. These appendices include some basic facts from several topics underlying Lie theory in infinite dimensions. Their titles are:

A1. Topological vector spaces
A2. Differential calculus
A2$\frac{1}{2}$. Basic differential equations of Lie theory
A3. Smooth manifolds and vector fields
A4. Topological groups

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A1. TOPOLOGICAL VECTOR SPACES

Abstract. We review some basic facts concerning topological vector spaces. A special emphasis is placed upon complexifications of topological vector spaces and upon continuous inverse algebras.

Locally convex spaces

Definition A1.1. A *topological vector space* over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a vector space X over $\mathbb{K}$ equipped with a Hausdorff topology such that both the vector addition

$$X \times X \to X, \qquad (x, y) \mapsto x + y,$$

and the scalar multiplication

$$\mathbb{K} \times X \to X, \qquad (\lambda, x) \mapsto \lambda x,$$

are continuous mappings. $\square$

Example A1.2.

(a) The usual topology makes $\mathbb{K}^n$ into a topological vector space whenever $n \geq 1$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(b) Every Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is in particular a topological vector space. $\square$

Definition A1.3. A topological vector space X is said to be *locally convex* if each point of X has a basis of convex neighborhoods.

Note that, to check that X is locally convex, it actually suffices to find a basis $\mathcal{V}$ of convex neighborhoods of $0 \in X$. In fact, for arbitrary $x \in X$, it then easily follows that $\{x + V \mid V \in \mathcal{V}\}$ is a basis of convex neighborhoods of x. $\square$

Example A1.4. Every Banach space is locally convex, since the open balls centered at some point $x \in X$ constitute a basis of convex neighborhoods of x. $\square$

The following characterization of locally convex spaces shows that Example A1.4 plays a central role among locally convex spaces. For this statement, we recall that a *seminorm* on a vector space X over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a function $p\colon X \to [0, \infty)$ such that for all $x, y \in X$ and $\alpha \in \mathbb{K}$ we have $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$.

Theorem A1.5. *Let X be a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then X is locally convex if and only if there exists a family of seminorms $\{p_i\}_{i \in I}$ defining the topology of X in the sense that, if for $n \geq 1$, $i_1, \ldots, i_n \in I$ and $\varepsilon > 0$ we denote*

$$V_{i_1, \ldots, i_n; \varepsilon} := \{x \in X \mid \max_{1 \leq k \leq n} |p_{i_k}(x)| < \varepsilon\},$$

then

$$\mathcal{V} := \{V_{i_1, \ldots, i_n; \varepsilon} \mid n \geq 1, i_1, \ldots, i_n \in I, \varepsilon > 0\}$$

is a basis of neighborhoods of $0 \in X$.

Proof. See e.g., §1 in Chapter II in [Sf66] for the connection between convex sets and seminorms in a topological vector space. $\square$

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Example A1.6. Let $X = \mathcal{C}^\infty[0,1]$ and for each $n \geq 0$ define

$$p_n\colon X \to [0,\infty), \qquad p_n(f) := \sup_{t\in[0,1]} |f^{(n)}(t)|.$$

Then $\{p_n\}_{n\geq 0}$ is a sequence of seminorms on X, hence Theorem A1.5 shows that X has a topology of locally convex space such that, if we define

$$N_{n,\varepsilon} := \{f \in X \mid \max_{0\leq k\leq n} \sup_{t\in[0,1]} |f^{(k)}(t)| < \varepsilon\} \qquad (n \in \mathbb{N}, \varepsilon > 0),$$

then $\{N_{n,\varepsilon} \mid n \in \mathbb{N}, \varepsilon > 0\}$ is a basis of (convex!) neighborhoods of $0 \in X$. □

Example A1.7. Let D be an open subset of $\mathbb{R}^m$ ($m \geq 1$) and $X = \mathcal{C}^\infty(D)$. For each compact subset K of D and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ we denote $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and define

$$p_{K,\alpha}\colon X \to [0,\infty), \qquad p_{K,\alpha}(f) := \sup_{t\in K} \Big| \frac{\partial^{\alpha_1+\cdots+\alpha_m} f}{\partial t^{\alpha_1}\cdots\partial t_m^{\alpha_m}}(t) \Big|,$$

(where we denote $t = (t_1, \dots, t_m) \in \mathbb{R}^m$ as usually). Then each $p_{K,\alpha}$ is a seminorm on X, and it follows by Theorem A1.5 that the sets

$$N_{K,n,\varepsilon} := \{f \in X \mid \max_{\alpha\in\mathbb{N}^m, |\alpha|\leq n} p_{K,\alpha}(f) < \varepsilon\} \qquad (K \text{ compact } \subseteq D, n \in \mathbb{N}, \varepsilon > 0)$$

constitute a basis of convex neighborhoods of $0 \in X$ in some topology making X into a locally convex space. □

Definition A1.8. Let X be a topological vector space. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is *convergent* to $x \in X$ if for every neighborhood V of x there exists $n_V \in \mathbb{N}$ such that for all $n \geq n_V$ we have $x_n \in V$. With the same notation, we say that $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if for every neighborhood W of $0 \in X$ there exists $m_V \in \mathbb{N}$ such that $x_n - x_m \in W$ whenever $n, m \geq m_V$.

Finally, we say that the topological vector space X is *sequentially complete* if every Cauchy sequence in X is convergent. □

Exercise A1.9. Let X be a topological vector space.

(a) Prove that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x \in X$ if and only if for every neighborhood W of $0 \in X$ there exists $n_W \in \mathbb{N}$ such that $x_n - x \in W$ whenever $n \geq n_W$.

(b) Prove that every convergent sequence in X is a Cauchy sequence.

(c) If X is locally convex and its topology is defined by a family of seminorms $\{p_i\}_{i\in I}$ as in Theorem A1.5, then a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is a Cauchy sequence if and only if for all $i \in I$ we have $\lim\limits_{m,n\to\infty} p_i(x_m - x_n) = 0$. □

We note that every Banach space is in particular a sequentially complete locally convex space. The next definition singles out a more general class of topological vector spaces of the latter type.

Definition A1.10. We say that a locally convex space X over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a *Fréchet space* if it is sequentially complete and its topology can be defined by a countable family of seminorms (see Theorem A1.5). □

Example A1.11. Every Banach space is in particular a Fréchet space, since it is (sequentially) complete and its topology is defined by a single (semi)norm. □

Exercise A1.12. Prove that the space $X = \mathcal{C}^\infty[0,1]$ in Example A1.6 is a Fréchet space. □

Exercise A1.13. Prove that the space $X = \mathcal{C}^\infty(D)$ in Example A1.7 is a Fréchet space. □

Theorem A1.14 (Hahn-Banach). *If X is a locally convex space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then for all $x, y \in X$, with $x \neq y$, there exists a continuous linear functional $l\colon X \to \mathbb{K}$ such that $l(x) \neq l(y)$.*

Proof. See e.g., Theorem 9.2 in Chapter II in [Sf66]. □

Proposition A1.15. *Let X be a sequentially complete locally convex space and $f\colon [0,1] \to X$ a continuous function. Then there exists the Riemann integral*

$$\int_0^1 f(t)\mathrm{d}t \in X.$$

Proof. The conclusion means that there exists $x \in X$ (to be denoted $\int_0^1 f(t)\mathrm{d}t$) such that for every neighborhood V of $0 \in V$ there exists $\delta > 0$ such that

$$(t_1 - t_0)f(\xi_1) + \cdots + (t_n - t_{n-1})f(\xi_n) \in x + V \tag{1}$$

whenever $0 = t_0 \le \xi_1 \le t_1 \le \xi_2 \le t_2 \le \cdots \le t_{n-1} \le \xi_n \le t_n = 1$ and $\sup\limits_{1 \le i \le n} < \delta$.

Since X is sequentially complete, it suffices (just as in the case $X = \mathbb{R}$) to show that for every sequence of subdivisions of $[0,1]$ with the mesh tending to 0, and for arbitrary choices of ξ's, the sequence of the corresponding Riemann sums (as in the left-hand side of (1)) is a Cauchy sequence.

Since X is a locally convex space, the latter property follows by Exercise A1.8(c) along with the fact that $f\colon [0,1] \to X$ is uniformly continuous in the following sense: for every neighborhood V of $0 \in X$ there exists $\varepsilon > 0$ such that, for all $s, t \in [0,1]$ with $|s - t| < \varepsilon$, we have $f(s) - f(t) \in V$. The proof of this latter fact is just an easy exercise (f is continuous on the compact $[0,1]$!). □

Complexifications

Definition A1.16. Let X be a real topological vector space. The *complexification* of X is the complex topological vector space $X_{\mathbb{C}} := X \times X$ equipped with the product topology, with the componentwise vector addition

$$X_{\mathbb{C}} \times X_{\mathbb{C}} \to X_{\mathbb{C}}, \quad (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$

and with the scalar multiplication defined by

$$\mathbb{C} \times X_{\mathbb{C}} \to X_{\mathbb{C}}, \quad (a + ib) \cdot (x_1, x_2) = (ax_1 - bx_2, ax_2 + bx_1).$$

We usually perform the identifications

$$X \simeq X \times \{0\} \hookrightarrow X_{\mathbb{C}} \text{ and } \mathrm{i}X \simeq \{0\} \times X \hookrightarrow X_{\mathbb{C}},$$

and thus

$$X_{\mathbb{C}} = X \dotplus \mathrm{i}X,$$

thinking of X as a real vector subspace of $X_{\mathbb{C}}$. In particular we write $x_1 + \mathrm{i}x_2$ instead of (x_1, x_2) whenever $x_1, x_2 \in X$. □

Exercise A1.17.

(a) If X is a real Banach space with the norm $\|\cdot\|_X$, then the complexification $X_{\mathbb{C}}(= X \times X)$ is a complex Banach space with the norm $\|\cdot\|_{X_{\mathbb{C}}}$ defined by

$$(\forall x_1, x_2 \in X) \qquad \|(x_1, x_2)\|_{X_{\mathbb{C}}} := \sup_{t\in[0,2\pi]} \|(\cos t)x_1 + (\sin t)x_2\|_X.$$

(b) If H is a real Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_H$, then the complexification $H_{\mathbb{C}}(= H \times H)$ is a complex Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_{H_{\mathbb{C}}}$ defined by

$$\langle(x_1, x_2), (y_1, y_2)\rangle_{H_{\mathbb{C}}} := \langle x_1, y_1\rangle_H + \langle x_2, y_2\rangle_H + \mathrm{i}(\langle x_2, y_1\rangle_H - \langle x_1, y_2\rangle_H)$$

whenever $x_1, x_2, y_1, y_2 \in H$. $\square$

Exercise A1.18. Let X and Y be real Banach spaces and denote by $\mathcal{B}_{\mathbb{R}}(X, Y)$ the real Banach space of all bounded $\mathbb{R}$-linear operators from X into Y. Also denote by $\mathcal{B}_{\mathbb{C}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ the complex Banach space of all bounded $\mathbb{C}$-linear operators from $X_{\mathbb{C}}$ into $Y_{\mathbb{C}}$. Then $\mathcal{B}_{\mathbb{C}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$ is the complexification of $\mathcal{B}_{\mathbb{R}}(X, Y)$, the natural embedding

$$\mathcal{B}_{\mathbb{R}}(X, Y) \hookrightarrow \mathcal{B}_{\mathbb{C}}(X_{\mathbb{C}}, Y_{\mathbb{C}})$$

being the one which associates to each $T \in \mathcal{B}_{\mathbb{R}}(X, Y)$ the operator

$$X_{\mathbb{C}} = X \times X \to Y_{\mathbb{C}} = Y \times Y, \quad (x_1, x_2) \mapsto (Tx_1, Tx_2). \quad \square$$

Continuous inverse algebras

Definition A1.19. A *topological algebra* A is a topological vector space equipped with a continuous bilinear mapping

$$A \times A \to A, \quad (a, b) \mapsto a \cdot b,$$

called the *multiplication* of A. If the topological vector space underlying A is locally convex, Fréchet, Banach or Hilbert, then we say that A is a *locally convex, Fréchet, Banach* or *Hilbert algebra*, respectively.

We say that the topological algebra A is *associative* if for all $a, b, c \in A$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, and that A is *unital* if there exists an element $\mathbf{1} \in A$ (called the *unit element* of A) such that $a \cdot \mathbf{1} = \mathbf{1} \cdot a = a$ for all $a \in A$. If A is both unital and associative, then an element $x \in A$ is said to be *invertible* if there exists $x_0 \in A$ such that $x \cdot x_0 = x_0 \cdot x = \mathbf{1}$. In this case, it is easy to check that x_0 is uniquely determined by x and we denote $x_0 =: x^{-1}$. The invertible elements of A are sometimes called the *units of* A. We denote

$$A^\times := \{x \in A \mid x \text{ is invertible}\}.$$

Note that $\mathbf{1} \in A^\times$ and $A^\times$ is a group with respect to the multiplication inherited from A.

If A is a topological algebra then a *subalgebra* of A is a vector subspace A_0 of A such that $a \cdot b \in A_0$ whenever $a, b \in A_0$. If moreover A is unital and the unit of A belongs to A_0, then we say that A_0 is a *unital subalgebra* of A.

If A is a real topological algebra, then the *complexification* $A_{\mathbb{C}}$ of the topological vector space underlying A has a natural structure of complex topological algebra with the multiplication defined by

$$(a_1 + \mathrm{i}a_2) \cdot (b_1 + \mathrm{i}b_2) = (a_1 \cdot b_1 - a_2 \cdot b_2) + \mathrm{i}(a_1 \cdot b_2 + a_2 \cdot b_1)$$

whenever $a_1, a_2, b_1, b_2 \in A$. If A is associative or unital, then so is $A_{\mathbb{C}}$. $\square$

Definition A1.20. Let A be a unital associative topological algebra. We say that A is a *continuous inverse algebra* if $A^\times$ is an open subset of A and the inversion mapping

$$\eta\colon A^\times \to A^\times, \quad x \mapsto x^{-1},$$

is continuous. $\square$

Lemma A1.21. *Let A be a unital associative topological algebra. If there exists an open subset W of A such that $\mathbf{1} \in W \subseteq A^\times$ and the inversion mapping*

$$\eta|_W\colon W \to A^\times$$

is continuous, then A is a continuous inverse algebra.

Proof. Fix $a \in A^\times$. Since a is invertible, it is easy to see that the mapping

$$L_a\colon A \to A, \quad b \mapsto ab,$$

is a homeomorphism and $L_a(A^\times) = A^\times$, hence $L_a(W) = aW$ is an open subset of $A^\times$ containing $L_a(\mathbf{1}) = a$. Since $a \in A^\times$ is arbitrary, it follows that $A^\times$ is open in A. Moreover, for all $c \in aW$ we have

$$\eta(c) = c^{-1} = \big(a(a^{-1}c)\big)^{-1} = (a^{-1}c)^{-1}a^{-1} = \eta(a^{-1}c)a^{-1}.$$

Since $a^{-1}c \in W$ and $\eta|_W$ is continuous, it then follows that η is continuous on the open neighborhood aW of a. Since $a \in A^\times$ is arbitrary, it follows that the inversion mapping $\eta\colon A^\times \to A^\times$ is continuous throughout on $A^\times$. $\square$

Proposition A1.22. *If A is a real continuous inverse algebra, then the complexification $A_{\mathbb{C}}$ is a complex continuous inverse algebra.*

Proof. Denote as usually by $\eta\colon A^\times \to A^\times$ the inversion mapping, and define

$$\psi\colon A^\times \times A \to A, \quad \psi(a,b) = \mathbf{1} + (a^{-1}b)^2.$$

Then $\psi(a,b) = \mathbf{1} + \big(\eta(a)\cdot b\big)^2$, hence ψ is a continuous mapping. Since $A^\times$ is open in A and $\psi(\mathbf{1},0) = \mathbf{1} \in A^\times$, it then follows that we can find an open neighborhood U of $\mathbf{1} \in A^\times$ and an open neighborhood V of $0 \in A$ such that $\psi(U \times V) \subseteq A^\times$. Hence

$$(\forall a \in U)\,(\forall b \in V) \quad \mathbf{1} + (a^{-1}b)^2 \in A^\times.$$

For arbitrary $a \in U$ and $b \in V$, we have

$$a + \mathrm{i}b = a(\mathbf{1} + \mathrm{i}\,\underbrace{a^{-1}b}_{=:c}) = a(\mathbf{1} + \mathrm{i}c).$$

On the other hand, we have $(\mathbf{1} + \mathrm{i}c)(\mathbf{1} - \mathrm{i}c) = \mathbf{1} + c^2 \in A^\times$, whence $\mathbf{1} + \mathrm{i}c$ is invertible, in fact $(\mathbf{1} + \mathrm{i}c)^{-1} = (\mathbf{1} - \mathrm{i}c)(\mathbf{1} + c^2)^{-1}$. Since a is also invertible, it then follows from the above equation that $a + \mathrm{i}b$ is also invertible and

$$\begin{aligned}
(a + \mathrm{i}b)^{-1} &= (\mathbf{1} + \mathrm{i}c)^{-1}a^{-1} \\
&= (\mathbf{1} - \mathrm{i}c)(\mathbf{1} + c^2)^{-1}a^{-1} \\
&= (\mathbf{1} - \mathrm{i}a^{-1}b)(\mathbf{1} + (a^{-1}b)^2)^{-1}a^{-1} \\
&= (\mathbf{1} - \mathrm{i}\eta(a)\cdot b)\cdot \eta(\mathbf{1} + (\eta(a)\cdot b)^2)\cdot \eta(a).
\end{aligned}$$

Hence $U + \mathrm{i}V \subseteq (A_{\mathbb{C}})^\times$ and, moreover, the inversion mapping

$$U + \mathrm{i}V \to A_{\mathbb{C}}, \quad z \mapsto z^{-1},$$

is continuous since η is a continuous mapping. Since $W := U + \mathrm{i}V$ is an open neighborhood of $\mathbf{1} \in A_{\mathbb{C}}$, the desired conclusion then follows by Lemma A1.21. $\square$

Exercise A1.23. Every unital associative Banach algebra is a continuous inverse algebra. □

Exercise A1.24. If B is a continuous inverse algebra and A is a unital subalgebra of B such that $A^\times = A \cap B^\times$, then A is in turn a continuous inverse algebra. □

Notes

The first part of the above review of topological vector spaces follows the lines of the corresponding section in the paper of J. Milnor [Mi84]. For further reading we refer to the books [Tr67] and [Sf66].

See Lemma 1.1 in [BS71b] for more details on the result contained in our Proposition A1.15. Further information on the complexifications of topological vector spaces can be found in section 2 of the paper [BS71a]. The results concerning continuous inverse algebras are taken from the paper [Gl02b]. The algebras of that type play an important role in K-theory, see e.g., [Swa77].

It is important to note that, from the point of view of operator theory, the Hilbert spaces constitute by far the most important class of topological vector spaces. See the celebrated book by P.R. Halmos [Ha82].

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A2. DIFFERENTIAL CALCULUS

Abstract. In this appendix we collect some basic elements of differential calculus in topological vector spaces, and particularly in locally convex spaces. We discuss real and complex analytic mappings on open subsets of locally convex spaces. As an important example, we prove that the inversion mapping of a continuous inverse algebra is analytic.

Differentiability

Definition A2.1. Let X be a topological vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and D an open subset of $\mathbb{K}$. We say that a continuous mapping $f\colon D \to X$ is of *class* C^1 if the limit

$$\dot{f}(t_0) := \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

exists in X for all $t_0 \in D$, and the mapping $\dot{f}\colon D \to X$ is continuous. Denote $f_1 := \dot{f}\colon D \to X$ and suppose that we have already defined $f_n\colon D \to X$ for some $n \geq 1$. If the mapping $f_{n+1} := \dot{f}_n\colon D \to X$ is defined and continuous, then we say that f is *of class* C^n. Finally, we say that f is *smooth* if it is of class C^n for all $n \geq 1$. $\square$

Definition A2.2. Let X be a real locally convex space, I an interval in $\mathbb{R}$ and $a, b \in I$. We say that a continuous function $f\colon I \to X$ is *weakly integrable from a to b* if there exists $x_0 \in X$ such that for every continuous linear functional $l\colon X \to \mathbb{R}$ we have

$$l(x_0) = \int_a^b (l \circ f)(t)\mathrm{d}t.$$

In this case we denote x_0 by $\int_a^b f(t)\mathrm{d}t$ and call it the *weak integral of f from a to b*. $\square$

Exercise A2.3. In the setting of Definition A2.2, prove that the vector x_0 is unique whenever it exists. $\square$

Exercise A2.4. In the setting of Definition A2.2, if X is moreover sequentially complete, prove that f is always weakly integrable from a to b. $\square$

Theorem A2.5. *Let X be a locally convex space, I an open interval in $\mathbb{R}$, and $a, b \in I$. If $f\colon I \to X$ is of class C^1, then the continuous function $\dot{f}\colon I \to X$ is weakly integrable from a to b and $\int_a^b \dot{f}(t)\mathrm{d}t = f(b) - f(a)$.*

Proof. See Theorem 1.5 in [Gl02a]. $\square$

Definition A2.6. Let X and Y be real topological vector spaces, U an open subset of X, and $f: U \to Y$ a continuous mapping. We say that f is *of class* C^1 if the limit

$$f'_x(h) := \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

exists in Y for all $x \in U$ and $h \in x$, and the mapping

$$df: U \times X \to Y, \qquad df(x,h) := f'_x(h)$$

is continuous.

Now suppose that $n \geq 1$ and

$$d^n f: U \times \underbrace{X \times \cdots \times X}_{n \text{ times}} \to Y, \qquad (x; h_1, \dots, h_n) \mapsto f_x^{(n)}(h_1, \dots, h_n)$$

was already defined and is continuous. If, for all $x \in U$ and $h_1, \dots, h_n, h_{n+1} \in X$, the limit

$$\begin{aligned} d^{n+1} f(x; h_1, \dots, h_n, h_{n+1}) &:= f_x^{(n+1)}(h_1, \dots, h_n, h_{n+1}) \\ &:= \lim_{t \to 0} \frac{f^{(n)}_{x+th_{n+1}}(h_1, \dots, h_n) - f_x^{(n)}(h_1, \dots, h_n)}{t} \end{aligned}$$

exists in X and the mapping $d^{n+1} f: U \times X^{n+1} \to Y$ is continuous, then we say that f is *of class* C^{n+1}. Furthermore, we say that f is *smooth* or *of class* $C^\infty(U,V)$ if it is of class C^n for all $n \geq 1$.

For $n = 1, 2, \dots, \infty$, we denote by $C^n(U,Y)$ the set of all mappings $U \to Y$ of class C^n.

Finally, if $X_1, \dots, X_n$ are real topological vector spaces, U is an open subset of $X_1 \times \cdots \times X_n$ and $g: U \to Y$ is a continuous mapping, then for every point $x = (x_1, \dots, x_n) \in U$ and $j \in \{1, \dots, n\}$ we define by

$$\partial_j g(x) := (g_j^x)'_{x_j}: X_j \to Y$$

the *j-th partial derivative of first order of g at x* (whenever it exists), where the mapping $g_j^x: U_j \to Y$ is defined by $g_j^x(z) := g(x_1, \dots, x_{j-1}, z, x_{j+1}, \dots, x_n)$ for all z in the open subset $U_j := \{\tilde{z} \in X_j \mid (x_1, \dots, x_{j-1}, \tilde{z}, x_{j+1}, \dots, x_n) \in U\}$ of X_j. □

Proposition A2.7. *Let X and Y be real locally convex spaces, U an open subset of X and $f: U \to Y$ a mapping of class C^1. Then for all $x \in U$ the mapping*

$$f'_x: X \to Y$$

is linear and continuous. If moreover f is of class C^n with $n \geq 2$, then for all $x \in U$ the mapping

$$f_x^{(n)}: \underbrace{X \times \cdots \times X}_{n \text{ times}} \to Y$$

is symmetric, continuous and n-linear.

Proof. See Lemma 1.9 and Proposition 1.13 in [Gl02a]. □

Remark A2.8. In the setting of Proposition A2.7, if moreover $X = \mathbb{R}$, then for all $x \in U$ and $h \in X = \mathbb{R}$ we have $f'_x(h) = h \cdot \dot{f}(x)$, where $\dot{f}(x) \in Y$ is introduced in Definition A2.1. Similarly, if moreover f is of class C^n then for all $x \in U$ and $h_1, \dots, h_n \in X = \mathbb{R}$ we have $f_x^{(n)}(h_1, \dots, h_n) = h_1 \cdots h_n \cdot f_n(x)$, where $f_n(x) \in Y$ is introduced in Definition A2.1. □

Proposition A2.9. *Let X and Y be real locally convex spaces, U an open subset of X and $f: U \to Y$ a mapping of class C^1. If for all $x \in U$ we have $f'_x = 0$, then f is constant on each connected component of U.*

Proof. See Proposition 1.11 in [Gl02a]. □

Proposition A2.10 (chain rule). *Let X, Y and Z be real locally convex spaces, U an open subset of X, V an open subset of Y and $U \xrightarrow{f} V \xrightarrow{g} Z$ mappings of class C^n, where $n \geq 1$. Then $g \circ f: U \to Z$ is of class C^n and for all $x \in U$ we have the commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "g'_{f(x)}"] & Z \\ X \arrow[u, "f'_x"] \arrow[ur, "(g\circ f)'_x"'] & \end{tikzcd}$$

that is, $(g \circ f)'_x = g'_{f(x)} \circ f'_x$.

Proof. See Propositions 1.15 and 1.12 in [Gl02a]. □

Theorem A2.11 (Taylor's formula). *Let X and Y be real locally convex spaces, U an open subset of X, and $f: U \to Y$ a mapping of class C^{n+1}, where $n \geq 0$. If $x \in U$ and $h \in X$ have the property that $x + th \in U$ whenever $0 \leq t \leq 1$, then*

$$\begin{aligned} f(x+h) =& f(x) + f'_x(h) + \frac{1}{2!} f''_x(h,h) + \cdots + \frac{1}{n!} f^{(n)}_x(h,\dots,h) \\ &+ \int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}_{x+th}(h,\dots,h)\mathrm{d}t. \end{aligned}$$

Proof. See Proposition A2.17 in [Gl02a]. □

Corollary A2.12. *Let X and Y be real Banach spaces, U an open subset of X, and $f: U \to Y$ a mapping of class C^{n+1}, where $n \geq 0$. If $x \in U$ and we denote $V_x := \{h \in X \mid (\forall t \in [0,1])\ x + th \in U\}$, then V_x is an open neighborhood of $0 \in X$ and the function $\theta: V_x \to Y$ defined for all $h \in V_x$ by*

$$f(x+h) = f(x) + f'_x(h) + \frac{1}{2!} f''_x(h,h) + \cdots + \frac{1}{n!} f^{(n)}_x(h,\dots,h) + \theta(h)$$

has the property

$$\lim_{h\to 0} \frac{||\theta(h)||}{||h||^n} = 0.$$

Proof. See either Corollary 4.4 in Chapter I in [La01], or Theorem 6 in Chapter 1 in [Nel69]. □

The converse to Corollary A2.12 holds under the following form.

Theorem A2.13. *Let X and Y be real Banach spaces, U an open subset of X and $f: X \to Y$. Suppose that for some positive integer n there exist, for $j = 0, 1, \dots, n$, the continuous mappings*

$$a_j: U \to \mathcal{B}^j(X, Y)$$

(where $\mathcal{B}^j(X,Y)$ stands for the real Banach space of symmetric multilinear mappings $X^j = X \times \cdots \times X \to Y$) such that the following assertion holds: For all $x \in U$ there exist $r > 0$ and $\theta\colon B_X(0,r) \to Y$ such that $B_X(x,r) \subseteq U$ and

$$(\forall h \in B_X(0,r)) \quad f(x+h) = a_0(x) + a_1(x)(h) + \cdots + \frac{1}{n!}a_n(x)(h,\dots,h) + \theta(h)$$

and $\lim\limits_{h\to 0} \|\theta(h)\|/\|h\|^n = 0$. Then f is of class C^n and $a_j = f^{(j)}$ for $j = 0,1,\dots,n$.

Proof. See Theorem 3 in Chapter 1 in [Nel69]. □

Analytic mappings

Definition A2.14. Let E and F be complex locally convex spaces, W an open subset of E, and $g\colon W \to F$. We say that g is *complex analytic* if it is of class $\mathcal{C}^1$ (when we view both E and F as real vector spaces) and for each $x \in W$ the mapping $g'_x\colon E \to F$ is $\mathbb{C}$-linear. (Recall from Proposition A2.7 that g'_x is always $\mathbb{R}$-linear.) The complex analytic mappings are sometimes called *holomorphic*.

If X and Y are real locally convex spaces and U is an open subset of X, then a mapping $f\colon U \to Y$ is said to be *real analytic* if there exist an open subset U_1 of the complexification $X_{\mathbb{C}}$ of X and a complex analytic mapping $f_1\colon U_1 \to Y_{\mathbb{C}}$ such that $U \subseteq U_1$ and $f_1|_U = f$. □

Proposition A2.15. *Every real or complex analytic mapping is smooth.*

Proof. See Proposition 2.4 in [Gl02a]. □

Theorem A2.16. *Let X and Y be real (respectively complex) locally convex spaces, U an open subset of X, and $f\colon U \to Y$ a smooth mapping. Then f is real (respectively complex) analytic if and only if for every $x \in U$ there exists a neighborhood V of $0 \in X$ such that $x + V \subseteq U$ and for all $h \in V$ we have*

$$f(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} f_x^{(n)}(h,\dots,h),$$

where $f_x^{(0)} := f(x)$.

Proof. See Lemma 2.5 and Definition 2.1 in [Gl02a]. □

Proposition A2.17. *Compositions of real or complex analytic mappings are real or complex analytic, respectively.*

Proof. See Propositions 2.7 and 2.8 in [Gl02a]. □

We now recall the definition of Fréchet differentiability.

Definition A2.18. Let X and Y be real Banach spaces, U an open subset of X and $f\colon U \to Y$. We say that f is *Fréchet differentiable* if for every $x_0 \in U$ there exists $T \in \mathcal{B}(X,Y)$ such that

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - T(x-x_0)\|}{\|x - x_0\|} = 0.$$

In this case T is uniquely determined by x_0 and we denote

$$df(x_0) := f'_{x_0} := T.$$

We say that f is *Fréchet smooth* if

$$df\colon U \to \mathcal{B}(X, \mathcal{B}(X,Y))$$

is Fréchet differentiable and moreover

$$d^2 f := d(df)\colon U \to \mathcal{B}(X, \mathcal{B}(X, \mathcal{B}(X,Y)))$$

is Fréchet differentiable ... and moreover

$$d^k f = d(d^{k-1} f)\colon U \to \underbrace{\mathcal{B}(X, \dots, \mathcal{B}(X}_{k+1 \text{ times}}, Y) \dots)$$

is Fréchet differentiable ... ad infinitum. □

Remark A2.19. It is clear that the notations f'_x, df etc. introduced in Definition A2.18 agree with the ones in Definition A2.6. Moreover, a mapping is Fréchet smooth if and only if it is smooth in the sense of Definition A2.6. (For a proof of this fact, see Theorem I.7 in [Ne01a].) □

Theorem A2.20. *Let X and Y be real Banach spaces, U an open subset of X, and for all $a, b \in X$ denote $D_{a,b} = \{t \in \mathbb{R} \mid a + tb \in U\}$. Then a smooth mapping $f\colon U \to Y$ is real analytic if and only if for all $a, b \in X$ the function*

$$D_{a,b} \to Y, \quad t \mapsto f(a + tb),$$

is real analytic.

Proof. See Theorem 7.5 in [BS71b]. □

Theorem A2.21. *Let E and F be complex Banach spaces, V an open subset of E and, for each $n \geq 1$, let $f_n\colon V \to F$ be a holomorphic mapping such that $\sup\limits_{x \in V} \|f_n(x)\| < \infty$. If moreover $\lim\limits_{m,n \to \infty} \Big(\sup\limits_{x \in V} \|f_n(x) - f_m(x)\|\Big) = 0$, then there exists a holomorphic mapping $f\colon V \to F$ such that $\lim\limits_{n \to \infty} \Big(\sup\limits_{x \in V} \|f_n(x) - f(x)\|\Big) = 0$.*

Proof. This is an easy application of Proposition 6.2 in [BS71b]. □

Proposition A2.22. *Let E and F be complex Banach spaces, V an open subset of E, $r > 0$ and $g\colon B_{\mathbb{C}}(0,r) \times V \to F$ a holomorphic mapping such that*

$$\sup\{\|g(t,x)\| \mid t \in B_{\mathbb{C}}(0,r), x \in V\} < \infty.$$

Define

$$f\colon B_{\mathbb{C}}(0,r) \times V \to F, \quad f(t,x) = \int_0^t g(s,x)\,ds.$$

Then f is holomorphic.

Proof. For all $t \in B_{\mathbb{C}}(0,r)$ and $x \in V$ we have $f(t,x) = \lim\limits_{n \to \infty} f_n(t,x)$, where

$$f_n t, x) = \frac{t}{n} \sum_{j=1}^{n} g(jt/n, x).$$

Then reason as in the proof of Proposition 6.3 in [BS71b], since $f_n\colon B_{\mathbb{C}}(0,r) \times V \to F$ is holomorphic for all $n \geq 1$. □

The following statement concerns the notion of continuous inverse algebra as introduced in Definition A1.20.

Lemma A2.23. *If A is a continuous inverse algebra, then the inversion mapping*

$$\eta: A^\times \to A^\times, \quad x \mapsto x^{-1},$$

is smooth and

$$d\eta: A^\times \times A \to A, (x,y) \mapsto -x^{-1}yx^{-1}.$$

Proof. For all $x, y \in A^\times$ we have

$$y^{-1} - x^{-1} = x^{-1}(x-y)y^{-1} = y^{-1}(x-y)x^{-1}.$$

Since $A^\times$ is an open subset of A, it follows that for all $x \in A^\times$ and $y \in A$ we have $x + ty \in A^\times$ whenever $|t|$ is small enough. For such t, we have by the above equation $\eta(x+ty) - \eta(x) = x^{-1}(-ty)(x+ty)^{-1}$. Since η is continuous, we get

$$\begin{aligned}\eta'_x(y) &= \lim_{t\to 0} \frac{\eta(x+ty) - \eta(x)}{t} = \lim_{t\to 0}\bigl(-\eta(x)\cdot y \cdot \eta(x+ty)\bigr) \\ &= -\eta(x)\cdot y\cdot \eta(x) = -x^{-1}yx^{-1}.\end{aligned}$$

Thus, if we consider the mapping $\tau: A\times A\times A \to A$, $(a,b,c) \mapsto abc$, and the natural projections $\mathrm{pr}_{A^\times}: A^\times \times A \to A^\times$ and $\mathrm{pr}_A: A^\times \times A \to A$, we get the following formula for the differential of η:

$$d\eta: A^\times \times A \to A, \quad d\eta = -\tau \circ \bigl((\eta\circ \mathrm{pr}_{A^\times}) \times \mathrm{pr}_A \times (\eta \circ \mathrm{pr}_A)\bigr).$$

Since all of the mappings η, $\mathrm{pr}_{A^\times}$, pr_A and τ are continuous, we deduce that $d\eta: A^\times \times A \to A$ is continuous, hence η is of class C^1. Then using the chain rule (Proposition A2.10) and the above formula for $d\eta$, we can prove by induction that η is of class C^k for $k = 1, 2, \ldots$, hence η is smooth. $\square$

Proposition A2.24. *If A is a complex continuous inverse algebra, then the inversion mapping $\eta: A^\times \to A^\times$ is complex analytic.*

Proof. We have seen in the proof of Lemma A2.23 that for each $x \in A^\times$ we have

$$\eta'_x: A \to A, \quad \eta'_x(y) = -x^{-1}yx^{-1},$$

hence η'_x is clearly $\mathbb{C}$-linear, and this is just the condition required in Definition A2.14. $\square$

Proposition A2.25. *If A is a real continuous inverse algebra, then the inversion mapping $\eta: A^\times \to A^\times$ is real analytic.*

Proof. First recall from Lemma A2.23 that η is smooth. On the other hand, it follows by Proposition A1.22 that the complexification $A_\mathbb{C}$ of A is a complex continuous inverse algebra. Hence the inversion mapping of $A_\mathbb{C}$,

$$\eta_\mathbb{C}: (A_\mathbb{C})^\times \to (A_\mathbb{C})^\times, \quad z \mapsto z^{-1},$$

is a complex analytic mapping according to Proposition A2.24.

Since $A^\times \subseteq (A_\mathbb{C})^\times$ and $\eta_\mathbb{C}|_A = \eta$, it then follows that η is real analytic. $\square$

Another example of analytic mappings is provided by the following exercise.

Exercise A2.26.
(a) Let X be a real Banach space and

$$\Theta\colon \mathbb{C} \to \mathbb{C}, \Theta(z) = \sum_{n=0}^{\infty} a_n z^n,$$

an entire function with $a_n \in \mathbb{R}$ for all $n \geq 0$. . Prove that for every $T \in \mathcal{B}(X)$ the series $\sum_{n=0}^{\infty} a_n T^n$ is convergent in the real Banach space $\mathcal{B}(E)$, and the mapping

$$\Theta\colon \mathcal{B}(X) \to \mathcal{B}(X), \Theta(T) := \sum_{n=0}^{\infty} a_n T^n,$$

is real analytic.
(b) Formulate and prove a version of assertion (a) where X is replaced by a complex Banach space and the entire series Θ has arbitrary coefficients. $\square$

Notes

We refer to the paper by J. Milnor [Mi84] for a quick review of the differential calculus in locally convex spaces. The detailed proofs of these results can be found in [Gl02a]. See also [Ke74], [Ht82] and [Ne01a].

The book [La01] contains a good exposition of the basic results in differential calculus in the framework of Banach spaces. See also [Nel69].

An introduction to analytic mappings on Banach spaces can be found in Chapter 1 in [Up85]. See [BS71a] and [BS71b] for analytic mappings on more general topological vector spaces. The analyticity of the inversion mapping in a continuous inverse algebra (Proposition A2.25) was proved in [Gl02b].

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A2$\frac{1}{2}$. BASIC DIFFERENTIAL EQUATIONS OF LIE THEORY

Abstract. The importance of this appendix should not be overlooked. We develop here the equations and formulas describing the most basic level of Lie theory, namely the theory of local Lie groups. It turns out that the theory of Banach-Lie groups heavily leans on the basic theorems concerning ordinary differential equations in Banach spaces.

Throughout this appendix we denote by Y a real Banach space, and for every $r > 0$ we denote by

$$B_Y(y_0, r) = \{y \in Y \mid \|y - y_0\| < r\}$$

the open ball with center at y_0 and radius r. Also, we denote by $\mathrm{GL}(Y)$ the set of all invertible bounded linear operators on Y.

Theorem A2$\frac{1}{2}$.1. *Let B be an open subset of Y such that $0 \in B$, J an open interval in $\mathbb{R}$, and $g\colon J \times B \to Y$ a smooth mapping. For $j \in \{1, 2\}$, consider an open interval I_j contained in J and a smooth function $\gamma_j\colon I_j \to Y$ such that for each $t \in I_j$ we have $\dot\gamma_j(t) = g(t, \gamma_j(t))$. If there exists $t_0 \in I_1 \cap I_2$ such that $\gamma_1(t_0) = \gamma_2(t_0)$, then $\gamma_1|_{I_1\cap I_2} = \gamma_2|_{I_1\cap I_2}$.*

Proof. See Theorem 1.3 in Chapter IV in [La01]. $\square$

Theorem A2$\frac{1}{2}$.2. *Let B be an open subset of Y such that $0 \in B$, and $g\colon B \times Y \to Y$ a smooth mapping. Then there exist $r > 0$ and $\varepsilon \in (0, 1)$ such that the following conditions are fulfilled.*

(a) *We have $B_Y(0, r) \subseteq B$.*

(b) *There exists a unique smooth mapping $\gamma\colon (-\varepsilon, \varepsilon) \times B_Y(0, r) \to B$ such that, for all $v \in B_Y(0, r)$, the mapping $\gamma_v(\cdot) := \gamma(\cdot, v)\colon (-\varepsilon, \varepsilon) \to B$ has the properties*

$$\gamma_v(0) = 0$$

and

$$(\forall t \in (-\varepsilon, \varepsilon)) \qquad \dot\gamma_v(t) = g(\gamma_v(t), v).$$

Proof. Define $E := Y \times Y$, $U := B \times Y$, $J = (-1, 1)$, and

$$f\colon J \times U \to E, \qquad f(t, (y, v)) := (g(y, v), 0) \text{ for } t \in J, y \in B, v \in Y.$$

Then use Theorem 1.11 in Chapter IV in [La01] to get $r > 0$, $\varepsilon \in (0, 1)$ and $\alpha\colon J_0 \times U_0 \to U$ such that $J_0 := (-\varepsilon, \varepsilon) \subseteq J$, $0 \in U_0$, U_0 is an open subset of U and α is the unique smooth mapping satisfying the conditions that for all $x \in U_0$ we have $\alpha(0, x) = x$ and the function $\alpha_x(\cdot) := \alpha(\cdot, x)\colon J_0 \to U$ has the property $\dot\alpha_x(t) = f(t, \alpha_x(t))$ for each $t \in J_0$.

On the other hand, since U_0 is an open subset of $U(\subseteq Y \times Y)$ and $0 \in U_0$, we can find $r > 0$ such that $B_Y(0,r) \times B_Y(0,r) \subseteq U_0$. Also, since $U = B \times Y$ and α takes values in U, it follows that there exist $\gamma_1 \colon J_0 \times U_0 \to B$ and $\beta \colon J_0 \times U_0 \to Y$ such that $\alpha(t,x) = (\gamma_1(t,x), \beta(t,x))$ for all $t \in J_0$ and $x \in U_0$.

It then follows that for each point $x = (y,v) \in U_0 \subseteq B \times Y$ we have

$$(y,v) = x = \alpha(0,x) = (\gamma_1(0,x), \beta(0,x))$$

and for all $t \in J_0$

$$(\dot{\gamma}_1(t,x), \dot{\beta}(t,x)) = (g(\gamma_1(t,x), \beta(t,x)), 0).$$

The relation $\dot{\beta}(t,x) = 0$ implies that the function $t \mapsto \beta(t,x)$ is constant, hence for all $t \in J_0$ we have $\beta(t,x) = \beta(0,x) = v$. Thus for all $t \in J_0$, $y, v \in B_Y(0,r)$, setting $x = (y,v)$ in the above relations we get

$$\dot{\gamma}_1(t,y,v) = g(\gamma_1(t,y,v), v) \text{ and } \gamma_1(0,y,v) = y.$$

Consequently, the function $\gamma(\cdot,\cdot) := \gamma_1(\cdot,0,\cdot) \colon J_0 \times B_Y(0,r) \to B$ has the desired properties. Its uniqueness follows by Theorem A$2\frac{1}{2}$.1. $\square$

Proposition A$2\frac{1}{2}$.3. *Let $r_1 > 0$ and*

$$\Psi \colon B_Y(0,r_1) \to \mathcal{B}(Y)$$

smooth. Then there exist $r \in (0,r_1)$ and $\varepsilon \in (0,1)$ such that there exists a unique smooth mapping $\gamma \colon (-\varepsilon,\varepsilon) \times B_Y(0,r) \to B_Y(0,r_1)$ with the following property: If $v \in B_Y(0,r)$ and we define $\gamma_v := \gamma(\cdot,v) \colon (-\varepsilon,\varepsilon) \to B_Y(0,r_1)$, then

$$\gamma_v(0) = 0$$

and

$$\dot{\gamma}_v(t) = \Psi(\gamma_v(t))v \text{ whenever } t \in (-\varepsilon,\varepsilon).$$

Proof. Just use Theorem A$2\frac{1}{2}$.2 for $B = B_Y(0,r_1)$ and $g \colon B \times Y \to Y$, $g(y,v) := \Psi(y)v$. $\square$

Exercise A$2\frac{1}{2}$.4. In the setting of Proposition A$2\frac{1}{2}$.3 we have

$$\gamma(ts,v) = \gamma(t,sv)$$

whenever $t, ts \in (-\varepsilon,\varepsilon)$ and $v, sv \in B_Y(0,r)$. $\square$

Proposition A$2\frac{1}{2}$.5. *Let $r_1 > 0$ and*

$$\Psi \colon B_Y(0,r_1) \to \mathcal{B}(Y)$$

smooth. Then there exists $r_2 \in (0,r_1)$ such that there exists a unique smooth mapping

$$\chi \colon B_Y(0,r_2) \to B_Y(0,r_1)$$

with the following properties:

(i) *We have* $\chi(0) = 0$ *and* $\chi_0' = \Psi(0)$.
(ii) *For all* $v \in B_Y(0, r_2) \setminus \{0\}$, *the function*

$$\chi_v \colon (-r_2/\|v\|, r_2/\|v\|) \to B_Y(0, r_1), \quad t \mapsto \chi(tv),$$

satisfies

$$\dot{\chi}_v(t) = \Psi(\chi_v(t))v \text{ whenever } t \in (-r_2/\|v\|, r_2/\|v\|).$$

Proof. Let $r \in (0, r_1)$, $\varepsilon \in (0,1)$ and $\gamma \colon (-\varepsilon, \varepsilon) \times B_Y(0,r) \to B_Y(0, r_1)$ given by Proposition A2$\frac{1}{2}$.3, and define $r_2 := r\varepsilon/2 (< r)$. Next define

$$\chi \colon B_Y(0, r_2) \to B_Y(0, r_1), \quad \chi(v) := \gamma(\varepsilon/2, (2/\varepsilon)v).$$

Then χ is smooth and $\chi(0) = 0$ according to Proposition A2$\frac{1}{2}$.3.

Now fix $v \in B_Y(0, r_2)$ and let $\chi_v \colon (-r_2/\|v\|, r_2/\|v\|) \to B_Y(0, r_1)$, $\chi_v(t) = \chi(tv)$, as in the statement. We then have by Exercise A2$\frac{1}{2}$.4 that

$$\chi_v(t) = \gamma(\varepsilon/2, (2/\varepsilon)tv) = \gamma((\varepsilon/2)t, (2/\varepsilon)v) = \gamma_{(2/\varepsilon)v}((\varepsilon/2)t)$$

whenever $|t| < r_2/\|v\|$, whence

$$\begin{aligned} \dot{\chi}_v(t) &= \dot{\gamma}_{(2/\varepsilon)v}((\varepsilon/2)t) \\ &= (\varepsilon/2)\Psi(\gamma_{(2/\varepsilon)v}(t))(2/\varepsilon)v \qquad \text{(by Proposition A2}\tfrac{1}{2}\text{.3)} \\ &= \Psi(\gamma_{(2/\varepsilon)v}(t))v \\ &= \Psi(\chi_v(t))v, \end{aligned}$$

as desired. For $t = 0$ we get $\chi_0' v = \Psi(0)v$ whenever $0 \neq v \in B_Y(0, r_2)$, hence the linear operators $\chi_0', \Psi(0) \in \mathcal{B}(Y)$ coincide.

To conclude the proof, let us note that the uniqueness of a smooth function $\chi \colon B_Y(0, r_2) \to B_Y(0, r_1)$ satisfying the conditions (i) and (ii) follows by Theorem A2$\frac{1}{2}$.1. □

Proposition A2$\frac{1}{2}$.6. *Let* $0 < r_2 < r_1$ *and*

$$\mu \colon B_Y(0, r_1) \times B_Y(0, r_1) \to Y$$

a smooth mapping such that

$$\mu(y, 0) = y \text{ for all } y \in B_Y(0, r_1),$$

$\mu(B_Y(0, r_2) \times B_Y(0, r_2)) \subseteq B_Y(0, r_1)$, *and*

$$\mu(\mu(y,z),v) = \mu(y, \mu(z,v))$$

whenever $y, z, v \in B_Y(0, r_2)$. *Furthermore, let* $\varepsilon > 0$, $y_0 \in Y$ *and a smooth mapping* $\gamma \colon (-\varepsilon, \varepsilon) \to B_Y(0, r_2)$ *such that* $\gamma(0) = 0$ *and*

$$\dot{\gamma}(t) = \partial_2 \mu(\gamma(t), 0) y_0 \text{ whenever } t \in (-\varepsilon, \varepsilon).$$

Then for all $t,s\in(-\varepsilon/2,\varepsilon/2)$ we have

$$\mu(\gamma(t),\gamma(s))=\gamma(t+s).$$

Proof. First remark that, for all $y,z,v\in B_Y(0,r_2)$ it follows by hypothesis that

$$\partial_2\mu(\mu(y,z),v)=\partial_2\mu(y,\mu(z,v))\partial_2\mu(z,v).$$

Now fix $s\in(-\varepsilon/2,\varepsilon/2)$ and define

$$\alpha,\beta\colon(-\varepsilon/2,\varepsilon/2)\to Y,\quad \alpha(t):=\mu(\gamma(s),\gamma(t)),\ \beta(t):=\gamma(s+t).$$

Then $\alpha(0)=\mu(\gamma(s),\gamma(0))=\mu(\gamma(s),0)=\gamma(s)=\beta(0)$. Also, for all $t\in(-\varepsilon/2,\varepsilon/2)$ we have

$$\dot\beta(t)=\dot\gamma(s+t)=\partial_2\mu(\gamma(s+t),0)y_0=\partial_2\mu(\beta(t),0)y_0,$$

and

$$\begin{aligned}\dot\alpha(t)&=\partial_2\mu(\gamma(s),\gamma(t))\dot\gamma(t)\\&=\partial_2\mu(\gamma(s),\gamma(t))\partial_2\mu(\gamma(t),0)y_0\\&=\partial_2\mu(\mu(\gamma(s),\gamma(t)),0)y_0\\&=\partial_2\mu(\alpha(t),0)y_0,\end{aligned}$$

where the second equality follows by the beginning remark. Thus Theorem A2$\frac12$.1 (with $B=B_Y(0,r_2)$, $J_1=J_2=(-\varepsilon/2,\varepsilon/2)$, $g(t,y)=\partial_2(0,y)y_0$, $\gamma_1=\alpha$ and $\gamma_2=\beta$) shows that $\alpha=\beta$ on $(-\varepsilon/2,\varepsilon/2)$, and this is just the desired equality. □

Corollary A2$\frac12$.7. *Let $0<r_2<r_1$ and*

$$\mu\colon B_Y(0,r_1)\times B_Y(0,r_1)\to Y$$

a smooth mapping such that

$$\mu(y,0)=y \text{ for all } y\in B_Y(0,r_1),$$

$\mu(B_Y(0,r_2)\times B_Y(0,r_2))\subseteq B_Y(0,r_1)$ *and*

$$\mu(\mu(y,z),v)=\mu(y,\mu(z,v))$$

whenever $y,z,v\in B_Y(0,r_2)$. Then there exists $r_3\in(0,r_2)$ such that there exists a unique smooth mapping

$$\chi\colon B_Y(0,r_3)\to B_Y(0,r_2)$$

with the following properties:

(i) *We have $\chi(0)=0$ and $\chi'_0=\Psi(0)$.*

(ii) *If $v\in B_Y(0,r_3)$ and $\max\{|t|,|s|\}<r_3/(2\|v\|)$, then*

$$\chi((t+s)v)=\mu(\chi(tv),\chi(sv)).$$

Proof. Construct r_3 and χ by using Proposition A2$\frac12$.5 for

$$\Psi\colon B_Y(0,r_2)\to\mathcal{B}(Y),\quad \Psi(y)=\partial_2\mu(y,0),$$

and then use Proposition A2$\frac12$.6 to get the desired property (ii). □

Proposition A2$\frac{1}{2}$.8. *Let $0 < r_2 < r_1$ and*

$$\mu\colon B_Y(0,r_1) \times B_Y(0,r_1) \to Y$$

a smooth mapping such that

$$\mu(y,0) = \mu(0,y) = y \text{ for all } y \in B_Y(0,r_1),$$

$\mu(B_Y(0,r_2) \times B_Y(0,r_2)) \subseteq B_Y(0,r_1)$ *and*

$$\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$$

whenever $x,y,z \in B_Y(0,r_2)$. For all $x \in B_Y(0,r_1)$ denote

$$a(x) := \partial_2\mu(x,0) \in \mathcal{B}(Y).$$

Then there exists $r_3 \in (0,r_2)$ such that $\mu\big(B_Y(0,r_3) \times B_Y(0,r_3)\big) \subseteq B_Y(0,r_2)$ and the following conditions are fulfilled:

(i) *For all $x \in B_Y(0,r_3)$ we have $a(x) \in \mathrm{GL}(Y)$.*
(ii) *For all $x \in B_Y(0,r_3)$ and $h,k \in Y$ we have*

$$(a^{-1})'_0(h,k) - (a^{-1})'_0(k,h) = (a^{-1})'_x(a(x)h, a(x)k) - (a^{-1})'_x(a(x)k, a(x)h).$$

Proof. For all $x,y \in B_Y(0,r_1)$ denote

$$a(x,y) := \partial_2\mu(x,y) \in \mathcal{B}(Y),$$

so that $a(x) = a(x,0)$.

Since $\mu(0,y) = y$ for all $x \in B_Y(0,r_1)$, it follows that

$$a(0,0) = \partial_2\mu(0,0) = \mathrm{id}_Y \in \mathrm{GL}(Y).$$

Since the mapping $a(\cdot,\cdot)\colon B_Y(0,r_1) \times B_Y(0,r_1) \to \mathcal{B}(Y)$ is continuous and $\mathrm{GL}(Y)$ is open in $\mathcal{B}(Y)$, we can find $r_3 \in (0,r_2)$ such that $a\big(B_Y(0,r_3) \times B_Y(0,r_3)\big) \subseteq \mathrm{GL}(Y)$, and in particular condition (i) is satisfied. Since $\mu(0,0) = 0 \in B_Y(0,r_2)$ and μ is also continuous, it follows that, maybe by shrinking r_3, we may assume that $\mu\big(B_Y(0,r_3) \times B_Y(0,r_3)\big) \subseteq B_Y(0,r_2)$ as well.

Now, to check condition (ii), differentiate the formula $\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$ with respect to z to get $\partial_2\mu(x,\mu(y,z))\partial_2\mu(y,z) = \partial_2\mu(\mu(x,y),z)$, that is,

$$a(x,\mu(y,z))a(y,z) = a(\mu(x,y),z)$$

for all $x,y,z \in B_Y(0,r_3)$. For $z = 0$ we get

$$\boxed{a(x,y)a(y) = a(\mu(x,y))}, \tag{1}$$

whence for $x,y \in B_Y(0,r_3)$ we have

$$a(\mu(x,y))^{-1}a(x,y) = a(y)^{-1}$$

and thus $a(\mu(x,y))^{-1}a(x,y)h = a(y)^{-1}h$ for all $h \in Y$. Differentiating the latter equation with respect to y, we get

$$(a^{-1})'_{\mu(x,y)}(a(x,y)k, a(x,y)h) + a^{-1}(\mu(x,y))\big(\partial_2 a(x,y)(k,h)\big) = (a^{-1})'_y(k,h)$$

whenever $h,k \in Y$. Setting $y = 0$ we get

$$(\forall h,k \in Y) \qquad (a^{-1})'_x(a(x)k, a(x)h) + (a^{-1})(x)\big(\partial_2^2\mu(x,0)(k,h)\big) = (a^{-1})'_0(k,h).$$

Similarly,

$$(\forall h,k \in Y) \qquad (a^{-1})'_x(a(x)h, a(x)k) + (a^{-1})(x)\big(\partial_2^2\mu(x,0)(h,k)\big) = (a^{-1})'_0(h,k).$$

Now, by subtracting the latter two equations and taking into account that the continuous bilinear mapping

$$\partial_2^2\mu(x,0)\colon Y \times Y \to Y$$

is symmetric, we get the desired formula in condition (ii). $\square$

Theorem A2$\frac{1}{2}$.9. *Let $0 < r_2 < r_1$ and*

$$\mu\colon B_Y(0,r_1) \times B_Y(0,r_1) \to Y$$

a smooth mapping such that

$$\mu(y,0) = \mu(0,y) = y \text{ for all } y \in B_Y(0,r_1),$$

$\mu\big(B_Y(0,r_2) \times B_Y(0,r_2)\big) \subseteq B_Y(0,r_1)$ *and*

$$\mu(\mu(x,y),z) = \mu(x,\mu(y,z))$$

whenever $x,y,z \in B_Y(0,r_2)$. For all $x \in B_Y(0,r_1)$ denote

$$a(x) := \partial_2\mu(x,0) \in \mathcal{B}(Y), \quad b(x) := \partial_1\mu(0,x) \in \mathcal{B}(Y).$$

Then there exists $r_3 \in (0,r_2)$ such that $\mu\big(B_Y(0,r_3) \times B_Y(0,r_3)\big) \subseteq B_Y(0,r_2)$ and the following conditions are fulfilled:

(i) *For all $x \in B_Y(0,r_3)$ we have $a(x), b(x) \in \mathrm{GL}(Y)$.*

(ii) *For all $x \in B_Y(0,r_3)$ and $h,k \in Y$ we have*

$$a(x)\big((a^{-1})'_x(h,k) - (a^{-1})'_x(k,h)\big) = -b(x)\big((b^{-1})'_x(h,k) - (b^{-1})'_x(k,h)\big).$$

Proof. As in the first part of the proof of Proposition A2$\frac{1}{2}$.8, we can find $r_3 \in (0,r_2)$ such that condition (i) is satisfied. We just have to take into account also the continuity of the mapping

$$b(x,y) = \partial_1\mu(x,y)$$

at $(0,0) \in B_Y(0,r_1) \times B_Y(0,r_1)$, along with the remark that $b(0,0) = \mathrm{id}_Y \in \mathrm{GL}(Y)$.

To check the equation in condition (ii), define

$$(\forall x \in B_Y(0,r_3)) \qquad \Gamma(x)\colon Y \times Y \to Y,\ \Gamma(x)(h,k) = -a'_x(h, a^{-1}(x)k),$$

and similarly

$$(\forall x \in B_Y(0,r_3)) \qquad \widetilde{\Gamma}(x)\colon Y \times Y \to Y,\ \widetilde{\Gamma}(x)(h,k) = -b'_x(h, b^{-1}(x)k).$$

By differentiating the equation $a(x)a^{-1}(x)k = k$ with respect to x, we get for all $h, k \in Y$

$$a'_x(h, a^{-1}(x)k) + a(x)(a^{-1})'_x(h,k) = 0$$

and similarly

$$a'_x(k, a^{-1}(x)h) + a(x)(a^{-1})'_x(k,h) = 0.$$

By subtracting the latter two equations we get for all $x \in B_Y(0,r_3)$, $h, k \in Y$,

$$a(x)\big((a^{-1})'_x(h,k) - (a^{-1})'_x(k,h)\big) = \Gamma(x)(h,k) - \Gamma(x)(k,h),$$

and similarly

$$b(x)\big((b^{-1})'_x(h,k) - (b^{-1})'_x(k,h)\big) = \widetilde{\Gamma}(x)(h,k) - \widetilde{\Gamma}(x)(k,h).$$

Hence it suffices to prove that

$$(2) \qquad (\forall x \in B_Y(0,r_3))\,(\forall h,k \in Y) \qquad \Gamma(x)(h,k) = \widetilde{\Gamma}(x)(k,h).$$

To this end, recall from formula (1) in the proof of Proposition A2$\frac{1}{2}$.8 that for all $x, y \in B_Y(0,r_3)$ and $k \in Y$ we have $a(\mu(x,y))k = a(x,y)a(y)k$. By differentiating the latter equation with respect to x, we get for all $h, k \in Y$,

$$a'_{\mu(x,y)}(b(x,y)h, k) = \partial_1 a(x,y)(h, a(y)k).$$

Since $\partial_1 a(x,y) = \partial_1\partial_2\mu(x,y) = \partial_2\partial_1\mu(x,y) = \partial_2 b(x,y)$, we further deduce that

$$a'_{\mu(x,y)}(b(x,y)h, k) = \partial_2 b(x,y)(h, a(y)k).$$

Setting $x = 0$ and taking into account that $\partial_2 b(0,y) = b'_y$, we get $a'_y(b(y)h,k) = b'_y(h, a(y)k)$, that is,

$$(\forall x \in B_Y(0,r_3))\,(\forall h,k \in Y) \quad a'_x(b(x)h,k) = b'_x(h, a(x)k).$$

Since $a(x), b(x) \in \mathcal{B}(Y)$ are invertible operators whenever $x \in B_Y(0,r_3)$, we can use the substitutions $h_0 = b(x)h$ and $k_0 = a(x)k$ to deduce from the above equation that

$$\underbrace{a'_x(h_0, a^{-1}(x)k_0)}_{=\Gamma(x)(h_0,k_0)} = b'_x(b^{-1}(x)h_0, k_0) = \underbrace{b'_x(k_0, b^{-1}(x)h_0)}_{=\widetilde{\Gamma}(x)(k_0,h_0)},$$

where the latter equality follows by the fact that the bilinear mapping

$$b'_x = \partial_1\partial_2\mu(x,0)\colon Y \times Y \to Y$$

is symmetric. Consequently $\Gamma(x)(h_0,k_0) = \widetilde{\Gamma}(x)(k_0,h_0)$, and (2) is proved. $\square$

Proposition A2$\frac{1}{2}$.10. *Let E be a complex Banach space, U an open subset of E, c a positive real number, $t_0 \in \mathbb{C}$ and*

$$f\colon B_{\mathbb{C}}(t_0, c) \times U \to E$$

a holomorphic function. Then for all $x_0 \in U$ there exist $b \in (0, c)$ and a unique holomorphic function

$$\alpha\colon B_{\mathbb{C}}(t_0, b) \to U$$

such that

$$(\forall t \in B_{\mathbb{C}}(t_0, b)) \quad \dot{\alpha}(t) = f(t, \alpha(t))$$

and $\alpha(t_0) = x_0$.

Proof. First pick $R \in (0, c)$ and $a \in (0, 1)$ such that $B_E(x_0, a) \subseteq U$ and there exist $L, K \geq 1$ with

$$\sup\{\|f(t, x)\| \mid t \in B_{\mathbb{C}}(t_0, R), x \in B_E(x_0, a)\} \leq K$$

and

$$\sup\{\|\partial_2 f(t, x)\| \mid t \in B_{\mathbb{C}}(t_0, R), x \in B_E(x_0, a)\} \leq L.$$

(The existence of R and a follows since both mappings $f\colon B_{\mathbb{C}}(0, b) \times U \to E$ and $\partial_2 f\colon B_{\mathbb{C}}(0, b) \times U \to \mathcal{B}(E)$ are continuous, hence bounded on some neighborhood of (t_0, x_0).) In particular, the condition on L implies that

$$(3) \quad (\forall t \in B_{\mathbb{C}}(t_0, R))\ (x_1, x_2 \in B_E(x_0, a)) \qquad \|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|,$$

according to the mean value theorem.

Now pick a real number b such that $0 < b < \min\{R, \frac{a}{LK}\}$, and define

$$M = \{\beta\colon B_{\mathbb{C}}(t_0, b) \to E \mid \beta \text{ holomorphic and } \sup_{t \in B_{\mathbb{C}}(t_0, b)} \|\beta(t) - x_0\| \leq 2a\}$$

and

$$(\forall \beta_1, \beta_2 \in M) \quad \operatorname{dist}(\beta_1, \beta_2) := \sup_{t \in B_{\mathbb{C}}(t_0, b)} \|\beta_1(t) - \beta_2(t)\|,$$

thus making M into a complete metric space (see Theorem A2.21).

On the other hand, for each $\beta \in M$, define

$$S\beta\colon B_{\mathbb{C}}(t_0, b) \to E, \quad (S\beta)(t) = x_0 + \int_{t_0}^{t} f(s, \beta(s))\mathrm{d}s.$$

Then Proposition A2.22 shows that $S\beta$ is a holomorphic function. Moreover, for all $t \in B_{\mathbb{C}}(t_0, b)$ we have

$$\|(S\beta)(t) - x_0\| \leq \int_{t_0}^{t} \|f(s, \beta(s))\|\mathrm{d}s \leq K|t - t_0| \leq Kb \leq 2a,$$

hence $S\beta \in M$. Furthermore, for all $\beta_1, \beta_2 \in M$ and all $t \in B_{\mathbb{C}}(t_0, b)$ we have

$$\begin{aligned}\|(S\beta_1)(t) - (S\beta_2)(t)\| &\leq \int_{t_0}^{t} \|f(s, \beta_1(s)) - f(s, \beta_2(s))\| \mathrm{d}s \\ &\leq |t - t_0| L \cdot \operatorname{dist}(\beta_1, \beta_2) \leq bL \cdot \operatorname{dist}(\beta_1, \beta_2) \\ &\leq a \cdot \operatorname{dist}(\beta_1, \beta_2).\end{aligned}$$

Hence, on the complete metric space M, we have a mapping $S: M \to M$ such that there exists $a \in (0,1)$ with $\operatorname{dist}(S\beta_1, S\beta_2) \leq a \cdot \operatorname{dist}(\beta_1, \beta_2)$ whenever $\beta_1, \beta_2 \in M$. It then follows that S has a unique fixed point $\alpha \in M$. The equation $S\alpha = \alpha$ is clearly equivalent to the required properties of α. □

Corollary A2$\frac{1}{2}$.11. *Let V be an open subset of the real Banach space X, J an open interval in $\mathbb{R}$ and*

$$g: J \times U \to X$$

a real analytic mapping. If $\gamma: J \to U$ is a smooth function such that

$$\dot{\gamma}(t) = g(t, \gamma(t)) \text{ whenever } t \in J,$$

then γ is real analytic.

Proof. It easily follows by Theorem A2.16 that, in order to prove that γ is real analytic, it suffices to show that γ is real analytic on some neighborhood of an arbitrary point $t_0 \in J$. Denote $x_0 = \gamma(t_0) \in U$.

It is clear that the complexification of the real Banach space $\mathbb{R} \times X$ is $\mathbb{C} \times X_{\mathbb{C}}$. Since g is real analytic, it then follows that there exists an open subset W of $\mathbb{C} \times X_{\mathbb{C}}$ and a holomorphic mapping $f: W \to X_{\mathbb{C}}$ such that $J \times V \subseteq W$ and $f|_{J \times U} = g$. Since $(t_0, x_0) \in W$ it follows that, by shrinking J and W, we may assume that there exist $c > 0$ and an open subset U of $X_{\mathbb{C}}$ such that $V \subseteq U$ and $W = B_{\mathbb{C}}(t_0, c) \times U$. It follows by Proposition A2$\frac{1}{2}$.10 that there exist $b \in (0, c)$ and a holomorphic mapping $\alpha: B_{\mathbb{C}}(t_0, b) \to U$ such that $\alpha(t_0) = x_0$ and

$$\alpha'(t) = f(t, \alpha(t)) \text{ whenever } t \in B_{\mathbb{C}}(t_0, b).$$

Then using Theorem A2$\frac{1}{2}$.1 for the functions

$$\alpha|_{(t_0 - c, t_0 + c)}: (t_0 - c, t_0 + c) \to X_{\mathbb{C}} \text{ and } \gamma: (t_0 - c, t_0 + c) \to X \hookrightarrow X_{\mathbb{C}},$$

we get $\gamma = \alpha|_{(t_0 - c, t_0 + c)}$, hence γ is real analytic. □

Theorem A2$\frac{1}{2}$.12. *Let $0 < r_2 < r_1$ and*

$$\mu: B_Y(0, r_1) \times B_Y(0, r_1) \to Y$$

a smooth mapping such that

$$\mu(y, 0) = \mu(0, y) = y \text{ for all } y \in B_Y(0, r_1),$$

$\mu(B_Y(0, r_2) \times B_Y(0, r_2)) \subseteq B_Y(0, r_1)$ *and*

$$\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$$

whenever $x, y, z \in B_Y(0, r_2)$. *Moreover assume that*

$$\boxed{\mu(tx, sx) = (t+s)x}$$

whenever $0 \neq x \in B_Y(0, r_1)$, $t, s \in \mathbb{R}$ *and* $\max\{|t|, |s|\} < \frac{r_1}{2\|x\|}$. *Then the mapping* μ *is real analytic on some neighborhood of* $(0,0) \in Y \times Y$.

Proof. The proof has several stages.

1° For all $x, y \in B_Y(0, r_1)$ denote as usually

$$a(x,y) = \partial_2\mu(x,y), \quad a(x) = a(x,0)$$
$$b(x,y) = \partial_1\mu(x,y), \quad b(y) = b(0,y),$$

so that $a(x), a(x,y), b(x), b(x,y) \in \mathcal{B}(Y)$.

It then follows by Proposition A2$\frac{1}{2}$.8 and Theorem A2$\frac{1}{2}$.9 that there exists $\tilde{r}_3 \in (0, r_2)$ such that $\mu(B_Y(0,\tilde{r}_3), B_Y(0,\tilde{r}_3)) \subseteq B_Y(0, r_2)$ and the following assertions hold.

(i) For all $x \in B_Y(0, \tilde{r}_3)$ we have $a(x), b(x) \in \mathrm{GL}(Y)$.

(ii) For all $x \in B_Y(0, \tilde{r}_3)$ and $h, k \in Y$ we have

$$\begin{aligned} S(h,k) :=& (a^{-1})'_0(h,k) - (a^{-1})'_0(k,h) \\ =& (a^{-1})'_x(a(x)h, a(x)k) - (a^{-1})'_x(a(x)k, a(x)h), \end{aligned}$$

and we thus get a skew-symmetric bounded bilinear mapping

$$S\colon Y \times Y \to Y.$$

(iii) For all $x \in B_Y(0, \tilde{r}_3)$ and $h, k \in Y$ we have

$$a(x)\big((a^{-1})'_x(h,k) - (a^{-1})'_x(k,h)\big) = -b(x)\big((b^{-1})'_x(h,k) - (b^{-1})'_x(k,h)\big).$$

Moreover, note that the boxed hypothesis implies that

$$(4) \qquad a(tx)x = b(tx)x = x \text{ whenever } 0 \neq x \in B_Y(0, r_1), |t| < \frac{r_1}{2\|x\|}.$$

2° We prove at this stage that the mappings

$$a(\cdot), b(\cdot)\colon B_Y(0, \tilde{r}_3) \to \mathcal{B}(Y)$$

are real analytic. Actually, we are going to consider only the case of $a(\cdot)$, since the case of $b(\cdot)$ can be treated similarly.

To prove that $a(\cdot)$ is real analytic, we denote

$$(\forall x \in Y) \quad S_x := S(x, \cdot) \in \mathcal{B}(Y)$$

and we will prove that

$$(5) \qquad (\forall x \in B_Y(0, \tilde{r}_3)) \quad a^{-1}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} S_x^{n-1} = \Theta(S_x),$$

where $\Theta\colon \mathbb{C} \to \mathbb{C}$ is the entire function defined by $\Theta(z) = \sum\limits_{n=1}^{\infty} \frac{1}{n!} z^{n-1} = (\mathrm{e}^z - 1)/z$. Since the mapping $Y \to \mathcal{B}(Y)$, $x \mapsto S_x$, is real analytic (being linear), while the mapping $\mathcal{B}(Y) \to \mathcal{B}(Y)$, $T \mapsto \Theta(T)$, is real analytic by Exercise A2.26 (a), it will follow by Proposition A2.17 that their composition is real analytic. But (5) shows that the corresponding composition is just $a^{-1}(\cdot)$. On the other hand, the inversion mapping $\eta\colon \mathrm{GL}(Y) \to \mathrm{GL}(Y)$, $T \mapsto T^{-1}$, is real analytic by Exercise A1.23 along with Proposition A2.25. Hence, by Proposition A2.17 again, $a(\cdot) = \eta \circ (a^{-1})(\cdot)$ is real analytic, as desired.

Now, to prove (5), fix $x, y \in B_Y(0, \tilde{r}_3)$ and an open interval $I \subseteq \mathbb{R}$ such that $0, 1 \in I$ and $tx \in B_Y(0, \tilde{r}_3)$ whenever $t \in I$. Then define

$$\varphi\colon I \to \mathcal{B}(Y), \quad \varphi(t) = ta^{-1}(tx),$$

and

$$\psi\colon I \to Y, \quad \psi(t) = \varphi(t)y = ta^{-1}(tx)y.$$

Then for all $t \in I$ we have

$$\dot{\psi}(t) = a^{-1}(tx)y + t(a^{-1})'_{tx}(x, y).$$

On the other hand, by (4) along with assertion (ii) in stage 1° of the present proof, we have

$$\begin{aligned}(a^{-1})'_{tx}(x, y) - (a^{-1})'_{tx}(y, x) =& (a^{-1})'_{tx}\big(a(tx)x, a(tx)(a^{-1}(tx)y)\big) \\ & - (a^{-1})'_{tx}\big(a(tx)(a^{-1}(tx)y), a(tx)x\big) \\ =& (a^{-1})'_0(x, a^{-1}(tx)y) - (a^{-1})'_0(a^{-1}(tx)y, x) \\ =& S(x, a^{-1}(tx)y) \\ =& S_x(a^{-1}(tx)y),\end{aligned}$$

so that

$$\dot{\psi}(t) = a^{-1}(tx)y + tS_x a^{-1}(tx)y = a^{-1}(tx)y + t(a^{-1})'_{tx}(y, x) + S_x\varphi(t)y.$$

If we write (4) under the form $a^{-1}(tx)x = x$ and then differentiate this equation with respect to x, we get $(a^{-1})'_{tx}(ty, x) + a^{-1}(tx)y = y$ for all $y \in Y$, hence

$$(\forall y \in Y) \quad \dot{\psi}(t) = y + S_x\varphi(t)y,$$

whence

$$(\forall t \in I) \quad \dot{\varphi}(t) = \mathrm{id}_Y + S_x\varphi(t).$$

Since $\varphi(0) = 0$, we get by Theorem A2$\frac{1}{2}$.1

$$(\forall t \in I) \quad \varphi(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} S_x^{n-1}.$$

We recall that $\varphi(t) = ta^{-1}(tx)$ and $1 \in I$, hence the above equality for $t = 1$ shows that (5) holds.

3° At this stage we prove that μ is real analytic on $B_Y(0,r_3)\times B_Y(0,r_3)$, where $r_3\in(0,\tilde{r_3})$ is chosen so that $\mu(B_Y(0,r_3)\times B_Y(0,r_3))\subseteq B_Y(0,\tilde{r}_3)$. Theorem A2.20 shows that it suffices to prove that, for arbitrary $u,v,w,z\in B_Y(0,r_3)$, the function

$$\gamma\colon D_{u,v,w,z}\to Y,\quad \gamma(t)=\mu(u+tv,w+tz),$$

is real analytic, where

$$D_{u,v,w,z}:=\{t\in\mathbb{R}\mid u+tv,w+tz\in B_Y(0,r_3)\}.$$

To this end, we will use Corollary A2$\frac{1}{2}$.11.

We have

$$\dot\gamma(t)=b(u+tv,w+tz)v+a(u+tv,w+tz)z.$$

On the other hand, by formula (1) (see the proof of Proposition A2$\frac{1}{2}$.8) we have $a(u+tv,w+tz)=a(\mu(u+tv,w+tz))a^{-1}(w+tz)$. Similarly to (1) we have $b(x,y)b(x)=b(\mu(x,y))$ whenever $x,y\in B_Y(0,\tilde{r}_3)$, whence $b(u+tv,w+tz)=b(\mu(u+tv,w+tz))b^{-1}(u+tv)$, so that

$$\begin{aligned}\dot\gamma(t)&=b(\mu(u+tv,w+tz))b^{-1}(u+tv)v+a(\mu(u+tv,w+tz))a^{-1}(w+tz)z\\&=b(\gamma(t))b^{-1}(u+tv)v+a(\gamma(t))a^{-1}(w+tz)z.\end{aligned}$$

We have seen at the beginning of stage 2° that all of the mappings $a(\cdot)$, $a^{-1}(\cdot)$, $b(\cdot)$ and $b^{-1}(\cdot)$ are real analytic, it follows that the mapping

$$f\colon D_{u,v,w,z}\times B_Y(0,\tilde{r}_3)\to Y, f(t,x)=b(x)b^{-1}(u+tv)v+a(x)a^{-1}(w+tz)z,$$

is real analytic as well. Since

$$(\forall t\in D_{u,v,w,z})\quad \dot\gamma(t)=f(t,\gamma(t)),$$

it then follows by Corollary A2$\frac{1}{2}$.11 that $\gamma\colon D_{u,v,w,z}\to Y$ is real analytic, and the proof ends. □

We now turn to some facts that hold in the more general context of locally convex spaces and are needed in Chapter 2.

Exercise A2$\frac{1}{2}$.13. Let X be a real locally convex space, V an open subset of X and $x_0,y_0\in V$.

(a) The *linear* mapping $\mu'_{(x_0,y_0)}\colon X\times X\to X$ has the property

$$(\forall u,v\in X)\quad \mu'_{(x_0,y_0)}(u,v)=\partial_1\mu(x_0,y_0)u+\partial_2\mu(x_0,y_0)v.$$

(b) The *bilinear* mapping $\mu''_{(x_0,y_0)}\colon(X\times X)\times(X\times X)\to X$ has the property

$$\begin{aligned}\mu''_{(x_0,y_0)}((u,v),(u,v))=&\partial_1^2\mu(x_0,y_0)(u,v)+2\partial_1\partial_2\mu(x_0,y_0)(u,v)\\&+\partial_2^2\mu(x_0,y_0)(u,v),\end{aligned}$$

whenever $u,v\in X$.

(c) If moreover V is convex, then the mapping $R\colon V\times V\to X$ defined by the equation

$$(\forall z\in V\times V)\quad \mu(z)=\mu(z_0)+\mu'_{z_0}(z-z_0)+\frac{1}{2!}\mu''_{z_0}(z-z_0,z-z_0)+R(z),$$

has the properties

$$R(z_0)=0,\ R'_{z_0}=0,\ R''_{z_0}=0,\ \partial_1R(x,y_0)=\partial_2R(x_0,y)=0,$$

whenever $x,y\in V$, where $z_0:=(x_0,y_0)\in V\times V$. □

Proposition A2$\frac{1}{2}$.14. *Let X be a real locally convex space, V_1 a convex open neighborhood of $0 \in X$, $\mu\colon V_1 \times V_1 \to X$ a smooth mapping such that*

$$(\forall x \in V_1) \quad \mu(x,0) = \mu(0,x) = x,$$

and $\eta\colon V_1 \to X$ a smooth mapping with the property that $\eta(0) = 0$ and

$$(\forall x \in V_1) \quad \mu(x,\eta(x)) = 0.$$

Moreover consider an open neighborhood V_2 of $0 \in X$ such that $V_2 \subseteq V_1$, $\eta(V_2) \subseteq V_1$ and $\mu(V_2 \times V_2) \subseteq V_1$, and define

$$\psi\colon V_2 \times V_2 \to X, \quad \psi(x,y) = \mu(\mu(x,y),\eta(x)).$$

For $u, v \in V_2$, define

$$\tilde{u}, \tilde{v}\colon V_2 \to X, \quad \tilde{u}(x) = \partial_2\mu(x,0)u, \ \tilde{v}(x) = \partial_2\mu(x,0)v.$$

Then

$$\boxed{\partial_1\partial_2\psi(0,0)(v,u) = \partial_1\partial_2\mu(0,0)(u,v) - \partial_1\partial_2\mu(0,0)(v,u) = (\tilde{v})'_0u - (\tilde{u})'_0v.}$$

Proof. The second of the desired equalities clearly follows by the very definition of $\tilde{u}$ and $\tilde{v}$. Next, we are going to prove that

$$\partial_1\psi(0,v)u = \partial_1\partial_2\mu(0,0)(u,v) - \partial_1\partial_2\mu(0,0)(v,u),$$

which implies the first of the asserted equalities.

To prove the above equality, first differentiate the equations $\mu(x,0) = \mu(0,x) = x$, to get

$$\partial_1\mu(x,0) = \partial_2\mu(0,x) = \mathrm{id}_X,$$

whence $\partial_1^2\mu(x,0) = \partial_2^2\mu(0,x) = 0$. It then follows by Exercise A2$\frac{1}{2}$.13 that

$$(\forall x, y \in V_1) \quad \mu(x,y) = x + y + \partial_1\partial_2\mu(0,0)(x,y) + R(x,y),$$

where $R(0,0) = 0$, $\partial_1 R(0,y) = \partial_2 R(0,y) = 0$. It then follows that for all $y \in V_1$ we have

$$\partial_1\mu(0,y) = \mathrm{id}_X + \partial_1\partial_2\mu(0,0)(\cdot,y) \text{ and } \partial_2\mu(y,0) = \mathrm{id}_X + \partial_1\partial_2\mu(0,0)(y,\cdot).$$

Then differentiate the equation $\mu(x,\eta(x)) = 0$ to get $\partial_1\mu(x,\eta(x)) + \partial_2\mu(x,\eta(x))\eta'_x = 0$. For $x = 0$, we get $\mathrm{id}_X + \mathrm{id}_X\eta'_0 = 0$, whence $\eta'_0 = -\mathrm{id}_X$.

Now the definition of ψ implies that for $v \in V_2$ we have

$$\begin{aligned}\partial_1\psi(0,v) &= \partial_1\mu(\mu(0,v),\eta(0))\partial_1\mu(0,v) + \partial_2\mu(\mu(0,v),\eta(0))\eta'_0 \\ &= \partial_1\mu(0,v) - \partial_2\mu(v,0) \\ &= \partial_1\partial_2\mu(0,0)(\cdot,v) - \partial_1\partial_2\mu(0,0)(v,\cdot),\end{aligned}$$

whence the first of the desired equalites clearly follows. $\square$

Notes

Several results contained in this appendix are updated versions of some of the basic facts underlying the paper by B. Maissen [Ma62], where the basic theory of Banach-Lie groups is developed following the pattern of finite-dimensional Lie theory. Our Proposition A2$\frac{1}{2}$.8 is inspired by Satz 4.2, while Theorem A2$\frac{1}{2}$.9 is essentially Satz 4.1 at page 241 in [Ma62]. Moreover, Theorem A2$\frac{1}{2}$.12 is the essential result contained in Satz 7.1 in [Ma62]. It says that every local Banach-Lie group is analytic.

Proposition A2$\frac{1}{2}$.14 contains some calculations carried out in section 5 of [Mi84].

For a good exposition of the needed elements of the theory of ordinary differential equations in Banach spaces, we refer to [La01]. See also Chapter 5 in [Up85] for an exposition of that theory in the context of analytic functions.

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A3. SMOOTH MANIFOLDS AND VECTOR FIELDS

ABSTRACT. We introduce the notion of smooth manifold modeled on a locally convex space, as well as the closely related notions of tangent vector, tangent bundle and vector field. The main result of this appendix (Theorem A3.18) shows that the set of all vector fields on a smooth locally convex manifold has a natural structure of Lie algebra.

Definition A3.1. A topological space X is called *regular* if it is Hausdorff and for every $x \in X$ and every neighborhood U of x there exists another neighborhood V of x such that $\overline{V} \subseteq U$. In other words, each point of x has a basis of closed neighborhoods. □

Exercise A3.2. In order for the Hausdorff topological space X to be regular, it suffices that it is of one of the following types:

(a) X is locally compact;
(b) X is a topological group;
(c) X is a locally convex topological vector space. □

Definition A3.3. A *smooth manifold* modeled on a locally convex topological vector space V is a regular topological space M equipped with a family of homeomorphisms $\{\varphi_\alpha: V_\alpha \to M_\alpha\}_{\alpha\in A}$ satisfying the following conditions.

(i) For every $\alpha \in A$, V_α is an open subset of V and M_α is an open subset of M.
(ii) We have $M = \bigcup\limits_\alpha M_\alpha$.
(iii) If $\alpha, \beta \in A$ and $M_\alpha \cap M_\beta \neq \emptyset$, then the corresponding change of coordinate function

$$\varphi_\beta^{-1} \circ \varphi_\alpha|_{\varphi_\alpha^{-1}(M_\alpha \cap M_\beta)}: \varphi_\alpha^{-1}(M_\alpha \cap M_\beta) \to \varphi_\beta^{-1}(M_\alpha \cap M_\beta)$$

is smooth. Note that both $\varphi_\alpha^{-1}(M_\alpha \cap M_\beta)$ and $\varphi_\beta^{-1}(M_\alpha \cap M_\beta)$ are open subsets of the locally convex topological vector space V.

In this case, the maps $\varphi_\alpha: V_\alpha \to M_\alpha$ will be called *local coordinate systems*, while the maps $\varphi_\alpha^{-1}: V_\alpha \to M_\alpha$ are called *local coordinate charts.*

A smooth manifold modeled on a locally convex, or Fréchet, or Banach, or Hilbert space will be called *locally convex, Fréchet, Banach,* respectively *Hilbert manifold.* □

Definition A3.4. Let M be a locally convex smooth manifold modeled on V with the family of local coordinate systems $\{\varphi_\alpha: V_\alpha \to M_\alpha\}_{\alpha\in A}$, and $\widehat{M}$ a locally convex smooth manifold modeled on $\widehat{V}$ with the family of local coordinate systems $\{\widehat{\varphi}_{\widehat{\alpha}}: \widehat{V}_{\widehat{\alpha}} \to \widehat{M}_{\widehat{\alpha}}\}_{\widehat{\alpha}\in\widehat{A}}$. Then a continuous function $f: M \to \widehat{M}$ is *smooth* if for every $x \in M$ there exist $\alpha \in A$ and $\widehat{\beta} \in \widehat{A}$ such that $x \in M_\alpha$ and the map

$$\widehat{\varphi}_{\widehat{\beta}}^{-1} \circ f \circ \varphi_\alpha|_{\varphi_\alpha^{-1}(M_\alpha \cap f^{-1}(\widehat{M}_{\widehat{\beta}}))}: \varphi_\alpha^{-1}(M_\alpha \cap f^{-1}(\widehat{M}_{\widehat{\beta}})) \to \widehat{V}_{\widehat{\beta}}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX is smooth. We will denote the set of all smooth mappings from M into $\widehat{M}$ by $\mathcal{C}^\infty(M, \widehat{M})$.

Note that $\varphi_\alpha^{-1}(M_\alpha \cap f^{-1}(\widehat{M}_{\widehat{\beta}}))$ is always an open (maybe empty) subset of the model space V.

The mapping $f\colon M \to \widehat{M}$ is called a *diffeomorphism* if it is bijective and both f and f^{-1} are smooth. □

Exercise A3.5. Prove that, if M, N and P are smooth manifolds, and $f\colon M \to N$ and $g\colon N \to P$ are smooth mappings, then $g \circ f\colon M \to P$ is in turn a smooth mapping.

This shows that there exists a category **Man** whose objects are the smooth manifolds and whose morphisms are the smooth mappings. □

Remark A3.6. Real (or complex) *analytic manifolds* and real (or complex) analytic mappings on smooth manifolds can be defined by replacing the word 'smooth' by 'real (or complex) analytic' in Definitions A.3 and A.4. □

Definition A3.7. Let M be a smooth manifold modeled on V, with the local coordinate systems $\{\varphi_\alpha\colon V_\alpha \to M_\alpha\}_{\alpha\in A}$. Fix a point $x_0 \in M$.

A *tangent vector* at x_0 is an equivalence class of parameterized paths through x_0 in the following sense. Let I_1 and I_2 be open intervals in $\mathbb{R}$, with $t_0 \in I_1 \cap I_2$, and $p_1\colon I_1 \to M$, $p_2\colon I_2 \to M$ smooth mappings (*paths*) with $p_1(t_0) = p_2(t_0) = x_0$. We say that p_1 and p_2 are *equivalent* at t_0 if there exists $\alpha \in A$ such that $x_0 \in M_\alpha$ and the two smooth mappings

$$\mathbb{R} \supseteq p_i^{-1}(M_\alpha) \ni t \mapsto \varphi_\alpha^{-1}(p_i(t)) \in V \qquad (i = 1, 2)$$

have the same derivative at t_0. If $p\colon I \to M$ is a smooth path and $t_0 \in I$, then the equivalence class of p at t_0 is denoted by $\dot{p}(t_0)$ and is called the *velocity vector* of p at t_0.

The set of all such tangent vectors at x_0 is denoted by $T_{x_0}M$ and is called the *tangent space* at x_0. Note that, if $x_0 \in M_\alpha$ as above, then there exists a natural bijective mapping

$$\Phi_\alpha\colon V \to T_{x_0}M$$

such that, for each $v \in V$, the tangent vector $\Phi_\alpha(v) \in T_{x_0}M$ is the equivalence class of the path $t \mapsto \varphi_\alpha(v_0 + tv)$, where $v_0 := \varphi_\alpha^{-1}(x_0) \in V_\alpha$. Using the bijection Φ_α we can equip $T_{x_0}M$ with the structure of a topological vector space isomorphic to V. □

Exercise A3.8. In the setting of Definition A3.7, prove the following assertions.

(a) The definition of the equivalence relation for paths through x_0 does not depend on the choice of the index α with $x_0 \in M_\alpha$.

(b) The mapping $\Phi_\alpha\colon V \to T_{x_0}M$ is indeed bijective.

(c) If $\beta \in A$ is another index with $x_0 \in M_\beta$, then $\Phi_\beta^{-1} \circ \Phi_\alpha\colon V \to V$ is an isomorphism of topological vector spaces.

(d) The structure of topological vector space of $T_{x_0}M$ is natural in the sense that it does not depend on the choice of $\alpha \in A$ with $x_0 \in M_\alpha$. □

Exercise A3.9. Let U be an open subset of a locally convex vector space V, viewed as a smooth manifold with the local coordinate system $\mathrm{id}_U\colon U \hookrightarrow V$. Prove that $TU = U \times V$. □

Definition A3.9. Let M be a smooth manifold modeled on the locally convex space V, with the local coordinate systems $\{\varphi_\alpha \colon V_\alpha \to M_\alpha\}_{\alpha\in A}$. The *tangent bundle* of M is defined as the disjoint union

$$TM := \bigcup_{x\in M} T_x M,$$

and its *canonical projection* $p \colon TM \to M$ is defined such that $v \in T_{p(v)}M$ for all $v \in TM$.

For every $\alpha \in A$ we also introduce the mapping

$$\psi_\alpha \colon V_\alpha \times V \to TM$$

such that, for $u \in V_\alpha$ and $v \in V$, the tangent vector $\psi_\alpha(u, v) \in T_{\varphi_\alpha(u)}M$ is by definition the equivalence class of the smooth path $t \mapsto \varphi_\alpha(u + tv)$ through $\varphi_\alpha(u) \in M$. If we denote the image of ψ_α by TM_α, then the tangent bundle TM has a natural structure of smooth manifold modeled on $V \times V$, with the local coordinate systems $\{\psi_\alpha \colon V_\alpha \times V \to TM_\alpha\}_{\alpha\in A}$. $\square$

Exercise A3.10. In the setting of Definition A3.9, prove the following assertions.

(a) The tangent bundle TM indeed has a structure of smooth manifold (in particular a topology) as indicated in Definition A3.9.

(b) For every $\alpha \in A$ and $u \in V_\alpha$ we have

$$(\forall v \in V) \qquad \Phi_\alpha(v) = \psi_\alpha(u, v),$$

where Φ_α is as in Definition A3.9. $\square$

Definition A3.11. Let M and $\widehat{M}$ be smooth manifolds modeled over the locally convex spaces V and $\widehat{V}$, with the local coordinate systems $\{\varphi_\alpha \colon V_\alpha \to M_\alpha\}_{\alpha\in A}$ and $\{\widehat{\varphi}_{\widehat{\alpha}} \colon \widehat{V}_{\widehat{\alpha}} \to \widehat{M}_{\widehat{\alpha}}\}_{\widehat{\alpha}\in\widehat{A}}$, respectively). If $f \colon M \to \widehat{M}$ is a smooth mapping and $x \in M$, then the *tangent* of f at x is the mapping

$$f'_x \colon T_x M \to T_{f(x)}\widehat{M}$$

defined in the following way. If $v \in T_x M$ and $p \colon I \to M$ is a smooth path such that $0 \in I$, $p(0) = x$ and $\dot{p}(0) = v$, then

$$f'_x(v) := \dot{q}(0) \in T_{f(x)}\widehat{M},$$

where $q := f \circ p \colon I \to \widehat{M}$. (Note that q is a smooth path and $q(0) = f(x)$.)

Then the *tangent* of f is the mapping

$$Tf \colon TM \to T\widehat{M}$$

defined by

$$(\forall x \in M) \qquad Tf|_{T_x M} := T_x f := f'_x.$$

Then Tf is a smooth mapping and for every $x \in M$ the restriction of Tf to $T_x M$ is a continuous linear operator $T_x M \to T_{f(x)}\widehat{M}$. $\square$

Exercise A3.12. In the setting of Definition A3.11, prove the following assertions.

(a) For each $x \in M$, the mapping $T_x f: T_x M \to T_{f(x)}\widehat{M}$ is correctly defined, and indeed linear and continuous.

(b) The mapping

$$Tf: TM \to T\widehat{M}$$

is indeed smooth. $\square$

Exercise A3.13. If **Man** stands for the category of smooth manifolds (see Exercise A3.5), prove that the correspondence

$$T: \mathbf{Man} \to \mathbf{Man}$$

which associates to each locally convex smooth manifold its tangent bundle, and to each smooth map its tangent mapping, has all of the properties (i)–(vi) in Remark 2.3. $\square$

Definition A3.14. Let M be a locally convex smooth manifold with the tangent bundle TM. A *smooth vector field* on M is a smooth map

$$v: M \to TM$$

such that $v(x) \in T_x M$ for al $x \in M$. The set of all vector fields on M clearly has a structure of vector space (with pointwise defined addition and scalar multiplication), and we denote that vector space by $\mathfrak{V}(M)$. $\square$

Definition A3.15. Let M be a locally convex smooth manifold, $v \in \mathfrak{V}(M)$ and Y a locally convex topological vector space. We define a linear operator

$$D_v: \mathcal{C}^\infty(M, Y) \to \mathcal{C}^\infty(M, Y)$$

in the following way. Let $\pi: Y \times Y \to Y$, $(y_1, y_2) \mapsto y_2$. Observing that the tangent bundle of Y is $TY = Y \times Y$ with the canonical projection $p: TY \to Y$, $(y_1, y_2) \mapsto y_1$ (which is different from π!), we define, for all $f \in \mathcal{C}^\infty(M, Y)$, a smooth function $D_v f \in \mathcal{C}^\infty(M, Y)$ by the commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TY \\ {\scriptstyle v}\uparrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{D_v f} & Y \end{array}$$

that is,

$$(\forall x \in M) \qquad \boxed{(Tf \circ v)(x) = \big(f(x), (D_v f)(x)\big)} \in Y \times Y = TY. \quad \square$$

Exercise A3.16. In the setting of Definition A3.15, prove that the mapping

$$\mathfrak{V}(M) \to \mathrm{End}(\mathcal{C}^\infty(M, Y)), \qquad v \mapsto D_v,$$

is linear. $\square$

Lemma A3.17. *Let V be a locally convex vector space and V_0 an open subset of V. Then for all $u, w \in \mathfrak{V}(V_0)$ there exists a unique vector field $[u,w] \in \mathfrak{V}(V_0)$ such that, for each open subset D of V_0 and each locally convex vector space Y we have*

$$(\forall f \in \mathcal{C}^\infty(D,Y)) \qquad D_{[u,w]}f = D_u(D_w f) - D_w(D_u f).$$

Proof. Note that $TV_0 = V_0 \times V$ with the canonical projection $V_0 \times V \to V_0$, $(x,t) \mapsto x$, so that

$$\mathfrak{V}(V_0) = \{v \colon V_0 \to V_0 \times V \mid (\exists \widetilde{v} \in \mathcal{C}^\infty(V_0, V)) \quad v(\cdot) = (\cdot, \widetilde{v}(\cdot))\}.$$

Then for $u, w \in \mathfrak{V}(V_0)$ fixed, let $\widetilde{u}, \widetilde{w} \in \mathcal{C}^\infty(V_0, V)$ with $u(\cdot) = (\cdot, \widetilde{u}(\cdot))$, $w(\cdot) = (\cdot, \widetilde{w}(\cdot))$, and define $[u,w] \in \mathfrak{V}(V_0)$ by

$$(1) \qquad [u,w] \colon V_0 \to V_0 \times V, \quad [u,w](\cdot) = (\cdot, (D_u\widetilde{w} - D_w\widetilde{u})(\cdot)).$$

Now, for each locally convex space Y, and all open subset D of V_0 and $f \in \mathcal{C}^\infty(D,Y)$, we clearly have

$$(2) \qquad (\forall v \in \mathfrak{V}(V_0))\,(\forall x \in V_0) \qquad (D_v f)(x) = f'_x \widetilde{v}(x),$$

whence

$$\begin{aligned}(D_{[u,w]}f)(x) &= f'_x(D_u\widetilde{w} - D_w\widetilde{u})(x) \\ &= f'_x((\widetilde{w})'_x\widetilde{u}(x) - (\widetilde{u})'_x\widetilde{w}(x)) \\ &= f'_x(\widetilde{w})'_x\widetilde{u}(x) - f'_x(\widetilde{u})'_x\widetilde{w}(x)\end{aligned}$$

On the other hand,

$$D_u(D_w f)(x) = (D_w f)'_x\widetilde{u}(x) = (f'_\bullet\widetilde{w}(\cdot))'_x\widetilde{u}(x) = f''_x(\widetilde{w}(x), \widetilde{u}(x)) + f'_x(\widetilde{w})'_x\widetilde{u}(x)$$

and similarly

$$D_w(D_u f)(x) = f''_x(\widetilde{u}(x), \widetilde{w}(x)) + f'_x(\widetilde{u})'_x\widetilde{w}(x).$$

Thus $D_{[u,w]}f = D_u(D_w f) - D_w(D_u f)$ by the symmetry property of f''_x (see Proposition A2.7).

To prove the uniqueness assertion, apply the property of $[v,w]$ in the special case when $D = V_0$, $Y = V$ and f is the inclusion mapping $V_0 \hookrightarrow V$. Then $f'_x = \mathrm{id}_V$ for all $x \in V_0$, and $f'' = 0$, hence by the above computations we get

$$D_u(D_w f)(x) - D_w(D_u f)(x) = (\widetilde{w})'_x\widetilde{u}(x) - (\widetilde{u})'_x\widetilde{w}(x) = (D_u\widetilde{w})(x) - (D_w\widetilde{u})(x),$$

while

$$(D_{[u,w]}f)(x) = \widetilde{[u,w]}(x).$$

Thus $\widetilde{[u,w]} = D_u\widetilde{w} - D_w\widetilde{u}$, that is, the vector field $[u,w] \in \mathfrak{V}(V_0)$ is necessarily given by (1), and the proof ends. $\square$

Theorem A3.18. *If M is a locally convex smooth manifold, then $\mathfrak{V}$ has a unique structure of Lie algebra such that, for each locally convex vector space Y and each open subset U of M, the linear map*

$$\mathfrak{V}(M) \to \mathrm{End}(\mathcal{C}^\infty(U,Y)), \quad v \mapsto D_v,$$

is a Lie algebra homomorphism.

Proof. Let $\{\varphi_\alpha \colon V_\alpha \to M_\alpha\}_{\alpha \in A}$ as in Definition A3.3. Also let $u, w \in \mathfrak{V}(M)$ fixed for the moment. Then for each $\alpha \in A$ there clearly exist $\bar{u}_\alpha, \bar{w}_\alpha \in \mathfrak{V}(V_\alpha)$ such that the diagrams

$$\begin{array}{ccc} T(V_\alpha) & \xrightarrow{T(\varphi_\alpha)} & T(M_\alpha) \\ {\scriptstyle \bar{u}_\alpha}\uparrow & & \uparrow{\scriptstyle u|_{M_\alpha}} \\ V_\alpha & \xrightarrow{\varphi_\alpha} & M_\alpha \end{array} \quad \text{and} \quad \begin{array}{ccc} T(V_\alpha) & \xrightarrow{T(\varphi_\alpha)} & T(M_\alpha) \\ {\scriptstyle \bar{w}_\alpha}\uparrow & & \uparrow{\scriptstyle w|_{M_\alpha}} \\ V_\alpha & \xrightarrow{\varphi_\alpha} & M_\alpha \end{array}$$

are commutative (since both $\varphi_\alpha \colon V_\alpha \to M_\alpha$ and $T(\varphi_\alpha)\colon T(V_\alpha) \to T(M_\alpha)$ are diffeomorphisms, and $T(\varphi_\alpha)$ maps the fiber $T_x(V_\alpha)$ into $T_{\varphi_\alpha(x)}(M_\alpha)$ for each $x \in V_\alpha$). Now define $[\bar{u}_\alpha, \bar{w}_\alpha] \in \mathfrak{V}(V_\alpha)$ by Lemma A3.17, and then denote

$$[u,w]_\alpha := T(\varphi_\alpha) \circ [\bar{u}_\alpha, \bar{w}_\alpha] \circ (\varphi_\alpha)^{-1} \in \mathfrak{V}(M_\alpha).$$

It then easily folows by the uniqueness assertion in Lemma A3.17 that, if $M_\alpha \cap M_\beta \neq \emptyset$, then $[u,w]_\alpha|_{M_\alpha \cap M_\beta} = [u,w]_\beta|_{M_\alpha \cap M_\beta}$. Hence there exists a unique vector field $[u,w] \in \mathfrak{V}(M)$ such that $[u,w]|_{M_\alpha} = [u,w]_\alpha$ for all $\alpha \in A$.

Now, for Y and U as in the statement, and $f \in \mathcal{C}^\infty(U,Y)$, we easily get by Lemma A3.17 that $D_{[u,w]}f = D_u(D_w f) - D_w(D_u f)$ on $U \cap V_\alpha$ for all α, whence $D_{[u,w]}f = D_u(D_w f) - D_w(D_u f)$ on U. In other words, the mapping

$$\mathfrak{V}(M) \times \mathfrak{V}(M) \to \mathfrak{V}(M), \quad (u,w) \mapsto [u,w],$$

has the property that for every open subset U of M we have

$$(\forall v, w \in \mathfrak{V}(M)) \qquad D_{[v,w]} = [D_v, D_w] \in \mathrm{End}(\mathcal{C}^\infty(U,Y)).$$

This easily implies that $[\cdot,\cdot]$ is a Lie algebra tructure on $\mathfrak{V}(M)$, in view of the fact that, if V stands for the model space of the manifold M, then the linear mapping

$$\mathfrak{V}(M) \to \prod_{\alpha \in A} \mathrm{End}(\mathcal{C}^\infty(M_\alpha, V)), \qquad v \mapsto (D_{v|_{M_\alpha}})_{\alpha \in A}$$

is injective. (In fact, if $v \in \mathfrak{V}(M)$ and $D_{v|_{M_\alpha}} = 0$, then $(D_{v|_{M_\alpha}})(\varphi_\alpha^{-1}) = 0$, whence $T(\varphi_\alpha^{-1})v|_{M_\alpha} = 0$. But the linear operator $T_x(\varphi_\alpha^{-1})$ is invertible for all $x \in M_\alpha$, hence $v|_{M_\alpha} = 0$.)

The uniqueness assertion follows by the uniqueness assertion in Lemma A3.17, taking $U = M_\alpha$ for arbitrary $\alpha \in A$. $\square$

Proposition A3.19. *Let M and N be two locally convex smooth manifolds, and $\varphi\colon M \to N$ a smooth mapping. If $v_1, v_2 \in \mathfrak{V}(M)$ and $w_1, w_2 \in \mathfrak{V}(N)$ satisfy $T\varphi \circ v_j = w_j \circ \varphi$ for $j = 1, 2$, then $T\varphi \circ [v_1, v_2] = [w_1, w_2] \circ \varphi$.*

Proof. It suffices to show that the desired equality holds on some neighborhood of an arbitrary point $p \in M$. Thus, replacing M by a suitably small neighborhood of p, and N by an appropriate neighborhood of $\varphi(p) \in N$, we may assume that there exist the locally convex vector spaces V and W such that M is an open subset of V and N is an open subset of W. Then there exists $\widetilde{v}_1, \widetilde{v}_2 \in \mathcal{C}^\infty(M, V)$ and $\widetilde{w}_1, \widetilde{w}_2 \in \mathcal{C}^\infty(N, W)$ such that for $j = 1, 2$ we have

$$(3)\qquad \begin{gathered} v_j(\cdot) = (\cdot, \widetilde{v}_j(\cdot))\colon M \to M \times V = TM \\ \text{and} \\ w_j(\cdot) = (\cdot, \widetilde{w}_j(\cdot))\colon N \to N \times W = TN \end{gathered}$$

(see the proof of Lemma A3.17). Now, using the fact that we have

$$(4)\qquad T\varphi\colon M \times V \to N \times W, \quad (m_0, v_0) \mapsto (\varphi(m_0), \varphi'_{m_0}(v_0)),$$

along with formula (2) in the proof of Lemma A3.17, we get for every $m_0 \in M$

$$\begin{aligned} (T\varphi \circ [v_1, v_2])(m_0) &= T\varphi(m_0, \widetilde{[v_1, v_2]}(m_0)) \\ &= (\varphi(m_0), \varphi'_{m_0}(\widetilde{[v_1, v_2]}(m_0))) \\ &= (\varphi(m_0), (D_{[v_1, v_2]}\varphi)(m_0)). \end{aligned}$$

We further deduce by Lemma A3.17 that

$$(5)\qquad (T\varphi \circ [v_1, v_2])(m_0) = (\varphi(m_0), (D_{v_1} D_{v_2}\varphi)(m_0) - (D_{v_2} D_{v_1}\varphi)(m_0)).$$

On the other hand, note that the hypothesis $T\varphi \circ v_j = w_j \circ \varphi$ implies by means of (3) and (4) that

$$(\forall x \in M) \qquad (\varphi(x), \varphi'_x(\widetilde{v}_j(x))) = (\varphi(x), (\widetilde{w}_j \circ \varphi)(x)),$$

hence by formula (2) in the proof of Lemma A3.17 we get for $j = 1, 2$

$$(6)\qquad (\forall x \in M) \qquad (D_{v_j}\varphi)(x) = \varphi'_x(\widetilde{v}_j(x)) = (\widetilde{w}_j \circ \varphi)(x).$$

Consequently

$$\begin{aligned} D_{v_1}(D_{v_2}\varphi)(m_0) &= D_{v_1}(\widetilde{w}_2 \circ \varphi)(m_0) \\ &= (\widetilde{w}_2 \circ \varphi)'_{m_0}(\widetilde{v}_1(m_0)) \qquad \text{(by (2) in the proof of Lemma A3.17)} \\ &= (\widetilde{w}_2)'_{\varphi(m_0)} \varphi'_{m_0} \widetilde{v}_1(m_0) \\ &= (\widetilde{w}_2)'_{\varphi(m_0)} \widetilde{w}_1(\varphi(m_0)) \qquad \text{(by (6))} \\ &= (D_{w_1}\widetilde{w}_2)(\varphi(m_0)), \end{aligned}$$

and, similarly,

$$D_{v_2}(D_{v_1}\varphi)(m_0) = (D_{w_2}\widetilde{w}_1)(\varphi(m_0)).$$

Thus

$$D_{v_1}(D_{v_2}\varphi)(m_0) - D_{v_2}(D_{v_1}\varphi)(m_0) = (D_{w_1}\widetilde{w}_2 - D_{w_2}\widetilde{w}_1)(\varphi(m_0)) = \widetilde{[w_1, w_2]}(\varphi(m_0)),$$

by formula (1) in the proof of Lemma A3.17. Consequently, by (5) we get

$$(T\varphi \circ [v_1, v_2])(m_0) = (\varphi(m_0), \widetilde{[w_1, w_2]}(\varphi(m_0))) = ([w_1, w_2] \circ \varphi)(m_0),$$

using again formula (1) in the proof of Lemma A3.17. $\square$

Exercise A3.20. Let M be an open subset of the real locally convex space V. For all $u, v \in \mathcal{C}^\infty(M, V)$ define $[u, v] \in \mathcal{C}^\infty(M, V)$ by

$$(\forall x \in M) \quad [u, v](x) = u'_x(v(x)) - v'_x(u(x)).$$

Then the bracket $[\cdot, \cdot]$ turns $\mathcal{C}^\infty(M, V)$ into a Lie algebra. $\square$

Notes

We refer to the book [Wa71] for an elementary introduction to the theory of finite-dimensional manifolds.

A quick introduction to infinite-dimensional manifolds modeled on locally convex spaces can be found in [Mi84]. For manifolds modeled on Banach spaces, see [La01]. The more special setting of analytic Banach manifolds is developed in [Up85].

In connection with Theorem A3.18, we note that an interesting property of the Lie algebras of vector fields in the case of finite-dimensional manifolds can be found in the paper [SP54].

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A4. TOPOLOGICAL GROUPS

ABSTRACT. In the present appendix, we develop the basic facts concerning topological groups that are needed in the main body of these lecture notes. The most important results contained here are the fact that the group topology is uniquely determined by its restriction to a neighborhood of $\mathbf{1}$ (Theorem A4.11), and the theorem concerning the construction of homomorphisms from simply connected topological groups (Theorem A4.19).

Definition A4.1. Let G be a group. A *group topology* on G is a topology τ on G making the map

$$G \times G \to G, \qquad (a,b) \mapsto ab^{-1}$$

into a continuous map.

A *topological group* is a group equipped with a group topology. □

Remark A4.2.

(a) In the framework of Definition A4.1, the condition that τ is a group topology is equivalent to the requirement that both the multiplication map

$$m\colon G \times G \to G, \qquad (a,b) \mapsto ab,$$

and the inversion map

$$\eta\colon G \to G, \qquad a \mapsto a^{-1},$$

are continuous.

(b) The discrete topology of any group is always a group topology. Thus, every group admits at least one group topology. On the other hand, there can exist several group topologies on a given group. For instance, the additive group $(\mathbb{R}, +)$ has at least two group topologies: the discrete topology and the usual one. □

Lemma A4.3. *Assume that E is a set and for every $x \in E$ we have singled out a set $\mathcal{V}(x)$ of subsets of E such that the following conditions are fulfilled.*

(V1) *If $x \in E$, $V \in \mathcal{V}(x)$, and $V \subseteq U \subseteq E$, then $U \in \mathcal{V}(x)$.*

(V2) *If $x \in E$ and $V_1, V_2 \in \mathcal{V}(x)$, then $V_1 \cap V_2 \in \mathcal{V}(x)$.*

(V3) *If $x \in E$ and $V \in \mathcal{V}(x)$, then $x \in V$.*

(V4) *If $x \in E$ and $V \in \mathcal{V}(x)$, then there exists $W \in \mathcal{V}(x)$ such that for all $y \in W$ we have $V \in \mathcal{V}(y)$.*

Next denote

$$\tau = \{D \mid D \subseteq E; (\forall x \in D)(\exists V \in \mathcal{V}(x))\, V \subseteq D\}.$$

Then τ is the unique topology on E such that, for all $x \in E$, $\mathcal{V}(x)$ is the set of neighborhoods of x with respect to τ.

Proof. See Proposition 2 in §1, no. 2, in Chapitre I in [Bo71]. □

Notation A4.4. If G is a group, $x \in G$, and $A, B \subseteq G$, then we denote

$$AB := \{ab \mid a \in A, b \in B\}, \quad xA := \{x\}A, \quad Ax := A\{x\}, \quad A^{-1} := \{a^{-1} \mid a \in A\}.$$

Whenever it is not otherwise stated, we denote by $\mathbf{1}$ the unit element of any group. □

The following statement is a first step towards the method to construct group topologies on a given group, starting from local structures around $\mathbf{1}$. (See Theorem A4.11.)

Proposition A4.5. *Assume that G is a group and $\mathcal{V}$ is a set of subsets of G such that the following conditions are satisfied.*

(GV0) *If $U_1, U_2 \in \mathcal{V}$ then $U_1 \cap U_2 \in \mathcal{V}$. If $V \in \mathcal{V}$ and $V \subseteq U \subseteq G$, then $U \in \mathcal{V}$.*
(GV1) *If $U \in \mathcal{V}$ then there exists $V \in \mathcal{V}$ with $VV \subseteq U$.*
(GV2) *For all $U \in \mathcal{V}$ we have $U^{-1} \in \mathcal{V}$.*
(GV3) *For all $U \in \mathcal{V}$ we have $\mathbf{1} \in U$.*
(GV4) *For all $U \in \mathcal{V}$ and $a \in G$ we have $aUa^{-1} \in \mathcal{V}$.*

Then there exists a unique group topology τ on G such that $\mathcal{V}$ is the set of all neighborhoods of $\mathbf{1} \in$ with respect to τ. Moreover, for each $a \in G$, we have

$$\mathcal{V}(a) := \{aV \mid V \in \mathcal{V}\} = \{Va \mid V \in \mathcal{V}\}$$

and this is the set of all neighborhoods of a with respect to τ.

Proof. The proof has several stages.

1° To prove the existence and uniqueness of the topology τ, we use Lemma A4.3 for $\mathcal{V}(x)$, $x \in G$, as in the statement. So we have to check that conditions (V1)–(V4) in Lemma A4.3 are satisfied.

To this end, note that both (V1) and (V2) follow from hypothesis (GV0), while (V3) follows from (GV3). To prove (V4), first note that, for arbitrary $x \in G$ and $V \subseteq G$, we have $V \in \mathcal{V}(x)$ if and only if $x^{-1}V \in \mathcal{V}$. This remark shows that condition (V4) in Lemma A4.3 is equivalent to the following: for all $x \in G$ and $V_0 \in \mathcal{V}$ there exists $W_0 \in \mathcal{V}$ such that for all $y \in xW_0$ we have $y^{-1}xV_0 \in \mathcal{V}$. To prove this, note that hypothesis (GV1) implies that there exists $W_0 \in \mathcal{V}$ with $W_0W_0 \subseteq V_0$. Then for each $y \in xW_0$ we have $x^{-1}y \in W_0$, hence $x^{-1}y \in W_0$, hence $x^{-1}yW_0 \subseteq W_0W_0 \subseteq V_0$. Thus $W_0 \subseteq y^{-1}xV_0$, which implies by hypothesis (GV2) that $y^{-1}xV_0 \in \mathcal{V}$, as desired.

2° We now show that the topology τ constructed at stage 1° by means of Lemma A4.3 is a group topology. To this end, first recall from Lemma A4.3 that $\mathcal{V}(x)$ $(= \{xV \mid V \in \mathcal{V}\}$ in the present situation) is the set of all neighborhoods of x, for all $x \in G$. Thus, in order to prove that the map

$$G \times G \to G, \qquad (a, b) \mapsto ab^{-1}$$

is continuous, it suffices to check that the following statement holds: for arbitrary $a, b \in G$ and $U \in \mathcal{V}$, there exists $W \in \mathcal{V}$ such that

$$(aW)(bW)^{-1} \subseteq (ab^{-1})U.$$

Note that the above inclusion is equivalent to $aWW^{-1}b^{-1} \subseteq ab^{-1}U$, and further to

$$WW^{-1} \subseteq b^{-1}Ub.$$

On the other hand, we have by hypothesis (GV4) that $b^{-1}Ub \in \mathcal{V}$, hence, according to hypothesis (GV1), there exists $W_1 \in \mathcal{V}$ such that

$$W_1 W_1 \subseteq b^{-1}Ub.$$

Now, for $W := W_1 \cap (W_1)^{-1} \in \mathcal{V}$ (use both hypotheses (GV0) and (GV2)), we get $WW^{-1} \subseteq W_1 W_1 \subseteq b^{-1}Ub$, as desired.

3° To conclude the proof, note that the second of the equalities in

$$\mathcal{V}(a) = \{aV \mid V \in \mathcal{V}\} = \{Wa \mid W \in \mathcal{V}\}$$

follows by hypothesis (GV4) for each $a \in G$. □

Conditions (GBV0)–(GBV4) in the following auxiliary result are usually easier to check than conditions (GV0)–(GV4) in Proposition A4.5.

Lemma A4.6. *Assume that G is a group and $\mathcal{B}$ is a set of subsets of G satisfying the following conditions:*

(GBV0) *For all $U_1, U_2 \in \mathcal{B}$ there exists $U_0 \in \mathcal{B}$ with $U_0 \subseteq U_1 \cap U_2$.*
(GBV1) *For each $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $VV \subseteq U$.*
(GBV2) *For each $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ with $V^{-1} \subseteq U$.*
(GBV3) *For all $U \in \mathcal{B}$ we have $\mathbf{1} \in U$.*
(GBV4) *If $U \in \mathcal{B}$ and $a \in G$, then there exists $V \in \mathcal{B}$ with $aVa^{-1} \subseteq U$.*

Then

$$\tau = \{D \mid D \subseteq G;\ (\forall a \in D)(\exists V \in \mathcal{B})\, aV \subseteq D\}$$

is the unique group topology on G such that $\mathcal{B}$ is a basis of neighborhoods of $\mathbf{1} \in G$ with respect to τ.

Proof. It is easy to check that

$$\mathcal{V} := \{U \mid U \subseteq G;\ (\exists V \in \mathcal{B})\, V \subseteq U\}$$

satisfies conditions (GV0)–(GV4) in Proposition A4.5. □

Notation A4.7. If G is a group and $A \subseteq G$, then we denote by $\langle A \rangle$ the subgroup of G generated by A, i.e., the smallest subgroup of G that contains A. □

Exercise A4.8. If G is a group and $A \subseteq G$, then

$$\langle A \rangle = \{\mathbf{1}\} \cup \bigcup_{n=1}^{\infty} \{a_1 \cdots a_n \mid a_1, \dots a_n \in A \cup A^{-1}\}. \quad \square$$

Exercise A4.9.

(a) If G is a connected topological group and U is a neighborhood of $\mathbf{1} \in G$, then $\langle U \rangle = G$.
(b) If G is a topological group and there exists a connected neighborhood U of $\mathbf{1} \in G$ such that $\langle U \rangle = G$, then G is connected. □

Definition A4.10. Let T_1 and T_2 be topological spaces. An *imbedding* (of topological spaces) of T_1 into T_2 is a mapping $f: T_1 \to T_2$ that induces a homeomorphism $T_1 \to f(T_1)$, provided we view $f(T_1)$ as a topological subspace of T_2. $\square$

The following theorem provides a useful way to endow a group with a group topology. The results of this type are particularly useful in Lie theory, inasmuch as they allow one to extend local structures to global ones.

Theorem A4.11. *Let G be a group with the multiplication map*

$$m: G \times G \to G, \qquad (x, y) \mapsto xy,$$

and $K \subseteq G$ such that

$$\mathbf{1} \in K = K^{-1} \text{ and } \langle K \rangle = G.$$

Assume that the subset K of G is equipped with a Hausdorff topology such that the inversion map

$$K \to K, \qquad x \mapsto x^{-1},$$

is continuous and there exists an open set $V_0 \subseteq K \times K$ satisfying the following conditions:

(a) *$m(V_0) \subseteq K$,*

(b) *$m|_{V_0}: V_0 \to K$ is continuous, and*

(c) *for all $x \in K$ we have $(x, x^{-1}), (x, 1), (1, x) \in V_0$.*

Then there exists a unique group topology on G making the inclusion map

$$K \hookrightarrow G$$

into an imbedding of topological spaces such that K is an open subset of G.

Proof. The proof has several stages.

1° To construct the group topology of G, we will make use of Lemma A4.6. To this end, we check conditions (GBV0)–(GBV4) in Lemma A4.6 for the set of subsets of G defined by

$$\mathcal{B} := \{W \mid W \subseteq K;\ W \text{ is a neighborhood of } \mathbf{1} \in K\}.$$

Conditions (GBV0) and (GBV3) are obvious.

For (GBV1), note that V_0 is a neighborhood of $(\mathbf{1}, \mathbf{1}) \in K \times K$. Since the mapping $m|_{V_0}: V_0 \to K$ is continuous and $m(\mathbf{1}, \mathbf{1}) = \mathbf{1}$, it then easily follows that for each $W \in \mathcal{B}$ there exists $W_1 \in \mathcal{B}$ with $W_1 W_1 \subseteq W$.

To see that (GBV2) holds, we use the fact that the inversion mapping $\eta: K \to K$, $x \mapsto x^{-1}$, is continuous. Since $\eta^2 = \mathrm{id}_K$, it follows that η is actually a homeomorphism of K onto itself, and thus for each $W \in \mathcal{B}$ we have $W^{-1}(= \eta(W)) \in \mathcal{B}$ as well.

In order to check condition (GBV3) in Lemma A4.6, we first note that, since $K = K^{-1}$ and $\langle K \rangle = G$, it follows by Exercise A4.8 that

$$G = \bigcup_{n=1}^{\infty} \underbrace{K \cdots K}_{n \text{ times}}.$$

One then easily shows by induction that it suffices to check (GBV3) only for $a \in K$. To do this, let $a \in K$ and $W \in \mathcal{B}$ arbitrary. By hypothesis (c) we have $(a, a^{-1}) \in V_0$. Since V_0 is open in $K \times K$ and $m|_{V_0}$ is continuous, it then follows that for some neighborhood U_0 of $a \in K$ we have both

$$U_1 \times \{a^{-1}\} \subseteq V_0 \text{ and } U_1 a^{-1} \subseteq W.$$

On the other hand, also by hypothesis (c) we have $(a, \mathbf{1}) \in V_0$. Again by the continuity of $m|_{V_0}$, there exists $W_1 \in \mathcal{B}$ such that

$$\{a\} \times W_1 \subseteq V_0 \text{ and } aW_1 \subseteq U_1.$$

Then $aW_1a^{-1} \subseteq U_1a^{-1} \subseteq W$, as desired in condition (GBV3) in Lemma A4.6.

Consequently, we can use Lemma A4.6 to make G into a topological group with the group topology τ defined by

$$\tau = \{D \mid D \subseteq G;\ (\forall a \in D)(\exists W \in \mathcal{B})\, aW \subseteq D\}.$$

Note that, in particular. we have $K \in \tau$. In fact, let $a \in K$. We have $(a, \mathbf{1}) \in V_0$ by hypothesis (c), hence, using as above the continuity of $m|_{V_0}$, we can find $W \in \mathcal{B}$ such that $(\{a\} \times W \subseteq V_0$ and$)$ $aW \subseteq K$.

2° We now prove that the inclusion mapping

$$\iota: K \hookrightarrow G$$

is an imbedding of topological spaces. To this end, we have to prove that it is both an open mapping (i.e., it maps every open subset of K onto an open subset of G) and a continuous mapping.

To see that ι is an open mapping, it clearly suffices to show that, for every neighborhood U of an arbitrary $k \in K$, the set $\iota(U)$ is a neighborhood of $\iota(k)(= k) \in G$. To this end, we repeat the proof of the fact that $K \in \tau$: we have $(k, \mathbf{1}) \in V_0$, hence the continuity of $m|_{V_0}$ shows that for some $W \in \mathcal{B}$ we have $kW \subseteq U = \iota(U)$. Thus $\iota(U)$ is a neighborhood of $\iota(k)$, according to the above definition of the topology τ.

Now, to prove that ι is continuous, let $D \in \tau$ arbitrary. We have $\iota^{-1}(D) = K \cap D$, hence we have to prove that $K \cap D \in \tau$. To this end, let $k \in K \cap D$ arbitrary. The fact that $k \in D$ shows that for some $W_1 \in \mathcal{B}$ we have

$$kW_1 \in \mathcal{B}.$$

On the other hand, the fact that $k \in K$ implies as above (using that $(k, \mathbf{1}) \in V_0$ along with the continuity of the multiplication map $m|_{V_0}$) that for some $W_2 \in \mathcal{B}$ we have both $\{k\} \times W_2 \subseteq V_0$ and

$$kW_2 \subseteq K.$$

Then

$$k \cdot (W_1 \cap W_2) = kW_1 \cap kW_2 \subseteq D \cap K.$$

Since $W_1, W_2 \in \mathcal{B}$, we have $W \subseteq W_1 \cap W_2$ for some $W \in \mathcal{B}$ (see condition (GBV0) in Lemma A4.6), hence $kW \subseteq D \cap K$, and this is just what is needed in order to have $D \cap K \in \tau$.

3° The uniqueness assertion is an easy consequence of the corresponding assertion in Lemma A4.6. □

Our next aim is to describe one of the basic methods to construct homomorphisms from simply connected groups to arbitrary groups (see Theorem A4.19 below). The notion of simply connected space (Definition A4.15 (c)) needs the idea of covering, in the sense of the following definition.

Definition A4.12. Let T and S be topological spaces and $f: T \to S$ a continuous mapping. We say that f is a *covering mapping* if for every $s \in S$ there exists an open neighborhood W of s such that, for some family $\{A_i\}_{i \in I}$ of pairwise disjoint open subsets of T, the following conditions are satisfied:

1° We have $f^{-1}(W) = \bigcup_{i \in I} A_i$.

2° For every $i \in I$ the mapping $f|_{A_i}: A_i \to W$ is a homeomorphism provided A_i and W are equipped with the topologies inherited from T and S, respectively. □

Exercise A4.13. Let G and H be topological groups and $\varphi: G \to H$ a group homomorphism.

(a) If there exists an open neighborhood U of $\mathbf{1} \in G$ such that $\varphi|_U$ is continuous, then φ is continuous.

(b) If there exists an open neighborhood V of $\mathbf{1} \in G$ such that $\varphi(W)$ is an open subset of $\mathbf{1} \in H$ and $\varphi|_W: W \to \varphi(W)$ is a homeomorphism when $\varphi(W)$ is equipped with the topology inherited from H, then φ is a covering map. □

Definition A4.14. Let T be topological space. If X is another topological space and $f_0, f_1: X \to T$ are two continuous mappings, we say that f_1 and f_2 are *homotopic* if there exists a continuous mapping $H: [0,1] \times X \to T$ such that for all $x \in X$ we have $H(0,x) = f_0(x)$ and $H(1,x) = f_1(x)$. In this case, H is said to be a *homotopy connecting f_0 and f_1*. □

Definition A4.15. Let T be topological space.

(a) We say that T is *connected* if $\emptyset$ and T are the only subsets of T which are simultaneously closed and open.

(b) We say that T is *locally connected* if every point of T has a basis of connected neighborhoods.

(c) We say that T is *simply connected* if it is connected and locally connected, and, whenever $h: P \to S$ is a covering mapping, $f: T \to S$ is continuous, $t_0 \in T$, $p_0 \in P$, $h(p_0) = f(t_0)$, it follows that there exists a unique continuous mapping $\tilde{f}: T \to P$ such that the diagram

$$\begin{tikzcd} P \arrow[d, "h"'] & T \arrow[l, "\tilde{f}"'] \arrow[dl, "f"] \\ S & \end{tikzcd}$$

is commutative and $\tilde{f}(t_0) = p_0$.

(d) We say that T is *pathwise connected* if for all $t_0, t_1 \in T$ there exists a continuous mapping (that is a *path*) $\gamma: [0,1] \to T$ such that $\gamma(0) = t_0$ and $\gamma(1) = t_1$.

(e) We say that T is *locally pathwise connected* if every point of T has a pathwise connected neighborhood.

(f) We say that T is *pathwise simply connected* if it is pathwise connected and locally pathwise connected, and every continuous path $\gamma: [0,1] \to T$ with $\gamma(0) = \gamma(1)$ is homotopic to a constant map $[0,1] \to T$. □

Exercise A4.16. Let G be a topological group. If $\mathbf{1} \in G$ has a basis of connected neighborhoods (respectively, a pathwise connected neighborhood), then G is locally connected (respectively, locally pathwise connected). □

The following theorem provides the main tool used to check that a certain space is simply connected.

Theorem A4.17. *Every pathwise simply connected space is simply connected.*

Proof. See Theorem 2.1 in Chapter IV in [Ho65]. □

The next theorem describes a very important property of simply connected spaces, and will play a key role in the proof of Theorem A4.19 below.

Theorem A4.18. *Let P and S be topological spaces and $h: P \to S$ a covering mapping. If*
(i) *P is connected and locally connected, and*
(ii) *S is simply connected,*
then f is a homeomorphism.

Proof. See Theorem 1.4 in Chapter IV in [Ho65]. The idea is to use condition (c) in Definition A4.15 for $T = S$ and $f = \mathrm{id}_S$, in order to construct a continuous inverse of h. □

We are now ready to describe the main method to construct group homomorphisms defined on simply connected groups.

Theorem A4.19. *Let G be a simply connected topological group and H an arbitrary group. Suppose that W is a connected open neighborhood of $\mathbf{1} \in G$ such that $W = W^{-1}$ and $f: W \to H$ is a mapping such that*

$$f(xy) = f(x)f(y) \text{ whenever } x, y, xy \in W.$$

Then there exists a unique group homomorphism $\varphi: G \to H$ such that $\varphi|_W = f$.

Proof. Denote
$$K := \{(g, f(g)) \mid g \in W\} \subseteq G \times H$$
and endow N with the unique topology making the bijection

$$\beta: W \to K, \quad g \mapsto (g, f(g))$$

into a homeomorphism. Then denote by E the subgroup of $G \times H$ generated by K.

Using Theorem A4.11, we are going to make E into a connected topological group such that K is an open neighborhood of $\mathbf{1} \in E$. To this end, denote by $m: G \times G \to G$ be the multiplication in G. Then $W_0 := m^{-1}(W) \cap (W \times W)$ is an open subset of $W \times W$ such that conditions (a)–(c) in Theorem A4.11 are satisfied (with K replaced by W and V_0 replaced by W_0). Since β is a homeomorphism, it then follows that $V_0 := \{(\beta(g_1), \beta(g_2)) \mid (g_1, g_2) \in W_0\}$ and K also satisfy conditions (a)–(c) in Theorem A4.11, hence the group $E = \langle K \rangle$ ($\subseteq G \times H$) has a unique structure of group topology such that K is an open neighborhood of $\mathbf{1} \in E$. Since W is connected and K is homeomorphic to W, it follows that K is connected, and it then follows by Exercise A4.9 (b) that the topological group E is connected.

Now consider the mapping

$$\pi: E \to G, (g, h) \mapsto g$$

which is the restriction to E of the natural projection $\mathrm{pr}_1: G \times H \to G$. Since the latter projection is a group homomorphism, it follows that π is a group homomorphism as well. Moreover, note that

$$\pi|_K = \beta^{-1}: K \to W$$

Since β is a homeomorphism and K is an open neighborhood of $\mathbf{1} \in E$, it then follows by Exercise A4.13 (b) that π is a covering map of E onto G. But E is connected and locally connected by Exercise A4.16, while G is simply connected, hence Theorem A4.18 shows that π is a homeomorphism.

In particular, π is bijective, and then $\pi^{-1}: G \to E$ is a group isomorphism. For every $g \in W$ we have

$$\pi^{-1}(g) = \beta(g) = (g, f(g)),$$

hence for the group homomorphism

$$\varphi := \mathrm{pr}_2 \circ \pi^{-1}: G \to H$$

we have $\varphi|_W = f$. (Here $\mathrm{pr}_2: G \times H \to H$ stands for the natural projection, which is a group homomorphism.)

The uniqueness of the group homomorphism φ follows since we have by Exercise A4.9 (a) that $\langle W \rangle = G$. $\square$

Notes

Theorem A4.11 appears explicitly as Lemma II.2 in the paper [Ne02]. See also pages 263–265 in [Hof68]. The basic idea underlying this result is that of local (topological) group. See e.g., page 209 in [Sw65].

Theorem A4.19 is sometimes called the "monodromy theorem". It appears e.g., as Proposition 5.60 in the notes by K.H. Hoffman [Hof68], or as Theorem 3.1 in [Ho65]. See Theorem 1.7 in Chapter IV in [Ho65] for a more general result of this type.

Among the basic references for the topic of topological groups, we mention the books [Bo71] and [Ho65].

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