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Stochastic differential equations associated with smooth mappings and non F_t-adapted solutions

Daniela Ijacu Constantin Vârsan

Abstract. Stochastic partial differential equations (s.p.d.e.) of Hamilton-Iacobi and parabolic type including non $\mathcal{F}_t$-adapted solutions are studied. They are associated with an orbit solution in a finite dimensional Lie algebra and a special type (Stratonovich type) of stochastic integral is introduced. Some applications from control theory are added.

Key words. Stochastic partial differential equations, non $\mathcal{F}_t$-adapted solutions and Stratonovich type stochastic integral.

1 Introduction

The analysis is concentrated both on first order stochastic differential equations and of parabolic type driven by an anticipating drift which is not adapted to the filtration $\{\{\mathcal{F}_t\}\uparrow, t \in [0,T]\} \subseteq \mathcal{F}$ generated by the given Wiener process of the stochastic perturbation. The usual martingale approch is not an appropriate one and we are obliged to decompose the non $\mathcal{F}_t$-adapted solutions into a continuous and $\mathcal{F}_t$-adapted process valued in the space of smooth mappings which are restricted to some solutions associated with the given anticipating drift. It is accomplished using a finite composition of local flows which are determined by a finite dimensional Lie Algebra associated with the smooth vector fields of the corresponding stochastic perturbation. In addition, the Fisk-Stratonovich integral suitable for the $\mathcal{F}_t$-adapted solutions is replaced by a special type stochastical integral "$\otimes$" allowing one to include non $\mathcal{F}_t$-adapted solutions with a special structure.

We write first order stochastic differential equations (s.p.d.e.) as follows:

$$\begin{cases} d_t u = g_0^\omega(x,u,\partial_x u)dt + \sum_{j=1}^m g_j(x,u,\partial_x u) \otimes dw_j(t), \ t \in [0,T] \\ u(0) = u_0^\omega(x), \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^n, \ \partial_x u \in \mathbb{R}^n, \ (u, \partial_x u) \in B(0,\rho) \subseteq \mathbb{R}^{n+1} \end{cases} \qquad (\alpha)$$

where $u_0^\omega \in C_b^2(\mathbb{R}^n)$ and $g_0^\omega \in C_b^2(\mathbb{R}^n \times B(0,\rho))$ are only $\mathcal{F}$-mesurable with respect to $\omega \in \Omega$ being non $\mathcal{F}_t$-adapted scalar functions and $w(t) \triangleq (w_1(t), \ldots, w_m(t)) \in \mathbb{R}^m$ is a standard

m-dimensional Wiener process on a given filtered probability space $\{\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}\uparrow\subseteq \mathcal{F}\}$. For simplicity, we shall omite to write the variable $\omega \in \Omega$ and the meaning of a solution for (s.p.d.e.) given in (α) is derived using a solution of the associated stochastic system of characteristics.

$$\begin{cases} d_t z = Z_0(z)dt + \sum_{j=1}^{m} \chi_\tau(t) Z_j(z) \otimes dw_j(t), \; t \in [0,T], \; z \in B(0,\rho) \times \mathbb{R}^n \\ z(0) = z_0(\lambda) \triangleq (u_0(\lambda), \partial_\lambda u_0(\lambda), \lambda) \in B(0,\rho_1) \times \mathbb{R}^n \triangleq D_1, \; \rho_1 \in (0,\rho) \end{cases} \qquad (\alpha_1)$$

where $z = (u, \partial_x u, x) \triangleq (u, p, x)$, and the smooth vector fields $Z_i(z) \in \mathbb{R}^{2n+1}$, for $i \in \{0, 1, \ldots, m\}$, are defined in the standard way.

$$\begin{aligned} Z_i(z) &\triangleq \begin{pmatrix} Y_i(z) \\ X_i(z) \end{pmatrix}, \quad X_i(z) \triangleq -\partial_p g_i(x,u,p) \in \mathbb{R}^n, \\ Y_i(z) &\triangleq \begin{pmatrix} g_i(x,u,p) - \langle p, \partial_p g_i(x,u,p)\rangle \\ \partial_x g_i(x,u,p) + p\partial_u g_i(x,u,p) \end{pmatrix} \in \mathbb{R}^{n+1} \end{aligned} \qquad (\alpha_2)$$

In the case of deterministic functions $u_0 \in C_b^2(\mathbb{R}^n)$, $g_0 \in C_b^2(D)$, $D = B(0,\rho) \times \mathbb{R}^n$, we are allowed to work with Fisk-Stratonovich integral "$\circ$" and the $\mathcal{F}_t$-adapted solutions associated with (α) and (α_1) are obtained. In our case, the use of a special type integral (Stratonovich type) is based on the Langevin's smooth approximations $w^\varepsilon(t)$ replacing the original Wiener process $w(t)$, $t \in (0,T)$.

A solution of the stochastic system of characteristics (α_1) is defined provided the Lie algebra $L(Z_1, \ldots, Z_m)$ determined by $\{Z_1, \ldots, Z_m\} \subseteq C^\infty(D, \mathbb{R}^{2n+1})$ is finite dimensional. Similar considerations apply to stochastic differential equations of parabolic type described by the following equations:

$$\begin{cases} d_t u = [\Delta_x u + f^\omega(t,x,u,\partial_x u)]dt + \sum_{j=1}^{m} g_j(x,u,\partial_x u) \otimes dw_j(t) \\ u(0) = u_0^\omega(x), \; t \in (0,T), \; x \in \mathbb{R}^n, \; u \in \mathbb{R}, \; \Delta_x u = \sum_{j=1}^{n} \partial_i^2 u \end{cases}$$

where the initial condition $u_0^\omega \in C_b^2(\mathbb{R}^n)$, and the continuous scalar function $f^\omega \in C([0,T]; C_b^2(D))$, may depend on the parameter $\omega \in \Omega$ in a non $\mathcal{F}_t$-adapted manner being only $\mathcal{F}$-mesurable.

From now on we shall not mention the dependence on $\omega \in \Omega$ and write $u_0(x)$, $f(t,x,u,p)$.

A local solution for (β) is derived from a continous and non $\mathcal{F}_t$-adapted solution $\hat{y}(t,\lambda) \triangleq (\hat{u}(t,\lambda), \hat{p}(t,\lambda)) \in B(0,\rho) \subseteq \mathbb{R}^{n+1}$ fulfilling the following system of parabolic stochastic differential equations.

$$
\begin{cases}
d_t\hat{y}(t,\lambda) = [\Delta_\lambda \hat{y}(t,\lambda) + Y_0(t,\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_\lambda \hat{y}(t,\lambda))]dt \\
\qquad\qquad + \sum_{i=1}^{m} \chi_\tau(t) Y_i(\hat{x}(t,\lambda),\hat{y}(t,\lambda)) \otimes dw_i(t),\ t \in (0,a] \\
\hat{y}(0,\lambda) = y_0(\lambda) \triangleq (u_0(\lambda), \partial_\lambda u_0(\lambda)) \in B(0,\rho_0),\ \rho_0 \in (0,\rho)
\end{cases} \qquad (\beta_1)
$$

with a fixed stopping time $\tau(\omega) : \Omega \to [0,T]$, and $\hat{x}(t,\lambda) = \lambda + \sum_{i=1}^{m} b_i w_i(t \wedge \tau)$ assuming $b_i \triangleq -\partial_p g_i(x,u,p) \triangleq X_i(z)$, $i \in \{1,\ldots,m\}$, are constant vectors in $\mathbb{R}^n$

Here, the smooth vector fields $Y_i(z) \in \mathbb{R}^{n+1}$, $i \in \{1,\ldots,m\}$, are the corresponding components of $Z_i(z) \in \mathbb{R}^{2n+1}$, defined in (α_2) and Y_0 is obtained from the original f as

$$
Y_0(t,x,y,\partial_x y) \triangleq \begin{pmatrix} f(t,x,u(t,x),p(t,x)) \\ \partial_x[f(t,x,u(t,x),p(t,x))] \end{pmatrix} \qquad (\beta_2)
$$

In both cases, the corresponding stochastic system of characteristics (α_1) (or (β_1)) allow one to get a continuous and non $\mathcal{F}_t$-adapted process $y(t,x) \triangleq (u(t,x),p(t,x)) = (u(t,x),\partial_x u(t,x))$, $t \in [0,a]$, valued in the space of smooth mappings $y \in C_b^1(\mathbb{R}^n; B(0,\rho))$, (or $y \in C_b^2(\mathbb{R}^n; B(0,\rho))$) for each $t \in (0,a]$, such that $\hat{y}(t,\lambda) = y(t,\hat{x}(t,\lambda))$, where $\hat{z}(t,\lambda) \triangleq (\hat{y}(t,\lambda),\hat{x}(t,\lambda))$, $\hat{y}(t,\lambda) \triangleq (\hat{u}(t,\lambda),\hat{p}(t,\lambda))$.

A local solution for the stochastical differential equation (α) has to be a continuous process $u = u(t,x)$, $t \in [0,a]$, valued in the space of smooth mappings $u \in C_b^2(\mathbb{R}^n)$ fulfilling the corresponding integral equation

$$
\int_0^t [d_s u(s,x)]_{x=\hat{x}(s,\lambda)} = \int_0^t g_0(\hat{x}(s,\lambda),\hat{y}(s,\lambda))ds + \sum_{j=1}^{m} \int_0^t \chi_\tau(s) g_j(\hat{x}(s,\lambda),\hat{y}(s,\lambda)) \otimes dw_j(s)
$$

where the stochastic differential $[d_s u(s,x)]_{x=\hat{x}(s,\lambda)}$ along to $x = \hat{x}(s,\lambda)$ is computed obeying to

$$
\int_0^t [d_s u(s,x)]_{x=\hat{x}(s,\lambda)} = \int_0^t d_s\hat{u}(s,\lambda) - \int_0^t \langle \hat{p}(s,\lambda), d_s\hat{x}(s,\lambda)\rangle
$$

As far as the parabolic stochastic equation (β) is concerned a special issue appears when the Laplacian operator $\Delta_x\, u(t,x)$ is computed along to $x = \hat{x}(t,\lambda)$ using the continuous process $y = \hat{y}(t,\lambda) \triangleq (\hat{u}(t,\lambda),\hat{p}(t,\lambda))$.

The simplest form we can get is relying on $X_i(z) \triangleq -\partial_p g_i(x,u,p) = b_i \in \mathbb{R}^n$, $i \in \{1,\ldots,m\}$, are constants.

Assuming that $f(t,x,u,\partial_x u)$ and $u_0(x)$ are some deterministic functions then the local solution associated with the parabolic equation (α) is constructed using the usual Fisk-Stratonovich integral as in [1].

The general procedure used here has its roots in [2] and [3] containing stochastic partial differential equations with diffusion part depending on some unknown vector functions. The work is divided into three parts following this introduction. In the first section we state the basic facts related to finite dimensional Lie algebras, gradient systems and orbit solution as in [4] and their implication in the definition of a Stratonovich type stochastic integral. In the second section we give some auxiliary and main results with proofs regarding both stochastic differential equations (α) and (β). In the last section we collect two applications from the control problems associated with stochastic differential equations and their non $\mathcal{F}_t$-adapted solutions.

§2. Preliminaries

Everywhere in this paper we assume that the smooth scalar deterministic function $g_j(x, u, p)$,are given such that $g_j \in C_b^\infty(\mathbb{R}^n \times B(0, \rho)), j \in \{1, ., m\}$,where the ball $B(0, \rho) \subseteq \mathbb{R}^{n+1}$is fixed.

Denote $z = (u, p, x) \in R^{2n+1}, D = B(0, \rho) \times \mathbb{R}^n$and define the smooth vector fields $Z_j \in C_b^\infty(D; \mathbb{R}^{2n+1}), j \in \{1, .., m\}$,as in ($\alpha_2$),

1) $Z_j(z) \triangleq \begin{pmatrix} Y_j(z) \\ X_j(z) \end{pmatrix}, X_j(z) \triangleq -\partial_p g_j(x, u, p) \in \mathbb{R}^n$

$$Y_j(z) \triangleq \begin{pmatrix} g_j(x, u, p) - \langle p, \partial_p g_j(x, u, p) \rangle \\ \partial_x g_j(x, u, p) + p\partial_u g_j(x, u, p) \end{pmatrix} \in \mathbb{R}^{n+1}$$

A solution for s.p.d.e (α)(or (β)) is derived usingthe corresponding stochastic system of characteristics defined in (α_1)(or(β_1)).In both cases we have to start with a local solution associated with the reduced stochastic differential system

2) $$\begin{cases} d_t z = \sum_{j=1}^{m} Z_j(z) \circ dw_j(t), t \in [0, T], z \in D \triangleq B(0, \rho) \times \mathbb{R}^n \\ z(0) = z_0 \in D_0 = B(0, \rho_0) \times \mathbb{R}^n, 0 < \rho_0 < \rho, \end{cases}$$

where the Fisk-Stratonovich integral " $\circ$ " is used and

$w(t) \triangleq (w_1(t), .., w_m(t)) \in \mathbb{R}^m$ is a standard m-dimensional Wiener process on a given filtred probability space $\{\Omega, \mathcal{F}, P, \{\mathcal{F}\} \uparrow \subseteq \mathcal{F}\}$.

A local solution of (2) is found as a continuous and $\mathcal{F}_t$ -adapted process valued in the space of smooth mappings $z \in C_b^\infty(D_0; \mathbb{R}^{2n+1})$ and it is done assuming

H_1) The Lie algebra $L(Z_1, .., Z_m) \subseteq C_b^\infty(D; \mathbb{R}^{2n+1})$ determined by the vector fields $\{Z_1, .., Z_m\}$ is finite dimensional.

The assumption (H_1) allows us to fixe a system of generators

$\{Z_1, .., Z_m, Z_{m+1}, .., Z_M\} \subseteq L(Z_1, .., Z_m)$ and to define the corresponding orbit of smooth mappings .

3) $S(p, z_0) \triangleq S_1(t_1) \circ .. \circ S_M(t_M)(z_0), p \triangleq (t_1, .., t_M) \in D_M = \prod_{j=1}^{M}[-a_j, a_j]$ for $z_0 \in D_0 \triangleq B(0, \rho_0) \times \mathbb{R}^n$,where

$S_j(t, z_0), t \in [-a_j, a_j], z_0 \in D_0$,is the local flow generated by the vector field $Z_j, j \in \{1, .., M\}$.

Using the nonsingular algebraic representation of the associated gradient system given in [4] we are able to recover the original vector fields $\{Z_1, .., Z_M\}$along to the orbit solution [3] and some analitic vector fields $q_j \in \mathcal{A}\ (D_M; \mathbb{R}^M), j \in \{1, .., M\}$are defined such that

$$4) \begin{cases} a)\ \dfrac{\partial S}{\partial p}(p; z_0)q_j(p) = Z_j(S(p; z_0)),\ j \in \{1, .., M\},\quad p \in D_M,\ z_0 \in D_0 \\ b)\ \text{the}(M \times M)\text{matrix } Q(p) \triangleq (q_1(p), .., q_M(p)), p \in D_M \\ \quad \text{is a nonsingular one} \end{cases}$$

A local solution for the stochastic differential system (2) is constructed using the mapping $S(p, z_0)$ in (3) provided an $\mathcal{F}_t$ -adapted continuous process $p = p(t) \in D_M, t \in [0, T]$,is defined as a solution for the following stochastic system

$$5)\ d_t p = \sum_{j=1}^{m} \alpha(p)q_j(p) \circ dw_j(t), p(0) = 0, p \in \mathbb{R}^M$$

Here the smooth scalar function $\alpha \in C^\infty(\mathbb{R}^M; [0, 1])$ is taken adequately and fulfilling.

$\alpha(p) = 0$ for $p \in \mathbb{R}^n \backslash B(0, 2\hat{\rho}), \alpha(p) = 1$ for $p \in B(0, \hat{\rho})$, where $\hat{\rho} > 0$ is fixed such that $B(0, 2\hat{\rho}) \subseteq D_M$.

Let $\tau(\omega) : \Omega \to [0, T]$ be a stopping time by $\tau(\omega) \triangleq \inf\{t \in [0, T]; \mid p(t) \mid > \hat{\rho}\}$ where the solution

$p = p(t), t \in [0, T]$ is defined in (5).

It is easily seen that $\hat{p}(t) \triangleq p(t \wedge \lambda) \in B(0, \hat{\rho}), t \in [0, T]$,is obeying to the following stochastic differential system :

6) $d_t p = \sum_{j=1}^{m} \chi_\tau(t) g_j(p) \circ dw_j(t), p(0) = 0, t \in [0, T]$ where $\chi_\tau(t) = 1$ for $\tau > t$ and $\chi_\tau(t) = 0$ for $\tau \leq t, t \in [0, T]$

The $\mathcal{F}_t$ -adapted and continuous process

$$7)\ z(t, z_0) \triangleq S(\hat{p}(t); z_0), t \in [0, T], z_0 \in D_0 = B(0, \rho_0) \times \mathbb{R}^n$$

will be a local solution for the stochastic system in [2] fulfilling the following system of integral equations.

$$8)\ z(t, z_0) = z_0 + \sum_{j=1}^{m} \int_0^t \chi_\tau(s) Z_j(z(s; z_0) \circ dw_j(s) =$$

$$= z_0 + \sum_{j=1}^{m} \int_0^{t\wedge\tau} Z_j(z(s;z_0) \circ dw_j(s), t \in [0,T], z_0 \in D_0.$$

It shows that the smooth orbit $S(p;z_0) \in D$ in (3) allows one to define a local solution of (2) using a stopping time τ which doesn't depend on the initial condition z_0 taken in an unbounded set D_0.

Now,a local solution for s.p.d.e.in (α) (or (β)) can be constructed provided a continously differentiable process $z_0 = z_0(t,\lambda) \in D_0, t \in (0,a], 0 < a < T$,is defined such that:

9) $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); z_0(t,\lambda)) \in D, t \in (0,a], \lambda \in \mathbb{R}^n$

is a local solution of an extended system of stochastic differential equations defined in (α_1)(or in (β_1))

.

Remark 1.

It is worth to mention that dealing with $\mathcal{F}_t$ -adapted drift and initial condition we are allowed to look for $z_0(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,as an $\mathcal{F}_t$ -adapted continuous process and the standard rule of stochastic differentiation applyed to the mapping in (9) will lead us to a differential equation without stochastic perturbation fulfilled by $z_0(t,\lambda), t \in [0,a]$.

In our case ,the use of non $\mathcal{F}_t$ -adapted drift and initial condition is not obeiyng to the usual Fisk-Stratonovich integral and the corresponding stochastic rule of differentiation.

A stochastic rule of differentiation for the non $\mathcal{F}_t$ -adapted solutions in (9) is derived provided the Fisk-Stratonovich integral is replaced by a special type stochastic integral "$\otimes$"(Stratonovich type) and using a Langevin's smooth approximation;

10) $w^{\varepsilon}(t) = \int_0^t y^{\varepsilon}(s)ds = w(t) - \eta(t,\varepsilon), t \in [0,T], 0 < \varepsilon \le 1,$

where $y^{\varepsilon}(t) = \frac{1}{\varepsilon}\int_0^t [\exp -(\frac{t-s}{\varepsilon})]dw(s), \eta(t,\varepsilon) = \int_0^t [\exp -(\frac{t-s}{\varepsilon})dw(s)$

we get the following integral equation

$$11)\ \hat{z}(t,\lambda) - \hat{z}(t',\lambda) = \int_{t'}^{t} \frac{\partial S}{\partial z_0}(\hat{p}(s); z_0(s,\lambda))\frac{dz_0}{ds}(s,\lambda)ds$$

$$+\sum_{j=1}^{m} \int_{t'}^{t} \chi_{\tau}(s) Z_j(\hat{z}(s,\lambda) \otimes dw_j(s)$$

fulfilled for any $t \in [t',t''] \subseteq (0,a]$.where $\hat{z}(t,\lambda)$ is defined in (9)and $z_0(t,\lambda), t \in [t'.t'']$,is continuously differentiable.

Here the Stratonovich type integral " $\otimes$ " is computed passing to the limit $\varepsilon \searrow 0$ in an ordinary rule of derivation applied to the smooth mapping.

12) $z^{\varepsilon}(t,\lambda) \triangleq S(p^{\varepsilon}(t); z_0(t,\lambda)), t \in [t', t''] \subset (0,a], 0 < a < T,$

where $p = p^{\varepsilon}(t), t \in [0,T]$,is fulfilling the following system of ordinary differential equations

13) $\frac{dp}{dt} = \sum_{j=1}^{m} \chi_{\tau}(t)\alpha(p)q_j(p)\frac{dw_j^{\varepsilon}}{dt}(t), t \in [0,T], p(0) = 0$ which coincides with the smooth Langevin's approximation associated with the stochasticdifferential equation in[6].

As a consequence we may and do write the following definition;

14) $$\int_{t'}^{t} \chi_{\tau}(\sigma) Z_j(\hat{z}(\sigma,\lambda)) \otimes dw_j(\sigma) \triangleq \left[\int_{t'}^{t} \chi_{\tau}(s) Z_j(S(\hat{p}(s); z_0)) \circ dw_j(s)\right]_{z_0 = z_0(t',\lambda)}$$
$$+\int_{t'}^{t}\left(\left[\int_{s}^{t} \frac{\partial}{\partial z_0}(\chi_{\tau}(\sigma) Z_j(S(\hat{p}(\sigma); z_0))) \circ dw_j(\sigma)\right]_{z_0 = z_0(s,\lambda)}\right) \frac{dz_0}{ds}(s,\lambda)ds, j \in \{1,..,m\}$$

where the Fisk-Stratonovich stochastic integral " $\circ$ " is used associated with a continuous $\mathcal{F}_t$ -adapted processes valued in the space of smooth mappings.

Based on the above given formula we may and do define a Stratonovich type integral associated with a continuous bounded scalar function.

Definition 1

Let $\varphi \in C_b^3(D,\mathbb{R})$ and $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); z_0(t,\lambda)) \in D, t \in [0,a], \lambda \in \mathbb{R}^n$,be defined as in [9].Then

$$\int_{t'}^{t'''} \chi_{\tau}(t)\varphi(\hat{z}(t,\lambda)) \otimes dw_j(t) \triangleq \left[\int_{t'}^{t''} \chi_{\tau}(t)\varphi(S(\hat{p}(t); z_0)) \circ dw_j(t)\right]_{z_0 = z_0(t',\lambda)} +$$
$$+\int_{t'}^{t''}\left(\left[\int_{t}^{t''} \frac{\partial}{\partial z_0}(\chi_{\tau}(s)\varphi(S(\hat{p}(s); z_0))) \circ dw_j(s)\right]_{z_0 = z_0(t,\lambda)}\right) \frac{dz_0}{dt}(t,\lambda)dt$$

where the Fisk-Stratonovich integral " $\circ$ " is used.

Remark 2.

Let $z_0(t,\lambda) \in D_0, t \in [0,a], \lambda \in \mathbb{R}^n$ be a continuous and $\mathcal{F}_t$ -adapted process being continuously differentiable for any t$\in [t', t''] \subset (0,a]$.Define the continuous and $\mathcal{F}_t$ -adapted process $\hat{z}(t,\lambda) = S(\hat{p}(t); z_0(t.\lambda)), t \in [0,a], \lambda \in \mathbb{R}^n$,as in [9].Then the stochastic integral of Stratonovich type" $\otimes$ " along to $\hat{z} = z(t,\lambda)$ coincide with the Fisk-Stratonovich integral " $\circ$ ".

Indeed,a direct computation which involves a change in the order of integration lead us tothe following

$$\int_{t'}^{t} \chi_T(s) Z_j\left(S(\hat{p}(s), z_0(s,\lambda))\right) \circ dw_j(s) = \int_{t'}^{t} \chi_T(s) Z_j\left(S(\hat{p}(s), z_0(t',\lambda))\right) \circ dw_j(s)$$

$$+\int_{t'}^{t} \chi_T(s)[Z_j\left(S(\hat{p}(s), z_0(s,\lambda))\right) - Z_j S(\hat{p}(s), z_0(t',\lambda)))] \circ dw_j(s) \triangleq T_1 + T_2$$

Using the continuous derivate $\frac{dz_0}{ds}(s,\lambda), s \in [t', t'']$ we write T_2 as

$$T_2 = \int_{t'}^{t} \chi_T(s) \left[\int_{t'}^{s} \frac{\partial}{\partial z_0}(Z_j(S(\hat{p}(s); z_0(\sigma,\lambda))) \frac{dz_0}{d\sigma}(\sigma,\lambda) d\sigma \right] \circ dw_j(s)$$

and changing the order of integration in the last integral we get

$$T_2 = \int_{t'}^{t} \left[\int_{\sigma}^{t} \frac{\partial}{\partial z_0}(\chi_T(s) Z_j(S(\hat{p}(s); z_0(\sigma,\lambda))) \circ dw_j(s) \right] \frac{dz_0}{d\sigma}(\sigma,\lambda) d\sigma$$

In conclusion

15) $T_1 + T_2 = \int_{t'}^{t} \chi_T(s) Z_j(\hat{z}(s,\lambda)) \otimes dw_j(s) = \int_{t'}^{t} \chi_T(s) Z_j(\hat{z}(s,\lambda)) \circ dw_j(s)$

where the stochastic integral " $\otimes$ " is defined as in (14) (see definition 1) and the Fisk-Stratonovich integral" $\circ$ "is linked with Ito's integral by the following;

16) $\int_{t'}^{t} \chi_T(s) Z_j(\hat{z}(s,\lambda)) \circ dw_j(s) \triangleq \frac{1}{2} \int_{t'}^{t} \chi_T(s) \frac{\partial Z_j}{\partial z}(\hat{z}(s,\lambda)) Z_j(\hat{z}(s,\lambda)) ds$

$$+\int_{t'}^{t} \chi_T(s) Z_j(\hat{z}(s,\lambda)) dw_j(s)$$

Remark 3

According to the non $\mathcal{F}_t$ -adapted solutions $z = \hat{z}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$, defined in (9) we may do and write the following rule of stochastic differentiation .

Let $f \in C_b^{1,3}([0,T] \times D; \mathbb{R})$ be given and the solution

$\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); z_0(t,\lambda)), t \in [0,a]$, is fulfilling the integral equations defined in [11].Then it holds

$$f(t'', \hat{z}(t'',\lambda)) - f(t', \hat{z}(t',\lambda)) \triangleq \lim_{\varepsilon \searrow 0} [f(t', z^\varepsilon(t',\lambda)) - f(t', z^\varepsilon(t',\lambda))]$$

$$= \lim_{\varepsilon \searrow 0} \left[\frac{\partial f}{\partial t}(t, z^\varepsilon(t,\lambda) + \langle \frac{\partial f}{\partial z}(t, z^\varepsilon(t,\lambda), \frac{dz^\varepsilon}{dt}(t,\lambda) \rangle \right] dt =$$

$$\int_{t'}^{t''} \left[\frac{\partial f}{\partial t}(t, \hat{z}(t,\lambda)) + \left\langle \frac{\partial f}{\partial z}(t, \hat{z}(t,\lambda)), \frac{\partial S}{\partial z_0}(\hat{p}(t); z_0(t,\lambda)) \frac{dz_0}{dt}(t,\lambda) \right\rangle \right] dt +$$

$$+\lim_{\varepsilon\searrow 0}\sum_{j=1}^{m}\int_{t'}^{t''}\chi_{\tau}(t)\alpha(p^{\varepsilon}(t))\left\langle\frac{\partial f}{\partial z}(t,z^{\varepsilon}(t,\lambda)),Z_j(z^{\varepsilon}(t,\lambda))\right\rangle\frac{dw_j^{\varepsilon}}{dt}(t)=$$

$= T_1 + \lim_{\varepsilon\searrow 0} T_2^{\varepsilon}$,where

$$T_2^{\varepsilon}\triangleq\sum_{j=1}^{m}\int_{t'}^{t''}\chi_{\tau}(t)\alpha(p^{\varepsilon}(t))\left\langle\frac{\partial f}{\partial z}(t,z^{\varepsilon}(t,\lambda)),Z_j(z^{\varepsilon}(t,\lambda))\right\rangle\frac{dw_j^{\varepsilon}}{dt}(t)dt$$

and $z=z^{\varepsilon}(t,\lambda)\triangleq S(p^{\varepsilon}(t);z_0(t,\lambda))$ is the Langevin's smooth approximation defined in (12)

Denote $\varphi_j(t,p,z)\triangleq\alpha(p)\left\langle\frac{\partial f}{\partial z}(t,z),Z_j(z)\right\rangle$and by a direct computation including a change of the order we express

$$T_2^{\varepsilon}\triangleq\sum_{j=1}^{m}\int_{t\prime}^{t\prime\prime}\chi_{\tau}(t)\varphi_j(t,p^{\varepsilon}(t),z^{\varepsilon}(t,\lambda))\frac{dw_j^{\varepsilon}}{dt}(t)dt=$$

$$=\sum_{j=1}^{m}\int_{t\prime}^{t\prime\prime}\chi_{\tau}(t)\varphi_j(t,p^{\varepsilon}(t),S(p^{\varepsilon}(t);z_0(t\prime,\lambda))\frac{dw_j^{\varepsilon}}{dt}(t)dt+$$

$$+\sum_{j=1}^{m}\int_{t\prime}^{t\prime\prime}dt[\int_{t\prime}^{t\prime\prime}\chi_{\tau}(s)\langle\frac{\partial\varphi_j}{\partial z}(s,p^{\varepsilon}(s),S(p^{\varepsilon}(s);z_0(t,\lambda)),$$
$$\frac{\partial S}{\partial z_0}(p^{\varepsilon}(s);z_0(t,\lambda))\frac{dz_0}{dt}z(t,\lambda)\rangle\frac{dw_j^{\varepsilon}}{dt}(s)ds$$

Letting $\varepsilon\searrow 0$ in the last equation we get

$$f(t'',\hat{z}(t'',\lambda))-f(t',\hat{z}(t',\lambda))=$$

$$=\int_{t'}^{t''}\left[\partial_t f(t,\hat{z}(t,\lambda))+\langle\partial_z f(t,\hat{z}(t,\lambda)),\partial_{z_0}S(\hat{p}(t);z_0(t,\lambda))\frac{dz_0}{dt}(t,\lambda)\rangle\right]dt+$$

$$+\sum_{j=1}^{m}\left[\int_{t'}^{t''}\chi_{\tau}(t)\,\langle\partial_z f(t,S(\hat{p}(t);z_0))\rangle\circ dw_j(t)\right]_{z_0=z_0(t',\lambda)}+$$

$$+\sum_{j=1}^{m}\int_{t'}^{t''}\left[\int_{t}^{t''}\chi_{\tau}(s)\frac{\partial}{\partial z}\,\langle\partial_z f(s,z),Z_j(s)\rangle\,(S(\hat{p}(s),z_0))\frac{\partial S}{\partial z_0}(\hat{p}(s);z_0)\circ dw_j(s)\right]\frac{dz_0}{dt}(t,\lambda)dt$$

where $z_0=z_0(t,\lambda)$

§ 3.Main results and proofs of some auxiliary lemmas

The auxiliary lemmas we need to prove are connected with the verification of the integral equation defined in (§2.11) provided the smooth approximation $w^{\varepsilon}(t),t\in[0,T]$,of the given standard Wiener process is used.

Lemma1. Let the smooth scalar functions $g_j \in C_b^\infty(\mathbb{R}^n \times B(0,\rho))$, $j \in \{1,..,m\}$,be given such that nthe corresponding vector fields $Z_j \in C_b^\infty(D, \mathbb{R}^{2n+1}), j \in \{1,..,m\}$,defined in (§2.1) fulfils the hypothesis (H_1).Define a smooth mapping $z = S(p, z_0) \in D, p \in D_M, z_0 \in D_0$,as in (§2.3).Let $p = \hat{p}(t), t \in [0,T]$ be the unique solution associated with stochastic differential equations in (§2.6) and consider a continuous process $z_0 = z_0(t,\lambda), t \in [0,a], 0 < a < T$.Denote $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); z_0(t,\lambda)), t \in [0,a]$.Then $z = \hat{z}(t,\lambda), t \in [t', t''] \subseteq (0,a]$,fulfills the following integral equation:

$$\hat{z}(t,\lambda) - \hat{z}(t',\lambda) = \int_{t'}^{t} \frac{\partial S}{\partial z_0}(\hat{p}(s); z_0(s,\lambda)\frac{dz_0}{ds}(s,\lambda)ds$$

$$+\sum_{j=1}^{m}\int_{t'}^{t} \chi_\tau(s) Z_j(\hat{z}(s,\lambda)) \otimes dw_j(s)$$

provided $z_0(t,\lambda), t \in [t', t'']$,is continuously differentiable and the stochastic integral " $\otimes$"is defined as in (§2.14).

Proof

For $\varepsilon \in (0,1]$ fixed,let $p = p^\varepsilon(t), t \in [0,T]$,be the smooth solution associated with the ordinary differential equation defined in (§2.13) and denote

1) $z^\varepsilon(t,\lambda) = S(p^\varepsilon(t), z_0(t,\lambda)), t \in [0,a]$.

By hypothesis the continuous process $z^\varepsilon(t,\lambda), t \in [0,a]$,is continuously differentiable for $t \in [t', t''] \subset [0,a]$ and using the standard rule of derivation we get

2) $\frac{dz^\varepsilon}{dt} = \frac{\partial S}{\partial p}(p^\varepsilon(t); z_0(t,\lambda))\frac{dp^\varepsilon}{dt}(t) + \frac{\partial S}{\partial z_0}(p^\varepsilon(t); z_0(t,\lambda))\frac{dz_0}{dt}(t,\lambda), t \in [t', t'')$.

Using the system (1) and the algebraic equations

$\frac{\partial S}{\partial p}(p; z_0)q_j(p) = Z_j(S(p, z_0)), j \in \{1,..,M\}$ (see §2,4) we rewrite (2) in integral form as follows

3) $$z^\varepsilon(t,\lambda) - z^\varepsilon(t',\lambda) = \int_{t\prime}^{t} \frac{\partial S}{\partial z_0}(p^\varepsilon(s); z_0(s,\lambda)\frac{dz_0}{ds}(s,\lambda)ds$$

$$+\sum_{j=1}^{m}\int_{t'}^{t} \chi_\tau(s)\alpha(p^\varepsilon(s))Z_j(z^\varepsilon(s,\lambda))\frac{dw_j^\varepsilon}{ds}(s)ds$$

By definition ,the smooth approximation $p^\varepsilon(t), t \in [0,T]$ is an $\mathcal{F}_t$ -adapted process fulfilling the conditions of approximation theorem in[4] and as a consequence we obtain

4) $\lim_{\varepsilon \searrow 0} p^\varepsilon(t) = \hat{p}(t)$,in $L_2(\Omega, P)$,uniformly in $t \in [0,a]$.

5) $\lim\limits_{\varepsilon\searrow 0} z^{\varepsilon}(t,\lambda) = \hat{z}(t,\lambda) \triangleq S(\hat{p}(t); z_0(t,\lambda))$ in $L_2(\Omega, P)$ uniformly in $t \in [0,a]$

On the other hand,the left hand side in (3) is rewritten as

$$6) z^{\varepsilon}(t,\lambda) - z^{\varepsilon}(t',\lambda) = S(p^{\varepsilon}(t); z_0(t,\lambda)) - S(p^{\varepsilon}(t); z_0(t',\lambda)) + S(p^{\varepsilon}(t); z_0(t',\lambda)) - S(p^{\varepsilon}(t'); z_0(t',\lambda)) =$$

$$= \int_{t'}^{t} \frac{\partial S}{\partial z_0}(p^{\varepsilon}(t); z_0(s,\lambda) \frac{dz_0}{ds}(s,\lambda)ds +$$

$$+ \sum_{j=1}^{m} \left[\int_{t'}^{t} \chi_{\tau}(s)\alpha(p^{\varepsilon}(s)) Z_j(S(p^{\varepsilon}(s); z_0)) \frac{dw_j^{\varepsilon}}{ds}(s)ds \right]_{z_0 = z_0(t',\lambda)} \triangleq T_1 + T_2$$

Using (6) in (3) we get by a direct computation

$$7) \sum_{j=1}^{m} \int_{t'}^{t''} \chi_{\tau}(s)\alpha(p^{\varepsilon}(s)) Z_j(z^{\varepsilon}(s,\lambda)) \frac{dw_j^{\varepsilon}}{ds}(s)ds = T_1 + T_2 -$$

$$- \int_{t'}^{t} \frac{\partial S}{\partial z_0}(p^{\varepsilon}(s); z_0(s,\lambda)) \frac{dz_0}{ds}(s,\lambda)ds = \; = \sum_{j=1}^{m}(T_1^j + T_2^j)$$

where

$$8) T_1^j = \int_{t'}^{t} \left(\left[\frac{\partial}{\partial z_0} \int_{s}^{t} \chi_{\tau}(\sigma)\alpha(p^{\varepsilon}(\sigma)) Z_j(S(p^{\varepsilon}(\sigma); z_0) \frac{dw_j^{\varepsilon}}{d\sigma}(\sigma) \right]_{z_0 = z_0(s,\lambda)} \right) \frac{dz_0}{ds}(s,\lambda)ds$$

$$T_2^j = \left[\int_{t\prime}^{t\prime\prime} \chi_{\tau}(s)\alpha(p^{\varepsilon}(s)) Z_j(S(p^{\varepsilon}(s); z_0)) \frac{dw_j^{\varepsilon}}{ds}(s)ds \right]_{z_0 = z_0(t\prime,\lambda)}, j \in \{1,..,m\}$$

As far as $\alpha(\hat{p}(t) = 1, t \in [0,T]$ (see §2.6) the following

$$9) T_1^j = \lim_{\varepsilon\searrow 0} T_1^j = \int_{t\prime}^{t} \left(\left[\frac{\partial}{\partial z_0} \int_{s}^{t} \chi_{\tau}(\sigma) Z_j(S(\hat{p}(\sigma); z_0) \circ dw_j(\sigma) \right]_{z_0 = z_0(s,\lambda)} \right) \frac{dz_0}{ds}(s,\lambda)ds$$

$$T_2^j = \lim_{\varepsilon\searrow 0} T_2^j = \left[\int_{t\prime}^{t} \chi_{\tau}(s) Z_j(S(\hat{p}(s); z_0)) \circ dw_j(s) \right]_{z_0 = z_0(t\prime,\lambda)}$$

are obtained using the approximation theorem given in appendix of (4) for the corresponding Fisk-Stratonovich integral.

According to (7) and (9) we may and do define the "⊗" type stochastic integral (Stratonovich type) as folows

10) $\int_{t'}^{t} \chi_\tau(s) Z_j(\hat{z}(s,\lambda) \otimes dw_j(s) = \lim_{\varepsilon \searrow 0} (T_1^j + T_2^j) = \hat{T}_1^j + \hat{T}_2^j, j \in \{1,..,m\}$,and it allows one to write the limit point in $L_2(\Omega, P)$ of the left hand side in (7) as

11) $\lim_{\varepsilon \searrow 0} \sum_{j=1}^{m} \int_{t'}^{t} \chi_\tau(s) \alpha(p^\varepsilon(s)) Z_j(z^\varepsilon(s,\lambda)) \frac{dw_j^\varepsilon}{ds}(s) ds = \sum_{j=1}^{m} (\hat{T}_1^j + \hat{T}_2^j)$

where $\hat{T}_1^j$ and $\hat{T}_2^j$are given in(9) using the Fisk-Stratonovich integral " $\circ$ ".Finally,using (4),(5) and (10) in (3) we rewrite the corresponding integral equation as

12) $\hat{z}(t,\lambda) - \hat{z}(t',\lambda) = \int_{t'}^{t} \frac{\partial S}{\partial z_0}(\hat{p}(s); z_0(s,\lambda) \frac{dz_0}{ds}(s,\lambda) ds$

$$+ \sum_{j=1}^{m} \int_{t'}^{t} \chi_\tau(s) Z_j(\hat{z}(s,\lambda)) \otimes dw_j(s), t \in [t', t''],$$

and the proof is complete.

A stochastical rule of derivation associated with the integral equation given in Lemma 1 is expressed in (§2,Remark3). Here we shall rewrite the mentioned rule using the stochastic integral " $\otimes$ "

Lemma 2 . Let $f \in C_b^3(D;\mathbb{R})$ be given and the non $\mathcal{F}_t$ -adapted solution $\hat{z}(t,\lambda) = S(\hat{p}(t); z_0(t,\lambda)), t \in [0,a]$ is fulfilling the integral equation in Lemma1.Then

$$f(\hat{z}(t'',\lambda)) - f(\hat{z}(t',\lambda)) = \int_{t'}^{t''} \left\langle \frac{\partial f}{\partial z}(\hat{z}(t,\lambda), \frac{\partial S}{\partial z_0}(\hat{p}(t); z_0(t,\lambda) \frac{dz_0}{dt}(t,\lambda) \right\rangle dt$$

$$+ \sum_{j=1}^{m} \int_{t'}^{t''} \chi_\tau(t) \left\langle \frac{\partial f}{\partial z}(\hat{z}(t,\lambda), Z_j(\hat{z}(t,\lambda)) \right\rangle \otimes dw_j(t)$$

where $\int_{t'}^{t''} \chi_\tau(t) h(\hat{z}(t,\lambda)) \otimes w_j(s) = \lim_{\varepsilon \searrow 0} \int_{t'}^{t''} \chi_\tau(t) h(\hat{z}(t,\lambda) \frac{dw_j^\varepsilon}{dt}(t)$

obeys to the definition (§2.1) for any $h \in C_b^2(D;\mathbb{R})$.

The verification of *Lemma* 2 is based on a direct computation already used in the proof of *Lemma* 1.

Now we are in position to state the main results regarding the stochastic partial differential equations (s.p.d.e) given in (α) and (β) of the introduction.

In both cases ,the meaning of a stochastic differential along to a fixed family of trajectories $x = \hat{x}(t,\lambda)$ has to be introduced relying on the smooth approximations used in the above given lemmas.Recalling the smooth mapping $S(p; z_0), p \in D_M, z_0 \in D_0$,defined in (§2.3) and using the properties stated in (§2.4,5) we rewrite

12) $S(p;z_0) = z_0 + \sum_{j=1}^{M} t_j \int_0^1 \hat{Z}_j(\theta p; S(\theta p; z_0)d\theta, p \in B(0,\hat{\rho}) \subseteq D_M$ where the smooth vector fields $\hat{Z}_j(p;z)$ are obtained as follows

13) $\hat{Z}_j(p;z) = \hat{Z}_j(z)Q^{-1}(p)\, , z \in D \triangleq B(0,\rho) \times R^n \subseteq R^{2n+1}, p \in B(0,\hat{\rho}), j \in \{1,..,M\}$

Denote

14) $\hat{Z}_j(p; S(p;z_{0)}) \triangleq \begin{pmatrix} \hat{Y}_j(p;z_0) \\ \hat{X}_j(p;z_0) \end{pmatrix}, j \in \{1,..,M\}$ and the equations in (12) are written accordingly

15) $$\begin{cases} G(p,x_0;y_0) = y_0 + \sum_{j=1}^{M} t_j \int_0^1 \hat{Y}_j(\theta p; z_0)d\theta, y_0 \in B(0,\rho_0) \subseteq \mathbb{R}^{n+1} \\ J(p,y_0;x_0) = x_0 + \sum_{j=1}^{M} t_j \int_0^1 \hat{X}_j(\theta p; z_0)d\theta, x_0 \in \mathbb{R}^n \end{cases}$$

for $p \in B(0,\hat{\rho}) \subseteq D_M$ and $z_0 \in D_0 \triangleq B(0,\rho_0) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n+1}$,where $\frac{\partial G}{\partial y_0}(p,x_0;y_0)$ and $\frac{\partial J}{\partial x_0}(p,y_0;x_0)$ are nonsingular matrices for any $(p,y_0;x_0) \in B(0,\hat{\rho}) \times B(0,\rho_0) \times \mathbb{R}^n$

provided $\hat{\rho} > 0$ is sufficently small.

In addition, we are looking for a continuous process.

$z_0 = \hat{z}_0(t,\lambda) \triangleq (\hat{y}_0(t,\lambda), \hat{x}_0(t,\lambda)) \in B(0,\rho_0) \times \mathbb{R}^n$ such that

16) $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); \hat{z}_0(t,\lambda)), t \in [0,a], \lambda \in \mathbb{R}^n$ is a local solution of the associated stochasticsystem of characteristics given in ($\S 1.\alpha_1$) relying on the integral equations defined in *Lemma* 1.

It implies to look for $z_0 = \hat{z}_0(t,\lambda), t \in [0,a], 0 < a < T$, as the unique solution of the following system of ordinary differential equations

17) $$\begin{cases} \frac{d\hat{z}_0}{dt}(t,\lambda) = \left[\frac{\partial S}{\partial z_0}(\hat{p}(t);\hat{z}_0(t,\lambda))\right]^{-1} Z_0\left(S(\hat{p}(t);\hat{z}_0(t,\lambda))\right) \\ \hat{z}_0(0,\lambda) = z_0(\lambda) \triangleq (y_0(\lambda),\lambda) \in B(0,\rho_1) \times \mathbb{R}^n, 0 < \rho_1 < \rho_0 \end{cases}$$

where the smooth vector field $Z_0 \in C_b^1(D;\mathbb{R}^{2n+1})$ is defined in ($\S 1.\alpha_2$).

As a consequence,the continuous process

$\hat{z}(t,\lambda) \triangleq (\hat{y}(t,\lambda), \hat{x}(t,\lambda))$,$t \in [0,a], \lambda \in \mathbb{R}^n$,is a loca lsolution of the corresponding system of characteristics (see $\S 1.\alpha_1$) and using (15) we write the equations fulfilled by $x = \hat{x}(t,\lambda)$ as follows

18) $\hat{x}(t,\lambda) \triangleq J(\hat{p}(t), \hat{y}_0(t,\lambda); \hat{x}_0(t,\lambda)) = \hat{x}_0(t,\lambda) +$

$$+\sum_{j=1}^{M} t_j(t)\int_0^1 \hat{X}_j(\theta\hat{p}(t);\hat{z}_0(t,\lambda))d\theta, \hat{p}(t) \triangleq (t_1(t),..,t_M(t))$$

Here,the continuously differentiable process

$x_0 = \hat{x}_0(t,\lambda)$, $y_0 = \hat{y}_0(t,\lambda)$ and $z_0 = \hat{z}_0(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,are obtained from (17) with the following integral form

19) $\hat{x}_0(t,\lambda) = \lambda + \int_0^t \hat{X}_0(\hat{p}(s), \hat{z}_0(s,\lambda)ds, t \in [0,a], \lambda \in \mathbb{R}^n$

20) $\hat{y}_0(t,\lambda) = y_0 + \int_0^t \hat{Y}_0(\hat{p}(s), \hat{z}_0(s,\lambda)ds, t \in [0,a], \lambda \in \mathbb{R}^n$ where

21) $\begin{pmatrix} \hat{X}_0(p;z_0) \\ \hat{Y}_0(p;z_0) \end{pmatrix} \triangleq \left[\frac{\partial S}{\partial z_0}(p;z_0)\right]^{-1} Z_0\left(S(p,z_0)\right)$

are smooth and bounded vector fields for $p \in B(0,\hat{\rho}) \subseteq D_M$ and

$z_0 \in D_0 \triangleq B(0,\rho_0) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n+1}$.

Using 19) in18) we see easily that the following algebraic equation

22) $\hat{x}(t,\lambda) = x$

has a unique solution $\lambda = \psi(t,x), t \in [0,a], x \in \mathbb{R}^n$, verifying

23)$\hat{x}(t,\psi(t,x)) = x, \quad \psi(t,\hat{x}(t,\lambda)) = \lambda, t \in [0,a],\ x,\lambda \in \mathbb{R}^n$,

Denote

24) $y(t,x) \triangleq \hat{y}(t,\psi(t,x)) \triangleq (u(t,x), p(t,x)$

and by definition (see $\psi(t,\hat{x}(t,\lambda)) = \lambda$)

25) $u(t,\hat{x}(t,\lambda)) = \hat{u}(t,\lambda), p(t,\hat{x}(t,\lambda)) = \hat{p}(t,\lambda)$

A local solution for s.p.d.e (α) is asimilated with the continiuous process in(24) provided we are able to show

26) $\frac{\partial u}{\partial x}(t,x) = p(t,x), t \in [0,a],\ x \in \mathbb{R}^n$ and

27)$\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \int_{t'}^{t''} d_t\hat{u}(t,\lambda) - \langle \hat{p}(t,\lambda), d_t\hat{x}(t,\lambda)\rangle$

for any $[t',t''] \subseteq [0,a]$

Here the left side in (27) is defined as

28) $\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \lim_{\varepsilon\searrow 0}\int_{t'}^{t''}[\frac{\partial}{\partial t}\tilde{u}^{\varepsilon}(t,x)]_{x=x^{\varepsilon}(t,\lambda)}dt$

where $\tilde{u}^\varepsilon(t,x) \triangleq u^\varepsilon(t,\psi^\varepsilon(t,x))$ with $x^\varepsilon(t,\psi^\varepsilon(t,x)) = x, \psi^\varepsilon(t,x^\varepsilon(t,\lambda)) = \lambda$

The smooth aproximation $z^\varepsilon(t,\lambda) \triangleq (\, y^\varepsilon(t,\lambda), x^\varepsilon(t,\lambda)) \triangleq S(p^\varepsilon(t); z_0(t,\lambda)).t \in [0,a], \lambda \in \mathbb{R}^n$ is constructed as in*Lemma*1 (see (1) or §2.13) according to the differential equations

$$29)\begin{cases} \frac{dz^\varepsilon}{dt}(t,\lambda) = \frac{\partial S}{\partial z_0}(\sigma^\varepsilon(t); z_0(t,\lambda))\frac{dz_0}{dt}(t,\lambda) + \\ \qquad + \sum_{j=1}^{m} \chi_\tau(t)\alpha(\sigma^\varepsilon(t)) Z_j(z^\varepsilon(t,\lambda))\frac{dw_j^\varepsilon}{dt}(t) \\ \frac{d\sigma^\varepsilon}{dt}(t) = \sum_{j=1}^{m} \chi_\tau(t)\alpha(\sigma^\varepsilon(t)) q_j(\sigma^\varepsilon(t))\frac{dw_j^\varepsilon}{dt}, t \in [0,a] \\ z^\varepsilon(0,\lambda) = z_0(\lambda) \triangleq (u_0(\lambda), \partial_\lambda u_0(\lambda), \lambda), \sigma^\varepsilon(0) = 0 \end{cases}$$

Relying on the proof of *Lemma* 1 we get

30) $\lim_{\varepsilon \searrow 0} z^\varepsilon(t,\lambda) = \hat{z}(t,\lambda) \triangleq (\hat{y}(t,\lambda), \hat{x}(t,\lambda))$ in $L_2(\Omega, P)$ for each $(t,\lambda) \in [0,a] \times \mathbb{R}^n$,

where $z = \hat{z}(t,\lambda)$ is a local solution of the integral equations given in *Lemma*1

.

Lemma 3. Under the same condition as in *Lemma* 1 define $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); \hat{z}_0(t,\lambda))$ where $\hat{z}_0(t,\lambda))$, $t \in [0,a], \lambda \in \mathbb{R}^n$, is the unique solution of the differential equation in(17). Then $z = \hat{z}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,is a local solution of the associated stochastic system of characteristics (α_1) i.e.

*) $\hat{z}(t,\lambda) = z_0(\lambda) + \int_0^t Z_0(\hat{z}(s,\lambda))\, ds + \sum_{j=1}^{m}\int_0^t \chi_\tau(s)\, Z_j(\hat{z}(s,\lambda)) \otimes dw_j(s)$ associated with s.p.d.e.(α) .

In addition ,let $u(t,x) \triangleq \hat{u}(t,\psi(t,x)), p(t,x) = \hat{p}(t,\psi(t,x))$ where

$y(t,\lambda) \triangleq (\hat{u}(t,\lambda), \hat{p}(t,\lambda))$ and $\lambda = \psi(t,x)$, is the unique solution of the algebraic equations(22). Then (26) and (27) hold true,i.e.

**) $\frac{\partial u}{\partial x}(t,x) = p(t,x), t \in [0,a], x \in \mathbb{R}^n$,

***) $\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(y,\lambda)} = \int_{t'}^{t''} d_t \hat{u}(t,\lambda) + \int_{t'}^{t''} \langle \hat{p}(t,\lambda), \partial_p g_0(\hat{z}(t,\lambda))\rangle\, dt +$

$+\sum_{j=1}^{m}\int_{t'}^{t''} \chi_\tau(t)\, \langle \hat{p}(t,\lambda), \partial_p g_j(\hat{z}(t,\lambda))\rangle \otimes dw_j(t)$ for any $[t',t''] \subseteq [0,a]$,where the left hand side in (***) is defined in (28).

Proof

By hypothesis the continuous process $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t);\hat{z}_0(t,\lambda)), t \in [0,a]$ is fulfilling the integral equations given in *Lemma* 1 and taking $z_0 \triangleq \hat{z}_0(t,\lambda), t \in [0,a]$ as the unique solution of the differential equations in (17) we get easily the conclusion (*) fulfilled.As far as the smooth approximation $z = z^{\varepsilon}(t,\lambda)$ given in (29) is used we may and do define $z_0 = \hat{z}_0(t,\lambda)$, $t \in [0,a]$,as the unique solution of the following ordinary differential equations

$$31)\begin{cases} \frac{d\hat{z}_0}{dt}(t,\lambda) = \left[\frac{\partial S}{\partial z_0}(\sigma^{\varepsilon}(t);\hat{z}_0(t,\lambda))\right]^{-1} Z_0\left(S(\sigma^{\varepsilon}(t);\hat{z}_0(t,\lambda))\right) t \in [0,a] \\ z_0(0,\lambda) \triangleq z_0(\lambda) \triangleq (u_0(\lambda), \partial_\lambda u_0(\lambda), \lambda) \end{cases}$$

Using (31)in (29) we get easily

$$32)\begin{cases} \frac{dz^{\varepsilon}}{dt}(t,\lambda) = Z_0(z^{\varepsilon}(t,\lambda)) + \sum_{j=1}^{m} \chi_{\tau}(t)\alpha(\sigma^{\varepsilon}(t)) Z_j(z^{\varepsilon}(t,\lambda))\frac{dw_j^{\varepsilon}}{dt}(t) \\ z^{\varepsilon}(0,\lambda) = z_0(\lambda), t \in [0,a], z^{\varepsilon}(t,\lambda) \triangleq (y^{\varepsilon}(t,\lambda), x^{\varepsilon}(t,\lambda)) \end{cases}$$ which can be assimilated with the characteristic system associated with the scalar equation

$$33)\begin{cases} \partial_t \tilde{u}^{\varepsilon}(t,x) = g_0(x,\tilde{u}^{\varepsilon}(t,x),\partial_x\tilde{u}^{\varepsilon}(t,x)) + \\ +\sum_{j=1}^{m} g_j(x,\tilde{u}^{\varepsilon}(t,x),\partial_x\tilde{u}^{\varepsilon}(t,x),\chi_{\tau}(t)\alpha(\sigma^{\varepsilon}(t))\frac{dw_j^{\varepsilon}}{dt}(t) \\ u^{\varepsilon}(0,x) = u_0(x), x \in \mathbb{R}^n \end{cases}$$

A direct computation used in the deterministic case and applied here allow one to see that

34) $\tilde{u}^{\varepsilon}(t,x) \triangleq u^{\varepsilon}(t,\psi^{\varepsilon}(t,x))$ with $x^{\varepsilon}(t,\psi^{\varepsilon}(t,x)) = x, \psi^{\varepsilon}(t,x^{\varepsilon}(t,\lambda)) = \lambda$ is obtaining to

35) $\partial_x\tilde{u}^{\varepsilon}(t,x) = \tilde{p}^{\varepsilon}(t,x) \triangleq p^{\varepsilon}(t,\psi^{\varepsilon}(t,x)), t \in [0,a], x \in \mathbb{R}^n$.

Rewrite (35) along to $x = x^{\varepsilon}(t,\lambda)$ for u^{ε} defined in (34) and obtain

36) $\frac{\partial u^{\varepsilon}(t,\lambda)}{d\lambda}\frac{\partial\psi^{\varepsilon}}{\partial x}((t,x^{\varepsilon}(t,\lambda)) = p^{\varepsilon}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$.where $\frac{\partial\psi^{\varepsilon}}{\partial\lambda}(t,x^{\varepsilon}(t,\lambda)) = \left[\frac{\partial x^{\varepsilon}}{\partial\lambda}(t,\lambda)\right]^{-1}$

It shows that

37) $\frac{\partial u^{\varepsilon}}{d\lambda}(t,\lambda) = p^{\varepsilon}(t,\lambda)\frac{\partial x^{\varepsilon}}{\partial\lambda}(t,\lambda), (t,\lambda) \in [0,a] \times \mathbb{R}^n$

and using$\lim_{\varepsilon\searrow 0} z^{\varepsilon}(t,\lambda) = \hat{z}(t,\lambda) \triangleq (\hat{y}(t,\lambda), \hat{x}(t,\lambda))$ in $L_2(\Omega,P)$ uniformly with respect to $t \in [0,a]$,we get

38) $\lim_{\varepsilon\searrow 0}\frac{\partial z^{\varepsilon}}{\partial\lambda}(t,\lambda) \triangleq \lim_{\varepsilon\searrow 0}\left(\frac{\partial y^{\varepsilon}}{\partial\lambda}(t,\lambda), \frac{\partial x^{\varepsilon}}{\partial\lambda}(t,\lambda)\right) = \left(\frac{\partial\hat{y}}{\partial\lambda}(t,\lambda), \frac{\partial\hat{x}}{\partial\lambda}(t,\lambda)\right)$in $L_2(\Omega,P)$,for each $(t,\lambda) \in [0,a] \times \mathbb{R}^n$.

By letting $\varepsilon \to 0$ in37) and using (38) one sees easily that it holds

39) $\frac{\partial\hat{u}}{d\lambda}(t,\lambda) = \hat{p}(t,\lambda)\frac{\partial\hat{x}}{\partial\lambda}(t,\lambda), (t,\lambda) \in [0,a] \times \mathbb{R}^n$ which shows the identity between $\partial_x u(t,x)$ and $p(t,x) \triangleq \hat{p}(t,\psi(t,x))$along to $x = \hat{x}(t,\lambda)$,i.e.

40) $\partial_x u(t,\hat{x}(t,\lambda)) = \hat{p}(t,\lambda), (t,\lambda) \in [0,a] \times \mathbb{R}^n$

Take $\lambda = \psi(t,x)$ in 40) and we get the conclusion (**) where$u(t,x) = \hat{u}(t,\psi(t,x))$ and $p(t,x) \overset{\triangle}{=} \hat{p}(t,\psi(t,x))$.

The last conclusion (***) is obtained by the following direct computation

$$41)\int_{t'}^{t''} [\partial_t \tilde{u}^\varepsilon(t,x)]_{x=x^\varepsilon(t,\lambda)}\, dt = \int_{t'}^{t''} \tfrac{du^\varepsilon}{dt}(t,\lambda)dt - \int_{t'}^{t''} \left\langle p^\varepsilon(t,\lambda), \tfrac{dx^\varepsilon}{dt}(t,\lambda)\right\rangle dt =$$

$$= u^\varepsilon(t'',\lambda) - u^\varepsilon(t',\lambda) + \int_{t'}^{t''} \langle p^\varepsilon(t,\lambda), \partial_p g_0(z^\varepsilon(t,\lambda))\rangle\, dt$$

$$+\sum_{j=1}^{m}\int_{t'}^{t''} \chi_\tau(t)\,\alpha\left(\sigma^\varepsilon(t)\,\langle p^\varepsilon(t,\lambda), \partial_p g_j(z^\varepsilon(t,\lambda))\rangle\right) \tfrac{dw_j^\varepsilon(t)}{dt}(t)dt$$

Letting $\varepsilon \to 0$ in(41) we get (***) and the proof is complete.

The following theorem is a direct consequence of the results stated in *Lemma* 3.

Theorem 1

Let $g_i(x,u,p), i \in \{1,..,m\})$ be given such that the hypothesis (H_1) is fulfilled.Let $\hat{z}(t,\lambda) \overset{\triangle}{=} (\hat{y}(t,\lambda), \hat{x}(t,\lambda)), (t,\lambda) \in [0,a] \times R^n$ be the local solution associated with the system of characteristics defined in Lemma 3. Let $u(t,x) \overset{\triangle}{=} \hat{u}(t,\psi(t,x)), p(t,x) \overset{\triangle}{=} \hat{p}(t,\psi(t,x))$ where $y(t,\lambda) \overset{\triangle}{=} (\hat{u}(t,\lambda), \hat{p}(t,\lambda))$ and $\lambda = \psi(t,x)$ is the solution in(22) .Then $\partial_x u(t,x) = p(t,x), (t,x) \in [0,a] \times \mathbb{R}^n$ and $u = u(t,x)$ is a local solution of the s.p.d.e.(α) along to $x = \hat{x}(t,\lambda)$, i.e. $u(0,x) = u_0(x), x \in \mathbb{R}^n$ and $[d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = g_0(\hat{z}(t,\lambda))\, dt + \sum_{j=1}^{m} \chi_\tau(t)\, g_j(\hat{z}(t,\lambda)) \otimes dw_j(t)$ for $t \in [0,a]$ where

$$\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \hat{u}(t'',\lambda) - \hat{u}(t',\lambda) + \int_{t'}^{t''} \langle \hat{p}(t,\lambda), \partial_p g_0(\hat{z}(t,\lambda))\rangle\, dt +$$

$$+\sum_{j=1}^{m}\int_{t'}^{t''} \chi_\tau(t)\,\langle \hat{p}(t,\lambda), \partial_p g_j(\hat{z}(t,\lambda))\rangle \otimes dw_j(t) \text{ for any } [t',t''] \subset [0,a]$$

As far as the s.p.d.e.(β) is concerned, relying on the local solution associated with the parabolic stochastic differential equation defined in (β_1) ,we may and do define a local solution fulfilling (β) along to $x = \hat{x}(t,\lambda) \overset{\triangle}{=} \lambda + \sum_{i=1}^{m} b_i w_i(t \wedge \tau)$ assuming in adition that $b_i \overset{\triangle}{=} -\partial_p g_i(x,u,p) \overset{\triangle}{=} X_i(z), i \in \{1,..,m\}$ are some constant vectors in $\mathbb{R}^n$ .The given smooth functions $g_j \in C_b^\infty(\mathbb{R}^n \times B(0,\rho)), j \in \{1,..,m\}$ are obeying to some hypothesis used in Theorem 1 and the general results proved in *Lemma* 1 and *Lemma* 2 are still valid where the continuous and non $\mathcal{F}_t$ -adapted process $z = \hat{z}_0(t,\lambda) \overset{\triangle}{=} (y_0(t,\lambda), \lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,has to be defined according to the parabolic stochastic differential equation given in(β_1). In this case ,a substitute for the conclusion given in *Lemma* 3 is the following

Lemma 4. Let $f \in C\left([0,T]; C_b^2\left(\mathbb{R}^n \times B(0,\rho)\right)\right)$ and

$g_j \in C_b^\infty\left(\mathbb{R}^n \times B(0,\rho)\right), j \in \{1,..,m\}$ be given such that the hypothesis(H_1) is fulfilled.

Assume $u_0 \in C_b^3\left(\mathbb{R}^n\right)$ with $\left(u_0(x), \partial_x u_0(x)\right) \in B(0,\rho_0) \subseteq \mathbb{R}^{n+1}$ for any $x \in \mathbb{R}^n$.

Then there exist continuous process

$\hat{y}_0(t,\lambda) : [0,a] \times \mathbb{R}^n \to B(0,\rho_1) \subseteq \mathbb{R}^{n+1}, 0 < \rho_0 < \rho_1 < \rho$,and $\hat{p}(t) : [0,a] \to B(0,\hat{\rho}) \subseteq D_M$ such that

C_1) $\hat{y}_0 \in C_b^{1,2}\left([t',t''] \times \mathbb{R}^n; B(0,\rho_1)\right)$ for any $[t',t''] \subset [0,a]$

C_2) $\hat{z}(t,\lambda) = S(\hat{p}(t); \hat{z}_0(t,\lambda)) \triangleq (\hat{y}(t,\lambda), \hat{x}(t,\lambda))$and $y = \hat{y}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,is a local solution of s.p.d.e.(β_1) where $\hat{z}_0(t,\lambda) \triangleq (\hat{y}_0(t,\lambda), \lambda)$

Proof

By hypotheses, the conclusion in *Lemma* 1 are fulfilled for the smooth mapping $\hat{z}(t,\lambda) \triangleq S(\hat{p}(t); z_0(t,\lambda))$ where $z_0(t,\lambda), t \in [t',t''] \subset (0,a]$ is continuously differentiable. On the other hand using (12) and (15) we express $\hat{z}(t,\lambda) = S(\hat{p}(t); z_0(t,\lambda) \triangleq (\hat{y}(t,\lambda), \hat{x}(t,\lambda))$,where

42) $\hat{y}(t,\lambda) = G(\hat{p}(t), \lambda; y_0(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,and the smooth mapping $y = G(p,\lambda,y_0), p \in B(0,\hat{\rho}) \subseteq D_M, \lambda \in \mathbb{R}^n, y_0 \in B(0,\rho_1) \subseteq \mathbb{R}^{n+1}$is defined such that

43) $\frac{\partial G}{\partial y_0}(p,\lambda,y_0)$ is a nonsingular matrix.

According to the integral equation in *Lemma* 1 and using the continuously differentiable process

$\hat{z}_0(t,\lambda) = (\hat{y}_0(t,\lambda), \lambda) \in B(0,\rho_1) \times \mathbb{R}^n, t \in (0,a]$,we get the following stochastic differential equations.

$$44) \begin{cases} d_t\hat{y}(t,\lambda) = \frac{\partial G}{\partial y_0}(\hat{p}(t), \lambda, \hat{y}_0(t,\lambda)) \frac{d\hat{y}_0}{dt}(t,\lambda)dt + \sum_{j=1}^{m} \chi_\tau(t) Y_j(\hat{z}(t,\lambda)) \otimes dw_j(t), \\ \hat{y}(0,\lambda) = y_0(\lambda) \triangleq (u_0(\lambda), \partial_\lambda u_0(\lambda)) \in B(0,\rho_0) \subseteq R^{n+1}, 0 < \rho_0 < \rho_1 \end{cases}$$

where the smooth vector fields $Z_j(z) \triangleq \begin{pmatrix} Y_j(z) \\ X_j(z) \end{pmatrix}, j \in \{1,..,m\}$ are defined in (§ 2.1)

We are looking for the unknown $y = \hat{y}_0(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$ such that the s.p.d.e.(β_1) in § 1 coincides with the stochastic differential equation (44) and it implies

$$45) \frac{\partial G}{\partial y_0}(\hat{p}(t), \lambda, \hat{y}_0(t,\lambda)) \frac{d\hat{y}_0}{dt}(t,\lambda) = \Delta_\lambda \hat{y}(t,\lambda)$$

$$+Y_0(t, \hat{x}(t,\lambda), \hat{y}(t,\lambda), \partial_\lambda \hat{y}(t,\lambda)) \left[\frac{\partial \hat{x}(t,\lambda)}{\partial \lambda}\right]^{-1}, t \in [0,a], \lambda \in \mathbb{R}^n$$

where the vector field $Y_0(t,x,y,\partial_x y)$ is defined in (§1.β_1) and

46) $\hat{y}(t,\lambda) = y(t,\hat{x}(t,\lambda)), \partial_x y(t,\hat{x}(t,\lambda)) = \partial_\lambda \hat{y}(t,\lambda)\left[\frac{\partial \hat{x}(t,\lambda)}{\partial \lambda}\right]^{-1}$ are used. Here the continuous process

$\hat{x}(t,\lambda) = J(\hat{p}(t), \hat{y}_0(t,\lambda);\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$.is expressed using the smooth mapping $J(p, y_0, x_0)$ defined in (15) and obeying to

47) $\frac{\partial \hat{x}}{\partial \lambda}(t,\lambda) \triangleq \left[\frac{\partial J}{\partial x_0}(\hat{p}(t), \hat{y}_0(t,\lambda);\lambda) + \frac{\partial J}{\partial y_0}(\hat{p}(t), \hat{y}_0(t,\lambda);\lambda)\, \partial_\lambda y_0(t,\lambda)\right]$

is a nonsingular matrix for any $t \in [0,a], \lambda \in \mathbb{R}^n$.

It allows one to find a unique smooth solution $\lambda = \psi(t,x)$ fulfilling the following algebraic equations.

48) $\hat{x}(t,\psi(t,x)) = x, \psi(t,\hat{x}(t,\lambda)) = \lambda,\ x,t \in [0,a], \lambda \in \mathbb{R}^n$.

By definition $\hat{x}(t,\lambda)$ is second order continuously differentiable with respect to $\lambda \in \mathbb{R}^n$ provided $y_0 \in C_b^{1,2}([t',t''] \times \mathbb{R}^n; B(0,\rho_1))$for any $[t',t'] \subset [0,a]$ and using(48) we obtain a second order continuously differentiable mapping $\lambda = \psi(t,x)$ with respect to $x \in \mathbb{R}^n$ provided $t \in [t',t'']$,define a smooth mapping of $x \in \mathbb{R}^n$.

49) $y(t,x) \triangleq \hat{y}(t,\psi(t,x)), t \in [t',t''] \subset [0,a]$ obeying to(46). Using (42)and (47) we rewrite (45) as a parabolic equation for the unknown $\hat{y}_0(t,\lambda)$

$$50)\ \begin{cases} \partial_t \hat{y}_0(t,\lambda) = \Delta_\lambda \hat{y}_0(t,\lambda) + \hat{Y}_0(t,\lambda,\hat{y}_0(t,\lambda), \partial_\lambda \hat{y}_0(t,\lambda)), t \in [0,a] \\ \hat{y}_0(0,\lambda) = y_0(\lambda) \triangleq (u_0(\lambda), \partial_\lambda u_0(\lambda)) \in B(0,\rho_0) \subseteq \mathbb{R}^{n+1} \end{cases}$$

where the Lipschitz continuous vector field $\hat{Y}_0(t,\lambda,y_0,\partial_\lambda \hat{y}_0)$with respect to $\hat{y}_0, \partial_i \hat{y}_0 \in B(0,\rho_1), 0 < \rho_0 < \rho_1$, is computed such that

51) $\Delta_\lambda \hat{y}_0(t,\lambda) + \hat{Y}_0(t,\lambda,\hat{y}_0(t,\lambda),\partial_\lambda \hat{y}_0(t,\lambda)) =$

$$= \left[\frac{\partial G}{\partial y_0}(\hat{p}(t),\lambda,\hat{y}_0(t,\lambda))\right]^{-1}\left(\Delta_\lambda \hat{y}(t,\lambda) + Y_0(t,\hat{x}(t,\lambda),\hat{y}(t,\lambda),\partial_\lambda \hat{y}(t,\lambda))\left[\frac{\partial \hat{x}(t,\lambda)}{\partial \lambda}\right]^{-1}\right)$$

Using (42) and a direct computation we get $\hat{Y}_0(t,x,\hat{y}_0,\partial_\lambda \hat{y}_0)$ as a continuous and bounded function of $(t,\lambda) \in [0,a] \times \mathbb{R}^n, \hat{y}_0, \partial_i \hat{y}_0 \in B(0,\rho_1) \subseteq \mathbb{R}^{n+1}, i \in \{1,..,n\}$,being Lipschitz continuous with respect to $\hat{y}_0, \partial_i \hat{y}_0 \in B(0,\rho_1), i \in \{1,..,n\}$.

The parabolic equation in(50) obeyes to the usual condition for writting its solution in integral form

$$52)\begin{cases} \hat{y}_0(t,\lambda) = \int\limits_{R^n} y_0(x)P(t,\lambda,x)dx + \\ \qquad + \int\limits_0^t ds \int\limits_{R^n} \hat{Y}_0(s,x,\hat{y}_0(s,x),\partial_x\hat{y}_0(s,x)P(t-s,\lambda,x)dx \\ \partial_\lambda\hat{y}_0(t,\lambda) = \int\limits_{R^n} \partial_x y_0(x)P(t,\lambda,x)dx + \\ \qquad + \int\limits_0^t ds \int\limits_{R^n} \hat{Y}_0(s,x,\hat{y}_0(s,x),\partial_x\hat{y}_0(s,x))\partial_\lambda P(t-s,\lambda,x)dx \end{cases}$$

for $t \in [0,a], \lambda \in \mathbb{R}^n$,where $0 < a < T$ is found independently of $\omega \in \Omega, \lambda \in \mathbb{R}^n$,and $P(\tau,\lambda,x), \tau > 0, \lambda, x \in \mathbb{R}^n$ is the fundamental solution solving the parabolic equation

53) $\partial_\tau P(\tau,\lambda,x) = \Delta_\lambda P(\tau,\lambda,x)$ obeing to $\int\limits_{\mathbb{R}^n} P(\tau,\lambda,x)dx = 1$

for any $\tau > 0, \lambda \in \mathbb{R}^n$.

It has the form

54) $P(\tau,\lambda,x) \triangleq (4\pi\tau)^{-\frac{n}{2}} \exp -\frac{|x-\lambda|^2}{4\tau}, \tau > 0, x,\lambda \in \mathbb{R}^n$,and induces a unique solution $\hat{y}_0(t,\lambda), \partial_\lambda\hat{y}_0(t,\lambda)$ solving the integral equation 52) as continuous process of $t \in [0,a]$ and satisfying

55) $y_0 \in C_b^{1,2}([t',t''] \times \mathbb{R}^n; B(0,\rho_1)), \hat{y}_0(0,\lambda) = y_0(\lambda), \hat{y}_0(t,\lambda)$obeyes to (50) for any $t \in [t',t''] \subset [0,a]$

As a consequence, $\hat{y}(t,\lambda) \triangleq G(\hat{p}(t),\lambda;\hat{y}_0(t,\lambda)), t \in [0,a], \lambda \in \mathbb{R}^n$ is a solution of s.p.d.e (β_1) and the proof is complete .

Remark

The conclusion in the above given *Lemma* 4 are obtained using the main hypothesis (H_1) fulfilled by the smooth function $g_j, j \in \{1,..,m\}$.A solution for s.p.d.e (β) is found using the continuous process $\hat{y}(t,\lambda), t \in [0,a], \lambda \in R^n$ and the smooth mapping $\lambda \triangleq \psi(t,x)$ satistying the algebraic equation (48) .

The computation of the Laplacian $[\Delta_x y(t,x)]_{x=\hat{x}(t,\lambda)}$ along to the continuous process$x = \hat{x}(t,\lambda)$ is not a simple one for $y(t,x) \triangleq \hat{y}(t,\psi(t,x))$.)

The simpliest form of the laplacian is available provided ,we assume,in adition that $\partial_p g_j(x,u,p) = b_i \in \mathbb{R}^n, i \in \{1,..,m\}$ are some constant vectors and we get $[\Delta_x y(t,x)]_{x=\hat{x}(t,\lambda)} = \Delta_\lambda \hat{y}(t,\lambda)$ by a direct computation.

A solution for s.p.d.e.(β) is constructed as in the following

Theorem 2

Let $f \in C([0,T]; C_b^2(\mathbb{R}^n \times B(0,\rho))) u_0 \in C^3(\mathbb{R}^n,\mathbb{R})$ be given such that $y_0(\lambda) = (u_0(\lambda), \partial_\lambda u_0(\lambda)) \in B(0,\rho_0) \subseteq \mathbb{R}^{n+1}$and $\partial_i y_0(\lambda) \in B(0,\rho_0) \subseteq \mathbb{R}^{n+1}, i \in \{1,..,n\}, \lambda \in \mathbb{R}^n$ .Let $g_j, j \in \{1,..,m\}$be given fufilling the hypothesis(H_1) and $\partial_p g_j(x,u,p) = b_j \in \mathbb{R}^n, j \in \{1,..,m\}$.Define $\hat{x}(t,\lambda) =$

$\lambda-\sum_{j=1}^{m} b_j w_j(t\wedge\lambda)$, $\psi(t,x)=x+\sum_{j=1}^{m} b_j w_j(t\wedge\lambda)$ and $y(t,x)\triangleq\hat{y}(t,\psi(t,x))\triangleq(u(t,x),p(t,x))$, $t\in[0,a]$, $x\in\mathbb{R}^n$, where $\hat{y}(t,\lambda)=(\hat{u}(t,x),\hat{p}(t,x))$ is the solution of the equation (β_1) given in *Lemma* 4.

Then $u=u(t,x)$, $t\in[0,a]$, $x\in\mathbb{R}^n$ is a local solution of s.p.d.e (β) along to $x=x(t,\lambda)$,i.e. $[\partial_x u(t,x)]_{x=\hat{x}(t,\lambda)}=\hat{p}(t,\lambda)$, $[\Delta_x u(t,x)]_{x=\hat{x}(t,\lambda)}=\Delta_\lambda\hat{u}(t,\lambda)$, $u(0,x)=u_0(x)$ and

$$*)\ [d_t u(t,x)]_{x=\hat{x}(t,\lambda)}=[\Delta_\lambda\hat{u}(t,\lambda)+f(t,\hat{z}(t,\lambda))]\,dt$$
$$+\sum_{j=1}^{m}\chi_\tau(t)\,g_j(\hat{z}(t,\lambda))\otimes dw_j(t),\ t\in[0,a] \text{ where } \hat{z}(t,\lambda)=(\hat{y}(t,\lambda),\hat{x}(t,\lambda))$$

and

$$**)\ \int_{t'}^{t''}[d_t u(t,x)]_{x=\hat{x}(t,\lambda)}=\hat{u}(t'',\lambda)-\hat{u}(t',\lambda)+$$
$$+\sum_{j=1}^{m}\int_{t'}^{t''}\chi_\tau(t)\,\langle\hat{p}(t,\lambda),b_j\rangle\otimes dw_j(t)$$

Proof

By hypotheses the conclusion in *Lemma* 4 hold true and let $\hat{y}_0(t,\lambda):[0,a]\times\mathbb{R}^n\to B(0,\rho_1)\subseteq\mathbb{R}^{n+1}$, $p(t):[0,a]\to B(0,\hat{\rho})\subset D_M=\prod_{i=1}^{M}[-a_i,a_i]$ be the continuous process fulfilling $(C_1)$and(C_2), of *Lemma* 4 with $\hat{z}(t,\lambda)=(\hat{y}(t,\lambda),\hat{x}(t,\lambda))=S(\hat{p}(t);\hat{z}_0(t,\lambda))$, $\hat{z}_0(t,\lambda)\triangleq(\hat{y}_0(t,\lambda),\lambda)$.Using the stochastic differential equation (β_1) fulfilled by $y=\hat{y}(t,\lambda)$ along to $x=\hat{x}(t,\lambda)\triangleq\lambda-\sum_{j=1}^{m} b_j w_j(t\wedge\lambda)$we rewrite the corresponding s.p.d.e for the scalar component $u=\hat{u}(t,\lambda)$ of $\hat{y}(t,x)\triangleq(\hat{u}(t,\lambda),\hat{p}(t,\lambda))$as follows

$$56)\ \begin{cases} d_t\hat{u}(t,\lambda)+\sum_{j=1}^{m}\chi_\tau(t)\,\langle\hat{p}(t,\lambda),b_j\rangle\otimes dw_j(t)= \\ [\Delta_\lambda\hat{u}(t,\lambda)+f(t,\hat{z}(t,\lambda))]\,dt+\sum_{j=1}^{m}\chi_\tau(t)\,g_j(\hat{z}(t,\lambda))\otimes dw_j(t)\end{cases}$$

for any $t\in[0,a]$, $u(0,\lambda)=u_0(\lambda)$, $\lambda\in\mathbb{R}^n$.

Here we have used the smooth vector fields $Z_i(z)\triangleq\begin{pmatrix}Y_i(z)\\-b_i\end{pmatrix}$, $i\in\{1,..,m\}$,as defined in (α_2) and the corresponding vector field $Y_0(t,x,y,\partial_x y)$ along to $x=\hat{x}(t,\lambda)\triangleq\lambda-\sum_{j=1}^{m} b_j w_j(t\wedge\lambda)$ relying on the hypothesis $\partial_p g_i(x,u,p)=b_i\in\mathbb{R}^n$, $i\in\{1,..,m\}$.According to the conclusion (C_1) in *Lemma* 4 we write the s.p.d.e fulfilled by $\hat{p}(t,\lambda)$, $t\in[0,a]$, $\lambda\in\mathbb{R}^n$as follows

$$57)\begin{cases} d_t\hat{p}(t,\lambda) = \{\Delta_\lambda \hat{p}(t,\lambda) + \partial_\lambda [f(t,\hat{x}(t,\lambda),\hat{u}(t,\lambda),\hat{p}(t,\lambda))]\} dt + \\ +\sum_{j=1}^{m} \chi_\tau(t)\, \partial_\lambda [g_j(\hat{x}(t,\lambda),\hat{u}(t,\lambda),\hat{p}(t,\lambda))] \otimes dw_j(t), t \in [0,a] \\ \hat{p}(0,\lambda) = p_0(\lambda), \lambda \in \mathbb{R}^n \end{cases}$$

where $y(t,x) \triangleq y(t,\psi(t,x)) = (\hat{u}(t,\psi(t,x),\hat{p}(t,\psi(t,x))$ and

$\psi(t,x) = x + \sum_{j=1}^{m} b_j w_j(t\wedge\lambda), t \in [0,a], x \in \mathbb{R}^n$,is fulfilling the following algebraic equations

58) $\hat{x}(t,\psi(t,x)) = x, \psi(t,\hat{x}(t,\lambda)) = \lambda, \frac{\partial \hat{x}}{\partial \lambda}(t,\lambda) = \frac{\partial \psi}{\partial x}(t,x) = I_n$ for any $t \in [0,a]$, $x, \lambda \in \mathbb{R}^n$.

The conclusion in Theorem 2 is proved provided the continuous process $u(t,x) \triangleq \hat{u}(t,\psi(t,x))$ and $p(t,x) \triangleq \hat{p}(t,\psi(t,x))$, $t \in [0,a]$, $x \in \mathbb{R}^n$. are obeying to the following equations

59) $[\partial_x u(t,x)]_{x=\hat{x}(t,\lambda)} = \hat{p}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$

60) $[\Delta_x u(t,x)]_{x=\hat{x}(t,\lambda)} = \Delta_\lambda \hat{u}(t,x), t \in [0,a], \lambda \in \mathbb{R}^n$

As far as(59) is concerned using (58) ,we rewrite it as follows

61) $\partial_\lambda \hat{u}(t,x) = \hat{p}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$

where $\hat{y}(t,\lambda) \triangleq (\hat{u}(t,\lambda), \hat{p}(t,\lambda))$is a solution of s.p.d.e.(56)and (57).

Taking the smooth approximation

$z^\varepsilon(t,\lambda) \triangleq S(\sigma^\varepsilon(t); z_0(t,\lambda)) = (y^\varepsilon(t,\lambda), x^\varepsilon(t,\lambda))$,where

$$62)\begin{cases} \frac{d\sigma^\varepsilon(t)}{dt} = \sum_{j=1}^{m} \chi_\tau(t)\, \alpha(\sigma^\varepsilon(t)) q_j(\sigma^\varepsilon(t)) \frac{dw_j^\varepsilon(t)}{dt}(t), t \in [0,a] \\ \sigma^\varepsilon(0) = 0 \end{cases}$$

and $z_0^\varepsilon(t,\lambda) = (y_0^\varepsilon(t,\lambda), \lambda)$, $t \in [0,a]$ we get the following system of parabolic equation

$$63)\begin{cases} \frac{dy^\varepsilon}{dt}(t,\lambda) = \Delta_\lambda y^\varepsilon(t,\lambda) + Y_0(t, x^\varepsilon(t,\lambda), y^\varepsilon(t,\lambda), \partial_\lambda y^\varepsilon(t,\lambda)) + \\ +\sum_{j=1}^{m} \chi_\tau(t)\, \alpha(p^\varepsilon(t)) Y_j(x^\varepsilon(t,\lambda), y^\varepsilon(t,\lambda), \partial_\lambda y^\varepsilon(t,\lambda)) \frac{dw_j^\varepsilon(t)}{dt}(t), \\ t \in [0,a], y^\varepsilon(0,\lambda) = y_0(\lambda) = (u_0(\lambda), \partial_\lambda u_0(\lambda)) \end{cases}$$

and

$$64)\begin{cases} \frac{dx^\varepsilon}{dt}(t,\lambda) = -\sum_{j=1}^{m} \chi_\tau(t)\, \alpha(\sigma^\varepsilon(t)) b_j \frac{dw_j^\varepsilon(t)}{dt}, t \in [0,a] \\ x^\varepsilon(0,\lambda) = \lambda \end{cases}$$

The solution of (63) is represented in integral form as follows

65) $y^\varepsilon(t,\lambda) = \int_{\mathbb{R}^n} y_0(x) P(t,\lambda,x) dx +$

$$+\int_0^t ds \int_{\mathbb{R}^n} F(s, x^\varepsilon(s,x), y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x)) P(t-s, \lambda, x) dx$$

where

$$66)\begin{cases} F(s, x^\varepsilon(s,x), y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x)) = Y_0(s, x^\varepsilon(s,x), y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x)) + \\ \quad + \sum_{j=1}^m \chi_\tau(s)\, \alpha(p^\varepsilon(s)) Y_j(s, x^\varepsilon(s,x), y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x)) \frac{dw_j^\varepsilon(s)}{ds}, \\ \quad P(\tau, \lambda, x) = (4\pi\tau)^{-\frac{n}{2}} \exp - \frac{|x-\lambda|^2}{4\tau}, \tau > 0, \lambda, x \in \mathbb{R}^n \end{cases}$$

$$67) \partial_\lambda y^\varepsilon(t,\lambda) \triangleq \int_{\mathbb{R}^n} (\partial_x y_0(x)) P(t, \lambda, x) dx +$$

$$+\int_0^t ds \int_{\mathbb{R}^n} F(s, x^\varepsilon(s,x), y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x)) \partial_\lambda P(t-s, \lambda, x) dx$$

The integral equation (65) and (67) have a unique continuous solution $(y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x)), t \in [0,a], \lambda \in \mathbb{R}^n$ and making the transformation $x - \lambda = \sqrt{4t}z$ for the first integral in(65), $x - \lambda = \sqrt{4(t-s)}z$ for the second integral in (65) we get $y^\varepsilon(t,\lambda)$ expressed as follows

$$68)\ y^\varepsilon(t,\lambda) = \{ \int_{R^n} y_0(\lambda + \sqrt{4t}z) \exp - |z|^2) dz +$$

$$+\int_0^t ds \int_{R^n} F_0(s, \lambda + \sqrt{4(t-s)}z) \exp - |z|^2) dz \} (\pi)^{-\frac{n}{2}}$$

where $F_0(s,x) \triangleq F(s, x^\varepsilon(s,x), y^\varepsilon(s,x), \partial_x y^\varepsilon(s,x))$

Using the special form of the vector fields Y_0, Y_j and

$y_0(x) \triangleq (u_0(x), \partial_x u_0(x))$ we get $y^\varepsilon(t,\lambda) \triangleq (u^\varepsilon(t,\lambda), p^\varepsilon(t,\lambda))$ in (68) with the property

69) $p^\varepsilon(t,\lambda) = \partial_\lambda u^\varepsilon(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$,for each $\varepsilon > 0$

On the oder hand,letting $\varepsilon \to 0$, we get

70) $\hat{y}(t,\lambda) \triangleq (\hat{u}(t,\lambda), \hat{p}(t,\lambda)) = \lim_{\varepsilon \searrow 0} (u^\varepsilon(t,\lambda), \partial_\lambda u^\varepsilon(t,\lambda))$ in $L_2(\Omega, P)$ uniformly with respect to $t \in [0,a]$ and λ is bounded set of $\mathbb{R}^n$,

It shows that $\partial_\lambda \hat{u}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$ exists as a continuous process and

71) $\partial_\lambda \hat{u}(t,\lambda) = \hat{p}(t,\lambda), t \in [0,a], \lambda \in \mathbb{R}^n$ where $\hat{u}(t,\lambda)$ and $\hat{p}(t,\lambda)$ are fulfilling the s.p.d.e .(56) and correspondingly (57)

Recalling the definition $u(t,x) \triangleq \hat{u}(t, \psi(t,x))$ where $\lambda = \psi(t,x)$ obeyes to (58) we see easily that the equations (60) also hold true and the conclusion (*) of Theorem 1 coincides with s.p.d.e(56) provided the equations (**) are proved.

In this respect, we use again the smooth version $u = \tilde{u}^{\varepsilon}(t,x) \triangleq u^{\varepsilon}(t, \psi^{\varepsilon}(t,x))$ and the definition (see (28))

72) $\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \lim_{\varepsilon \searrow 0} \int_{t'}^{t''} [\partial_t \tilde{u}^{\varepsilon}(t,x)]_{x=\hat{x}(t,\lambda)}\, dt$

As far as $\tilde{u}^{\varepsilon}(t, x^{\varepsilon}(t,\lambda)) = u^{\varepsilon}(t,x)$ and using (69) we rewrite

73) $[\partial_t \tilde{u}^{\varepsilon}(t,x)]_{x=x^{\varepsilon}(t,\lambda)} = \partial_t u^{\varepsilon}(t,\lambda) - \langle p^{\varepsilon}(t,\lambda), \frac{dx^{\varepsilon}}{dt}(t,\lambda)\rangle =$

$$= \partial_t u^{\varepsilon}(t,x) + \sum_{j=1}^{m} \chi_{\tau}(t)\, \alpha(\sigma^{\varepsilon}(t))\, \langle p^{\varepsilon}(t,\lambda), b_j\rangle \frac{dw_j^{\varepsilon}}{dt}(t), t \in [0,a]$$

where $\sigma = \sigma^{\varepsilon}(t)$, $t \in [0,a]$,is defined as in (62) .

According to 73) and $\lim y^{\varepsilon}(t,\lambda) = \hat{y}(t,\lambda)$ in $L_2(\Omega, P)$ we rewrite (72) as follows

74) $\int_{t'}^{t''} [d_t u(t,x)]_{x=\hat{x}(t,\lambda)} = \hat{u}(t'',\lambda) - \hat{u}(t',\lambda) + \sum_{j=1}^{m} \int_{t'}^{t''} \chi_{\tau}(t)\, \langle \hat{p}(t,\lambda), b_j\rangle \otimes dw_j(t)$

for any $[t', t''] \subset (0,a]$ and the proof is complete.

§4.Aplications. Control problems associated with non F_t-adapted solutions

Here are included two applications regarding Pontryagin's principle associated with stochastic differential equations and non $\mathcal{F}_t$-adapted solutions subjected to some vector valued cost functions.

§4.1 Usualy a stochastic optimal control problem is described by a stochastic differential system with a control function.

$$*) \begin{cases} d_t x = f(t,x,u)dt + \sum_{j=1}^{m} \chi_{\tau}(t)\, g_j(t,x) \otimes dw_j(t) \\ x(0) = x_0, (t,x,u) \in [0,T] \times X \times U \end{cases}$$

where $X \subseteq \mathbb{R}^n, U \subseteq \mathbb{R}^m$ are some fixed closed sets and $w(t) = (w_1(t), .., w_d(t))$,

$t \in [0,T]$, is a standard Wiener process over a complete filtered probability space $\{\Omega, \mathcal{F}, P, \{\mathcal{F}_t\} \uparrow \mathcal{F}\}$

Here $\tau(\omega) : \Omega \to [0, t_f]$ is a stopping time used for getting a bounded solution $x(t,\omega) \in B(x_0, \rho) \subseteq X$ .The control function $u(\cdot) \in L_{\infty}([0,T] \times \Omega; U)$ is taken in a class $\mathcal{A}$ of piecewise continuous trajectories $u(t,\omega), t \in [0,T]$ for each $\omega \in \Omega$.

For each $u(\cdot) \in \mathcal{A}$ we define the coresponding solution $x = x(t,\omega;u), (t,\omega) \in [0,T] \times \Omega$, of (*) and associate the following pathwise functional.

**) $J^{\omega}(x;u) = F(x(T,\omega;u)) + \int_0^T f_0(t, x(t,\omega;u), u(t,\omega)dt$ for each $\omega \in \Omega$

Assume that $(\tilde{x}(t,\omega); \tilde{u}(t,\omega)) \in X \times U, (t,\omega) \in [0,T] \times \Omega$, is minimizing the functional (**) for each $\omega \in \Omega$ and as far as the nacessary conditions are concerned it is meaningfull to solve an associated Hamilton -Iacobi system of stochastic differential equations (see$(C_1)-(C_3)$) provided the given $f, g_j \in \mathbb{R}^n$ and $F, f_0 \in \mathbb{R}$ are continuously differentiable with respect to $z \in B(x_0, \rho)$.

Noticing that the control function $u(\cdot) \in \mathcal{A}$ is not an $\mathcal{F}_t$-adapted one (see $u(t,\cdot)$ is non $\mathcal{F}_t$-mesurable),we are forced to use some special type of stochastic integral "$\otimes$" appearing in (*) and the associated Hamilton-Iacobi system as well. The stochastic integral " $\otimes$ " coincides with the standard Fisk-Stratonovich integral provided the both control function $u(\cdot)$ and the solution of(*) are $\mathcal{F}_t$-adapted processes.The meaning of a stochastic integral " $\otimes$ " associated with a non$\mathcal{F}_t$-adapted solution is clarified working with Langevin's approximarion $w^{\varepsilon}(t), t \in [0,T]$ of the original Wiener process $w(t) \in \mathbb{R}^d$ and representing a solution of (*) as

$x(t,\omega;u) = G(p(t,\omega); y(t,\omega;u)), t \in [0,T], \omega \in \Omega$ for each $u(\cdot) \in \mathcal{A}$

Here the smooth mapping $G(p,\lambda) : D_M \times B(x_0,\rho_0) \to X \subseteq \mathbb{R}^n$ is generated as an orbit solution associated with a finite dimensional Lie algebra determined by the smooth (C^∞) difusion vector fields $\{g_1(t,\cdot),..,g_d(t,\cdot)\}$ for each $t \in [0,T]$ where $D_M = \prod_{i=1}^{M} [-a_i, a_i]$ and $B(x_0,\rho_0) \subseteq X$ is a fixed ball .

In adition $p(t,\omega) : [0,T]\times\Omega \to D_M$ is a continuous and $\mathcal{F}_t$-adapted process,while $y(t,\omega;u)$, is a picewise continuously differentiable and non $\mathcal{F}_t$-adapted process of $t \in [0,T]$ for each $u(\cdot) \in \mathcal{A}$.The above given clues allow us to convert the stochastic control problem into a detrministic one with respect to the new state variable $y \in B(x_0,\rho_0) \subseteq X$ fulfilling the following control system.

$$\square) \begin{cases} \frac{dy}{dt} = \left[\frac{\partial G}{\partial \lambda}(p(t,\omega);y)\right]^{-1} f(t, G(p(t,\omega);y), u) \triangleq f^{\omega}(t,y,u) \\ y(0) = x_0 \in X \end{cases}$$

Let $y = y^{\omega}(t,u), t \in [0,T]$, be the picewise continuously differentible solution associated with $u(\cdot) \in \mathcal{A}$ for each $\omega \in \Omega$ and define the corresponding functional

$$\square\square) \; I^{\omega}(y;u) = F^{\omega}(y^{\omega}(T;u)) + \int_0^T f_0^{\omega}(t, y^{\omega}(t;u), u(t,\omega))dt$$

where $F^{\omega}(y) = F(G(p(t_f,\omega);y))$ and $f_0^{\omega}(t,y,u) = f_0(t, G\left(p(t,\omega);y\right), u)$

Denote $\tilde{y}^{\omega}(t) = y^{\omega}(t,\tilde{u}), t \in [0,T], \omega \in \Omega$, and we get that $(\tilde{y}^{\omega}(t), \tilde{u}(t,\omega))$ is an optimal pair for the optimal control problem determined by the dynamic given in ($\square$) and the functional $I^{\omega}(y;u)$ defined as in ($\square\square$) .Write the corresponding necessary optimaly conditions and we get the associated Hamilton-Iacobi system

$$C) \begin{cases} \frac{dy^{\omega}}{dt}(t) = \frac{\partial H^{\omega}}{\partial \psi}(t, \tilde{y}^{\omega}(t), \tilde{u}(t,\omega), \psi^{\omega}(t)), \tilde{y}^{\omega}(0) = x_0 \\ \frac{d\psi^{\omega}}{dt}(t) = -\frac{\partial H^{\omega}}{\partial y}(t, \tilde{y}^{\omega}(t), \tilde{u}(t,\omega), \psi^{\omega}(t)), \psi^{\omega}(T) = \partial_y F^{\omega}(\tilde{y}^{\omega}(T)) \\ \min\limits_{u \in U} H^{\omega}(t, \tilde{y}^{\omega}(t), u, \psi^{\omega}(t)), \psi^{\omega}(t)) = H^{\omega}(t, \tilde{y}^{\omega}(t), \tilde{u}(t,\omega), \psi^{\omega}(t)), \psi^{\omega}(t)) \end{cases}$$

for $t \in [0,T]$, where the augmented Lagrangean

$H^\omega(t,y,u,\psi) = \psi f^\omega(t,y,u) + f_0^\omega(t,y,u)$is used and $\psi \in R^n$ is a row vector.

The explicit form of the Hamilton-Iacobi equation (C) lead us to the corresponding stochastic differential equation associated with the original control problem (*) and(**) via an associated stochastic differential form

$$H(t,y,u,\psi;dt,dw(t))) = [\psi f(t,x,u) + f_0(t,x,u)]\,dt + \sum_{j=1}^{d} \chi_\tau(t)\psi g_j(t,x) \otimes dw_j(t)$$

Recall that $\tilde{x}(t,\omega) = G(p(t,\omega);\tilde{y}^\omega(t))$ and define

$\tilde{\psi}(t,\omega) = \psi^\omega(t)\left[\frac{\partial G}{\partial \lambda}(p(t,\omega);\tilde{y}^\omega(t)\right]^{-1}, t \in [0,T], \omega \in \Omega$

Then the following stochastic Hamilton-Iacobi system stands for the corresponding Pontryagin principle

$$C_1)\begin{cases} d_t\tilde{x} = \frac{\partial H}{\partial \psi}(t,\tilde{u}(t,\omega),\psi^\omega(t);dt,dw(t)) = f(t,\tilde{x},\tilde{u}(t,\omega)dt + \\ \qquad\qquad + \sum_{j=1}^{d}\chi_\tau(t)g_j(t,\tilde{x}) \otimes dw_j(t) \\ \tilde{x}(0) = x_0, t \in [0,T], \omega \in \Omega \end{cases}$$

$$C_2)\begin{cases} d_t\psi = -\frac{\partial H}{\partial x}(t,\tilde{x}(t,\omega),\tilde{u}(t,\omega),\tilde{\psi});dt,dw(t)) = \\ = -\left[\psi\frac{\partial f}{\partial x}(t,\tilde{x}(t,\omega),\tilde{u}(t,\omega)) + \frac{\partial f_0}{\partial x}(t,\tilde{x}(t,\omega),\tilde{u}(t,\omega)\right]dt_- \\ \qquad\qquad - \sum_{j=1}^{d}\chi_\tau(t)\tilde{\psi}\frac{\partial g_j}{\partial x}(t,\tilde{x}(t,\omega)) \otimes dw_j(t) \\ \tilde{\psi}(t_f,\omega) = \frac{\partial F}{\partial x}(\tilde{x}(t_f,\omega)), t \in [0,T], \omega \in \Omega \end{cases}$$

where $\frac{\partial f_0}{\partial x}, \frac{\partial F}{\partial x} \in R^n$ are row vectors and

$$C_3)\begin{cases} \min H_0(t,\tilde{x}(t,\omega),u,\tilde{\psi}(t,\omega)) = H_0(t,\tilde{x}(t,\omega),\tilde{u}(t,\omega),\tilde{\psi}(t,\omega)) \\ \text{a.e } t \in [0,T], \text{for each } \omega \in \Omega, \text{where} \\ H_0(t,x,u,\psi) = \psi f(t,x,u) + f_0(t,x,u) \end{cases}$$

The conclusion $((C_1) - (C_3))$are a direct consequence of the deterministic Pontryagin's principle (C) provided we notice that the $(n \times n)$ matrices $\tilde{M}(t,\omega) = \frac{\partial G}{\partial \lambda}(p(t,\omega);\tilde{y}^\omega(t))$ and $\tilde{N}(t,\omega) = \left[\tilde{M}(t,\omega)\right]^{-1}$ are fulfilling the following linear stochastic differential equations

$$C_4)\begin{cases} d_t\tilde{M} = \frac{\partial f}{\partial x}(t,\tilde{x}(t,\omega),\tilde{u}(t,\omega))\tilde{M}dt + \sum_{j=1}^{d}\chi_\tau(t)\frac{\partial g_j}{\partial x}(t,\tilde{x}(t,\omega))\tilde{M} \otimes dw_j(t) \\ d_t\tilde{N} = -\tilde{N}\frac{\partial f}{\partial x}(t,\tilde{x}(t,\omega),\tilde{u}(t,\omega))dt - \sum_{j=1}^{d}\chi_\tau(t)\tilde{N}\frac{\partial g_j}{\partial x}(t,\tilde{x}(t,\omega)) \otimes dw_j(t) \end{cases}$$

$\tilde{M}(0) = \tilde{N}(0) = I_n$

Applying a stochastic rule of derivation associated with the stochastic differential equations (C_1), (C_4) and (C) we get the conclusions (C_2) and (C_3) fulfilled .

In adition,the equation appearing in (C_4) are obtained from the original system $(*)$ by a straight derivation with respect to the initial value $x_0 \in X$.

§4.2 **Dynamical game theory associated with Nash-equilibrum and stochastic perturbation**

A differential game with stochastic perturbation is determined by a dynamic of the state variable $x \in X \subseteq \mathbb{R}^n$ defined by a system of stochastic differential equations

$$1)\begin{cases} d_t x = f\left(t, x, u_1, .., u_N\right) dt + \sum_{j=1}^{d} g_j(t, x) \otimes dw_j(t) \\ x(0) = x_0, (t, x, u_1, .., u_N) \in [0, t_f] \times X \times U_1 \times .. \times U_N \end{cases}$$

where $U_i \subseteq \mathbb{R}^{m_i}$ is a fixed set and is a standard d-dimensional Wiener process over a filtered complete probanility space$\{\Omega, \mathcal{F}, P, \{\mathcal{F}_t\} \uparrow \mathcal{F}\}$.

The control function $u(\cdot) \triangleq (u_1(\cdot), .., u_N(\cdot))$ is taken in the class $\mathcal{A}$ of admissible controls which are defined by bounded and measurable functions

$u(\cdot) : \Omega \to U = \prod_{i=1}^{N} U_i$ with piecewise continuous trajectories $u(t, \omega), t \in [0, t_f]$, over the product measurable space $\{[0, t_f] \times \Omega, \mathcal{B} \otimes \mathcal{F}, dt \otimes P\}$

For each admissible control $u(\cdot) \in \mathcal{A}$ we define $x = x(t, \omega; u), (t, \omega) \in [0, t_f] \times \Omega$ as the corresponding solution fulfilling the stochastic differential equation given in (1) and associate the following functionals

2) $J_i(\omega, u) = F^i\left(x\left(t_f, \omega; u\right)\right) + \int f_0^i\left(t, x\left(t, \omega; u\right), u\left(t, \omega\right)\right) dt, i \in \{1, 2, .., N\}$,for each $\omega \in \Omega$.

Denote $\mathcal{A}_i$ the corresponding class of admisible controls $u_i(\cdot), i \in \{1, .., N\}$ and write $\mathcal{A} = \prod_{i=1}^{N} \mathcal{A}_i$

The following object

3)$\Gamma_N(\omega) \triangleq \{[0, t_f], X = \mathbb{R}^n, U_i, \mathcal{A}_i, f, x_0, J_i(\omega, \cdot)\}_{i=1,2,..,N}, \omega \in \Omega$ is called a stochastic differential game with N -players and open loop strategies

To be sure that a solution $x(t, \omega; u), (t, \omega) \in [0, t_f] \times \Omega$ satisfying (1) exists and the functionals $J_i(\omega, \cdot), i \in \{1, .., N\}$ are well defined we need and do assume the following hypothesis

i_0) $g_j(t, x) \triangleq A_j(t)x + b_j(t)$,where the $(n \times n)$ matrix $A_j(t)$ and $b_j(t) \in R^n, j \in \{1, .., d\}$ are continuous function;

i_1) $f(t, x, u) \in \mathbb{R}^n, F^i(x) \in R, f_0^i(t, x, u) \in \mathbb{R}, i \in \{1, .., N\}$are continuous function on $[0, t_f] \times \mathbb{R}^n \times U$ and $\mid f(t, x, u) \mid \leq k_R(1 + \mid x \mid)$, for any $x \in \mathbb{R}^n, t \in [0, t_f]$ and $u \in$

$B(0,\mathbb{R}) \subseteq \prod_{i=1}^{N} \mathbb{R}^{m_i} = \mathbb{R}^m$ where $k_R > 0$ is a constant.

$i_2)$ $| f(t,x'',u) - f(t,x',u) | \leq L_R^{\rho} | x'' - x' |$,for any $t \in [0,t_f], u \in B(0,R) \subseteq \mathbb{R}^m$, $x', x'' \in B(x_0,\rho) \subseteq R^n$,where $L_R^{\rho} > 0$ is a constant.

An admissible solution $x(t,\omega;u)$ for (1) is represented as follows

4) $x(t,\omega;u) = G(t,\omega)(y(t,\omega;u)) + \eta(t,\omega), t \in [0,t_f], \omega \in \Omega$ where the nonsingular $(n \times n)$ matrix G

and $\eta \in \mathbb{R}^n$ are defined as continuous and $\mathcal{F}_t$-adapted process fulfilling the following stochastic differential equation

5) $d_t G = \sum_{j=1}^{d} A_j(t) G \circ dw_j(t), G(0,\omega) = I_n, t \in [0,t_f],$

$$\eta(t,\omega) = \sum_{j=1}^{d} \int_0^t b_j(s), dw_j(s), t \in [0,t_f],$$

where " $\circ$ " means Fisk-Stratonovich integral and stands for the standard Ito's integral.

The vector value function $y(t,\omega;u) \in \mathbb{R}^n$ is defined as a differentiable and non $\mathcal{F}_t$-adapted process fulfilling the following system of differentiable equation

$$6)\begin{cases} \frac{dy}{dt} = [G(t,\omega)]^{-1} f(t, G(t,\omega)(y)) + \eta((t,\omega), u(t,\omega)) \triangleq \\ \qquad \triangleq \tilde{f}(\omega,t,y,u(t,\omega)), t \in [0,t_f] \\ y(0) = x_0 \in \mathbb{R}^n \end{cases}$$

Remark 1

The $(n \times n)$ matrix $G(t,\omega)$ is invertible and its inverse $K(t,\omega) \triangleq [G(t,\omega)]^{-1}$ obeyes to the following linear equations

7) $d_t K = -\sum_{j=1}^{d} K A_j(t) \circ dw_j(t), K(0) = I_n, t \in [0,t_f].$

Definition 1 ($\tilde{u}_i(\cdot) \in \mathcal{A}_i$ is the best replay)

Let $\Gamma_N(\omega), \omega \in \Omega$, be a N-players differential game defined as in (3).Denote $u_{(-i)}(\cdot) = (u_1(\cdot), .., u_{i-1}(\cdot), u_{i+1}(\cdot), .., u_N(\cdot)) \in \prod_{j \neq i} \mathcal{A}_j$ for each. $i \in \{1,..,N\}$ We say that $\tilde{u}_i(\cdot) \in \mathcal{A}_i$ is the best replay against $u_{(-i)}(\cdot)$ if

$J_i(\omega; (u_{(-i)}, \tilde{u}_i)) \leq J_i(\omega; (u_{(-i)}, u_i))$ for any $u_i \in \mathcal{A}_i, \omega \in S_i$

Denote $R_i(u_{(-i)})$ as the set consisting of all best replays against

$u_{(-i)}(\cdot) \in \prod_{j \neq i} \mathcal{A}_j$.

Definition 2 (Nash-equilibrum).

Let $\Gamma_N(\omega), \omega \in \Omega$, be a N-players differential game defined as in (3). We say that $u(\cdot) \triangleq (u_1(\cdot), .., u_N(\cdot)) \in \prod_{i=1}^{N} \mathcal{A}_i$ is a Nash eqilibrum if $u_i(\cdot) \in R_i\left(u_{(-i)}\right)$ for each $i \in \{1, .., N\}$ and $\omega \in \Omega$.

Remark 2

A Nash equilibrum solution $(x^*(t, \omega), u^*(t, \omega)), (t, \omega) \in [0, t_f] \times \Omega$ of the differential game $\Gamma_N(\omega), \omega \in \Omega$, defined in (3) leed us to a Nash equilibrum solution $(y^*(t, \omega), u^*(t, \omega))$ associated with anN-players differential game defined as follows

$_N(\omega) = \left\{[0, t_f], Y = R^n, U_i, \mathcal{A}_i, \tilde{f}(\omega, \cdot)x_0, Ji(\omega, \cdot)\right\}_{i \in \{1,..,N\}}, \omega \in \Omega$,where $\tilde{f}(\omega, t, y, u) \triangleq K(t, \omega) f(t, G(t, \omega)(y) + \eta(t, \omega), u)$ and

2) $\tilde{J}_i(\omega; u) = \tilde{F}^i(\omega, y(t_f, \omega; u)) + \int_0^t \tilde{f}_0^i(\omega, t, y(t, \omega; u), u(t, \omega))dt$ with

$\tilde{F}^i(\omega, y) \triangleq \tilde{F}^i(\omega, G(t_f, \omega)(y) + \eta(t_f, \omega))$ and

$\tilde{f}_0^i(\omega, t, y; u) \triangleq \tilde{f}_0^i(t, G(t, \omega)(y) + \eta(t, \omega), u), i \in \{1, .., N\}$

The corresponding deterministic dynamic system is described the evolution of the new state variable $y \in Y = \mathbb{R}^n$ as follows

$$\tilde{1})\begin{cases} \frac{dy}{dt} = \tilde{f}(\omega, t, y, u), t \in [0, t_f], \omega \in \Omega \\ y(0) = x_0 \end{cases}$$

As is known, the corresponding necessary conditions with the deterministic differential game $\tilde{\Gamma}_N(\omega)$ has the following content .

Theorem 1

Let $(y^*(\cdot), u^*(\cdot))$ be a Nesh equilibrum solution associated with the deterministic differential game $\Gamma_N(\omega)$ for each $\omega \in \Omega$.

Assume that the given functions $f(t, x, u) \in \mathbb{R}^n, F^i(x), f_0^i(t, \cdot, u) \in \mathbb{R}, i \in \{1, .., N\}$ are fulfilling the hypothesis $(i_1)$and (i_2) and in addition $f(t, \cdot, u) \in C^1(\mathbb{R}^n, \mathbb{R})$ for each $(t, u) \in [0, t_f] \times U, i \in \{1, .., N\}$ .Then with

$\tilde{H}^i(\omega, t, y, u, \tilde{\psi}_i) \triangleq \psi_i^T \tilde{f}(\omega, t, y, u) + \tilde{f}_0^i(\omega, t, y, u)$ the following equations hold

$\tilde{C}_1)\ \frac{dy^*}{dt}(t, \omega) = \frac{\partial \tilde{H}^i}{\partial \tilde{\psi}_i}(\omega, t, y^*(t, \omega), u^*(t, \omega), \tilde{\psi}_i(t, \omega)), t \in [0, t_f]$

$\tilde{C}_2)\ \frac{d\tilde{\psi}_i}{dt}(t, \omega) = -\frac{\partial \tilde{H}^i}{\partial y}(\omega, t, y^*(t, \omega), u^*(t, \omega), \tilde{\psi}_i(t, \omega)), t \in [0, t_f]$

$\tilde{C}_3)$ $\tilde{\psi}_i(t_f,\omega) = \nabla_y F^i(\omega, y^*(t_f,\omega)), \omega \in \Omega.$

$\tilde{C}_4)\tilde{H}^i(\omega, t, y^*(t,\omega), u^*(t,\omega), \tilde{\psi}_i(t,\omega)) =$

$$= \min \tilde{H}^i(\omega, t, y^*(t,\omega), (u^*_{(-i)}(t,\omega), u_i), \tilde{\psi}_i(t,\omega))$$

for each continuity point $t \in [0, t_f]$ of $u_i^*(\cdot,\omega)$ and any $\omega \in \Omega, i \in \{1,..,N\}$

The proof of this theorem is a rewritting of the Pontrjagine principle associated with the corresponding optimal control problem for each $i \in \{1,..,N\}, \omega \in \Omega$ relying on the property that $u_i^*(\cdot)$ is an optimal conrol with respect to the functional $J_i(\omega, u_i) = \tilde{J}_i\left(\omega; \left(u^*_{(-i)}, u_i\right)\right)$,and $u_i(\cdot) \in \mathcal{A}_i$ for each $i \in \{1,..,N\}$ and $\omega \in \Omega$ .As expected,the corresponding necessary condition associated with the original Nash-equilibrum solution $(x^*(\cdot), u^*(\cdot))$ are a direct consequence of the above given conclusions $\left(\tilde{C}_1\right) - \left(\tilde{C}_2\right)$.

Theorem 2

Let $(x^*(\cdot), u^*(\cdot))$ be a Nash-equilibrum solution associated with the stochastic differential game $\Gamma_N(\omega)$ defined in (3).Assume that the given function $g_j(t,x), f(t,x,u) \in \mathbb{R}^n, F^i(x), f_0^i(t,x,u) \in \mathbb{R}, j \in \{1,..,d\}, i \in \{1,..,N\}$ are fulfilling the hypothesis $(i_0), (i_1), (i_2)$.In adition,suppose $f(t,\cdot,u) \in C^1(\mathbb{R}^n, \mathbb{R})$ and $F^i(\cdot), f_0^i(t,\cdot,u) \in C^1(\mathbb{R}^n,\mathbb{R})$ for any $(t,u) \in [0,t_f] \times U$ and $i \in \{1,..,N\}$.Then with a stochastic differential form

$$H^i(t,x,u,\psi_i; dt, dw) = \left[\psi_i^T f(t,x,u) + f_0^i(t,x,u)\right] dt +$$
$$+ \sum \psi_i^T g_j(t,x) \otimes dw_j(t)$$ the

following equation hold

$C_1)$ $d_t\, x^*(t,\omega) = \frac{\partial H^i}{\partial \psi_i}(t, x^*(t,\omega), u^*(t,\omega), \psi_i(t,\omega); dt, dw)$

$C_2)$ $d_t \psi_i(t,\omega) = -\frac{\partial H^i}{\partial x}(t, x^*(t,\omega), u^*(t,\omega), \psi_i(t,\omega);; dt, dw)$

$C_3)$ $\psi_i(t_f,\omega) = \nabla_x F^i(x^*(t_f,\omega))$

$C_4)$ $\int_0^{t_f} H^i(t, x^*(t,\omega), u^*(t,\omega), \psi^i(t,\omega); dt, dw) \leq$

$$\leq \int_0^{t_f} H^i(t, x^*(t,\omega), (u^*_{(-i)}(t,\omega), u_i(t,\omega), \psi_i(t,\omega); dt, dw)$$

for any $u_i(\cdot) \in \mathcal{A}_i, \omega \in \Omega, i \in \{1,..,N\}$

<u>Remark 3</u>

We notice that the conclusion (C_4) is equivalent to

$\tilde{C}_4)$ $H_0^i(t, x^*(t,\omega), u^*(t,\omega), \psi_i(t,\omega) = \min_{u_i \in U_i} H_0^i(t, x^*(t,\omega), (u^*_{(-i)}(t,\omega), u_i), \psi_i(t,\omega))$ for each continuity point $t \in [0,t_f]$ of $u_i^*(\cdot,\omega)$ and any $\omega \in \Omega, i \in \{1,..,N\}$, where $H_0^i(t,x,u,\psi_i) =$

$\psi_i^T f(t,x,u) + f_0^i(t,x,u)$ is the drift part of the stochastic differential form H^i.

In addition, the stochastic Hamilton-Iacobi equations appearing in (C_1) and (C_2) can be converted into a deterministic form using the same drift part $H_0^i(t,x,u,\psi_i)$ provided we represent

$x^*(t,\omega) = G(t,\omega)(y^*(t,\omega)) + \eta(t,\omega)$

$\psi_i^T(t,\omega) = \tilde{\psi}_i^T(t,\omega)K(t,\omega), t \in [0,t_f],\ i \in \{1,..,N\}$,where the $(n \times n)$ matrices G and K are the continuous and solutions associated with linear stochastic differential equations(5)and (7) correspondingly.

Here $(y^*(\cdot), u^*(\cdot))$ is the induced Nash equilibrum solution associated with the deterministic differential game $\tilde{\Gamma}_N(\omega)$ whose necessary ncondition are described in Theorem1. The corresponding Hamilton -Iacobi equations (see(C_1), (C_2) and (C_3)) can be rewritten using $H_0^i(t,x,u,\psi_i)$ as follows :

$$\tilde{C_1}) \begin{cases} \frac{dy^*}{dt}(t,\omega) = K(t,\omega)\frac{\partial H_0^i}{\partial \psi_i}(t,x^*(t,\omega),\psi_i(t,\omega)), t \in [0,t_f] \\ y^*(0,\omega) = x_0 \end{cases}$$

$$\tilde{C_2})\ \frac{d\tilde{\psi}_i}{dt}(t,\omega) = -\frac{\partial H^i}{\partial x}(t,x^*(t,\omega),u^*(t,\omega),\psi_i(t,\omega))G(t,\omega), t \in [0,t_f]$$

$$\tilde{C_3})\ \psi_i^T(t,\omega) = (\nabla_x F^i(x^*(t_f,\omega)))^T G(t_f,\omega)$$

Proof of Theorem 2

The arguments of the proof are contained in the above given remark provided we represent the equilibrum soution as

$x^*(t,\omega) = G(t,\omega)(y^*(t,\omega)) + \eta(t,\omega)$ where G and η are given in (5),and $(y^*(\cdot), u^*(\cdot))$ is a Nash equilibrum solution associated with the deterministic differential game $\tilde{\Gamma}_N(\omega), \omega \in \Omega$,defined as in theorem 1.The conclusions $\left(\tilde{C_1}\right) - \left(\tilde{C_4}\right)$ of theorem1 are true and rewrite them as in the remark 3 using the drift part H_0^i of the stochastic differential form H^i.Define $\psi_i^T(t,\omega) \triangleq \tilde{\psi}_i^T(t,\omega)K(t,\omega), t \in [0,t_f], i \in \{1,..,N\}$, and using a stochastic rule of differentiation associated with (5),(7),and $(\tilde{C_1})$ we get the conclusions (C_1), (C_2) and (C_3) in the theorem.The conclusion (C_4) is a direct consequece of the pointwise form given in $\left(\tilde{C_4}\right)$ and the proof is complete.

Final conclusion (again about the conclusions $(C_1) - (C_4)$ in the Theorem 2)

The admissible class $u_i(\cdot) \in \mathcal{A}_i, i \in \{1,..,N\}$ accepted for the control variable is too restricted (see $u_i(\cdot)$ is a bounded function)when dealing with linear problems.(differential games) and cuadratique cost functionals.

If it is the case,the control set $U_i = \mathbb{R}^{m_i}$ is an unbounded one and from the conclusion $\left(\tilde{C}_4\right)$ in Remark3 we get $\frac{\partial H_0^i}{\partial u_i}(t, x^*(t,\omega), u^*(t,\omega), \psi_i(t,\omega)) = 0$ for any $\omega \in \Omega, t \in [0, t_f]$,which allows to express $u_i^*(\cdot)$ as a function of $\psi_i(\cdot)$ in the following form

$\tilde{C}_4)$ $u_i^*(t,\omega) = \left(R_i(t) + R_i^T(t)\right)^{-1} B_i^T(t)\, \psi_i(t,\omega), i \in \{1,..,N\}$

provided $f(t,x,u) \triangleq f_0(t,x) + \sum_{i=1}^{N} B_i(t)\, u_i$ and the functional $J_i(\omega, u)$ is defined by

$$J_i(\omega, u) = F^i\left(x(t_f, \omega; u)\right) + \int_0^{t_f} \left[h_0^i\left(x(t,\omega;u)\right) + \langle R_i(t)\, u_i(t,\omega), u_i(t,\omega)\rangle\right]$$

where $R_i(t)$ is a continuous and nonsingular $(m_i \times m_i)$ matrix.

Here $\psi_i(t,\omega) = K^T(t,\omega)\, \tilde{\psi}_i(t,\omega), (t,\omega) \in [0,t_f] \times \Omega$ and by the definition of $K(t,\omega) \triangleq G^{-1}(t,\omega)$ fulfilling the linar stochastic differential equations given in (7) we notice that is not a bounded on

It is useful,when is necessary,to work with a bounded covector function $\psi_i(t,\omega)$ and it will be accomplished using a stopping time $\tau(\omega) : \Omega \to [0, t_f]$ with respect to the matrix solution $G(t,\omega)$ and the continuous process $\eta(t,\omega)$ defined in (5) .In this respect we fixe a ball $B(I_n, R) \subseteq M_n$ in the linear space of $(n \times n)$ matrices and let $B(0,R) \subseteq \mathbb{R}^n$ be a ball with the radius R and centered at the origin in $\mathbb{R}^n$ .Define

$\tau(\omega) = \inf\{t \in [0,t_f] : (G(t,\omega), \eta(t,\omega)) \in B(I_n, R) \times B(0,R)\}$ and we get that the corresponding characteristic function $\chi_\tau(t) = \begin{cases} 1 & \tau \geq t \\ 0 & \tau < t \end{cases}$ is an $\mathcal{F}_t$-adapted measurable process.

Write $\hat{G}(t,\omega) \triangleq G(t \wedge \lambda, \omega), \hat{\eta}(t,\omega) \triangleq \eta(t \wedge \lambda, \omega)$,and they are bounded local solutions associated with the stochastic differential equations in (5),i.e.

$$\begin{cases} d_t\hat{G}(t,\omega) = \sum_{j=1}^{d} \chi_\tau(t)\, A_j(t)\, \hat{G}(t,\omega) \circ dw_j(t), \hat{G}(0,\omega) = I_n \\ d_t\hat{\eta}(t,\omega) = \sum_{j=1}^{d} \chi_\tau(t)\, b_j(t)\, dw_j(t), \hat{\eta}(0,\omega) = 0, t \in [0,t_f] \end{cases}$$

Associate the corresponding stochastic differential form

$\hat{H}^i\left(t, x, u, \hat{\psi}_i; dt, dw\right) = \left[\hat{\psi}_i^T f(t,x,u) + f_0^i(t,x,u)\right] dt + \sum_{j=1}^{d} \chi_\tau(t)\, \hat{\psi}_i^T g_j(t,x) \otimes dw_j(t), i \in \{1,..,N\}$ and the stochastic dunamical system

$$\hat{1})\begin{cases} d_t x = f(t,x,u)\, dt + \sum_{j=1}^{d} \chi_\tau(t)\, g_j(t,x) \otimes dw_j(t), t \in [0,t_f] \\ x(0) = x_0 \end{cases}$$

With the same functionals $J_i(\omega, u), i \in \{1,..,N\}$ defined in (2) we may and do write the corresponding necessary condition for a Nash equilibrum solution $(\hat{x}^*(\cdot), u^*(\cdot))$ associated with the corresponding stochastic differential game $\Gamma_N(\omega)$ defined bythe dynamic $(\hat{1})$ and the functionals

$\hat{J}_i(\omega, u) = F^i(\hat{x}(t_f, \omega; u)) + \int_0^{t_f} f_0^i(t, \hat{x}(t_f, \omega; u), u(t, \omega))\, dt, i \in \{1,..,N\}$ where $\hat{x}(t, \omega; u), t \in [0, t_f]$ stands for a solution of the stochastic differential system in $(\hat{1})$ corresponding to the admisible control u$(\cdot) \in \mathcal{A}$ .Under the same hypotheses as in theorem 2 we gewt the following necessary conditions

$\hat{C}_1)$ $d_t\hat{x}^*(t, \omega) = \frac{\partial \hat{H}_0^i}{\partial \hat{\psi}_i}(t, \hat{x}^*(t, \omega), u^*(t, \omega), \hat{\psi}_i(t, \omega); dt, dw),$

$\hat{C}_2)$ $d_t\hat{\psi}_i(t, \omega) = \frac{\partial \hat{H}^i}{\partial x}(t, \hat{x}^*(t, \omega), u^*(t, \omega), \hat{\psi}_i(t, \omega);; dt, dw)$

$\hat{C}_3)$ $\hat{\psi}_i(t_f, \omega) = \nabla_x F^i(\hat{x}^*(t_f, \omega))$

$\hat{C}_4)$ $\hat{H}_0^i(t, \hat{x}^*(t, \omega), u^*(t, \omega), \hat{\psi}^i(t, \omega)) =$

$$= \min \hat{H}_0^i(t, \hat{x}^*(t, \omega), (u^*_{(-i)}(t, \omega), u_i), \hat{\psi}_i(t, \omega))$$

for each continuity point $t \in [0, t_f]$ of $u_i^*(\cdot, \omega)$ and any $\omega \in \Omega$ where

$\hat{H}_0^i(t, x, u, \hat{\psi}_i) = \hat{\psi}_i^T f(t, x, u) + f_0^i(t, x, u)$ is the drift part of the stochastic differential form $\hat{H}^i$.

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