INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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Preprint nr. 10/2004

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March, 2004

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Qusiregularity in metric spaces

MIHAI CRISTEA

Abstract: We show that if $f : X \to Y$ is a continuous, open and discrete map of finite multiplicity $N(f)$ between two p-regular metric spaces, then f satisfies the modular inequality $M_p(\Gamma) \leq K \cdot N(f) \cdot M_p(f(\Gamma))$ for every path family Γ from X if and only if $H(x, f) \leq H$ for every $x \in X$.
AMS 1991 Subject Classification: 30C65.
Keywords: quasiregular maps in metric spaces.

1 Introduction.

If X, Y are metric spaces and $f : X \to Y$ is continuous, open, discrete, $x \in X$ and $r > 0$, we let

$$L(x, f, r) = \sup_{y \in S(x,r)} d(f(y), f(x)), l(x, f, r) = \inf_{y \in S(x,r)} d(f(y), f(x))$$

and we put $H(x, f) = \limsup_{r \to 0} \frac{L(x,f,r)}{l(x,f,r)}$, the linear dilatation of f at x.

We also put for $\alpha \geq 1, h_\alpha(x, f) = \liminf_{r \to 0} \sup_{r \leq t \leq \alpha r} \frac{L(x,f,t)}{l(x,f,t)}, h(x, f) = h_1(x, f)$ and if $A \subset X$, we let $N(f, A) = \sup_{y \in Y} Card\ f^{-1}(y) \cap A$. We let $N(f) = N(f, X)$.

If $f : X \to Y$ is a map, we say that f is open if f carries open sets into open sets, and we say that f is discrete if $f^{-1}(y)$ is an isolated set for every $y \in Y$.

If $D \subset I\!R^n$ is open, $n \geq 2$, a map $f : D \to I\!R^n$ is quasiregular if $f \in W^{1,n}_{loc}(D, I\!R^n) \cap C(D, I\!R^n)$ and $\|f'(x)\|^n \leq K \cdot J_f(x)$ a.e. for some $K \geq 1$, and this it is known as the analytic definition of the quasiregularity. If f is continuous, open, discrete, with $N(f) < \infty$, then f is quasiregular if and only if there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in D$ (see [MRV] Th.4.5 and Th.4.13). For this reason, at least for mappings of finite multiplicity, we can say that a continuous, open and discrete map $f : X \to Y$ between two metric spaces is quasiregular (considering the metric definition) if there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in X$. Homeomorphisms between metric spaces are called quasiconformal if there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in X$ (the metric definition of the quasiconformality) and such maps are recently considered in [HK 1,2], [T], [BK], [H].

If (X, μ) is a metric measure space and Γ is a family of nonconstant paths in X, we let $F(\Gamma) = \{\rho : X \to [0, \infty]$Borel maps $| \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable $\}$, and if $p > 0$, we let the p-modulus of Γ by $M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p(x) d\mu$.

If D, D' are domains in $\mathbb{R}^n, K \geq 1$ and $f : D \to D'$ is a homeomorphism, we say that f is K-quasiconformal if $\frac{1}{K} \cdot M_n(\Gamma) \leq M_n(f(\Gamma)) \leq K \cdot M_n(\Gamma)$ for every path family Γ from D (the geometric definition of the quasiconformality). We also say that f is K-quasiconformal, considering the analytic definition of the quasiconformality, if $f \in W^{1,n}_{loc}(D, \mathbb{R}^n)$ and $\|f'(x)\|^n \leq K \cdot J_f(x)$ a.e. It is known that for D, D' domains in $\mathbb{R}^n$ and $f : D \to D'$ a homeomorphism, this three definitions of the quasiconformality are equivalent (see [Va], Th. 34.1 and Th.34.6). Recently it is shown in [HKST] that this definitions of the quasiconformality are also equivalent on arbitrary metric spaces satisfying just a few conditions of regularity.

If $D \subset \mathbb{R}^n$ is open, $n \geq 2, f : D \to \mathbb{R}^n$ is continuous, open, discrete with $N(f) < \infty$, then f is quasiregular with $K_0(f) \leq K$ if and only if f satisfies the so called $K_0(f)$ inequality, i.e. if $M_n(\Gamma) \leq K \cdot N(f, D) \cdot M_n(f(\Gamma))$ for every path family Γ from D (see [Ri], Th.6.7, page 44). Here $K_0(f)$ is the smallest $K \geq 1$ such that $\|f'(x)\|^n \leq K \cdot J_f(x)$ a.e. We shall say that a continuous, open and discrete map $f : X \to Y$ between two metric measure spaces with $N(f) < \infty$ is K-quasiregular, (considering the geometric definition) if there exists $p > 0$ such that $M_p(\Gamma) \leq K \cdot N(f) \cdot M_p(f(\Gamma))$ for every path family Γ from D.

In [HH] are considered quasiregular maps $f : U \to G$ using the analytic definition of the quasiregularity, where G is a Carnot group and $U \subset G$ is open. Their methods seems to be enough difficult to be transported on arbitrary metric spaces, where the metric and the geometric definition of the quasiregularity are very natural, at least for mappings of finite multiplicity.

However, a theory of quasiregular maps on arbitrary metric spaces is necessary, since there are plenty of such mappings defined on sets which are not Riemannian manifolds. Indeed, if D, D' are domains in $\mathbb{R}^n$, $f : D \to D'$ is quasiregular and surjective with $N(f, D) < \infty$, $H(x, f) \leq H$ for every $x \in D$, $A \subset D$, $B \subset D'$ are such that $A = f^{-1}(B)$, then $f|A : A \to B$ is open, discrete, with $N(f|A, A) < \infty, H(x, f|A) \leq H(x, f) \leq H$ for every $x \in A$ and the sets A and B can be taken enough complicated. Probably this is the easyest way to produce quasiregular maps defined on some sets which are not manifolds.

In [BK] are constructed examples of quasiconformal mappings in metric spaces by taking sets $A \subset B \subset \mathbb{R}^n$, spaces $X = B \cup_A B$ and mappings $f : X \to X$, $f = Id_X$. We can use this technique of gluing spaces and mappings to obtain some other examples of quasiregular mappings. First, if X, Y are metric spaces, A is a closed subset of X and Y, we let $X \cup_A Y$ the disjoint union of X and Y, with points in the two copies of A identified. Then $X \cup_A Y$ is a metric space, where $d(x, y)$ is the distance from X if $x, y \in X, d(x, y)$ is the distance from Y if $x, y \in Y$ and $d(x, y) = \inf_{a \in A} d(x, a) + d(a, y)$ if x, y are in two different parts of the union $X \cup_A Y$.

Let D, D' be domains in $\mathbb{R}^n, A$ closed in D and in $D', f : D \to \mathbb{R}^n, g : D' \to \mathbb{R}^n$ be quasiregular and nonconstant such that $f|A = g|A, f(A) = g(A)$ is a closed subset of $f(D)$ and $g(D'), A = f^{-1}(f(A)) = g^{-1}(g(A))$ and let $X = D \cup_A D', Y = f(D) \cup_{f(A)} g(D')$. Then D and D' are the two parts of $X, f(D)$

and $g(D')$ are the two parts of Y and we define $F: X \to Y$ by $F|D = f, F|D' = g$. (We can take for instance $D = D' = I\!R^2, A = \{(x,y) \in I\!R^2 | y = 0\}, f(z) = g(z) = z^2$ for $z \in I\!R^2$). We can easy see that F is discrete and we show that F is open. Let $x \in D \backslash A$. Then, for small $r, B_X(x,r) = B_D(x,r)$, where $B_X(x,r)$ is the ball of center x and radius r from X and $B_D(x,r)$ is the ball of center x and radius r from D. Let $\delta > 0$ be small enough such that $B_Y(f(x), \delta) = B_{f(D)}(f(x), \delta)$ and $B_{f(D)}(f(x), \delta) \subset f(B_D(x,r))$. Then $F(B_X(x,r)) = f(B_D(x,r)) \supset B_{f(D)}(f(x), \delta) = B_Y(f(x)\delta)$, hence F is open at x, and in the same way we show that F is open at x if $x \in D' \backslash A$. If $x \in A$, then $B_X(x,r) = B_D(x,r) \cup_A B_{D'}(x,r)$ for $r > 0$ and if $\delta > 0$ is taken small enough such that $B_{f(D)}(f(x)\delta) \subset f(B_D(x,r)), B_{g(D')}(g(x), \delta) \subset g(B_{D'}(x,r))$, then $F(B_X(x,r)) = F(B_D(x,r)) \cup_A B_{D'}(x,r) = f(B_D(x,r)) \cup_{f(A)} g(B_{D'}(x,r))$ $\supset B_{f(D)}(f(x), \delta) \cup_{f(A)} B_{g(D')}(g(x), \delta) = B_Y(F(x), \delta)$ hence F is also open in x. We proved that F is an open map on X, and we see that F is continuous on X, that $N(F) \leq \max\{N(f), N(g)\}$ and that $H(x,F) \leq \max\{H(x,f), H(x,g)\}$ for every $x \in X$. It results that if $N(f) < \infty, N(g) < \infty$, then there exists $H_1, H_2 \geq 1$ such that $H(x,f) \leq H_1$ for $x \in D$ and $H(x,g) \leq H_2$ for $x \in D'$, hence $H(x,F) \leq \max\{H_1, H_2\}$ for every $x \in X$ and $N(F) < \infty$, hence F is quasiregular considering the metric definition. We see that F is not a local homeomorphism if f and g are not local homeomorphisms, that X is not a manifold and it is a Loewner space (see [HK2], 6.14, page 42), and this construction is also valid for our main theorems, Theorem 1 and Theorem 2.

This procedure allows us to construct a lot of quasiregular maps on rather general metric spaces, using in a canonical way two arbitrary quasiregular maps defined on some open subsets from $I\!R^n$.

In this way we can also produce a lot of open, discrete mappings with uniformly bounded linear dilatation without having finite multiplicity, so our paper may be a starting point for some further researches of this kind of mappings.

A metric space X endowed with a Borel measure μ is called an Ahlfors Q - regular space if there exists a constant $C \geq 1$ such that $C^{-1} \cdot r^Q \leq \mu(B_r) \leq C \cdot r^Q$ for every ball B_r of radius r from X.

If (X, μ) is a metric measure space and E, F, G are subsets from X, we let $\Delta(E,F,G) = \{\gamma : [a,b] \to G | \gamma \text{ is a path and } \gamma(a) \in E, \gamma(b) \in F\}$ and if $G = X$, we put $\Delta(E,F) = \Delta(E,F,X)$. We say that (X,μ) is a Loewner space if it is a connected p-regular space with $p > 1$ and there exists a function $\Phi : (0,\infty) \to (0,\infty)$ such that $\Phi(t) \leq M_p(\Delta(E,F))$ for every nondegenerate continua E and F in X with $\dfrac{d(E,F)}{\min\{d(E), d(F)\}} \leq t$.

A metric space X is called linearly locally connected of constant $c \geq 1$ ($c - LLC$) if there exists $c \geq 1$ such that any two points in $B(x,r)$ can be joined by a path in $B(x,cr)$, and any two points in $X \setminus \overline{B}(x,r)$ can be joined by a path in $X \setminus \overline{B}(x, \frac{r}{c})$, for every ball $B(x,r)$ in X.

We shall prove in our main results (Theorem 1 and Theorem 2) that the metric and geometric definition of the quasiregularity are equivalent for continuous, open and discrete maps of finite multiplicity between p-regular spaces.

Theorem 1. Let X, Y be locally compact metric spaces, $c \geq 1$, Y a

$c-LLC$ space, μ a Borel measure on X , ν a Borel measure on Y such that there exist constants C_0, C_1 and $p>1$ such that $C_0^{-1} \cdot r^p \leq \mu(B_r) \leq C_0 \cdot r^p$ and $C_1^{-1} \cdot r^p \leq \nu(B'_r) \leq C_1 \cdot r^p$ for every ball B_r of radius r in X and every ball B'_r of radius r in Y. Let $f : X \to Y$ be continuous, open, discrete such that $N(f) < \infty$, and there exists $H \geq 1$ such that $H(x,f) \leq H$ for every $x \in X$. Then there exists a constant K depending on C_0, C_1, c, p, H such that $M_p(\Gamma) \leq K \cdot N(f) \cdot M_p(f(\Gamma))$ for every path family Γ from X.

Theorem 2. Let X be a locally compact p-Loewner space, Y a $c-LLC$ p-regular space such that there exists C_1 such that $\mu(B(y,r)) \leq C_1 \cdot r^p$ for every ball $B(y,r)$ from Y and let $f : X \to Y$ be continuous, open and discrete. Suppose that $D \subset X$ is open, $N(f,D) < \infty$ and there exists $K \geq 1$ such that $M_p(\Gamma) \leq K \cdot N(f,D) \cdot M_p(f(\Gamma))$ for every path family Γ from D.

Then there exists a constant $H = H(K, p, N(f,D), C_1, c)$ such that $H(x,f) \leq H$ for every $x \in D$.

A known theorem of H. Renggli [R] and P. Caraman [Ca] says that if $n \geq 2$, D, D' are domains in $\mathbf{R}^n$ and $f : D \to D'$ is a homeomorphism, then f is quasiconformal if and only if f carries path families Γ from D of infinite modulus into path families from D' of infinite modulus and a generalization of this result for open discrete maps is given by M. Cristea in [Cr 2,3]. We give the following version of the theorem of Renggli and Caraman for open, discrete maps on metric spaces:

Theorem 3. Let X be a locally compact p-Loewner space, Y a $c-LLC$ p-regular space, and let $f : X \to Y$ be continuous, open and discrete and suppose that there exists $\delta > 0$ such that $M_p(f(\Gamma)) \geq \delta$ for every path family Γ from D with $M_p(\Gamma) = \infty$. Then there exists $H \geq 1$ such that $H(x,f) \leq H$ for every $x \in X$.

As a corollary, we obtain that an open, discrete map $f : X \to Y$ between metric spaces which satisfies the modular inequality $M_p(\Gamma) \leq K \cdot M_p(f(\Gamma))$ for every path family Γ from X is such that the linear dilatation $H(x,f)$ is uniformly bounded on X. We cannot estimate the constant $H \geq 1$ such that $H(x,f) \leq H$ for every $x \in X$ in terms of K and the spatial constants of X and Y.

Theorem 4. Let X be a locally compact p-Loewner space, Y a $c-LLC$ p-regular space and let $f : X \to Y$ be continuous , open and discrete such that there exists $K \geq 1$ such that $M_p(\Gamma) \leq K \cdot M_p(f(\Gamma)))$ for every path family Γ from X. Then there exists $H \geq 1$ such that $H(x,f) \leq H$ for every $x \in X$.

For open, discrete mappings $f : X \to Y$ between metric spaces satisfying the modular inequality $M_p(\Gamma) \leq K \cdot M_p(f(\Gamma))$ for every path family Γ from X, we have the following estimate of the multiplicity function $N(f)$:

Theorem 5. Let X be a locally compact p-Loewner space, Y a $c-LLC$ p-regular space such that there exits $C_1 > 0$ such that $\mu(B(y,r)) \leq C_1 \cdot r^p$ for every ball $B(y,r)$ from Y and let $f : X \to Y$ be continuous, open and discrete and suppose that there exists $K \geq 1$ such that $M_p(\Gamma) \leq K \cdot M_p(f(\Gamma))$ for every path family Γ from X.

Then, for every $\lambda > 1$, there exists at most $n = [\frac{2K \cdot Q(p,C_1)}{\Phi(1) \cdot (\log \lambda)^{p-1}}]$ points

$x_1, ..., x_n$ from X such that $f(x_k) = f(x_1)$ and $h(x_k, f) > c^2\lambda$ for every $k = 1, ..., n$.

Here $Q(p, C_1)$ is the constant from [HK2] Lemma 3.14 and Φ is the function from the definition of X as a Loewner space.

2 Notations and preliminaries.

If (X, μ) is a metric measure space and Γ_1, Γ_2 are path families such that $\Gamma_1 \subset \Gamma_2$ and $p \geq 1$, then $M_p(\Gamma_1) \leq M_p(\Gamma_2)$ and if Γ_n are path families in X, then $M_p(\bigcup_{n=1}^{\infty} \Gamma_n) \leq \sum_{n=1}^{\infty} M_p(\Gamma_n)$. We say that the paths from Γ_2 are longer than the paths from Γ_1 (and we write $\Gamma_2 > \Gamma_1$) if for every path $\gamma \in \Gamma_2$, there exists a subpath $\gamma^* \in \Gamma_1$. If $\Gamma_2 > \Gamma_1$ and $p \geq 1$, then $M_p(\Gamma_2) \leq M_p(\Gamma_1)$. We say that the path families $\Gamma_1, ..., \Gamma_n, ...$ from X are separate if there exists disjoint Borel sets $E_1, ..., E_n, ...$ in X such that if $g_n = X_{CE_n}, n \in \mathbf{N}$, it results that $\int_\gamma g_n ds = 0$ for every $\gamma \in \Gamma_n$ and $n \in \mathbf{N}$. As in the classical case, we prove that if $\Gamma_1, ..., \Gamma_n, ...$ are separate in $X, \Gamma_n > \Gamma$ for $n \in \mathbf{N}$ and $p \geq 1$, then $M_p(\Gamma) \geq \sum_{n=1}^{\infty} M_p(\Gamma_n)$. As is underlined in [T], the following result of Ziemer [Z] holds on metric spaces: "Let $\Gamma_1 \subset \Gamma_2 \subset ... \subset \Gamma_n \subset ...$ be path families in X and $p > 1$. Then $M_p(\bigcup_{n=1}^{\infty} \Gamma_n) = \lim_{n \to \infty} M_p(\Gamma_n)$."

The following $\frac{1}{5}$−covering theorem will be used in our paper:

Theorem A. (Basic covering theorem) Let X be a metric space and $\mathcal{F}$ a family of balls of uniformly bounded diameter. Then $\mathcal{F}$ contains a disjoint subfamily $\mathcal{G}$ such that $\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B$. If X is locally compact, we can take $\mathcal{G}$ countable.

If X is a metric space, $\gamma : I = [a, b] \to X$ is a path, B_i are sets from X, we say that $(I_i, B_i)_{i \in J}$ is a parametrized cover of γ if $J \subset \mathbf{N}, I_i \subset I, \bigcup_{i \in J} I_i = I$ and $\gamma(I_i) \subset B_i$ for every $i \in J$. If $\mathcal{B}$ is a base covering of $X, \varphi : \mathcal{B} \to [0, \infty]$ is a map and $\delta > 0$, we let $\delta - \varphi_{\mathcal{B}}(\gamma) = \inf \sum_{i \in I} \Phi(A_i)$, where the infimum is taken over all parametrized coverings $(I_i, A_i)_{i \in I}$ of γ with elements $A_i \in \mathcal{B}$ such that $d(A_i) < \delta$ for $i \in I$. Then the map $\delta \to \delta - \varphi_{\mathcal{B}}(\gamma)$ is decreasing and we put $\varphi_{\mathcal{B}}(\gamma) = \lim_{\delta \to 0} \delta - \varphi_{\mathcal{B}}(\gamma)$. This definition was inspired by the definition given by J. Tyson in [T], §3.14. We can take the collection $\mathcal{B}$ as an union $\mathcal{B} = \mathcal{B}^1 \cup, ..., \cup \mathcal{B}^k$ of collections and the map $\varphi = (\varphi_1, ..., \varphi_k)$ with $\varphi_l : \mathcal{B}^l \to [0, \infty], l = 1, ..., k$. We can also see that $\varphi_{\mathcal{B}}(\gamma)$ is invariant to a increasing reparametrization of γ. The following proposition follows closely the ideas from [T]:

Proposition 1. Let X be a locally compact metric space, Γ a path family in $X, h : X \to \mathbf{R}_+$ a map such that $\inf_{x \in K} h(x) > 0$ for every $K \subset X$ compact, $\rho \in F(\Gamma)$ lower semicontinuous and let $\mathcal{B} = \mathcal{B}^1 \cup, ..., \cup \mathcal{B}^k$ be a base covering of X. Let $\varepsilon > 0$ and $\varphi_l : \mathcal{B}^l \to [0, \infty], \varphi_l(A) = \inf_{a \in A}(\rho(a) + \varepsilon h(a)) \cdot d(A)$ for $A \in \mathcal{B}^l, l = 1, ..., k, \varphi = (\varphi_1, ..., \varphi_k)$. Then $\varphi_{\mathcal{B}}(\gamma) \geq 1$ for every $\gamma \in \Gamma$.

Proof: Let $\mathcal{B}^l = (B_{li})_{i \in I_l}, l = 1, ..., k$ and let $\gamma \in \Gamma, \gamma : I \to X$ be such that γ is not locally rectifiable and let $I_0 \subset I$ be compact such that $\gamma | I_0$ is not rectifiable and let $K \subset X$ be compact such that $\gamma(I_0) \subset \text{Int } K$ and let $m = \inf_{x \in K} h(x) > 0$ and $\delta_0 = d(\gamma(I_0), C \text{ Int } K) > 0$.

Let $t > 0$ and $\Delta = (a = t_0 < t_1, ..., < t_p = b)$ be a partition of I_0 such that $V_\Delta(\gamma_0) = \sum_{q=0}^{p-1} d(\gamma(t_{q+1}), \gamma(t_q)) > t$. By removing terms if necessary, we may assume that $\gamma(t_q) \neq \gamma(t_{q+1}), q = 0, 1, ..., p-1$. Let δ_1 be the minimum of the quantities $d(\gamma(t_q), \gamma(t_{q+1})), q = 0, 1, ..., p-1$. Let $0 < \delta < \min\{\delta_0, \delta_1\}$ and $(I_{li}, B_{li})_{i \in J_l}, l = 1, ..., k$ be a parametrized covering of γ with $d(B_{li}) < \delta$ for $i \in J_l, l = 1, ..., k$. Let $J'_l = \{i \in J_l | B_{li} \cap \gamma(I_0) \neq \emptyset\}, l = 1, ..., k$.

Then $I_0 \subset \bigcup_{l=1}^k \bigcup_{i \in J'_l} I_{li}$ and $B_{li} \subset \mathrm{Int}K$ for $i \in J'_l, l = 1, ..., k$. Also, we see that every interval I_{li} contains at most one point $t_q, q = 0, 1, ..., p$. We have $\sum_{l=1}^k \sum_{i \in J_l} \varphi_l(B_{li}) \geq \sum_{l=1}^k \sum_{i \in J'_l} \varphi_l(B_{li}) \geq \varepsilon \cdot m \sum_{l=1}^k \sum_{i \in J'_l} d(B_{li}) \geq \varepsilon \cdot m(V_\Delta(\gamma_0) - (p+1)\delta)$, hence $\delta - \varphi_{\mathcal{B}}(\gamma) \geq \varepsilon \cdot m(t - (p+1)\delta)$. Letting δ tends to zero, we see that $\varphi_{\mathcal{B}}(\gamma) \geq \varepsilon m t$, and letting t tends to infinite, we see that $\varphi_{\mathcal{B}}(\gamma) = \infty \geq 1$. We proved that $\varphi_{\mathcal{B}}(\gamma) \geq 1$ if $\gamma \in \Gamma$ is not locally rectifiable.

Let now $\gamma \in \Gamma$ be locally rectifiable. We prove that $\int_\gamma \rho ds \leq \varphi_{\mathcal{B}}(\gamma)$.We suppose first that ρ is continuous and we can also presume that γ is parametrized by its arc length. Let $I_0 \subset I$ be compact and $\gamma_0 = \gamma | I_0 : I_0 \to X$.

Then, if $\Delta = (0 = t_0 < t_1 < ... < t_n = l(\gamma_0))$ is a partition of $[0, l(\gamma_0)]$ and $c_i \in [t_i, t_{i+1}], i = 0, 1, ..., n-1$, we see that $\int_{\gamma_0} \rho ds = \int_0^{l(\gamma_0)} \rho(\gamma_0(t)) dt = \lim_{||\Delta|| \to 0} \sum_{i=0}^{n-1} \rho(\gamma(c_i)) \cdot d(\gamma(t_i), \gamma(t_{i+1}))$.

Let $U \subset X$ be open such that $\bar{U}$ is compact, $Im\gamma_0 \subset U, m = \inf_{x \in \bar{U}} h(x) > 0$, and let $\alpha > 0$ be such that $|\rho(x) - \rho(y)| < \frac{\varepsilon \cdot m}{2}$ if $x, y \in \bar{U}$ and $d(x, y) \leq \alpha$. We take $\eta > 0$ and let $\Delta = (0 = t_0 < t_1 < ... < t_n = l(\gamma_0))$ be a partition of $[0, l(\gamma_0)]$ such that $||\Delta|| < \frac{\alpha}{4}$ and $\int_{\gamma_0} \rho ds - \eta < \sum_{q=0}^{n-1} \rho(\gamma(t_q)) \cdot d(\gamma(t_q), \gamma(t_{q+1}))$. By removing terms if necessary, we can suppose that $\gamma(t_q) \neq \gamma(t_{q+1})$ for $q = 0, 1, ..., n-1$ and let $\delta_1 > 0$ be the infimum of the quantities $d(\gamma(t_q), \gamma(t_{q+1})), q = 0, 1, ..., n-1$.

Let $\delta_0 = d(\gamma(I_0), CU) > 0$ and let $\delta < \min\{\delta_0, \frac{\alpha}{4}, \delta_1\}$.

Let $(I_{li}, B_{li})_{i \in J_l}, l = 1, ..., k$ be a parametrized cover of γ such that $d(B_{li}) < \delta, i \in J_l, l = 1, ..., k$, and $M = \sup_{x \in \bar{U}} \rho(x)$.

Then every interval I_{li} contains at most one point $t_q, q = 0, 1, ..., n-1$. We take $J_{lq} = \{i \in J_l | I_{li} \subset (t_q, t_{q+1})\}, l = 1, ..., k, q = 0, 1, ..., n-1$. Then $J_{lq} \cap J_{l(q+1)} = \emptyset, l = 1, ..., k, q = 0, 1, ..., n-1, \bigcup_{q=0}^{n-1} J_{lq} \subset J_l, l = 1, ..., k$ and $\sum_{l=1}^k \sum_{i \in J_{lq}} d(B_{li}) \geq d(\gamma(t_q), \gamma(t_{q+1})) - 2\delta$ for $q = 0, 1, ..., n-1$.

If $t \in [t_q, t_{q+1}], i \in \bigcup_{q=0}^{n-1} J_{lq}, \gamma(t) \in B_{li}$ and $b \in B_{li}$, then $d(b, \gamma(t_q)) \leq d(b, \gamma(t)) + d(\gamma(t), \gamma(t_q)) \leq d(B_{li}) + d(t, t_q) \leq \delta + ||\Delta|| < \frac{\alpha}{4} + \frac{\alpha}{4} = \frac{\alpha}{2}$ for $q = 0, 1, ..., n-1$, and this implies that $|\rho(\gamma(t_q) - \rho(b)| \leq \frac{\varepsilon \cdot m}{2}, q = 0, 1, ..., n-1$. Then we have that $\rho(\gamma(t_q)) \cdot d(B_{li}) < \inf_{a \in B_{li}} (\rho(a) + \varepsilon \cdot h(a)) \cdot d(B_{li}) = \varphi_l(B_{li})$ for every $l = 1, ..., k, q = 0, 1, ..., n-1, i \in \bigcup_{q=0}^{n-1} J_{lq}$. We obtain that

$$\sum_{l=1}^k \sum_{i \in J_l} \varphi_l(B_{li}) \geq \sum_{l=1}^k \sum_{i \in \bigcup_{q=0}^{n-1} J_{lq}} \varphi_l(B_{li}) = \sum_{l=1}^k \sum_{q=0}^{n-1} \sum_{i \in J_{lq}} \varphi_l(B_{li}) =$$

$$\sum_{q=0}^{n-1} \sum_{l=1}^k \sum_{i \in J_{lq}} \varphi_l(B_{li}) \geq \sum_{q=0}^{n-1} \sum_{l=1}^k \sum_{i \in J_{lq}} \rho(\gamma(t_q)) \cdot d(B_{li}) =$$

$$\sum_{q=0}^{n-1}\rho(\gamma(t_q))\cdot\sum_{l=1}^{k}\sum_{i\in J_{lq}}d(B_{li})\geq\sum_{q=0}^{n-1}\rho(\gamma(t_q))(d(\gamma(t_q),\gamma(t_{q+1}))-2\delta)=$$

$$\sum_{q=0}^{n-1}\rho(\gamma(t_q))\cdot d(\gamma(t_q),\gamma(t_{q+1}))-2\delta\cdot\sum_{q=0}^{n-1}\rho(\gamma(t_q))\geq\int_{\gamma_0}\rho ds-\eta-2M\delta n.$$

It results that $\delta-\varphi_{\mathcal{B}}(\gamma)\geq\int_{\gamma_0}\rho ds-\eta-2Mn\delta$. We fix $\eta>0$ and $\Delta\in\mathcal{D}([0,l(\gamma_0)])$ as before and we let δ tends to zero. We find that $\varphi_{\mathcal{B}}(\gamma)\geq\int_{\gamma_0}\rho ds-\eta$ and letting now η tends to zero, we find that $\varphi_{\mathcal{B}}(\gamma)\geq\int_{\gamma_0}\rho ds$. Since $\int_{\gamma}\rho ds=\sup_{I_0\subset I,I_0}$ compact $\int_{\gamma|I_0}\rho ds$, we obtain that $\varphi_{\mathcal{B}}(\gamma)\geq\int_{\gamma}\rho ds$.

We proved that if γ is locally rectifiable and ρ is continuous, then $\varphi_{\mathcal{B}}(\gamma)\geq\int_{\gamma}\rho ds$. Since ρ is lower semicontinuous, we can use a theorem of Baire to find continuous maps $\rho_n:X\to\mathbf{R}_+$ such that $\rho_n\nearrow\rho$, and we see that $\int_{\gamma}\rho ds=\sup\int_{\gamma}\rho' ds$, where the supremum is taken over all continuous functions $0\leq\rho'\leq\rho$. We therefore proved that if ρ is lower semicontinuous, then $\varphi_{\mathcal{B}}(\gamma)\geq\int_{\gamma}\rho ds$ if $\gamma\in\Gamma$ is locally rectifiable, hence, if $\rho\in F(\Gamma)$, we see that $\varphi_{\mathcal{B}}(\gamma)\geq 1$ for every $\gamma\in\Gamma$ locally rectifiable. We proved that if $\rho\in F(\Gamma)$ is lower semicontinuous, then $\varphi_{\mathcal{B}}(\gamma)\geq 1$ for every $\gamma\in\Gamma$.

Lemma 1. Let E be a Hausdorff space, $U\subset E$ open, $Q\subset E$ connected such that $Q\cap U\neq\emptyset, Q\cap CU\neq\emptyset$. Then $Q\cap\partial U\neq\emptyset$.

Proof: Suppose that $Q\cap\partial U=\emptyset$. Then $Q\cap C\overline{U}\neq\emptyset$, and since $Q\cap U\neq\emptyset, Q$ is connected , $Q\cap U$ and $Q\cap C\overline{U}$ are open in Q and $Q=(Q\cap U)\cup(Q\cap C\overline{U})$, we obtained a contradiction. It results that $Q\cap\partial U\neq\emptyset$.

Lemma 2. Let E,F be Hausdorff spaces, $D\subset E$ open such that $\overline{D}$ is compact and let $f:E\to F$ be continuous and open. Then $\partial f(D)\subset f(\partial D)$.

Proof: We see that $f(\bar{D})=\overline{f(D)}$. Let $y\in\partial f(D)\subset\overline{f(D)}=f(\bar{D})$. Then there exists $x\in\bar{D}$ such that $y=f(x)$. If $x\in D$, we use the fact that f is an open map to see that $f(D)$ is open, nonempty and $f(x)\in f(D)$, hence $f(x)\in Intf(D)$, which contradicts the fact that $y\in\partial f(D)$. It results that $x\in\partial D$, hence $y=f(x)\in f(\partial D)$ and since y was arbitrary chosen in $\partial f(D)$, it results that $\partial f(D)\subset f(\partial D)$.

Lemma 3. Let E,F be locally compact metric spaces, F $c-LLC, f: E\to F$ be continuous, open, discrete and let $x\in E$. Then there exists $r_x>0$ such that $\overline{B}(x,r_x)\subset D,\overline{B}(x,r_x)\cap f^{-1}(f(x))=\{x\}$ and $B(f(x),\frac{l(x,f,r)}{c})\subset f(B(x,r))\subset\overline{B}(f(x),c\cdot L(x,f,r))$ for $0<r\leq r_x$.

Proof: Let $r>0$ be such that $\overline{B}(x,r)\subset D$. We show that $B(f(x),\frac{l(x,f,r)}{c})\subset f(B(x,r))$. Indeed, if this thing is not true, we can find a point $b\in B(f(x),\frac{l(x,f,r)}{c})\setminus f(B(x,r))$. Since F is $c-LLC$, we can find $Q\subset B(f(x),l(x,f,r))$ connected such that $f(x)\in Q, b\in Q$. Then $f(B(x,r))$ is open, $Q\cap f(B(x,r))\neq\emptyset, Q\cap Cf(B(x,r))\neq\emptyset$ and from Lemma 1 we see that $Q\cap\partial f(B(x,r))\neq\emptyset$. Using now Lemma 2, we see that $Q\cap f(S(x,r))\neq\emptyset$. Let $y\in S(x,r)$ be such that $f(y)\in Q\subset B(f(x),l(x,f,r))$. Then $d(f(x),f(y))<$

$l(x,f,r)$ and $d(x,y)=r$, which contradicts the definition of $l(x,f,r)$. We therefore proved that $B(f(x),\frac{l(x,f,r)}{c})\subset f(B(x,r))$ if $r>0$ is such that $\overline{B}(x,r)\subset D$.

Let now $r_x>0$ be such that $\overline{B}(x,r_x)\subset D, \overline{B}(x,r_x)\cap f^{-1}(f(x))=\{x\}$ and $Cf(B(x,r))\cap C\overline{B}(f(x),cL(x,f,r))\neq\emptyset$ for $0<r\leq r_x$. We take $0<r\leq r_x$ and suppose that $f(B(x,r))\not\subset\overline{B}(f(x),cL(x,f,r))$ and let $a\in B(x,r)$ be such that $d(f(a),f(x))>c\cdot L(x,f,r)$. Let $b\in Cf(B(x,r))\cap C\overline{B}(f(x),cL(x,f,r))$.

Since F is $c-LLC$, we can find $Q\subset C\overline{B}(f(x),L(x,f,r))$ connected such that $f(a)\in Q, b\in Q$. We see that $Q\cap f(B(x,r))\neq\emptyset, Q\cap Cf(B(x,r))\neq\emptyset, f(B(x,r))$ is open, and from Lemma 1 we find that $Q\cap\partial f(B(x,r))\neq\emptyset$. Using Lemma 2, we see that $Q\cap f(S(x,r))\neq\emptyset$ and let $y\in S(x,r)$ be such that $f(y)\in Q$. Then $d(x,y)=r$ and $d(f(x),f(y))>L(x,f,r)$, which contradicts the definition of $L(x,f,r)$. We therefore proved that $f(B(x,r))\subset\overline{B}(f(x),cL(x,f,r))$ for $0<r\leq r_x$.

Lemma 4. Let E,F be locally compact metric spaces, $f:E\to F$ continuous and open and $r>0$ be such that there exists $c>0$ such that $B(f(x),r)\subset f(B(x,c))$. Then, if $\lambda_r=\inf\{\delta>0|f(B(x,\delta))\supset B(f(x),r)\}$, it results that $l(x,f,\lambda_r)\leq r$.

Proof: We show first that $B(f(x),r)\subset f(\overline{B}(x,\lambda_r))$. Let $y\in B(f(x),r)$ and $\lambda_r<c_j\leq c, c_j\searrow\lambda_r$. Since $B(f(x),r)\subset f(B(x,c_j))$ for every $j\in\mathbf{N}$, we can find $x_j\in B(x,c_j)$ such that $y=f(x_j)$ for every $j\in\mathbf{N}$. Since $\overline{B}(x,c)$ is compact, we can find $x_0\in\overline{B}(x,c)$ and a subsequence $(x_{j_k})_{k\in\mathbf{N}}$ of $(x_j)_{j\in\mathbf{N}}$ such that $x_{j_k}\to x_0$, and then $f(x_0)=\lim_{k\to\infty}f(x_{j_k})=y$. Letting k tends to infinite in the inequality $d(x,x_{j_k})\leq c_{j_k}$, we obtain that $d(x,x_0)\leq\lambda_r$, hence $x_0\in\overline{B}(x,\lambda_r), y=f(x_0)\in f(\overline{B}(x,\lambda_r))$. Since y was chosen arbitrary in $B(f(x),r)$, we proved that $B(f(x),r)\subset f(\overline{B}(x,\lambda_r))$.

Let now $c_j<\lambda_r, c_j\nearrow\lambda_r$. Using the definition of λ_r, it results that $B(f(x),r)\not\subset f(B(x,c_j))$ for every $j\in\mathbf{N}$, and let $y_j\in B(f(x),r)\setminus f(B(x,c_j))$ for every $j\in\mathbf{N}$. Since $B(f(x),r)\subset f(\overline{B}(x,\lambda_r))$, we can find $a_j\in\overline{B}(x,\lambda_r)$ such that $f(a_j)=y_j, j\in\mathbf{N}$, and we see that $c_j\leq|x-a_j|\leq\lambda_r$ for every $j\in\mathbf{N}$. Since $\overline{B}(x,\lambda_r)$ is compact, we can find $a_0\in\overline{B}(x,\lambda_r)$ and $(a_{j_k})_{k\in\mathbf{N}}$ a subsequence of $(a_j)_{j\in\mathbf{N}}$ such that $a_{j_k}\to a_0$, and letting k tends to infinite in the inequality $c_{j_k}\leq|x-a_{j_k}|\leq\lambda_r$, we see that $a_0\in S(x,\lambda_r)$. Then $f(a_0)=\lim_{k\to\infty}f(a_{j_k})=\lim_{k\to\infty}y_{j_k}$, and since $y_{j_k}\in B(f(x),r)$ for $k\in\mathbf{N}$, we see that $d(f(x),f(a_0))\leq r$. We obtain that $l(x,f,\lambda_r)\leq d(f(x),f(a_0))\leq r$.

3 Proofs of the main results.

Proof of Theorem 1. Let $\lambda>1, D=5^p\cdot H^{2p}\cdot C_1^2\cdot c^{4p}\cdot\lambda^p$ and $A=\{x\in X|$ there exists $I_x\subset\mathbf{R}_+$ with $0\in I_x'$ such that $\nu(f(B(x,5r))<D\cdot\nu(f(B(x,r))$ for every $r\in I_x\}$. Let $x\in CA$. Since $H(x,f)<H$, we can find $r_x>0$ such that $L(x,f,r)\leq H\cdot l(x,f,r)$ and $\nu(f(B(x,5r))\geq D\cdot\nu(f(B(x,r))$ for every $0<r<\frac{r_x}{5}$. We put $I_x=(0,\frac{r_x}{5})$.

We have

$$L(x,f,r)^p\leq H^p\cdot l(x,f,r)^p\leq H^p\cdot C_1\cdot c^p\cdot\nu(B(f(x),\frac{l(x,f,r)}{c}))\leq$$

$$H^p \cdot C_1 \cdot c^p \cdot \nu(f(B(x,r)) \leq \frac{H^p \cdot C_1 \cdot c^p}{D} \nu(f(B(x,5r))) \leq$$

$$\leq \frac{H^p \cdot C_1 \cdot c^p}{D} \cdot \nu(\overline{B}(f(x), cL(x,f,5r))) \leq \frac{H^p \cdot C_1^2 \cdot c^{2p}}{D} \cdot L(x,f,5r)^p \leq$$

$$\leq \frac{H^{2p} \cdot C_1^2 \cdot c^{2p}}{D} \cdot l(x,f,5r)^p = \frac{l(x,f,5r)^p}{c^{2p} \cdot 5^p \cdot \lambda^p},$$

hence

$$5c^2 \lambda L(x,f,r) \leq l(x,f,5r) \text{ if } 0 < 5r < r_x, x \in CA \qquad (1).$$

We have that $B(f(x), 5c\lambda L(x,f,r)) \subset B(f(x), \frac{l(x,f,5r)}{c}) \subset f(B(x,5r))$ for $0 < 5r \leq r_x$. Let $0 < 5r \leq r_x$ and $Q_r = \{s > 0 | \overline{B}(f(x), 5c\lambda L(x,f,r)) \subset f(B(x,s))\}$ and $\lambda_r = \inf Q_r$. From Lemma 3, we see that $f(B(x,r)) \subset \overline{B}(f(x), cL(x,f,r))$, hence $r \leq \lambda_r$, and we also see that $\lambda_r \leq 5r$. If $\lambda_r = 5r$, we take $\rho_r = \lambda_r$ and from Lemma 4 we see that $l(x,f,\rho_r) \leq 5c\lambda L(x,f,r)$ and $B(f(x), 5c\lambda L(x,f,r) \subset f(B(x,\rho_r))$. If $\lambda_r < 5r$, suppose that there exists $\rho_0 > 0$ such that $5c\lambda^2 \cdot L(x,f,r) < l(x,f,\alpha)$ for every $\lambda_r < \alpha < \lambda_r + \rho_0 < 5r$. Since the map $y \to d(f(x), f(y))$ is a continuous map defined on the compact set $S(x,\lambda_r)$, we can find $x_r \in S(x,\lambda_r)$ such that $d(f(x), f(x_r)) = l(x,f,\lambda_r)$. Let now $x_k \in B(x, \lambda_r + \rho_0) \setminus \overline{B}(x, \lambda_r)$ be such that $x_k \to x_r$. Then $d(f(x), f(x_k)) \geq l(x,f,d(x,x_k)) \geq 5c\lambda^2 L(x,f,r)$ for every $k \in \mathbb{N}$ and letting k tends to infinite, we obtain that $d(f(x), f(x_r)) \geq 5c\lambda^2 L(x,f,r)$. We find that $5c^2\lambda L(x,f,r) \leq d(f(x), f(x_r)) = l(x,f,\lambda_r) \leq 5c\lambda L(x,f,r)$, which represents a contradiction.

It results that if $\lambda_r < 5r$, we can find $c_k \searrow \lambda_r$ such that $l(x,f,c_k) \leq 5c\lambda^2 L(x,f,r)$ and $B(f(x), 5c\lambda L(x,f,r)) \subset f(B(x,c_k))$ for every $k \in \mathbb{N}$. We take in this case $\rho_r = c_k$ for some $\lambda_r < c_k < 5r$ as before, and we see that $l(x,f,\rho_r) \leq 5c\lambda^2 L(x,f,r)$ and $B(f(x), 5c\lambda L(x,f,r)) \subset f(B(x,\rho_r))$.

We also take $D_1 = 5^p \cdot c^{3p} \cdot H^{2p} \cdot c_1^2 \cdot \lambda^{2p}$ and we have $\nu(f(B(x,\rho_r))) \leq \nu(\overline{B}(f(x), cL(x,f,\rho_r))) \leq C_1 \cdot c^p \cdot L^p(x,f,\rho_r) \leq C_1 \cdot c^p \cdot H^p \cdot l^p(x,f,\rho_r) \leq 5^p \cdot c^{2p} \cdot H^p \cdot C_1 \cdot \lambda^{2p} \cdot L^p(x,f,r) \leq C_1 \cdot \lambda^{2p} \cdot c^{2p} \cdot H^{2p} \cdot l^p(x,f,r) \leq 5^p \cdot c^{3p} \cdot H^{2p} \cdot \lambda^{2p} \cdot C_1^2 \cdot \nu(B(f(x), \frac{l(x,f,r)}{c})) \leq 5^p \cdot c^{3p} \cdot H^{2p} \cdot \lambda^{2p} \cdot C_1^2 \cdot \nu(f(B(x,r))) = D_1 \cdot \nu(f(B(x,r)))$.

We therefore found for every $x \in CA, r_x > 0$ such that for every $0 < 5r \leq r_x$, there exists $r \leq \rho_r \leq 5r$ such that

$$l(x,f,\rho_r) \leq 5c\lambda^2 L(x,f,r), B(f(x), 5c\lambda L(x,f,r)) \subset f(B(x,\rho_r))$$

and

$$\nu(f(B(x,\rho_r))) \leq D_1 \cdot \nu(f(B(x,r))) \qquad (2).$$

We also take $\rho_r < \mu_r < 6r$ so that $d(f(B(x,\mu_r))) \leq 2 \cdot d(f(B(x,\rho_r)))$ for $0 < 5r \leq r_x$.

Let $\mathcal{B}^1 = (B(x,5r))_{x \in A, r \in I_x}, \mathcal{B}^2 = (B(x,\mu_r))_{x \in CA, r \in I_x}$ and $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$. Then $\mathcal{B}$ is a base of X and $f(\mathcal{B})$ is a base in $f(X)$. Let Γ be a path family in X and let $\rho' \in F(f(\Gamma))$ be lower semicontinuous and let $\varepsilon > 0$ and $h : Y \to [0,\infty], h \in L^p(Y,\nu)$ be such that $\inf_{y \in K} h(y) > 0$ for every $K \subset Y$ compact. We define $\varphi = (\varphi_1, \varphi_2), \varphi_l : f(\mathcal{B}^l) \to [0, \infty], \varphi_l(A) = \inf_{a \in A}(\rho'(a) + \varepsilon h(a)) \cdot d(A)$ for $A \in f(\mathcal{B}^l), l = 1, 2$. Using Proposition 1, we see that

$$\varphi_{f(\mathcal{B})} f \circ \gamma \geq 1 \quad \text{for every } \gamma \in \Gamma \qquad (3).$$

Let $\Gamma_n = \{\gamma \in \Gamma, \gamma : I \to X | \gamma$ is locally rectifiable and there exists $I_n \subset I$ compact such that $d(\gamma | I_n) \geq \frac{1}{n}$ and $\delta - \varphi_{f(\mathcal{B})} f \circ \gamma \geq \frac{1}{4}$ for $0 < \delta \leq \frac{1}{4n}\}$ for $n \in \mathbb{N}$. We fix $n \in \mathbb{N}$ and let $0 < \delta < \frac{1}{4n}$. Since $\mathcal{A} = (B(x,r))_{x \in A, r \in I_x}, 0 < r < \delta/6, d(f(B(x, 6r)) < \delta$ is a covering of A, we can find I_1 countable such that $A \subset \bigcup_{x \in A, r \in I_x, r < \delta/6} B(x,r) \subset \bigcup_{i \in I_1} B(x_i, 5r_i)$, with $x_i \in A, r_i \in I_{x_i}, 6r_i < \delta, d(f(B(x_i, 6r_i))) < \delta, \nu(f(B(x_i, 5r_i))) < D \cdot \nu(f(B(x_i, r_i)), i \in I_1, B(x_i, r_i) \cap B(x_j, r_j) = \emptyset, i \neq j, i, j \in I_1$.

Let $B = X \setminus \bigcup_{i \in I_1} B(x_i, r_i)$. Then $B \subset CA$ and B is closed. Let $Q_1, ..., Q_k, ...$ be compact such that $Q_k \subset IntQ_{k+1}, k \in \mathbb{N}$ and $X = \bigcup_{k=1}^{\infty} Q_k$ and let $\delta_k = d(Q_k, CIntQ_{k+1}) > 0, k \in \mathbb{N}$ and $Q_0 = \emptyset$. Since $B \subset \bigcup_{x \in B, r \in I_x} B(x,r)$ and $B \cap (Q_{k+1} \setminus IntQ_k)$ is compact for $k \in \mathbb{N}$, we can find S_k finite such that $B \cap (Q_{k+1} \setminus IntQ_k) \subset \bigcup_{i \in S_k} B(x_i, r_i), x_i \in B, r_i \in I_{x_i}, i \in S_k$ with $6r_i < \min\{\delta, \delta_1, ..., \delta_k\}, d(f(B(x_i, 6r_i))) < \delta, i \in S_k, k \in \mathbb{N}$. It results that if $k \in \mathbb{N}$ is fixed and $j \in S_q$ with $q \geq k+2$, then $\overline{B}(x_j, \rho_{r_j}) \cap Q_k = \emptyset$ for every $j \in S_q$, hence if $M_1 = \bigcup_{q=1}^{\infty} S_q$, the set $L_k = \{i \in M_1 | Q_k \cap B \cap \overline{B}(x_i, \rho_{r_i}) \neq \emptyset\}$ is included in $\bigcup_{q=1}^{k+1} S_q$ and is therefore finite for every $k \in \mathbb{N}$. We can find $M_1 \subset \mathbb{N}$ such that $B \subset \bigcup_{i \in M_1} B(x_i, r_i), x_i \in B, r_i \in I_{x_i}, i \in M_1$ and the sets $L_k = \{i \in M_1 | Q_k \cap B \cap \overline{B}(x_i, \rho_{r_i}) \neq \emptyset\}$ are finite for every $k \in \mathbb{N}$. Then $f(B) \subset \bigcup_{i \in M_1} f(B(x_i, r_i)) \subset \bigcup_{i \in M_1} B(f(x_i), \lambda c L(x_i, f, r_i))$ is a covering of $f(B)$ and substracting a $\frac{1}{5}$-covering of $f(B)$, we can find $I_2 \subset M_1$ such that $f(B) \subset \bigcup_{i \in M_1} B(f(x_i), \lambda c L(x_i, f, r_i)) \subset \bigcup_{i \in I_2} B(f(x_i), 5c\lambda L(x_i, f, r_i)) \subset \bigcup_{i \in I_2} f(B(x_i, \rho_{r_i})))$.

We have that $\nu(f(B(x_i, \rho_{r_i})) \leq D_1 \cdot \nu(f(B(x_i, r_i)))$ for $i \in I_2$ and since $f(B(x_i, r_i)) \subset B(f(x_i), c\lambda L(x_i, f, r_i)), f(B(x_j, r_j)) \subset B(f(x_j), c\lambda L(x_j, f, r_j))$ for $i \neq j, i, j \in I_2$, it results that $f(B(x_i, r_i)) \cap f(B(x_j, r_j)) = \emptyset$ for $i \neq j, i, j \in I_2$.

We let $B_1 = \bigcup_{i \in I_2} \overline{B}(x_i, \rho_{r_i})$ and we prove that B_1 is closed. Indeed, let $a_k \in B_1, a_k \to a$. If there exists $i \in I_2$ and $(a_{k_q})_{q \in \mathbb{N}}$ a subsequence of $(a_k)_{k \in \mathbb{N}}$ such that $a_{k_q} \in \overline{B}(x_i, \rho_{r_i})$ for $q \in \mathbb{N}$, we use the fact that $\overline{B}(x_i, \rho_{r_i})$ is a closed set to see that $a \in \overline{B}(x_i, \rho_{r_i}) \subset B_1$. If this case it is not possible, we can find $(a_{k_q})_{q \in \mathbb{N}}$ a subsequence of $(a_k)_{k \in \mathbb{N}}$ such that $a_{k_q} \in \overline{B}(x_q, \rho_{r_q}), q \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $a \in IntQ_k$ and $r > 0$ be such that $B(a, r) \subset Q_k$. Since $a_{k_q} \to a$, we can find $q_0 \in \mathbb{N}$ such that $a_{k_q} \in B(a, r) \subset Q_k$ for $q \geq q_0$, hence $Q_k \cap \overline{B}(x_q, \rho_{r_q}) \neq \emptyset$ for $q \geq q_0$, and this contradicts the way we chosed the balls $B(x_i, r_i), i \in M_1$. It results that B_1 is a closed set, and we see that $f(B) \subset f(B_1)$.

Let $A_1 = \bigcup_{i \in I_2} B(x_i, \mu_{r_i})$. Then A_1 is open, $B_1 \subset A_1$ and suppose that $B \not\subset A_1$. Then $B \setminus A_1$ is a closed and nonempty set , and we cover this set with balls $B(x, r), x \in B \setminus A_1, r \in I_x$ such that every ball $\overline{B}(x, \rho_r)$ is disjoint from B_1 and $6r < \delta, d(f(B(x, 6r)) < \delta$. As before, we can find $M_2 \subset N$

such that $B \setminus A_1 \subset \bigcup_{i\in M_2} B(x_i, r_i), x_i \in B, r_i \in I_{x_i}, i \in M_2$, and the sets $\{i \in M_2 | Q_k \cap (B \setminus A_1) \cap \overline{B}(x_i, \rho_{r_i}) \neq \emptyset\}$ are finite for every $k \in \mathbb{N}$.

Then $f(B \setminus A_1) \subset \bigcup_{i\in M_2} f(B(x_i, r_i)) \subset \bigcup_{i\in M_2} f(B(f(x_i), \lambda c L(x_i, f, r_i)))$ is a covering of $f(B \setminus A_1)$ and substracting a $\frac{1}{5}$-subcovering from it, we can find $I_3 \subset M_2$ such that:

$$f(B \setminus A_1) \subset \bigcup_{i\in M_2} f(B(x_i, r_i)) \subset \bigcup_{i\in M_2} B(f(x_i), c\lambda L(x_i, f, r_i)) \subset$$

$$\bigcup_{i\in I_3} (B(f(x_i), 5c\lambda L(x_i, f, r_i))) \subset \bigcup_{i\in I_3} f(B(x_i, \rho_{r_i})).$$

As before, we see that $f(B(x_i, r_i) \cap f(B(x_j, r_j) = \emptyset$ for $i \neq j, i, j \in I_3$ and let $B_2 = \bigcup_{i\in I_3} \overline{B}(x_i, \rho_{r_i})$. Then B_2 is closed, $B_1 \cap B_2 = \emptyset, f(B \setminus A_1) \subset f(B_2)$ and let $A_2 = \bigcup_{i\in I_2 \cup I_3} B(x_i, \mu_{r_i})$.

If $B \subset A_2$, the process is finished. If not, we continue this process and at the step k we obtain sets $I_2, ..., I_{k+1}$ from $\mathbb{N}$, disjoint closed sets $B_1, ..., B_k, B_q = \bigcup_{i\in I_{q+1}} \overline{B}(x_i, \rho_{r_i})$, sets $A_q = \bigcup_{i\in \bigcup_{l=2}^{q+1} I_l} B(x_i, \mu_{r_i}), x_i \in B, r_i \in I_{x_i}, i \in I_{q+1}, q = 1, ..., k$ and $f(B_q) \supset f(B \setminus A_{q-1}), q = 2, ..., k, 6r_i < \delta, d(f(B(x_i, 6r_i)) < \delta$, $d(f(B(x_i, \mu_{r_i}) < 2d(f(B(x_i, \rho_{r_i})), f(B(x_i, r_i)) \cap f(B(x_j, r_j)) = \emptyset, i \neq j, i, j \in I_q, q = 2, ..., k+1$. Suppose that $k > N(f)$. Then $f(B_q) \supset f(B \setminus A_{q-1}) \supset ... \supset f(B \setminus A_{k-1}) \neq \emptyset$ for $q = 1, ..., k$, and since $B_1, ..., B_k$ are disjoint and $k > N(f)$, we obtained a contradiction. It results that the process must stop at a step $k \leq N(f)$.

We therefore found $k \leq N(f)$, sets $I_2, ..., I_{k+1}$ from $\mathbb{N}$, balls $B_{li} = B(x_{li}, \rho_{li})$, $x_{li} \in B, r_{li} \in I_{x_{li}}, \rho_{li} = \rho_{r_{li}}, r_{li} < \rho_{li} < \mu_{li} < 6r_{li}$ such that $d(f(B(x_{li}, \mu_{li})) < 2d(f(B(x_{li}, \rho_{li})), \nu(f(B(x_{li}, \rho_{li})) \leq D_1 \cdot \nu(f(B(x_{li}, r_{li})), 6r_i < \delta, d(f(B(x_{li}, 6r_{li}))) < \delta, f(B(x_{li}, r_{li})) \cap f(B(x_{lj}, r_{lj})) = \emptyset, i \neq j, i, j \in I_l, l = 2, ..., k+1$, and $B \subset \bigcup_{l=2}^{k+1} \bigcup_{i\in I_l} B(x_{li}, \mu_{li})$.

We take $B_{1i} = B(x_{1i}, 5r_{1i}), x_{1i} = x_i, r_{1i} = r_i, i \in I_1, A_{1i} = B_{1i}, u_{1i} = 5r_{1i}, i \in I_1$, and we let $A_{li} = B(x_{li}, \mu_{li}), l = 2, ..., k+1, i \in I_l$. We see that the balls $B(x_{li}, r_{li}), i \in I_l$, are disjoint for every $l \in \{1, ..., k+1\}$ fixed.

We define a Borel map $\rho : X \to [0, \infty]$ by $\rho(x) = \sum_{l=1}^{k+1} \sum_{i\in I_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} \chi_{B(x_{li}, 10r_{li})}$. Let $\gamma \in \Gamma_n, \gamma : I \to X$ and $I_n \subset I$ be compact such that $d(\gamma | I_n) \geq \frac{1}{n}$. Then $Im\gamma \subset X = \bigcup_{l=1}^{k+1} \bigcup_{i\in I_l} A_{li}$. We denote by $(I_{liq})_{q\in N_{li}}$ all the maximal intervals J such that $\gamma(J) \cap B(x_{li}, \mu_{li}) \neq \emptyset, \gamma(J) \not\subset B(x_{li}, 10r_{li}), i \in I_l, l = 1, ..., k+1$.

We see that $Im\gamma$ is not contained in a single ball $B(x_{li}, 10r_{li})$ for some $i \in I_l, l = 1, ..., k+1$, since $d(B(x_{li}, 10r_{li})) \leq 20r_{li} < \frac{20}{6}\delta < 4\delta < \frac{1}{n}$ and $d(\gamma) \geq \frac{1}{n}$. Let $A_{liq} = A_{li}$ for $l = 1, ..., k+1, i \in I_l, q \in N_{li}$. Let $K_l = \{i \in I_l | Im\gamma \cap A_{li} \neq \emptyset\}, l = 1, ..., k+1$. Then $(I_{liq}, f(A_{liq}))_{l=1,...,k+1, i\in K_l, q\in N_{li}}$ is a parametrized cover of $f \circ \gamma$ such that $d(f(A_{liq})) < \delta$ for every $l = 1, ..., k+1, i \in K_l, q \in N_{li}$.

We have

$$\int_\gamma \rho ds = \int_\gamma \sum_{l=1}^{k+1} \sum_{i\in I_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} \chi_{B(x_{li}, 10r_{li})} ds \geq$$

$$\int_0^{l(\gamma)} \sum_{l=1}^{k+1} \sum_{i \in K_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} \chi_{B(x_{li},10r_{li})}(\gamma^0(t))dt \geq$$

$$\sum_{l=1}^{k+1} \sum_{i \in K_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} (\sum_{q \in N_{li}} 4r_{li}) \geq$$

$$4 \cdot \sum_{l=1}^{k+1} \sum_{i \in K_l} \varphi_l(f(A_{li})) Card N_{li} = 4 \cdot \sum_{l=1}^{k+1} \sum_{i \in K_l} \sum_{q \in N_{li}} \varphi_l(f(A_{liq})) \geq$$

$$4 \cdot (\delta - \varphi_{f(\mathcal{B})} f \circ \gamma) \geq 1,$$

since $\gamma \in \Gamma_n$.

We proved that $\rho \in F(\Gamma)$. We have, denoting by $C(\lambda, p, \mu)$ the constant from [H], Ex 2.10, page 13, that

$$M_p(\Gamma_n) \leq \int_X \rho^p(x) d\mu \leq \int_X (\sum_{l=1}^{k+1} \sum_{i \in I_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} \chi_{B(x_{li},10r_{li})}(x))^p d\mu \leq$$

$$(k+1)^{p-1} \sum_{l=1}^{k+1} \int_X (\sum_{i \in I_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} \chi_{B(x_{li},10r_{li})}(x))^p d\mu \leq$$

$$(k+1)^{p-1} C(10, p, \mu) \cdot \sum_{l=1}^{k+1} \int_X \sum_{i \in I_l} \frac{\varphi_l(f(A_{li}))}{r_{li}} \chi_{B(x_{li},r_{li})}(x))^p d\mu =$$

$$(k+1)^{p-1} \cdot C(10, p, \mu) \cdot \sum_{l=1}^{k+1} \sum_{i \in I_l} \frac{\varphi_l(f(A_{li}))^p}{r_{li}^p} \mu(B(x_{li}, r_{li})) \leq$$

$$(k+1)^{p-1} \cdot C_1 \cdot C(10, p, \mu) \cdot \sum_{l=1}^{k+1} \sum_{i \in I_l} \varphi_l(f(A_{li}))^p =$$

$$(k+1)^{p-1} \cdot C_1 \cdot C(10, p, \mu) \cdot \sum_{l=1}^{k+1} \sum_{i \in I_l} \inf_{a \in f(A_{li})} (\rho'(a) + \varepsilon h(a))^p \cdot d(f(A_{li}))^p \leq$$

$$(k+1)^{p-1} \cdot 2^p \cdot C_1 \cdot C(10, p, \mu) \cdot (\sum_{l=1}^{k+1} \sum_{i \in I_l} \inf_{a \in f(B_{li})} (\rho'(a) + \varepsilon h(a))^p \cdot d(f(B_{li}))^p \leq$$

$$(k+1)^{p-1} \cdot 4^p \cdot C_1 \cdot c^p C(10, p, \mu) (\sum_{i \in I_l} \inf_{a \in f(B(x_{1i}, 5r_{1i}))} (\rho'(a) + \varepsilon h(a))^p L^p(x_{1i}, f, 5r_{1i}) +$$

$$\sum_{l=2}^{k+1} \sum_{i \in I_l} \inf_{a \in f(B(x_{li}, \rho_{li}))} (\rho'(a) + \varepsilon h(a))^p \cdot L^p(x_{li}, f, \rho_{li})) \leq$$

$$(k+1)^{p-1} \cdot (4Hc)^p \cdot C_1 \cdot C(10, p, \mu) \cdot (\sum_{i \in I_l} \inf_{a \in f(B(x_{1i}, 5r_{1i}))} (\rho'(a) + \varepsilon h(a))^p \cdot l^p(x_{1i}, f, 5r_{1i}) +$$

$$\sum_{l=2}^{k+1}\sum_{i\in I_l}\inf_{a\in f(B(x_{li},\rho_{li}))}(\rho'(a)+\varepsilon h(a))^p\cdot l^p(x_{li},f,\rho_{li}))\leq$$

$$(k+1)^{p-1}\cdot 4^p\cdot H^p\cdot c^{2p}\cdot C_1^2\cdot C(10,p,\mu)\cdot(\sum_{i\in I_1}\inf_{a\in f(B(x_{1i},5r_{1i}))}(\rho'(a)+\varepsilon h(a))^p\cdot$$

$$\nu(B(f(x_{1i},\frac{l(x_{1i},f,5r_{1i})}{c})$$

$$+\sum_{l=2}^{k+1}\sum_{i\in I_l}\inf_{a\in f(B(x_{li},\rho_{li}))}(\rho'(a)+\varepsilon h(a))^p\cdot\nu(B(f(x_{li},\frac{l(x_{li},f,\rho_{li})}{c}))\leq$$

$$(k+1)^{p-1}\cdot 4^p\cdot H^p\cdot c^{2p}\cdot C_1^2\cdot C(10,p,\mu)\cdot(\sum_{i\in I_1}\inf_{a\in f(B(x_{1i},r_{1i}))}(\rho'(a)+\varepsilon h(a))^p\cdot$$

$$\nu(B(f(x_{1i},5r_{1i}))+$$

$$\sum_{l=2}^{k+1}\sum_{i\in I_l}\inf_{a\in f(B(x_{li},r_{li}))}(\rho'(a)+\varepsilon h(a))^p\cdot\nu(f(B(x_{li},\rho_{li})))\leq$$

$$(k+1)^{p-1}\cdot 4^p\cdot H^p\cdot c^{2p}\cdot C_1^2\cdot C(10,p,\mu)\cdot\max\{D,D_1\}\cdot(\sum_{l=1}^{k+1}\sum_{i\in I_l}\inf_{a\in f(B(x_{li},r_{li}))}(\rho'(a)+\varepsilon h(a))^p\cdot$$

$$\nu(B(f(x_{li},r_{li})))\leq$$

$$(k+1)^{p-1}\cdot 20^p\cdot H^{3p}\cdot c^{6p}\cdot C_1^4\cdot\lambda^{2p}\cdot C(10,p,\mu)\cdot\sum_{l=1}^{k+1}\sum_{i\in I_l}\int_{f(B(x_{li},r_{li}))}(\rho'+\varepsilon h)^p(y)d\nu\leq$$

$$(k+1)^{p-1}\cdot 20^p\cdot H^{3p}\cdot c^{6p}\cdot C_1^4\cdot\lambda^{2p}\cdot C(10,p,\mu)\cdot(N(f)\cdot\int_{\bigcup_{i\in I_1}f(B(x_{1i},r_{1i}))}(\rho'+\varepsilon h)^p(y)d\nu+$$

$$\sum_{l=2}^{k+1}\int_{\bigcup_{i\in I_l}f(B(x_{li},r_{li}))}(\rho'+\varepsilon h)^p(y)d\nu)\leq$$

$$2\cdot(k+1)^{p-1}\cdot 20^p\cdot H^{3p}\cdot c^{6p}\cdot\lambda^{2p}\cdot C_1^4\cdot C(10,p,\mu)N(f)\cdot\int_Y(\rho'+\varepsilon h)^p(y)d\nu$$

$$\leq\lambda^{2p}\cdot Q_1\cdot N(f)\cdot\int_Y(\rho'+\varepsilon h)^p(y)d\nu \text{ (with } Q_1=$$

$$2\cdot(k+1)^{p-1}\cdot 20^p\cdot H^{3p}\cdot c^{6p}\cdot C_1^4\cdot C(10,p,\mu))\leq$$

$$2^{p-1}\cdot\lambda^{2p}\cdot Q_1\cdot N(f)(\int_Y\rho'^p(y)d\nu+\varepsilon^p\int_Y h^p(y)d\nu).$$

We put $K=2^{p-1}Q_1$ and letting first λ tends to 1 and then ε tends to zero we obtain that $M_p(\Gamma_n)\leq K\cdot N(f)\cdot\int_Y\rho'^p(y)d\nu$ for every $\rho\in F(f(\Gamma))$ lower semicontinuous (4).

We want to show that $M_p(\Gamma_n) \leq K \cdot N(f) \cdot M_p(f(\Gamma))$ (5).

Of course, (5) is true if $M_p(f(\Gamma)) = \infty$, hence we can suppose that $M_p(f(\Gamma)) < \infty$. We use the Vitali-Carathéodory theorem to see that every Borel map $\rho' \in L^p(Y, \nu)$ can be approximated in $L^p(Y, \nu)$ by lower semicontinuous functions $g \geq \rho'$ to see that (4) implies (5).

Let now $\Gamma_0 = \{\gamma \in \Gamma | \gamma$ is locally rectifiable $\}$. Then $M_p(\Gamma) = M_p(\Gamma_0)$, $\Gamma_n \subset \Gamma_{n+1}$ for every $n \in \mathbb{N}$ and $\Gamma_0 = \bigcup_{n=1}^{\infty} \Gamma_n$, hence $M_p(\Gamma_n) \nearrow M_p(\Gamma_0) = M_p(\Gamma)$. Letting n tends to infinite in (5), we obtain that $M_p(\Gamma) \leq K \cdot N(f) \cdot M_p(f(\Gamma))$.

Remark 1. The condition we really used in the proof of Theorem 1 was "there exists $H \geq 1$ such that $h_5(x, f) \leq H$ for every $x \in X$", which is a priori a weaker requirment than the condition "there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in X$". In fact, if $\alpha > 1$ is arbitrary, using the covering theorem from [BK], Lemma 3.2 instead of the basic $\frac{1}{5}$ covering theorem, we see that we can replace in Theorem 1 the condition " there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in X$" by the condition "there exists $\alpha > 1$ and $H \geq 1$ such that $h_\alpha(x, f) \leq H$ for every $x \in X$", and the conclusion of Theorem 1 will be the same. However, the constant K from Theorem 1, depending on C_0, C_1, c, p, H will be modified. We obtain in this way a result related to Theorem 1.3 from [BK].

We can also see that if the modular inequality $M_p(\Gamma) \leq K \cdot N(f) \cdot M_p(f(\Gamma))$ holds for every path family Γ in X, then there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in X$.

Proof of Theorem 2. Let $x \in D, \alpha > 0$ and $U \in \mathcal{V}(x)$ be such that $\bar{U}$ is compact, $\bar{U} \subset B(x, \alpha) \subset D$ and $\bar{U} \cap f^{-1}(f(x)) = \{x\}$. Then $f(x) \notin f(\partial U)$ and let $\rho' = d(f(x), f(\partial U)) > 0$. Then $f(\partial U)$ is compact, hence is a closed set, and since f is continuous , we see that $f^{-1}(f(\partial U))$ is a closed set and $\rho = d(x, f^{-1}(f(\partial U))) > 0$.

Let $g = f|U \setminus f^{-1}(f(\partial U)) : U \setminus f^{-1}(f(\partial U)) \to f(U) \setminus f(\partial U)$. Then g is an open, discrete and proper map, and using Theorem 2 from [Cr1] (which extends some results of S. Stoilow [St], page 109 and E.E. Floyd [F]), we see that for every path $p : [0,1] \to f(U) \setminus f(\partial U)$ and every $\alpha \in U \setminus f^{-1}(f(\partial U))$ with $f(\alpha) = p(0)$, we can find $q : [0,1] \to U \setminus f^{-1}(f(\partial U))$ a path such that $q(0) = \alpha$ and $p = f \circ q$.

Let $0 < \rho_2' < \rho_1' < \frac{\rho'}{c}$. Then $f^{-1}(S(f(x), \rho_k'))$ is a closed set and $\rho_k = d(x, f^{-1}(S(f(x), \rho_k'))) > 0$ for $k = 1, 2$. We show that $B(f(x), \rho_1') \subset f(U)$.

Indeed, we see from Lemma 2 that $\partial f(U) \subset f(\partial U)$. Let $y \in B(f(x), \rho_1')$ and Q connected such that $f(x) \in Q, y \in Q$ and $Q \subset B(f(x), c\rho_1')$. Since $B(f(x), c\rho_1') \cap f(\partial U) = \emptyset$, we see that $B(f(x), c\rho_1') \cap \partial f(U) = \emptyset$, hence $Q \cap \partial f(U) = \emptyset$. Since Q is connected, $Q \cap f(U)$ is open and nonempty in Q and $Q = (Q \cap f(U)) \cup (Q \cap Cf(\bar{U}))$, we see that $Q = Q \cap f(U)$, hence $Q \subset f(U)$. It results that $y \in f(U)$, and since y was choosed arbitrary in $B(f(x), \rho_1')$, we proved that $B(f(x), \rho_1') \subset f(U)$.

We take $0 < r < \frac{\rho_2}{2}$ such that $\overline{B}(x, r) \subset U$ and $cL(x, f, r) < \rho_2'$. Since the map $y \to d(f(x), f(y))$ is continuous and $S(x, r)$ is compact, we can take

$a_r, b_r \in S(x,r)$ such that $L(x,f,r) = d(f(x), f(a_r)).l(x,f,r) = d(f(x), f(b_r))$.

Since X is a Loewner space, there exists $\Phi : (0,\infty) \to (0,\infty)$ such that $M_p(\Delta(M,N)) \geq \Phi(t)$ for every nondegenerate continua M and N in X with $\frac{d(M,N)}{\min\{d(M),d(N)\}} \leq t, t \in (0,\infty)$. Let $\lambda > 1$ and suppose that $L(x,f,r) > 2c^2\lambda l(x,f,r)$. Since Y is $c-LLC$, we can find a path $p : [0,1] \to B(f(x), cl(x,f,r))$ such that $p(0) = f(b_r), p(1) = f(x)$. Then $B(f(x), cl(x,f,r)) \subset Img$, hence we can find a path $q : [0,1] \to U \setminus f^{-1}(f(\partial U))$ such that $q(0) = b_r$ and $f \circ q = p$. Then $f(q(1)) = p(1) = f(x)$, hence $q(1) \in \bar{U} \cap f^{-1}(f(x)) = \{x\}$, hence $q(1) = x$. Let $E' = Imp, E = Imq$. Then $d(E) \geq d(x,b_r) \geq r$.

Let now $d \in B(f(x), \rho_1') \setminus \overline{B}(f(x), \rho_2')$. Since Y is $c - LLC$ we can find a path $p_2 : [0,1] \to Y \setminus \overline{B}(f(x), \frac{L(x,f,r)}{\lambda c})$ such that $p_2(0) = f(a_r), p_2(1) = d$. Let $A = \{t \in [0,1], p_2(t) \in S(f(x), \rho_2')\}$. Using Lemma 2, we see that $A \neq \emptyset$ and let $t_1 = \inf A$ and $p_1 : [0,t_1] \to B(f(x), \rho_1'), p_1 = p_2|[0,t_1]$. We can find $q_1 : [0,t_1] \to U \setminus f^{-1}(f(\partial U))$ a path such that $q_1(0) = a_r, f \circ q_1 = p_1$. Since $Imp_1 \cap S(f(x), \rho_2') \neq \emptyset$, we see that $Imq_1 \cap f^{-1}(S(f(x), \rho_2')) \neq \emptyset$, and using Lemma 2, we see that $Imq_1 \cap S(x, \rho_2) \neq \emptyset$. Let $F' = Imp_1, F = Imq_1$. Then $E, F \subset B(x,\alpha), d(F) \geq r, d(E,F) \leq r$, hence $\frac{d(E,F)}{\min\{d(E),d(F)\}} \leq \frac{r}{r} = 1$. It results that $M_p(\Delta(E,F)) \geq \Phi(1) > 0$. Let $k > 1$ and $\Gamma = \Delta(E,F,B(x,k\alpha))$ and $\Gamma_1 = \Delta(E,F) \setminus \Gamma$. If $\gamma \in \Gamma_1$, there exists γ^* a subpath of γ such that $\gamma^* \in \Delta(\overline{B}(x,\alpha), X \setminus B(x,k\alpha))$, hence if $\Gamma_2 = \{\gamma^* | \gamma \in \Gamma_1\}$, we use Lemma 3.14 from [HK2] to see that $M_p(\Gamma_1) \leq M_p(\Gamma_2) \leq Q(p,C_1) \cdot [logk]^{1-p} < \frac{\Phi(1)}{2}$ if we take $k \in \mathbb{N}$ great enough. Here $Q(p,C_1)$ is the constanat from Lemma 3.14 [HK2]. We obtain that $M_p(\Gamma) + M_p(\Gamma_1) \geq M_p(\Delta(E,F)) \geq \Phi(1)$, hence $M_p(\Gamma) \geq \frac{\Phi(1)}{2}$.

Let $\Gamma' = f(\Gamma)$. Then $\Gamma' \subset \Delta(\overline{B}(f(x), cl(x,f,r), Y \setminus \overline{B}(f(x), \frac{L(x,f,r)}{c\lambda}))$ and since $2cl(x,f,r) < \frac{L(x,f,r)}{\lambda c}$, we use Lemma 3.14 from [HK2] to see that $M_p(\Delta(\overline{B}(f(x), cl(x,f,r), Y \setminus \overline{B}(f(x), \frac{L(x,f,r)}{\lambda c})) \leq Q(p,C_1)(\log(\frac{L(x,f,r)}{\lambda c^2 l(x,f,r)}))^{1-p}$. Then $\frac{\Phi(1)}{2} \leq M_p(\Gamma) \leq K \cdot N(f,D) \cdot M_p(\Gamma') \leq K \cdot N(f,D) \cdot M_p(\Delta(\overline{B}(f(x), cl(x, f,r)), Y \setminus \overline{B}(f(x), \frac{L(x,f,r)}{c\lambda})) \leq K \cdot N(f,D) \cdot Q(p,C_1) \cdot (\log(\frac{L(x,f,r)}{\lambda c^2 l(x,f,r)}))^{1-p}$ for every $0 < r \leq \frac{\rho_2}{2}$. Let $H_1 = c^2 \cdot exp(\frac{2K \cdot N(f,D) \cdot Q(p,C_1)}{\Phi(1)}))^{\frac{1}{p-1}}$.

We proved that if $2\lambda c^2 l(x,f,r) < L(x,f,r)$ and $0 < r < \frac{\rho_2}{2}$, then $\frac{L(x,f,r)}{l(x,f,r)} \leq \lambda \cdot H_1$, hence $\frac{L(x,f,r)}{l(x,f,r)} \leq \max\{\lambda H_1, 2\lambda c^2\}$ for every $0 < r \leq \frac{\rho_2}{2}$. We take $H = \max\{2c^2, H_1\}$ and keeping $0 < r \leq \frac{\rho_2}{2}$ fixed and letting λ tends to 1, we see that $\frac{L(x,f,r)}{l(x,f,r)} \leq H$ for every $0 < r \leq \frac{\rho_2}{2}$. We therefore proved that $H(x,f) \leq H$ for every $x \in D$.

Proof of Theorem 3. Suppose that we cannot find $H \geq 1$ such that $H(x,f) \leq H$ for every $x \in X$. Then there exists $x_m \in X$ such that $H(x_m,f) \geq C = c^3 exp(Q(p,C_1)^{\frac{1}{p-1}} \cdot 2^{\frac{m}{p-1}})$ (where $Q(p,C_1)$ is the constant from [HK2] Lemma 3.14) for every $m \in \mathbb{N}$. Let $\Phi : (0,\infty) \to (0,\infty)$ be such that $M_p(\Delta(E,F)) \geq \Phi(t)$ for every nondegenerate continua M and N in X with $\frac{d(M,N)}{\min\{d(M),d(N)\}} \leq t, t \in (0,\infty)$. As in Theorem 2, we can find $0 < r_m < \delta_m$ such that $\overline{B}(x_m,\delta_m) \cap \overline{B}(x_p,\delta_p) = \emptyset, p \neq m, p,m \in \mathbb{N}$ and continua E_m, F_m

in X, E'_m, F'_m in Y such that $d(E_m) \geq r_m, d(F_m) \geq r_m, d(E_m, F_m) \leq r_m, x_m \in E_m$, there exists $a_m \in S(x_m, r_m), b_m \in S(x_m, r_m)$ such that $a_m \in E_m, b_m \in F_m, F_m \cap S(x_m, 2r_m) \neq \emptyset, E_m, F_m \subset B(x_m, \delta_m), f(x_m) \in E'_m, f(E_m) = E'_m, f(F_m) = F'_m, E'_m \in B(f(x_m), cl(x_m, f, r_m)), F'_m \in Y \setminus \overline{B}(f(x_m), \frac{L(x_m, f, r_m)}{c^2}))$, and $M_p(\Delta(E_m, F_m, B(x_m, \delta_m)) \geq \frac{\Phi(1)}{2}$ and $L(x_m, f, r_m) \geq Cl(x_m, f, r_m)$ for $m \in \mathbf{N}$. Let $\Gamma_m = \Delta(E_m, F_m, B(x_m, \delta_m))$ and $\Gamma'_m = f(\Gamma_m), m \in \mathbf{N}$. Then $\frac{\Phi(1)}{2} \leq M_p(\Gamma_m)$ and $M_p(\Gamma'_m) \leq Q(p, C_1) \cdot (\log(\frac{L(x_m, f, r_m)}{c^3 l(x_m, f, r_m)}))^{1-p} \leq \frac{1}{2^m}$ for every $m \in \mathbf{N}$.

Let $m_0 \in \mathbf{N}$ be such that $\sum_{m \geq m_0} \frac{1}{2^m} < \delta$ and $\Gamma = \bigcup_{m=m_0}^{\infty} \Gamma_m, \Gamma' = \bigcup_{m=m_0}^{\infty} \Gamma'_m$. Since the path families Γ_m are separate, we have that $M_p(\Gamma) = \sum_{m=m_0}^{\infty} M_p(\Gamma_m) = \infty$ and $M_p(\Gamma') \leq \sum_{m=m_0}^{\infty} M_p(\Gamma'_m) \leq \sum_{m=m_0}^{\infty} \frac{1}{2^m} < \delta$, which contradicts the hypothesis. It results that there exists $H \geq 1$ such that $H(x, f) \leq H$ for every $x \in X$.

Proof of Theorem 5. Suppose that we can find $n+1$ distinct points $x_1, ..., x_{n+1}$ such that $f(x_k) = f(x_1)$ and $h(x_k, f) > c^2 \lambda, k = 1, ..., n+1$. Let $\mu > 1$. Then we can find $\rho_0 > 0$ such that $\frac{L(x_k, f, r)}{l(x_k, f, r)} > c^2 \lambda \mu$ for $0 < r < \rho_0, k = 1, ..., n+1$. Let $\Phi : (0, \infty) \to (0, \infty)$ be such that $M_p(\Delta(M, N)) \geq \Phi(t)$ for every nondegenerate continua M and N from X with $\frac{d(M,N)}{\min\{d(M), d(N)\}} \leq t, t \in (0, \infty)$, and let $Q(p, C_1)$ be the constant from [HK2], Lemma 3.14. As in Theorem 2 and Theorem 3, we can find $0 < r_m < \delta_m < \rho_0$ such that $\overline{B}(x_m, \delta_m) \cap \overline{B}(x_p, \delta_p) = \emptyset, m \neq p, m, p \in \{1, .., n+1\}$ and continua E_{mr}, F_{mr} in X, E'_{mr}, F'_{mr} in Y such that $d(E_{mr}) \geq r, d(F_{mr}) \geq r, d(E_{mr}, F_{mr}) \leq r, E_{mr} \cup F_{mr} \subset B(x_m, \delta_m), F_{mr} \cap S(x_m, 2r) \neq \emptyset$, there exists points $a_{mr} \in S(x_m, r), b_{mr} \in S(x_m, r)$ such that $a_{mr} \in E_{mr}, b_{mr} \in F_{mr}, x_{mr} \in E_{mr}, f(E_{mr}) = E'_{mr}, f(F_{mr}) = F'_{mr}, E'_{mr} \subset B(f(x_m), cl(x_m, f, r)), F'_{mr} \subset Y \setminus \overline{B}(f(x_m), \frac{L(x_m, f, r)}{\mu c}), M_p(\Delta(E_{mr}, F_{mr}, B(x_m, \delta_m)) \geq \frac{\Phi(1)}{2}$ for $0 < r \leq r_m, m = 1, ..., n+1$.

Since the maps $r \to L(x_m, f, r)$ are continuous for $m = 1, ..., n+1$, we can suppose that we can find $0 < \rho$ and $0 < \rho_m < r_m$ such that $L(x_m, f, \rho_m) = L(x_1, f, \rho_1) = \rho$ for $m = 1, ..., n+1$. Let $\Gamma_m = \Delta(E_{m\rho_m}, F_{m\rho_m}, B(x_m, \delta_m))$ and $\Gamma'_m = f(\Gamma_m), m = 1, ..., n+1$. We see that $M_p(\Gamma_m) \geq \frac{\Phi(1)}{2}$ and $\Gamma'_m \subset \Delta(\overline{B}(f(x_m), c \cdot l(x_m, f, \rho_m)), Y \setminus \overline{B}(f(x_m), \frac{L(x_m, f, \rho_m)}{\mu c}) \subset \Delta(\overline{B}(f(x_m), \frac{\rho}{c \lambda \mu}), Y \setminus B(f(x_m), \frac{\rho}{\mu c}))$ for $m = 1, ..., n+1$. Let $\Gamma = \bigcup_{m=1}^{n+1} \Gamma_m, \Gamma' = f(\Gamma)$. Since the families $\Gamma_1, ..., \Gamma_{n+1}$ are separate, we see that $\frac{n+1}{2} \cdot \Phi(1) \leq \sum_{m=1}^{n+1} M_p(\Gamma_m) = M_p(\Gamma)$ and we have that $\frac{(n+1)}{2} \Phi(1) \leq M_p(\Gamma) \leq K \cdot M_p(\Gamma') \leq K \cdot Q(p, C_1) \cdot (\log(\frac{\rho}{\mu c} / \frac{\rho}{\lambda \mu c}))^{1-p} = K \cdot Q(p, C_1) \cdot (\log \lambda)^{1-p}$.

We obtain that $n+1 \leq \frac{2K \cdot Q(p, C_1)}{\Phi(1) \cdot (\log \lambda)^{p-1}}$, which represents a contradiction. The theorem is now proved.

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