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# MAPPINGS WITH FINITE DISTORTION AND ARBITRARY JACOBIAN SIGN

MIHAI CRISTEA

Abstract. We consider some classes of mappings $f : D \to I\!R^n$ which are $ACL^n$ and $\|f'(x)\|^n \le K \cdot |J_f(x)|$ a.e. Such mappings generalize quasiregular mappings, but are not necessary open, and the Jacobian has not necessary a constant sign and satisfy the $K_0(f)$ inequality. We also consider non open and nonsingular mappings $f : D \to I\!R^n$ with bounded dilatation $K_\alpha$ and we show that such mappings satisfy a strong ACL condition, extending in this way some results from [9].

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## 1. INTRODUCTION.

In 1966, Yu.G.Reshetnyak introduced in [20] the class of continuous mappings $f \in W^{1,n}_{\text{loc}}(D, I\!R^n)$ for which $\|f'(x)\|^n \le K \cdot J_f(x)$ a.e. for some $1 \le K < \infty$ (where $D \subset I\!R^n$ is a domain, $f'(x)$ is the distributional derivative of $f$ at $x$, $J_f(x) = \det f'(x)$ and $W^{1,p}_{\text{loc}}(D, I\!R^n)$ denotes the Sobolev space of all functions $f : D \to I\!R^n$ which are locally in $L^p$, together with it's distributional derivatives). Reshetnyak called them mappings of bounded distortion, and proved that such mappings are a.e. diœerentiable, satisfy condition $(N)$ and are either constant on $D$, or are open, discrete on $D$. They are also called quasiregular mappings, and if $f$ is an embedding, they are called quasiconformal mappings. A lot of properties of analytic functions remain true for this larger class of mappings (see the monographs[19] and [21]).

Recently, were considered some larger class of mappings $f : D \to I\!R^n$, called mappings with ønite dilatation (distortion). They are mappings in $W^{1,n}_{\text{loc}}(D, I\!R^n)\left(W^{1,1}_{\text{loc}}(D, I\!R^n)\right)$ such that there exists a measurable map $K(\cdot, f) : D \to [0, \infty]$, ønite a.e. and such that $\|f'(x)\|^n \le K(x, f) \cdot J_f(x)$ a.e. Mappings with ønite dilatation are a.e. diœerentiable and satisfy condition $(N)$, and if $f$ is not constant and $K(\cdot, f)$ is in $L^p$ for some $p > n-1$, then $f$ is also open, discrete (see [5, 6, 10, 11, 12, 14, 15]), and of course, $J_f(x) \ge 0$ a.e.

An important tool in studying quasiregular mappings is the modulus of a path family $\Gamma$, i.e. $M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{I\!R^n} \rho^n(x)\, dx$, where $F(\Gamma) = $–$\rho : \overline{I\!R}^n \to [0, \infty] \mid$ Borel maps such that $\int_\gamma \rho ds \ge 1$ for every $\gamma \in \Gamma$". If $f : D \to I\!R^n$ is $K$-quasiregular, $A \in \mathcal{B}(D)$ is such that $N(f, A) < \infty$, then it is valid the so called $K_0(f)$ inequality, i.e. $M(\Gamma) \le K \cdot N(f, A) \cdot M(f(\Gamma))$ for every path family $\Gamma$ from $A$.

Here $N(y,f,A) = \text{Card } f^{-1}(y) \cap A$ and $N(f,A) = \sup_{y\in \mathbb{R}^n} N(y,f,A)$. This modular inequality is essential in studying for instance the boundary behaviour of quasiregular mappings, see [25].

If $D \subset \mathbb{R}^n$ is open, $0 < \alpha \leq 1$, $f \in C(D,\mathbb{R}^n)$ is discrete and $r > 0$ is such that $\overline{B}(x,r) \subset D$, we can consider $L(x,f,r) = \sup_{y\in S(x,r)} \|f(y) - f(x)\|$, $\ell(x,f,r) = \inf_{y\in S(x,r)} \|f(y) - f(x)\|$, $H_\alpha(x,f) = \limsup_{r\to 0} \frac{L(x,f,\alpha r)}{\ell(x,f,r)}$. $H_1(x,f) = H(x,f)$ is the usual dilatation of $f$ at $x$.

It is known (see [16] Th.4.5 and Th.4.13) that if $f \in C(D,\mathbb{R}^n)$ is open, discrete and $N(f,D) < \infty$, then $f$ is quasiregular if and only if there exists $H \geq 1$ such that $H(x,f) \leq H$ for every $x \in D$.

If $D \subset \mathbb{R}^n$ is open, $f: D \to \mathbb{R}^n$ is a map, we say that $f$ is light if $\dim f^{-1}(y) \leq 0$ for every $y \in \mathbb{R}^n$, and we say that $f$ is nonsingular if $\text{Int } f(Q) \neq \varnothing$ for every $Q \subset D$ open, $Q \neq \varnothing$. If $f$ is continuous and light, then $f$ is nonsingular, see [7], page 92. For nonsingular mappings, $0 < \alpha \leq 1$ and $x \in D$, we can define $K_\alpha(x,f) = \limsup_{r\to 0} \frac{d(f(B(x,\alpha r)))^n}{\mu_n(f(B(x,r)))}$.

Here $\mu_n$ is the Lebesgue measure from $\mathbb{R}^n$. In [1] it is proved that if $f \in C(D,\mathbb{R}^n)$ is open, discrete and $N(f,D) < \infty$, then $f$ is quasiregular if and only if $K_\alpha(x,f) \leq K$ for every $x \in D$, for some $0 < \alpha \leq 1$ and $K > 0$.

Another direction for relaxing the conditions characteristic to quasiregular mappings is considered recently in [8] and [9] for open, discrete mappings $f: D \to \mathbb{R}^n$ for which the linear dilatation $H(x,f)$ depends on the points $x \in D$. In [9] it is proved that if $D \subset \mathbb{R}^n$ is open, $f \in C(D,\mathbb{R}^n)$ is open, discrete, $E \subset D, s > \frac{n}{n-1}$, $H(x,f) < \infty$ on $D\backslash E$, $H(\cdot f) \in L^s_{\text{loc}}(D)$ and $E$ has $\sigma$-finite $n-1$ dimensional Hausdorff measure, then $f$ is ACL, a.e. differentiable and $f \in W^{1,p}_{\text{loc}}(D,\mathbb{R}^n)$, with $p = \frac{ns}{n+s-1}$. We denoted here by $m_p$ the $p$ Hausdorff measure in $\mathbb{R}^n$ and we say that $F \subset \mathbb{R}^n$ is of $\sigma$-finite $p$ Hausdorff measure if $F = \bigcup_{i=1}^{\infty} F_i$, with $m_p(F_i) < \infty$ for every $i \in \mathbb{N}$.

The common point of all this research directions is that all the maps considered are open, discrete, with constant positive Jacobian sign, and generalizes quasiregular mappings. Let us consider the following example.

Example 1. Let $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z$ if $\text{Im } z \geq 0$, $f(z) = \bar{z}$ if $\text{Im } z < 0$. Then $f \in \mathbf{C}(\mathbb{C},\mathbb{C})$, $f$ is a $C^\infty$ map on $\mathbb{C} \setminus d$, where $d = \{z \in \mathbb{C} \mid \text{Im } z = 0\}$, $\|f'(z)\|^2 = |J_f(z)|$ for every $z \in \mathbb{C} \setminus d, J_f(z) = 1$ if $\text{Im } z > 0, J_f(z) = -1$ if $\text{Im } z < 0, H(z,f) = 1$ for every $z \in \mathbb{C}, K_\alpha(z,f) \leq \frac{2^{n+1}\cdot\alpha^n}{V_n}$ for every $z \in \mathbb{C}, f$ is a discrete map with $N(f,\mathbb{C}) \leq 2$ and $f$ is not open. Here $V_n$ denotes the volume of the unit ball from $\mathbb{R}^n$.

This example enable us to consider some classes of mappings which also generalizes quasiregular mappings, but are not necessary open, and the Jacobian has not necessary a constant sign.

We consider first the class of mappings $f: D \to \mathbb{R}^n, D \subset \mathbb{R}^n$ open, such that $f \in C(D,\mathbb{R}^n) \cap W^{1,n}_{\text{loc}}(D,\mathbb{R}^n)$ and $\|f'(x)\|^n \leq K \cdot |J_f(x)|$ a.e. for some $K \geq 1$, and we prove in Theorem 1 that the $K_0(f)$ inequality holds for such mappings.

Theorem 1. Let $D \subset I\!R^n$ be open, $f \in C(D, I\!R^n) \cap W^{1,n}_{\text{loc}}(D, I\!R^n)$ be such that there exists $K \geq 1$ such that $\|f'(x)\|^n \leq K \cdot |J_f(x)|$ a.e. and there exists $A \in \mathcal{B}(D)$ and $B \subset I\!R^n$ with $\mu_n(B) = 0$ such that $q = N(f, A, B) < \infty$. Then $M(\Gamma) \leq K \cdot q \cdot M(f(\Gamma))$ for every path family $\Gamma$ from $A$.

Here $N(f, A, B) = \sup\limits_{y \in I\!R^n \setminus B} \text{Card}\, f^{-1}(y) \cap A$.

We consider maps $f \in C(D, I\!R^n) \cap W^{1,1}_{\text{loc}}(D, I\!R^n)$, $D \subset I\!R^n$ open, for which there exists a measurable function $K(\cdot, f) : D \to [0, \infty]$ ønite a.e. and such that $\|f'(x)\|^n \leq K(x, f) \cdot |J_f(x)|$ a.e., and we call them mappings with ønite distortion and arbitrary Jacobian sign. Of course, this class of mappings contains the mappings with ønite distortion mentioned before. If $K(x, f) \leq K$ a.e., we say that $f$ is a map with bounded distortion and arbitrary Jacobian sign.

We say that a closed set $E \subset D$ with $\mu_n(E) = 0$ is removable for the $K$-quasiregular map $f$, if $f \in C(D, I\!R^n)$, $f$ is $K$-quasiregular on $D \setminus E$, and $f$ extends to a $K$-quasiregular map on $D$. A classical eliminability result of J.Visl [23] says that if $m_{n-1}(E) = 0$ and $f$ is $K$-quasiconformal on $D \setminus E$, then $E$ is removable for $f$. M.Vuorinen [25] generalized J.Visl's result, showing that if $m_{n-1}(E) = 0, f$ is $K$-quasiregular on $D \setminus E$ and $N(f, D \setminus E) < \infty$, then $E$ is removable for $f$. We give some eliminability results for the maps with ønite distortion and arbitrary Jacobian sign, generalizing M.Vuorinen's result from [25] and an earlier result of M.Cristea [2]. Of course, the ørst question is what it means removability for mappings with ønite distortion and arbitrary Jacobian sign. We say that a closed set $E \subset D$ with $\mu_n(E) = 0$ is removable for $f$ if $f \in C(D, I\!R^n)$, $f$ is a map with ønite distortion and arbitrary Jacobian sign on $D \setminus E$, and $f$ extends to a map with ønite distortion and arbitrary Jacobian sign on $D$. We prove:

Theorem 2. Let $D \subset I\!R^n$ be open, $n \geq 2, s \geq \frac{1}{n-1}, F \subset D$ closed in $D, f \in C(D, I\!R^n) \cap W^{1,1}_{\text{loc}}(D \setminus F, I\!R^n)$ such that there exists a measurable map $K(\cdot, f) : D \to [0, \infty]$ ønite a.e. and in $L^s_{\text{loc}}(D)$ such that $\|f'(x)\|^n \leq K(x, f) \cdot |J_f(x)|$ a.e. and there exists $B \subset I\!R^n$ with $\mu_n(B) = 0$ such that for every $x \in D$, there exists $U_x \in \mathcal{V}(x)$ with $N(f, U_x, B) < \infty$. Suppose that either $F$ is of $\sigma$-ønite $n-1$ dimensional Hausdorœ measure, or that $\mu_n(F) = 0$ and $m_1(f(F)) = 0$,and let $p = \frac{ns}{s+1}$. Then $f \in W^{1,p}_{\text{loc}}(D, I\!R^n)$, hence $E$ is removable for $f$ and $f$ is ACL$^{\text{p}}$ on $D$.

Another way to extend the class of quasiregular mappings is the method used in [8] and [9], i.e. to consider mappings $f : D \to I\!R^n$ with $H(x, f)$ or $K_\alpha(x, f)$ depending on $x$ and ønite a.e. The minimal assumption for the existence of $K_\alpha(x, f)$ is that $\mu_n(f(B(x, r))) > 0$ for small $r$, which is satisøed if $f$ is nonsingular, and the minimal assumption for the existence of $H_\alpha(x, f)$ is that $f$ is a discrete map. If $D \subset I\!R^n$ is open, $x \in D, 0 < \alpha \leq 1, f \in C(D, I\!R^n)$ is open, discrete, then $B(f(x), \ell(x, f, r)) \subset f(B(x, r))$ for small $r$, and we have

$$(*) \qquad K_\alpha(x, f) \leq \frac{2^n}{V_n} \cdot H_\alpha(x, f)^n .$$

As we can see from Example 1, if $f$ is not open, it is possible that $B(f(x), \rho) \not\subset fB(x, r)$ for every $\rho > 0$, hence $(*)$ may not hold. We can overcome this diŒculty for non open, but nonsingular maps. Indeed, if $D \subset I\!R^n$ is open, $f \in C(D, I\!R^n)$ is nonsingular, $x \in D, \bar{B}(x, r) \subset D$, we put $\widetilde{L}(x, f, r) = \inf\{\rho > 0 | f(B(x, r))$

is included in some ball of radius $\rho\}$, $\widetilde{\ell}(x,f,r) = \sup\{\rho > 0 | f(B(x,r))$ contains some ball of radius $\rho\}$, and for $0 < \alpha \leq 1$, we set $\widetilde{H}_\alpha(x,f) = \limsup_{r \to 0} \frac{\widetilde{L}(x,f,\alpha r)}{\widetilde{\ell}(x,f,r)}$, and we easily see that

$$(**) \qquad K_\alpha(x,f) \leq \frac{2^n}{V_n} \cdot \widetilde{H}_\alpha(x,f)^n .$$

We show in Theorem 3 that mappings with locally ønite multiplicity and with $K_\alpha(x,f) < \infty$ a.e., or with $\widetilde{H}_\alpha(x,f) < \infty$ a.e. are a.e. diœerentiable.

Theorem 3. Let $D \subset I\!R^n$ be open, $0 < \alpha \leq 1, f \in C(D, I\!R^n)$ be nonsingular such that there exists $B \subset I\!R^n$ with $\mu_n(B) = 0$ and such that for every $x \in D$, there exists $U_x \in \mathcal{V}(x)$ with $N(f, U_x, B) < \infty$. Suppose that one of the following conditions are satisøed:

a) $K_\alpha(x,f) < \infty$ a.e.

b) $f$ is open, discrete and $H_\alpha(x,f) < \infty$ a.e.

c) $\widetilde{H}_\alpha(x,f) < \infty$ a.e.

Then $f$ is a.e. diœerentiable on $D$.

We shall prove that mappings with locally ønite multiplicity and with integrable dilatation $K_\alpha(\cdot,f)$ or $\widetilde{H}_\alpha(\cdot,f)$, satisfy a strong ACL condition, generalizing in this way some results from [9], established for open, discrete mappins in $I\!R^n$.

If $Q = \prod_{i=1}^{m}[a_i,b_i] \subset I\!R^m$ and $f \in C(Q, I\!R^n)$, we say that $f$ is $m$-ACH ($m$-absolute continuous) if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if $\Delta_1, \ldots, \Delta_k$ are closed, disjoint intervals in $Q$ with $\sum_{i=1}^{k} m_m(\Delta_i) \leq \delta_\varepsilon$, then $\sum_{i=1}^{k} m_m(f(\Delta_i)) \leq \varepsilon$. If $m = 1$, it is obviously that a $1-$ AC map is absolutely continuous.

Let $n \geq 2, k \in \{1,\ldots,n-1\}$ and $I_{n-k} = \{\alpha = (\alpha_1,\ldots,\alpha_{n-k}) | \alpha_i \in \{1,\ldots,n\}$, $\alpha_i \neq \alpha_j$ for $i \neq j$, $i,j \in \{1,\ldots,n-k\}\}$. If $I \in I_{n-k}, I = (\alpha_1,\ldots,\alpha_{n-k})$ and $x = (x_1,\ldots,x_n) \in I\!R^n$, we denote by $x_I = (x_{\alpha_1},\ldots,x_{\alpha_{n-k}})$ and we put $\Pi_I : I\!R^n \to I\!R^n$ the projection given by $\Pi_I(x) = (x_I, 0)$ for $x \in I\!R^n$ and we let $q(I) = \{\alpha_1, \ldots \alpha_{n-k}\}$ and $r(I) = \{1, \ldots, n\} \backslash q(I)$. If $D \subset I\!R^n$ is open, $f \in C(D, I\!R^n)$ and $Q = \prod_{i=1}^{n}[a_i,b_i]$ is such that $\bar{Q} \subset D$, we let for $I \in I_{n-k}$ and $x \in I\!R^n$ such that $H = \Pi_I^{-1}(\Pi_I(x)) \cap Q \neq \oslash$, the map $f^Q_{x,I} : H \to I\!R^n$ given by $f^Q_{x,I} = f|H$.

If $D \cap I\!R^n$ is open, $f \in C(D, I\!R^n), k \in \{1,\ldots,n-1\}$, we say that $f$ is $k$-ACH (absolute continuous on $k$ hyperplanes) if for every interval $Q = \prod_{i=1}^{n}[a_i,b_i]$ with $\bar{Q} \subset D$ with the sydes parallel to coordinate axes and every $I \in I_{n-k}$, it results that $m_{n-k}(E_I) = 0$, where $E_I = -x \in \Pi_I(Q) |$ the map $f^Q_{x,I} : \Pi_I^{-1}(x) \cap Q \to I\!R^n$ is not $k-$AC" for $I \in I_{n-k}$. Of course, a $1-$ ACH map is ACL, i.e. is continuous and for every interval $Q = \prod_{i=1}^{n}[a_i,b_i]$ with $\bar{Q} \subset D$, it results that the maps $f^Q_{x,I}$ are absolutely continuous on $\Pi_I^{-1}(x) \cap Q$ for every $I \in I_{n-1}$ and for a.e. $x \in \Pi_I(Q)$. ACL mappings have a.e. classical partial derivatives, and if this classical partial derivatives are locally in $L^p$, with $p \geq 1$, we say that $f$ is ACL$^p$ on $D$. In [21], Prop.1.2, page 6, it is proved that $f$ is ACL$^p$ on $D$ if and only $f \in W^{1,p}_{\text{loc}}(D, I\!R^n) \cap C(D, I\!R^n)$. We proved:

Theorem 4. Let $D \subset I\!R^n$ be open, $n \geq 2, m \in \{1, \ldots, n-1\}, f \in C(D, I\!R^n)$ be nonsingular such that there exits $B \subset I\!R^n$ with $m_m(B) = 0$ and such that for every $x \in D$, there exists $U_x \in \mathcal{V}(x)$ with $N(f, U_x, B) < \infty$ and let $0 < \alpha \leq 1$ and $E \subset D$ of $\sigma$-finite $n-m$ dimensional Hausdorff measure. Suppose that one of the following condition is satisfied:

a) $K_\alpha(x, f) < \infty$ on $D \backslash E$ and $K_\alpha \in L^s_{\text{loc}}(D)$, with $s > \frac{m}{n-m}$.

b) $f$ is open, discrete, $H_\alpha(x, f) < \infty$ on $D \backslash E$ and $H_\alpha \in L^s_{\text{loc}}(D)$, with $s > \frac{mn}{n-m}$.

c) $\widetilde{H}_\alpha(x, f) < \infty$ on $D \backslash E$ and $\widetilde{H}_\alpha \in L^s_{\text{loc}}(D)$, with $s > \frac{mn}{n-m}$.

Then $f$ is $m$-ACH.

We also prove:

Theorem 5. Let $D \subset I\!R^n$ be open, $n \geq 2$, $f \in C(D, I\!R^n)$ be nonsingular such that there exists $B \subset I\!R^n$ with $\mu_n(B) = 0$ and such that for every $x \in D$ there exists $U_x \in \mathcal{V}(x)$ with $N(f, U_x, B) < \infty$, and let $0 < \alpha \leq 1, s > 0$ and $E \subset D$ with $\mu_n(E) = 0$. Suppose that one of the following condition is satisfied:

a) $K_\alpha(x, f) < \infty$ on $D \backslash E$ and $K_\alpha(\cdot, f) \in L^s_{\text{loc}}(D)$.

b) $f$ is open, discrete, $H_\alpha(x, f) < \infty$ on $D \backslash E$ and $H_\alpha(\cdot, f) \in L^s_{\text{loc}}(D)$.

c) $\widetilde{H}_\alpha(x, f) < \infty$ on $D \backslash E$ and $\widetilde{H}_\alpha(\cdot, f) \in L^s_{\text{loc}}(D)$.

Then $f$ is a.e. differentiable on $D$, and if $s = \infty$, then $f' \in L^n_{\text{loc}}(D)$.

If $s < \infty$, then $f' \in L^p_{\text{loc}}(D)$, where $p = \frac{ns}{s+1}$ in case a), $p = \frac{ns}{n+s-1}$ in case b) and $p = \frac{ns}{n+s}$ in case c). If $E$ is of $\sigma$-finite $n-1$ dimensional Hausdorff measure and $m_1(B) = 0$, then $f$ is ACL$^{\text{n}}$ if $s = \infty$, and $f$ is ACL$^{\text{p}}$ on $D$ if $s < \infty$, where $p = \frac{ns}{s+1}$ if $\frac{1}{n-1} < s < \infty$ and condition a) is satisfied, $p = \frac{ns}{n+s-1}$ if $\frac{n}{n-1} < s < \infty$ and condition b) is satisfied, and $p = \frac{ns}{n+s}$ if $\frac{n}{n-1} < s < \infty$ and condition c) is satisfied.

We prove the similar result which holds for open, discrete mappings with finite multiplicity in $I\!R^n$, i.e. the fact that mappings with locally finite multiplicity in $I\!R^n$ and with bounded $K_\alpha(\cdot, f)$ dilatation, or with bounded $\widetilde{H}_\alpha(\cdot, f)$ dilatation, are mappings with bounded distortion and arbitrary Jacobian sign.

Theorem 6. Let $D \subset I\!R^n$ be open, $n \geq 2, f \in C(D, I\!R^n)$ be nonsingular such that there exists $B \subset I\!R^n$ with $m_1(B) = 0$ and such that for every $x \in D$, there exists $U_x \in \mathcal{V}(x)$ such that $N(f, U_x, B) < \infty$, and let $0 < \alpha \leq 1, K > 0$ and $E \subset D$ of $\sigma$-finite $n-1$ dimensional Hausdorff measure. Suppose that one of the following condition is satisfied:

a) $K_\alpha(x, f) \leq K$ on $D \backslash E$.

b) $f$ is open, discrete and $H_\alpha(x, f) \leq K$ on $D \backslash E$.

c) $\widetilde{H}_\alpha(x, f) \leq K$ on $D \backslash E$.

Then $f$ is a.e. differentiable and ACL$^{\text{n}}$ and is a map with bounded distortion and arbitrary Jacobian sign and $\|f'(x)\|^n \leq \frac{V_n \cdot K}{\alpha^n} \cdot |J_f(x)|$ a.e. if condition a) is satisfied, $\|f'(x)\|^n \leq \left(\frac{K}{\alpha}\right)^{n-1} \cdot |J_f(x)|$ a.e. if condition b) is satisfied, and $\|f'(x)\|^n \leq \left(\frac{2K}{\alpha}\right)^n \cdot |J_f(x)|$ a.e. if condition c) is satisfied. If $A \in \mathcal{B}(D)$ is such that $q = N(f, A, B) < \infty$ and $\Gamma$ is a path famility from $A$, then $M(\Gamma) \leq \frac{q \cdot V_n \cdot K}{\alpha^n} \cdot M(f(\Gamma))$ if condition a) is satisfied, $M(\Gamma) \leq q \cdot \left(\frac{K}{\alpha}\right)^{n-1} \cdot M(f(\Gamma))$ if condition b) is satisfied, and $M(\Gamma) \leq q \cdot \left(\frac{2K}{\alpha}\right)^n \cdot M(f(\Gamma))$ if condition c) is satisfied.

## 2. CHANGE OF VARIABLE FORMULAE AND $q$-SUBADDITIVE FUNCTIONS

We let for $D \subset \mathbb{R}^n$ open $\mathcal{B}(D) = \{A \subset D | A \text{ is a Borel set}\}$ and $\mathcal{L}(D) = \{A \subset D | A \text{ is Lebesgue measurable}\}$. The following change of variable formula for Sobolev mappings is presented in [13], Th.6.32, page 104 for $g \equiv 1$ and the proof for arbitrary $g$ is standard.

Lemma 1. Let $D \subset \mathbb{R}^n$ be open, $f \in W^{1,1}_{\text{loc}}(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$, $A \in \mathcal{L}(D)$ and let $g : \mathbb{R}^n \to [0, \infty]$ be a Borel map. Then there exists $E \subset D$ with $\mu_n(E) = 0$ and $\int_A g(f(x)) \cdot |J_f(x)| \, dx = \int_{\mathbb{R}^n} g(y) \cdot N(y, f, A \backslash E) dy$, and if $f$ satisfies condition $(N)$, we can take $E = \emptyset$.

If $D \subset \mathbb{R}^n$ is open, $f \in C(D, \mathbb{R}^n)$, $x \in D$ and there exists $\lim\limits_{r \to 0} \frac{\mu_n(f(B(x,r)))}{\mu_n(B(x,r))}$, we denote it by $\mu'_f(x)$, and from [18], page 325-334, we see that if $f$ is differentiable in $x$, then there exists $\mu'_f(x)$ and $\mu'_f(x) = |J_f(x)|$.

If $D \subset \mathbb{R}^n$ is open, $q \geq 1$ and $\varphi : \mathcal{B}(D) \to [0, \infty]$, we say that $\varphi$ is a $q$-subadditive function if $\varphi(A) \leq \varphi(B)$ for $A, B \in \mathcal{B}(D)$, $A \subset B$, $\varphi(A) < \infty$ if $A$ is compact, $A \subset D$, and $\sum\limits_{i=1}^{k} \varphi(A_i) \leq q \cdot \varphi(A)$ for every $A_1, \ldots, A_k$ disjoint Borel sets from $A$ with $A_i \subset A$, $i = 1, \cdots, k$. We define for $x \in D$ $\underline{\varphi}'(x) = \liminf\limits_{r \to 0} \limits_{d(Q) < r} \frac{\varphi(Q)}{\mu_n(Q)}$, $\bar{\varphi}'(x) = \limsup\limits_{r \to 0} \limits_{d(Q) < r} \frac{\varphi(Q)}{\mu_n(Q)}$ where $Q$ runs through all open cubes and all open balls such that $x \in Q \subset D$. As in [18], page 204-209 (see also [16] Lemma 2.3), if $\varphi$ is a $q$-subadditive function, then $\bar{\varphi}'(x) \leq q \cdot \underline{\varphi}'(x) < \infty$ a.e., $\bar{\varphi}'$ and $\underline{\varphi}'$ are Borel functions and $\int_U \underline{\varphi}'(x) \, dx \leq q \cdot \varphi(U)$ for every $U \subset D$ open. We have

Lemma 2. Let $D \subset \mathbb{R}^n$ be open, $f \in C(D, \mathbb{R}^n)$ be a.e. differentiable such that there exists $B \subset \mathbb{R}^n$ with $\mu_n(B) = 0$ and $q = N(f, D, B) < \infty$. Then, if $G \subset\subset D$ is open, it results that $\int\limits_G |J_f(x)| \, dx < \infty$.

Proof. We define $\varphi : \mathcal{B}(D) \to [0, \infty]$ by $\varphi(K) = \mu_n(f(K))$ for $K \in \mathcal{B}(D)$. Since $f$ is continuous, it maps Borel sets into measurable sets (see [3], Th.22.13, page 69), hence $\varphi$ is well defined and takes finite values on the compact sets from $D$ and we can easily see that $\varphi$ is a $q$-subadditive function on $\mathcal{B}(D)$. We have $\int_G |J_f(x)| \, dx = \int_G \mu'_f(x) \, dx \leq \int_G \bar{\varphi}'(x) \, dx \leq q \cdot \int_G \underline{\varphi}'(x) \, dx \leq q^2 \cdot \varphi(G) =$
$= q^2 \cdot \mu_n(f(G)) < \infty$ for every $G \subset\subset D$ open.

## 3. MAPPINGS WITH FINITE DISTORTION AND ARBITRARY JACOBIAN SIGN.

The important $K_0(f)$ inequality for ACL$^n$ mappings with bounded distortion and arbitrary Jacobian sign will be proved in Theorem 1, and the proof follows the line of the classical one for quasiregular mappings (see [24], Th.5.3, page 12 and [21], Th.2.4, page 31).

If $\alpha : [a, b] \to \mathbb{R}^n$ is a rectifiable path, we denote by $s_\alpha$ it's lenght function, and by $\alpha^0 : [0, \ell(\alpha)] \to \mathbb{R}^n$ it's normal representation (see [24], page 4-5). If $D \subset \mathbb{R}^n$ is open, $x \in D, f : D \to \mathbb{R}^n$ is a map, an $n \times n$ matrix $L$ is called the approximate differential of $f$ at $x$ if the maps $f_h : B(0, 1) \to \mathbb{R}^n$, defined by $f_h(y) = \frac{f(x+hy) - f(x)}{h}$ for $y \in B(0, 1)$ and $h > 0$, converges to $L$ in measure on

$B(0,1)$. Then $L$ is unique and we denote $L = (\mathrm{app})\mathrm{d}f_x$. An $n \times n$ matrix is called a quasidiœerential of $f$ at $x$ if there exists $r_i \to 0$ such that for every $\varepsilon > 0$, there exists $i_\varepsilon \in I\!N$ such that $\sup\limits_{\|z-x\|=r_i} \|f(z) - f(x) - L(z-x)\| \le \varepsilon \cdot r_i$ for $i \ge i_\varepsilon$. We say that $f$ is weakly diœerentiable at $x$, and that $L$ is the weak diœerential of $f$ at $x$, if $L = (\mathrm{app})\mathrm{d}f_x$ and $L$ is a quasidiœerential of $f$ at $x$.

Proof of Theorem 1: We see from [21], Prop.1.2, page 6, that the distributional partial derivatives and the classical partial derivatives coincides a.e., and from [4], Th.5.21 page 129, we see that $f$ is a.e. weakly diœerentiable. We denote by $\left[\frac{\partial f_i}{\partial x_j}(x)\right]$ the distributional partial derivative of $f$ at $x$, by $\frac{\partial f_i}{\partial x_j}(x)$ the classical partial derivative of $f$ at $x$, by $f'(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right]_{i,j=1,\cdots,n}$ and by $Df(x) = \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j=1,\cdots,n}$. We see from [4], Th.5.21 page 129 that the weak derivative of $f$ at $x$ acts as the distributional derivative of $f$ at $x$, i.e. $< f'(x)(y), e_i >= \sum\limits_{j=1}^{n} \left[\frac{\partial f_i}{\partial x_j}(x)\right] \cdot y_j$ for $y \in I\!R^n$. The classical partial derivatives $\frac{\partial f_i}{\partial x_j}$ are Borel maps on their existence domain, hence the map $x \to Df(x)$ is a Borel map on it's existence domain.

We see that there exists $F \in \mathcal{B}(D)$ with $\mu_n(F) = 0$ and the map $x \to \|Df(x)\|$ is a Borel map deøned on $D \backslash F$, and there exists $M \in \mathcal{L}(D)$ such that $F \subset M$, $\mu_n(M) = 0$, $f$ is weakly diœerentiable on $D \backslash M$ and $f'(x) = Df(x)$ on $D \backslash M$.

Let now $\Gamma$ be a path family in $A, \Gamma_0 = \{\gamma \in \Gamma | \gamma$ is rectiøable and $f$ is not absolutely continuous on some subpath of $\gamma\}$, $\Gamma_1 = \{\gamma \in \Gamma | \gamma$ is rectiøable and $m_1(\{t \in [0, \ell(\gamma)] \, | \gamma^0(t) \in M\} > 0\}, \Gamma_2 = \{\gamma \in \Gamma | \gamma$ is locally rectiøable and is not rectiøable". Using Fuglede's theorem (see [24], Th.28.2, page 95), we see that $M(\Gamma_0) = 0$, from [24], Th.33.1, page 111 we see that $M(\Gamma_1) = 0$, and from [24], Th.6.9, page 19, we see that $M(\Gamma_2) = 0$. Let $\widetilde{\Gamma} = \Gamma \backslash (\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$ and we take $Df(x) = 0$ on $F \cup CD$.

Let $\rho' \in F(f(\Gamma))$ be a Borel map on $\overline{I\!R}^n$. We deøne a Borel map $\rho : \overline{I\!R}^n \to [0, \infty]$ by $\rho(x) = \rho'(f(x)) \cdot \|Df(x)\|$ if $x \in A, \rho(x) = 0$ if $x \in CA$.

Let $\alpha : [a,b] \to A, \alpha \in \widetilde{\Gamma}$. Then $\alpha$ is rectiøable and let $p = \ell(\alpha)$. We have $f \circ \alpha = f \circ \alpha^0 \circ s_\alpha$, and since $f \circ \alpha^0$ is absolutely continuous, we see that $f \circ \alpha$ is rectiøable, and let $q = \ell(f \circ \alpha) = \ell(f \circ \alpha^0), s : [0,p] \to [0,q]$ the length function of $f \circ \alpha^0$ and let $\beta = (f \circ \alpha^0)^0$. Then $s$ is absolutely continuous, $\beta = (f \circ \alpha)^0, \beta \circ s = (f \circ \alpha^0)^0 \circ s = f \circ \alpha^0$ and since $f \circ \alpha^0$ is absolutely continuous, we see that $s'(u) = \left\|(f \circ \alpha^0)'(u)\right\|$ a.e. in $[0,p]$. We have

$\int\limits_{f\circ\alpha} \rho' \mathrm{d}s = \int_0^q \rho'(\beta(t))\,\mathrm{d}t = \int_0^p \rho'(\beta(s(u))) \cdot s'(u)\,\mathrm{d}u = \int_0^p \rho'(f(\alpha^0(u))) \cdot \left\|(f \circ \alpha^0)'(u)\right\| \mathrm{d}u.$

Let now $u \in [0,p]$ be such that there exists $(f \circ \alpha^0)'(u)$, $(\alpha^0)'(u)$ and $\alpha^0(u) \notin M$, and let $\varepsilon > 0$. Since $\alpha^0$ is a normal representation, we ønd $s_j \to 0$ such that $\alpha^0(u + s_j) \neq \alpha^0(u)$ and since $\alpha^0(u) \notin M$, $f$ is weakly diœerentiable in $\alpha^0(u)$. We therefore can ønd $\delta_j \to 0$ and $j_\varepsilon \in I\!N$ such that

$$\|f(y) - f(\alpha^0(u)) - f'(\alpha^0(u))(y - \alpha^0(u))\| \le \varepsilon \cdot \|y - \alpha^0(u)\| \tag{1}$$

if $\|y-\alpha^0(u)\|=\delta_j, j\geq j_\varepsilon$.

We set $a_k=\alpha^0(u+s_k)-\alpha^0(u)$ for $k\in I\!N$ and we see that $a_k\to 0$ and $a_k\neq 0$ for $k\in I\!N$. Since $\delta_j\to 0$, we can ønd $p_\varepsilon\geq j_\varepsilon$ and $(a_{k_j})_{j\in I\!N}$ a subsequence of $(a_k)_{k\in I\!N}$ such that $\left|\|a_{k_j}\|-\delta_j\right|\leq\left|\|a_{k_{j+1}}\|-\|a_{k_j}\|\right|$ for $j\geq p_\varepsilon$. Using the continuity of the map $t\to\|\alpha^0(u+t)-\alpha^0(u)\|$, we can ønd $q_\varepsilon\geq p_\varepsilon$ and a sequence $(r_j)_{j\in I\!N}$ such that $\|\alpha^0(u+r_j)-\alpha^0(u)\|=\delta_j$ and $|s_{k_j}-r_j|\leq|s_{k_{j+1}}-s_{k_j}|$ for $j\geq q_\varepsilon$. Then $r_j\to 0$ and from (1) we see that

(2) $\|f(\alpha^0(u+r_j))-f(\alpha^0(u))\|\leq(\|f'(\alpha^0(u))\|+\varepsilon)\cdot\|\alpha^0(u+r_j)-\alpha^0(u)\|$

for $j\geq q_\varepsilon$.

It results that $\left\|(f\circ\alpha^0)'(u)\right\|=\lim_{j\to\infty}\frac{\|f(\alpha^0(u+r_j))-f(\alpha^0(u))\|}{\|\alpha^0(u+r_j)-\alpha^0(u)\|}\cdot\frac{\|\alpha^0(u+r_j)-\alpha^0(u)\|}{r_j}\leq$ (using (2)) $(\|f'(\alpha^0(u))\|+\varepsilon)\cdot\left\|(\alpha^0)'(u)\right\|\leq(\|f'(\alpha^0(u))\|+\varepsilon)$ and letting $\varepsilon$ tends to zero,we see that

(3) $$\left\|(f\circ\alpha^0)'(u)\right\|\leq\|f'(\alpha^0(u))\|\text{ for a.e. }u\in[0,p].$$

Using (3) and the fact that $f'(\alpha^0(u))=Df(\alpha^0(u))$ for a.e. $u\in[0,p]$, we see that $\int_{f\circ\alpha}\rho'ds\leq\int_0^p\rho'(f(\alpha^0(u)))\cdot\|f'(\alpha^0(u))\|du=\int_0^p\rho'(f(\alpha^0(u)))\cdot\|Df(\alpha^0(u))\|du$ $=\int_\alpha\rho'(f(x))\cdot\|Df(x)\|dx$. We proved that

(4) $$\int_{f\circ\alpha}\rho'ds\leq\int_\alpha\rho'(f(x))\cdot\|Df(x)\|ds\text{ for every }\alpha\in\widetilde{\Gamma}.$$

Using (4), we see that $1\leq\int_{f\circ\alpha}\rho'ds\leq\int_\alpha\rho'(f(x))\cdot\|Df(x)\|ds=\int_\alpha\rho ds$,

i.e. $\rho\in F(\widetilde{\Gamma})$, hence $M(\Gamma)=M(\widetilde{\Gamma})=\int_{I\!R^n}\rho^n(x)dx=\int_A\rho'^n(f(x))\cdot\|Df(x)\|^n dx=$ $\int_A\rho'^n(f(x))\cdot\|f'(x)\|^n dx\leq K\cdot\int_A\rho'^n(f(x))\cdot|J_f(x)|dx=$ (using Lemma 1) $K\cdot\int_{I\!R^n}\rho'^n(y)\cdot N(y,f,A\backslash E)dy=K\cdot\int_{I\!R^n\backslash B}\rho'^n(y)\cdot N(y,f,A\backslash E)dy$ $\leq K\cdot q\cdot\int_{I\!R^n}\rho'^n(y)dy$. Here $E\subset D$ is the set from Lemma 1 such that $\mu_n(E)=0$.

We ønally proved that $M(\Gamma)\leq K\cdot q\cdot M(f(\Gamma))$.

Remark 1.The proof of Theorem 1 is much more simpliøed if we additionally suppose that $f$ is a.e. diœerentiable on $D$.

As in [24], page 104, we can prove:

Lemma 3. Let $m\in\{1,\ldots,n-1\}, E\subset I\!R^n$ of $\sigma$-ønite $n-m$ dimensional Hausdorœ measure, $I\in I_{n-m}$ and let $\Pi_I: I\!R^n\to I\!R^n, \Pi_I(x)=\sum_{i\in q(I)}x_ie_i$. Then $E\cap\Pi_I^{-1}(y)$ is at most countable for a.e. $y\in\Pi_I(I\!R^n)=I\!R^{n-m}\times\{0\}$.

We prove now in Theorem 2 a removability result for mappings with ønite distortion and arbitrary Jacobian sign, and the proof also follows the basic ideas from the similar proof for quasiregular mappings.

Proof of Theorem 2: Let $x_0\in D$ and $Q=\prod_{i=1}^n[a_i,b_i]$ be a parallepiped with the sides parallel to coordinate axes such that $x_0\in\operatorname{Int}Q\subset\bar{Q}\subset U_{x_0}$, and let

$I \in I_{n-1}$. If $x \in \Pi_I(Q)$, we set $F_{x,I} = \{t \in [a_i, b_i], i \in r(I) \,|x + te_i \in F\}$. Let $f_{x,I} : [a_i, b_i] \to \mathbb{R}^n$ be deøned by $f_{x,I}(t) = f(x + te_i)$ for $t \in [a_i, b_i], i \in r(I)$. If $F$ is of $\sigma$-ønite $n-1$ dimensional Hausdorœ measure, we see from Lemma 3 that $F_{x,I}$ is at most countable for a.e. $x \in \Pi_I(Q)$, and this implies that $m_1(f_{x,I}(F_{x,I})) = 0$ for a.e. $x \in \Pi_I(Q)$. If $m_1(f(F)) = 0$, then automatically $m_1(f_{x,I}(F_{x,I})) = 0$ for every $x \in \Pi_I(Q)$. Since $\mu_n(F) = 0$, we apply Fubini's theorem to see that also $m_1(F_{x,I}) = 0$ for a.e. $x \in \Pi_I(Q)$. Since $f \in W^{1,1}_{\text{loc}}(D \backslash F) \cap C(D, \mathbb{R}^n)$, we see from [21], Prop.1.2, page 6 that $f_{x,I}$ is absolutely continuous on every closed interval from $[a_i, b_i] \backslash F_{x,I}, i \in r(I)$, and we obtain that

$$(1) \qquad m_1(f_{x,I}(A)) = 0 \text{ if } A \subset [a_i, b_i], i \in r(I)$$

and $m_1(A) = 0$, for a.e. $x \in \Pi_I(Q)$.

Let $E \subset D$ be such that $\mu_n(E) = 0$ be as in Lemma 1. Then $\int_Q \left\| \frac{\partial f}{\partial x_i}(x,t) \right\|^p dxdt \leq \int_Q \|f'(x,t)\|^p dxdt \leq$ (using Hlder's inequality) $\left(\int_Q K(z,f)^{\frac{p}{n-p}} dz\right)^{\frac{n-p}{n}} \cdot \left(\int_{Q\backslash F} |J_f(z)| dz\right)^{\frac{p}{n}} = \left(\int_Q K(z,f)^s dz\right)^{\frac{n-p}{n}} \cdot \left(\int_{\mathbb{R}^n} N(y, f, Q \backslash (E \cup F)) dy\right)^{\frac{p}{n}} \leq \left(\int_Q K(z,f)^s dz\right)^{\frac{n-p}{n}} \cdot [N(f, U_{x_0}, B) \cdot \mu_n(f(Q))]^{\frac{p}{n}} < \infty$, hence $f' \in L^p_{\text{loc}}(D)$. Since $p \geq 1$, we have $\int_Q \left\| \frac{\partial f(z)}{\partial x_i} \right\| dz < \infty$, and using Fubini's theorem, we obtain that

$$(2) \qquad \int_{a_i}^{b_i} \left\| f'_{x,I}(t) \right\| dt < \infty, \; i \in r(I) \text{ for a.e. } x \in \Pi_I(Q).$$

We use now (1) and (2) and Barry's theorem (see [22], page 285) to see that the map $f_{x,I}$ is absolutely continuous on $[a_i, b_i], i \in r(I)$, for a.e. $x \in \Pi_I(Q)$. We therefore proved that $f$ is ACL, and since we showed that $f'$ is locally in $L^p$, it results that $f$ is ACL$^p$ on $D$, and from , [21], page 6, we see that $f \in W^{1,p}_{\text{loc}}(D, \mathbb{R}^n)$.

## 4. MAPPINGS WITH $K_\alpha(x,f) < \infty$ A.E.

We show ørst in Theorem 3 that mappings with locally ønite multiplicity and with $K_\alpha(x,f) < \infty$ a.e., or with $\widetilde{H}_\alpha(x,f) < \infty$ a.e., are a.e. diœerentiable, extending in this way Lemma 2.2 from [9], which is proved for open, discrete mappings in $\mathbb{R}^n$.

Proof of Theorem 3: Let $\varphi : \mathcal{B}(D) \to [0,\infty]$ be deøned by $\varphi(K) = \mu_n(f(K))$ for $K \in \mathcal{B}(D)$. As in Lemma 2 we show that $\bar{\varphi}'(x) < \infty$ a.e., and let $x \in D$ be such that $K_\alpha(x,f) < \infty$ and $\bar{\varphi}'(x) < \infty$. Then, if $\lambda > 1$ and $r > 0$ is such that $\bar{B}(x,r) \subset D$ we have $\frac{L(x,f,r)^n}{r^n} \leq \frac{V_n \cdot d(f(B(x,\lambda r)))^n}{\left(\frac{\alpha}{\lambda}\right)^n \cdot \mu_n\left(f\left(B\left(x,\frac{\lambda r}{\alpha}\right)\right)\right)} \cdot \frac{\mu_n\left(f\left(B\left(x,\frac{\lambda r}{\alpha}\right)\right)\right)}{\mu_n\left(B\left(x,\frac{\lambda r}{\alpha}\right)\right)}$.

Letting ørst $r$ tends to zero and then $\lambda$ tends to 1, we obtain $\limsup_{y\to x} \frac{\|f(y)-f(x)\|}{\|y-x\|} \leq \frac{V_n}{\alpha^n} \cdot K_\alpha(x,f) \cdot \bar{\varphi}'(x) < \infty$. We apply now the theorem of Rademacher and Stepanov to see that $f$ is a.e. diœerentiable on $D$. If condition b) or condition c) is satisøed , we use $(*)$ or $(**)$ to see that $K_\alpha(x,f) < \infty$ a.e. and we use the preceeding argument.

By usual covering arguments, we obtain the following $n$-dimensional version of Lemma 31.1, page 106 from [24]:

Lemma 4. Let $K \subset I\!R^n$ be compact and $0 < \alpha \leq 1$. Then there exists a constant $C(\alpha, n)$ depending only on $\alpha$ and $n$ such that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for every $0 < r \leq \delta_\varepsilon$, there exists a ønite set $I$ depending on $r$ and $\varepsilon$ such that $K \subset \underset{i\in I}{\sqcup} B(x_i, \alpha r), x_i \in K, i \in I$, every point from $\underset{i\in I}{\sqcup} B(x_i, r)$ belongs to at most $C(\alpha, n)$ balls $B(x_i, r)$ and $\sum\limits_{i\in I} \mu_n(B(x_i, r)) \leq C(\alpha, n) \cdot (\mu_n(K) + \varepsilon)$.

We prove now Theorem 4, and the strong ACL property from Theorem 4 (the $m$-ACH property) seems to be new even for quasiregular mappings.

Proof of Theorem 4: Suppose that condition a) is satisøed. Let $Q = \prod\limits_{i=1}^{n} [a_i, b_i]$ be a cube with the sydes parallel to coordinate axes with $\bar{Q} \subset D$ and suppose that $q = N(f, \bar{Q}, B) < \infty$. Let $I \in I_{n-m}, I = (\alpha_1, \ldots, \alpha_{n-m})$, with $\alpha_j \in \{1, \ldots, n-1\}$ for $j = 1, \ldots, n-m$, $\Pi_I : I\!R^n \to I\!R^n$ given by $\Pi_I(x) = (x_I, 0)$ for $x \in I\!R^n$, and we set for $A \in \mathcal{B}(\Pi_I(Q)), E_A = \Pi_I^{-1}(A) \cap Q$ and let $\varphi : \mathcal{B}(\Pi_I(Q)) \to I\!R_+$ be given by $\varphi(A) = \mu_n(f(E_A))$ for $A \in \mathcal{B}(\Pi_I(Q))$. Then $\varphi$ is a $q$-subadditive function, hence $\bar{\varphi}'(x)$ exists a.e. and $\bar{\varphi}'(x) < \infty$ for a.e. $x \in \Pi_I(Q)$. Let us øx such a point $x \in \Pi_I(Q)$, and let $H = \Pi_I^{-1}(x) \cap Q$, and using Lemma 3 and Fubini's theorem, we can also suppose that the map $t \to K_\alpha((x,t), f)^s \in L(H)$ and that $H \cap E$ is at most countable.

Let $F \subset H \backslash E$ be compact. We take $a_k \nearrow \infty$ and $F_k = \{z \in F | a_k \leq K_\alpha(z, f) < a_{k+1}\}, F_{kj} = \{z \in F_k | \frac{\mathrm{d}(f(B(z,\alpha r)))^n}{\mu_n(f(B(z,r)))} < a_{k+1}$ for $0 < r \leq \frac{1}{j}\}$for $k, j \in I\!N$,and we see that $F_{kj} \nearrow F_k$ for $k \in I\!N$. We øx $k, j \in I\!N$ and let $K \subset F_{kj}$ be compact, $\varepsilon, t > 0$ and $\delta = \min\left\{\mathrm{d}(K, CD), \frac{1}{j}\right\}$. Using the uniform continuity of $f$ on $\bar{B}\left(K, \frac{\delta}{2}\right) \subset D$, we can ønd $0 < \rho < \frac{\delta}{2}$ such that $\|f(y) - f(z)\| \leq t$ if $y, z \in \bar{B}\left(K, \frac{\delta}{2}\right)$ and $\|y - z\| \leq \rho$. Using Lemma 4, we can ønd a ȷgoodȷ covering of $K$, i.e. we ønd $0 < \delta_\varepsilon < \frac{\rho}{2}$ and a constant $C(\alpha, n)$ depending only on $n$ and $\alpha$ such that for every $0 < r \leq \delta_\varepsilon$, we can ønd a ønite set $J$ depending on $r$ and $\varepsilon$ such that $K \subset \underset{i\in J}{\sqcup} B(x_i, \alpha r), x_i \in K$ for $i \in J$, every point from $\underset{i\in J}{\sqcup} B(x_i, r)$ belongs to at most $C(\alpha, n)$ balls $B(x_i, r)$ and $\sum_{i\in J} \mu_m(B_m(x_i, r)) \leq C(\alpha, n) \cdot (\mu_m(K)) + \varepsilon)$. Here $B_m(x_i, r)$ denotes the balls of center $x_i$ and radius $r$ from $H$ for $i \in J$.

We øx such $0 < r \leq \delta_\varepsilon$ and the corresponding set $J$ and points $x_i \in K, i \in J$, and let $\ell = \mathrm{Card}\, J$. Then $f(K) \subset \underset{i\in J}{\sqcup} f(B(x_i, \alpha r))$ and $\mathrm{d}(f(B(x_i, \alpha r))) \leq t$ for every $i \in J$. We have $m_m^t(f(K)) \leq \sum_{i\in J} \mathrm{d}(f(B(x_i, \alpha r)))^m = \sum_{i\in J} r^{\frac{m(n-m)}{n}}$
$\cdot r^{\frac{-m(n-m)}{n}} \cdot \mathrm{d}(f(B(x_i, \alpha r)))^m \leq$ (using Hlder's inequality)$\leq (\ell \cdot r^m)^{\frac{n-m}{n}} \cdot$
$\left(\sum_{i\in J} d(f(B(x_i, \alpha r)))^n \cdot r^{-(n-m)}\right)^{\frac{m}{n}} \leq V_m^{\frac{m-n}{n}} \cdot \left(\sum_{i\in J} \mu_m(B_m(x_i, r))\right)^{\frac{n-m}{n}} \cdot$
$\left(a_{k+1} \cdot \sum_{i\in J} \mu_n(f(B(x_i, r))) \cdot r^{-(n-m)}\right)^{\frac{m}{n}} \leq (a_{k+1})^{\frac{m}{n}} \cdot V_m^{\frac{m-n}{n}} \cdot$
$C(\alpha, n)^{\frac{n-m}{n}} \cdot (\mu_m(K) + \varepsilon)^{\frac{n-m}{n}} \cdot \left(\frac{V_{n-m} \cdot C(\alpha,n) \cdot q \cdot \mu_n\left(f\left(E_{B_{n-m}(x,r)}\right)\right)}{\mu_{n-m}(B_{n-m}(x,r))}\right)^{\frac{m}{n}}.$

Here we denoted by $B_{n-m}(x, r)$ the ball of center $x$ and radius $r$ from $\Pi_I(Q)$. Letting $C = V_m^{\frac{m-n}{n}} \cdot V_{n-m}^{\frac{m}{n}} \cdot C(\alpha, n) \cdot q^{\frac{m}{n}} \cdot \bar{\varphi}'(x)^{\frac{m}{n}}$ and letting ørst $r \to 0$, then $\varepsilon \to 0$ and ønally $t \to 0$ we obtain
$m_n(f(K)) \leq (a_{k+1})^{\frac{m}{n}} \cdot C \cdot \mu_m(K)^{\frac{n-m}{n}}$.

Since $K$ was arbitrary compact in $F_{kj}$, we see that $m_m(f(F_{kj})) \leq C\cdot(a_{k+1})^{\frac{m}{n}}\cdot \mu_m(F_{kj})^{\frac{n-m}{n}}$ for every $k,j \in I\!N$. Since $F_{kj} \nearrow F_k$, we let $j$ tends to inønite and we obtain that

$$(1) \qquad m_m(f(F_k)) \leq C\cdot(a_{k+1})^{\frac{m}{n}}\cdot\mu_m(F_k)^{\frac{n-m}{n}} \text{ for } k\in I\!N.$$

Since $f(F) = \bigsqcup\limits_{k=1}^{\infty} f(F_k)$, we have

$m_m(f(F)) \leq \sum_{k=1}^{\infty}(f(F_k)) \leq C\cdot\sum_{k=1}^{\infty}(a_{k+1})^{\frac{m}{n}}\cdot\mu_m(F_k)^{\frac{n-m}{n}} =$
$C\cdot\sum_{k=1}^{\infty}(a_{k+1})^{\frac{m}{n}}\cdot a_k^{\frac{-s(n-m)}{n}}\cdot a_k^{\frac{s(n-m)}{n}}\cdot\mu_m(F_k)^{\frac{n-m}{n}} \leq$ (using Hlder's inequality) $\leq$
$C\cdot\left(\sum_{k=1}^{\infty}a_k^s\cdot\mu_m(F_k)\right)^{\frac{n-m}{n}}\cdot\left(\sum_{k=1}^{\infty}a_{k+1}\cdot a_k^{\frac{-s(n-m)}{m}}\right)^{\frac{m}{n}}$

$\leq C{\cdot}S{\cdot}\left(\sum_{k=1}^{\infty}\int\limits_{F_k}K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}}$ =(since $F_k$ are disjoint and $F=\bigsqcup\limits_{k=1}^{\infty}F_k$) $=$
$C\cdot S\cdot\left(\int_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}}$ where $\widetilde{S}=\sum_{k=1}^{\infty}a_{k+1}\cdot a_k^{\frac{-s(n-m)}{m}}$ is convergent if we take $a_k = 2^k$ for $k\in I\!N$, and $S=\left(\widetilde{S}\right)^{\frac{m}{n}}$.

Indeed, if $a_k=2^k$ for $k\in I\!N$, then $\widetilde{S}=\sum\limits_{k=1}^{\infty}\frac{2^{k+1}}{2^{\frac{ks(n-m)}{m}}} = 2{\cdot}\sum_{k=1}^{\infty}\left(\frac{1}{2^{\frac{s(n-m)}{m}-1}}\right)^k$ is convergent if $\frac{s(n-m)}{m}-1>0$, i.e. if $s>\frac{m}{n-m}$. We proved that

$$(2) \qquad m_m(f(F)) \leq C\cdot S\cdot\left(\int\limits_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}} \quad \text{for } F\subset H\backslash E \text{ compact}$$

Let now $F\subset H$ be compact. Since $f(E\cap H)$ is countable, we see that $f(F)\backslash f(E\cap H)$ is a Borel set. Suppose ørst that $m_m(f(F))<\infty$ and let $\varepsilon>0$. Using [17], page 114, Th.8.13, we can ønd $M\subset f(F)\backslash f(E\cap H)$ compact such that $m_m(f(F)\backslash M)<\varepsilon$. Let $K=f^{-1}(M)\cap F$. Then $K$ is compact, $K\subset H\backslash E$, $f(K)=M$, and using (2) we have
$m_m(f(F)) \leq \varepsilon + m_m(f(K)) \leq \varepsilon + C\cdot S\left(\int_K K_\alpha((x,t)\,f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}} \leq \varepsilon + C\cdot S\left(\int_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}}$ and letting $\varepsilon$ tends to zero, we obtain that $m_m \leq C{\cdot}S\left(\int_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}}$.

If $m_m(f(F))=\infty$, since $m_m(f(E\cap H))=0$, we use again [17], page 114, Th.8.13 to ønd $M_p\subset f(F)\backslash f(E\cap H)$ compact sets such that $p\leq m_m(M_p)<\infty$ for every $p\in I\!N$, and let $K_p=f^{-1}(M_p)\cap F$. Then $K_p\subset H\backslash E$ are compact sets such that $f(K_p)=M_p$ for $p\in I\!N$, and using (2) we have $p\leq m_m(M_p)= m_m(f(K_p)) \leq C{\cdot}S\left(\int_{K_p}K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}} \leq C{\cdot}S\left(\int_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}}$ and letting $p$ tends to inønite, we ønd that $\infty = m_m(f(F)) \leq C\cdot S\left(\int_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}}$ .We proved in both cases that

$$(3) \quad m_m(f(F)) \leq C\cdot S\left(\int_F K_\alpha((x,t),f)^s\,\mathrm{d}t\right)^{\frac{n-m}{n}} \quad \text{for every } F\subset H \text{ compact.}$$

Let now $\varepsilon > 0$. Since the map $A \to \int_A K_\alpha((x,t),f)^s \,\mathrm{d}t$ is an absolutely continuous measure on $\mathcal{B}(H)$ with respect to the Lebesgue measure $\mu_m$ from $H$, we can ønd $\delta_\varepsilon > 0$ such that $\left(\int_A K_\alpha((x,t),f)^s \,\mathrm{d}t\right)^{\frac{n-m}{n}} < \frac{\varepsilon}{q \cdot C \cdot S}$ if $m_m(A) \leq \delta_\varepsilon$. Let $\Delta_1, \ldots, \Delta_k$ be closed, disjoint intervals in $H$ such that $\sum_{i=1}^{k} m_m(\Delta_i) \leq \delta_\varepsilon$. Then $\sum_{i=1}^{k} m_m(f(\Delta_i)) \leq q \cdot m_m\left(f\left(\bigsqcup_{i=1}^{k} \Delta_i\right)\right) \leq$ (using (3) $\leq q \cdot C \cdot S \cdot$
$\left(\int_{\bigsqcup_{i=1}^{k}\Delta_i} K_\alpha((x,t),f)^s \,\mathrm{d}t\right)^{\frac{n-m}{n}} \leq \varepsilon$.

We proved that $f$ is $m$-ACH if condition a) is satisøed. Suppose that condition b) is satisøed. Using (*), we see that if $H_\alpha \in L^s_{\mathrm{loc}}(D)$, with $s > \frac{mn}{n-m}$, then $K_\alpha(x,f)^{\frac{s}{n}} \leq C \cdot H_\alpha(x,f)^s$ on $D \backslash E$ for some constant $C$ and $\frac{s}{n} > \frac{m}{n-m}$. From what we have proved before, it results that $f$ is $m$-ACH. We apply the same argument if we suppose that condition c) is satisøed.

Remark 2. The $m$-ACH property from Theorem 4 is also valid for intervals $Q$ in $D$ with the sydes parallel to an arbitrary orthonormal system.

Proof of Theorem 5: Suppose that condition a) is satisøed. We see from Theorem 3 that $f$ is a.e. diœerentiable and we also see that $\mu'_f(x)$ exists a.e. and $\mu'_f(x) = |J_f(x)|$ a.e. Let $x \in D$ be such that $f$ is diœerentiable in $x$ and let $\varepsilon > 0$. Then there exists $\delta_\varepsilon > 0$ such that $\|f(z) - f(x) - f'(x)(z-x)\| \leq \varepsilon \cdot \|z - x\|$ for $\|z - x\| \leq \delta_\varepsilon$. We can take $\delta_\varepsilon$>0 such that we also have $\mu_n(f(B(x,r))) \leq \left(\mu'_f(x) + \varepsilon\right) \cdot \mu_n(B(x,r))$ and $\mathrm{d}(f(B(x,\alpha r)))^n \leq K_\alpha(x,f) \cdot \mu_n(f(B(x,r)))$ for $0 < r \leq \delta_\varepsilon$. Let $1 < \lambda < 2$ and $0 < r \leq \frac{\delta_\varepsilon}{2}$ and let $z \in S(x,\alpha r)$ be such that $\|f'(x)(z-x)\| = \|f'(x)\| \cdot \|z - x\|$. We have
$\alpha r \cdot (\|f'(x)\| - \varepsilon) = (\|f'(x)\| - \varepsilon) \cdot \|z - x\| = \|f'(x)(z-x)\| - \varepsilon \cdot \|z - x\|$
$\leq \|f(z) - f(x)\| + \|f(z) - f(x) - f'(x)(z-x)\| - \varepsilon \cdot \|z - x\| \leq \|f(z) - f(x)\| \leq \mathrm{d}(f(B(x,\lambda\alpha r)))$. We obtain
$(\|f'(x)\| - \varepsilon)^n \leq \frac{\mathrm{d}(f(B(x,\lambda\alpha r)))^n}{\alpha^n \cdot r^n} \leq \frac{\lambda^n \cdot V_n}{\alpha^n} \cdot K_\alpha(x,f) \cdot \frac{\mu_n(f(B(x,\lambda r)))}{\mu_n(B(x,\lambda r))} \leq \frac{\lambda^n \cdot V_n}{\alpha^n} \cdot K_\alpha(x,f) \cdot \left(\mu'_f(x) + \varepsilon\right) = \frac{\lambda^n \cdot V_n}{\alpha^n} \cdot K_\alpha(x,f) \cdot (|J_f(x)| + \varepsilon)$.

Letting $\varepsilon \to 0$ and $\lambda \to 1$, we ønd that

$$\|f'(x)\|^n \leq \frac{V_n}{\alpha^n} \cdot K_\alpha(x,f) \cdot |J_f(x)| \text{ a.e.} \tag{1}$$

Let now $G \subset\subset D$ be open. Suppose that $s < \infty$ and let $p = \frac{ns}{s+1}$. Then $0 < p < n$ and $\int_G \|f'(x)\|^p \,\mathrm{d}x \leq C \cdot \left(\int_G K_\alpha(x,f)^{\frac{p}{n-p}} \,\mathrm{d}x\right)^{\frac{n-p}{n}} \cdot \left(\int_G |J_f(x)| \,\mathrm{d}x\right)^{\frac{p}{n}} = C \cdot \left(\int_G K_\alpha(x,f)^s \,\mathrm{d}x\right)^{\frac{n-p}{n}} \cdot \left(\int_G |J_f(x)| \,\mathrm{d}x\right)^{\frac{p}{n}} < \infty$,where $C$ is a constant.

We used here Lemma 2. If $s = \infty$, we prove in the same way that $f' \in L^n_{\mathrm{loc}}(D)$. If $E$ is of $\sigma$-ønite $n-1$ dimensional Hausdorœ measure and $m_1(B) = 0$, we apply Theorem 4 to see that $f$ is ACL$^{\mathrm{p}}$ on $D$, where $p = n$ if $s = \infty$ and $p = \frac{ns}{s+1}$ if $\frac{1}{n-1} < s < \infty$.

If condition b) is satisøed, then $K_\alpha(x,f)^{\frac{1}{n}} \leq C \cdot H_\alpha(x,f) < \infty$ on $D \backslash E$, and from Theorem 3 we see that $f$ is a.e. diœerentiable on $D$. Let $x \in D \backslash E$ be such that $f$ is diœerentiable at $x$ and let $\ell(A) = \inf_{\|h\|=1} \|A(h)\|$ if $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$. Then there exists $\delta_\varepsilon > 0$ such that $\|f(z) - f(x) - f'(x)(z-x)\| \leq \varepsilon \cdot \|z - x\|$ if $\|z - x\| \leq \delta_\varepsilon$ and we have $(\|f'(x)\| - \varepsilon) \cdot r \leq L(x, f, r) \leq (\|f'(x)\| + \varepsilon) \cdot r, \ell(x, f, r) \leq (\ell(f'(x)) + \varepsilon) \cdot r$ if $0 < r \leq \delta_\varepsilon$. If $J_f(x) = 0$, then $\frac{L(x,f,\alpha r)}{\ell(x,f,r)} \geq \frac{(\|f'(x)\|-\varepsilon)\cdot\alpha}{\ell(f'(x))+\varepsilon}$, and since $H_\alpha(x, f) < \infty$, this implies, letting $\varepsilon$ tend to zero, that $\|f'(x)\| = 0$, i.e. $f'(x) = 0$. If $J_f(x) \neq 0$, we also have $(\ell(f'(x)) - \varepsilon) \cdot r \leq \ell(x, f, r)$ for $0 < r \leq \delta_\varepsilon$, where $\varepsilon > 0$ is choosed such that $\varepsilon < \ell(f'(x))$, hence $\frac{(\|f'(x)\|-\varepsilon)\cdot\alpha}{\ell(f'(x))+\varepsilon} \leq \frac{L(x,f,\alpha r)}{\ell(x,f,r)} \leq \frac{(\|f'(x)\|+\varepsilon)\cdot\alpha}{\ell(f'(x))-\varepsilon}$ for $0 < r \leq \delta_\varepsilon$,and letting $r$ tends to zero, and then $\varepsilon$ tends to zero, we ønd that $H_\alpha(x, f) = \alpha \cdot \frac{\|f'(x)\|}{\ell(f'(x))}$. We therefore obtain that

$$\|f'(x)\|^n \leq \frac{H_\alpha(x, f)^{n-1}}{\alpha^{n-1}} \cdot |J_f(x)| \text{ a.e. in } D. \tag{2}$$

Using (2) instead of (1), we ønd as before that $f' \in L^n_{\text{loc}}(D)$ if $s = \infty$ and that $f' \in L^p_{\text{loc}}(D)$, where $p = \frac{ns}{n+s-1}$ if $s < \infty$. Suppose now that $E$ is of $\sigma$-ønite $n-1$ dimensional Hausdorœ measure and $m_1(B) = 0$ and that $\frac{n}{n-1} < s < \infty$. Then $p = \frac{ns}{n+s-1} \geq 1$ and from Theorem 4 we see that $f$ is $\text{ACL}^{\text{p}}$ on $D$. If $s = \infty$, we see from Theorem 4 that $f$ is $\text{ACL}^{\text{n}}$ on $D$.

Suppose that condition c) is satisøed. Then $K_\alpha(x, f)^{\frac{s}{n}} \leq C \cdot \widetilde{H}_\alpha(x, f)^s < \infty$ on $D \setminus E$, hence from Theorem 3 we see that $f$ is a.e. diœerentiable, and if $t = \frac{s}{n}$, we see that $K_\alpha(\cdot, f) \in L^t_{\text{loc}}(D)$. If $s = \infty$, then $t = \infty$, hence $f' \in L^n_{\text{loc}}(D)$. If $s < \infty$, let $p = \frac{nt}{t+1} = \frac{ns}{n+s}$. Then we apply the preceeding arguments to see that $f' \in L^p_{\text{loc}}(D)$. If $E$ is of $\sigma$-ønite $n-1$ dimensional Hausdorœ measure and $m_1(B) = 0$, then $p = \frac{ns}{s+n} \geq 1$ if $\frac{n}{n-1} < s < \infty$, and from Theorem 4 we see that $f$ is $\text{ACL}^{\text{p}}$ on $D$. If $s = \infty$, we also see from Theorem 4 that $f$ is $\text{ACL}^{\text{n}}$ on $D$.

Proof of Theorem 6: Suppose that condition a) is satisøed. We see from Theorem 5 that $f$ is a.e. diœerentiable and $\text{ACL}^{\text{n}}$ on $D$ and relation (1) from Theorem 5 implies that $\|f'(x)\|^n \leq \frac{V_n \cdot K}{\alpha^n} \cdot |J_f(x)|$ a.e., hence $f$ is a map with bounded distortion and arbitrary Jacobian sign on $D$. If $\Gamma$ is a path family from $A$, we see from Theorem 1 that $M(\Gamma) \leq \frac{q \cdot V_n \cdot K}{\alpha^n} \cdot M(f(\Gamma))$.

If condition b) holds, then $f$ is a.e. diœerentiable and relation (2) from Theorem 5 implies that $\|f'(x)\|^n \leq \left(\frac{K}{\alpha}\right)^{n-1} \cdot |J_f(x)|$ a.e., and from Theorem 5 we see that $f$ is $\text{ACL}^{\text{n}}$ on $D$.

If condition c) holds, then $K_\alpha(x, f) \leq \frac{2^n}{V_n} \cdot \widetilde{H}_\alpha(x, f)^n$ on $D \setminus E$, hence $\|f'(x)\|^n \leq \left(\frac{2K}{\alpha}\right)^n \cdot |J_f(x)|$ a.e. and Theorem 4 implies that $f$ is $\text{ACL}^{\text{n}}$ on $D$.

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