INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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Preprint nr. 17/2004

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Decembrie, 2004

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MAPPINGS OF FINITE DISTORTION: ZORIC'S THEOREM, EQUICONTINUITY RESULTS

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ABSTRACT: We generalize some known results from the theory of quasiregular mappings, as Zoric's theorem and some equicontinuity results. This generalizations hold for a special class of mappings with finite distortion, satisfying condition $(\mathcal{A})$, which extends the known class of quasiregular mappings and for which are valid some recent modular and capacities inequalities established in [7].

AMS 1991 *Classification No:* 30C65

Keywords: Zoric's theorem, Montel's theorem, mappings of finite distortion.

1. INTRODUCTION.

A mapping $f : D \to \mathbb{R}^n$, where $D \subset \mathbb{R}^n$ is a domain, is said to have finite distortion if the following conditions are satisfied:

1) $f \in W^{1,1}_{\text{loc}}(D, \mathbb{R}^n)$.

2) The Jacobian determinant $J_f(x)$ is locally integrable.

3) There exists a measurable function $K : D \to [0, \infty]$, finite a.e. such that $|f'(x)|^n \leq K(x) \cdot J_f(x)$ a.e.

Notice that when $K \in L^\infty(D)$, we obtain the known class of quasiregular mappings, and we refer the reader to [10],[11] for the monographs dedicated to this subject. Quasiregular mappings are either constant, or open, discrete, satisfies condition (N) and a Hölder condition.

If the dilatation map $K \in L^p_{\text{loc}}(D)$, $p > n-1$ and $f \in W^{1,n}_{\text{loc}}(D, \mathbb{R}^n)$, then it is shown in [9] that a map with finite distortion is open, discrete.

Recently were considered in [7], [8] mappings $f : D \to \mathbb{R}^n$ of finite distortion for which there exists $\mathcal{A} : [0, \infty) \to [0, \infty)$ smooth, strictly increasing, with $\mathcal{A}(0) = 0$, $\lim_{t \to \infty} \mathcal{A}(t) = \infty$ and satisfying the conditions:

$(\mathcal{A}_0)\ \exp(\mathcal{A}(K)) \in L^1_{\text{loc}}(D)$.

$(\mathcal{A}_1)\ \int_1^\infty \frac{\mathcal{A}'(t)}{t} dt = \infty$.

$(\mathcal{A}_2)$ there exists $t_0 > 0$ such that $\mathcal{A}'(t) \cdot t$ increases to infinity for $t \geq t_0$.

As in [7], we say that a mapping of finite distortion satisfies condition $(\mathcal{A})$ if f satisfies condition $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$. In [4], Theorem 1.3 it is shown that such nonconstant mappings are either constant or open, discrete if $\mathcal{A}(t) = \lambda t$ for some $\lambda > 0$. We shall show in Theorem 1 that mappings $f : D \to \mathbb{R}^n$ with finite distortion, satisfying condition $(\mathcal{A})$ are such the dilatation map K is in $L^p_{\text{loc}}(D)$ for every $p > 0$ and $f \in W^{1,p}_{\text{loc}}(D, \mathbb{R}^n)$ for every $1 \leq p < n$. This ensures (see [5], Prop.2.5, [6], Theorem 1.1 or Theorem 2.1 [7]) that such mappings are continuous and either constant, or open, discrete. Our proofs are based on the following basic ingredients, valid for mappings $f : D \to \mathbb{R}^n$ of finite distortion and satisfying condition $(\mathcal{A})$:

(i) $M(f(\Gamma)) \leq M_{K^{n-1}}(\Gamma)$ for every path family Γ from D.

(ii) $\operatorname{cap} f(E) \leq \operatorname{cap}_{K^{n-1}}(E)$, where $E = (G, C)$ is a capacitor with $G \subset\subset D$.

(iii) $\operatorname{cap}_{K^{n-1}}\left(B(x,R), \overline{B}(x,r)\right) \to 0$ when $r \to 0$ and $R > 0$ is keeped fixed established in [7], Corollary 4.2, Corollary 5.2 and Theorem 5.3. Here K is the dilatation map of f. We also find in Lemma 1 a condition in order to ensure that $\operatorname{cap}_{K^{n-1}}\left(B(x,R), \overline{B}(x,r)\right) \to 0$ when $r > 0$ is keeped fixed and $R \to \infty$. This last condition is necesary in the proof of Zoric's theorem (Theorem 2 and Theorem 3).

We generalize Zoric's theorem for mappings with finite distortion which satisfies condition $(\mathcal{A})$:

Theorem 3. (Zoric's theorem). *Let $n \geq 3, f : \mathbb{R}^n \to \mathbb{R}^n$ a local homeomorphism and a map of finite distortion, satisfying condition $(\mathcal{A})$ such that there exists $r > 0$ such that $\int\limits_{CB(0,r)} \exp\left(\mathcal{A}\left(K(x)\right)\right) \cdot \frac{1}{|x|^{2n}} dx < \infty$. Then $f : \mathbb{R}^n \to \mathbb{R}^n$ is a global homeomorphism.*

We also give in Theorem 2 a version of Zoric's theorem with some "singular" sets K and B, extending some results from [1] and [2]. We give a Picard type theorem in Theorem 4, in connection with a similar result from [7].

Theorem 4. (Picard's theorem) *Let $F \subset \mathbb{R}^n$ be closed, $f : \mathbb{R}^n \backslash F \to \mathbb{R}^n$ a non-constant map of finite distortion, satifying condition $(\mathcal{A})$, such that $M_{K^{n-1}}(F) = 0$ and there exists $r > 0$ such that $\int\limits_{CB(0,r)} \exp\left(\mathcal{A}\left(K(x)\right)\right) \cdot \frac{1}{|x|^{2n}} dx < \infty$. Then* $\operatorname{cap} Cf\left(\mathbb{R}^n \backslash F\right) = 0$.

We generalize in Theorem 8 a known equicontinuity result for quasiregular mappings:

Theorem 5. *Let $D \subset \mathbb{R}^n$ be a domain, $M \subset \mathbb{R}^n$ with* cap $M > 0$ *and let W be a family of mappings $f : D \to \mathbb{R}^n \backslash M$ of finite distortion, satisfying condition $(\mathcal{A})$ and having the same dilatation* map K. *Then W is equicontinuous and we take the euclidean distance on D and the chordal distance on $\mathbb{R}^n$.*

We immediately obtain a Montel's theorem for mappings of finite distortion satisfying condition $(\mathcal{A})$:

Theorem 6. (Montel's theorem). Let $D \subset \mathbb{R}^n$ be a domain, W be a bounded family of mappings $f : D \to \mathbb{R}^n$ of finite distortion, having the same dilatation map K and satisfying condition $(\mathcal{A})$. Then W is a normal family.

2. Preliminaries

We call a path $q : [0,1) \to \mathbb{R}^n$ an open path and a point $x \in \overline{\mathbb{R}^n}$ will be called a limit point of q if there exists $t_p \nearrow 1$ such that $q(t_p) \to x$. If Γ is a path family in $\overline{\mathbb{R}^n}$, we define $F(\Gamma) = \{\rho : \overline{\mathbb{R}^n} \to [0,\infty]$ Borel maps $\mid \int\limits_{\gamma} \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. Let now $D \subset \mathbb{R}^n$ be open and $\omega : D \to [0,\infty]$ be measurable and finite a.e. Then, if $\widetilde{\omega} : \mathbb{R}^n \to [0,\infty]$ is defined by $\widetilde{\omega}(x) = \omega(x)$ if $x \in D$, $\widetilde{\omega}(x) = 0$ if $x \notin D$, we define the ω-modulus of Γ by $M_\omega(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{\mathbb{R}^n} \rho^n(x) \cdot \widetilde{\omega}(x) dx$ and for $\omega \equiv 1$ we obtain the usual modulus $M(\Gamma)$. If Γ_1, Γ_2 are path families in $\overline{\mathbb{R}^n}$, we say that $\Gamma_1 > \Gamma_2$ if every path from

Γ_1 has a subpath in Γ_2 and as in the classical case, we prove that if $\Gamma_1 > \Gamma_2$, then $M_\omega(\Gamma_1) \leq M_\omega(\Gamma_2)$. Also, we prove that $M_\omega\left(\overset{\infty}{\underset{i=1}{\sqcup}}\Gamma_i\right) \leq \sum_{i=1}^{\infty} M_\omega(\Gamma_i)$, and if $\omega_1 \leq \omega_2$, then $M_{\omega_1}(\Gamma) \leq M_{\omega_2}(\Gamma)$. We define for $D \subset \mathbb{R}^n$ open, $E,\ F \subset \overline{D}$ by $\Delta(E, F, D)$ the family of all paths, open or not, which joins E with F in D.

We say that $E = (D, C)$ is a condenser if C is compact, D is open, $C \subset D \subset \mathbb{R}^n$, and let $\omega : D \to [0, \infty]$ be measurable and finite a.e. We define the ω-capacity of E by $\text{cap}_\omega E = \inf \int_{\mathbb{R}^n} |\nabla u|^n(x) \cdot \omega(x)\, dx$, where $u \in C_0^\infty(D)$ and $u \geq 1$ on C, and if $\omega = 1$ we obtain the usual capacity of E, cap E. We see that if u is a test function for $\text{cap}_\omega E$, the $\rho = |\nabla u| \in F(\Gamma_E)$, and this implies that $M_\omega(\Gamma_E) \leq \text{cap}_\omega(E)$. Here $\Gamma_E = \{\gamma : [a, b] \to D \text{ path } |\gamma(a) \in C$ and γ has a limit point in $\partial D\}$ and from Prop. 10.2, [11], page 54, we have cap $E = M(\Gamma_E)$. If $C \subset \mathbb{R}^n$ is compact, we say that cap $C = 0$ if $\text{cap}(A, C) = 0$ for every $C \subset A \subset \mathbb{R}^n$ open and from [11], Lemma 2.2, page 64, the definition is independent on the open set A such that $C \subset A$. If $C \subset \mathbb{R}^n$ is arbitrary, we say that cap $C = 0$ if cap $K = 0$ for every $K \subset C$ compact.

Let now $D \subset \mathbb{R}^n$ be open, $\omega : D \to [0, \infty]$ be measurable and finite a.e., $A \subset D$ a set. We say that A is of zero ω-modulus, and we write $M_\omega(A) = 0$, if the ω-modulus of all paths having some limit point in A is zero. If $\omega \geq 1$, we see that $M(\Gamma) \leq M_\omega(\Gamma)$ hence, if $M_\omega(A) = 0$, then cap $A = 0$.

If $A \subset D$ is countable and $\lim_{r \to 0} M_\omega\left(\Delta\left(\overline{B}(x, r), CB(x, R), D\right)\right) = 0$ for every $x \in A$ and every $0 < R$ with $\overline{B}(x, R) \subset D$, we prove as in the classical case that $M_\omega(A) = 0$. Using Theorem 5.3 [7], we see that this thing holds for instance if $\omega = K^{n-1}$, where $K : D \to [0, \infty]$ is measurable and finite a.e. and for every $x \in A$ there exists $\rho_x > 0$ such that $\overline{B}(x, \rho_x) \subset D$ and $\int_{B(x,\rho_x)} \exp(\mathcal{A}(K(y)))\, dy < \infty$, where $\mathcal{A}$ satisfies conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Also, Lemma 1 gives a condition which ensures that $M_{K^{n-1}}(\infty) = 0$.

If $D \subset \mathbb{R}^n$ is open and $b \in \partial D$, we define $C(f, b) = \{w \in \overline{\mathbb{R}^n} \mid$ there exists $b_p \in D$, $b_p \to b$ such that $f(b_p) \to w\}$ and if $B \subset \partial D$, we let $C(f, B) = \underset{b \in B}{\sqcup} C(f, b)$. If E, F are Hausdorff spaces, $f : E \to F$ is a map, $p : [0, 1] \to F$ is a path, $x \in E$ is such that $f(x) = p(0)$, we say that $q : [0, a) \to E$ is a maximal lifting of p from x if $q(0) = x, 0 < a \leq 1$, $f \circ q = p|[0, a)$ and q is maximal with this property. If q is defined on $[0, 1]$, we say that q is a lifting of p. If E, F are domains in $\mathbb{R}^n$ and f is continuous, open, discrete, there exists allways a maximal lifting. We let by μ_n the Lebesgue measure in $\mathbb{R}^n$. We shall denote by q the chordal metric in $\overline{\mathbb{R}^n}$ given by $q(a, b) = |a - b| \cdot \left(1 + |a|^2\right)^{-\frac{1}{2}} \cdot \left(1 + |b|^2\right)^{-\frac{1}{2}}$ if $a, b \in \mathbb{R}^n, q(a, \infty) = \left(1 + |a|^2\right)^{-\frac{1}{2}}$ if $a \in \mathbb{R}^n$ and by $|a - b|$ the euclidean distance between a and b in $\mathbb{R}^n$. We denote by $B_q(x, r)$, respectively $B(x, r)$ the ball of center x and radius r if we consider on $\mathbb{R}^n$ the chordal metric, respectively the euclidean metric and in the same way we denote for a set $A \subset I\!R^n$ by $q(A)$, respectively $d(A)$ the diameter of A. We denote by $W^{1,p}_{\text{loc}}(D, I\!R^n)$ the Sobolev space of all functions $f : D \to I\!R^n$ which are locally in $L^p(D)$, together with their first order weak partial derivatives.

If W is a family of mappings $f : D \to I\!R^n$, we say that W is bounded if for every $K \subset D$ compact, there exists $M(K) > 0$ such that $|f(x)| \le M(K)$ for every $x \in K$ and every $f \in W$. If X, Y are metric spaces and W is a family of mappings $f : X \to Y$ we say that W is equicontinuous at x if for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $\mathrm{d}(f(x), f(y)) \le \varepsilon$ if $\mathrm{d}(y, x) \le \delta_\varepsilon$ for every $f \in W$, and we say that W is equicontinuous if it is equicontinuous at every point $x \in X$. We say that W is a normal family if every sequence of W has a subsequence which converges uniformly on the compact subsets of X to a map $f : X \to Y$.

Lemma 1. *Let $n \ge 2, D \subset I\!R^n$ be an unbounded domain, $K : D \to [0, \infty]$ measurable and finite a.e., $\mathcal{A} : [0, \infty) \to [0, \infty)$ be smooth, strictly increasing, with $\mathcal{A}(0) = 0$, $\lim\limits_{t\to\infty} \mathcal{A}(t) = \infty$, satisfying condition $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ such that there exists $r > 0$ such that $\int\limits_{D\cap CB(0,r)} \exp(\mathcal{A}(K(x))) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x < \infty$. Then $M_{K^{n-1}}(\Delta(\overline{B}(0, r) \cap D, CB(0, s) \cap D, D) \to 0$ when $s \to \infty$ and $r > 1$ is keeped fixed.*

Proof. Let $g : \overline{I\!R^n} \to \overline{I\!R^n}$ be defined by $g(x) = \frac{x}{|x|^2}$ if $x \in \overline{I\!R^n}$ and let $\Gamma_{rs} = \Delta(\overline{B}(0, r) \cap D, CB(0, s) \cap D, D)$, $\Lambda_{rs} = \Delta(g(D) \cap \overline{B}(0, \frac{1}{s}), g(D) \cap CB(0, \frac{1}{r}), g(D))$ for $1 < r < s$. We show that $M_{K^{n-1}}(\Gamma_{rs}) = M_{K^{n-1}og}(\Lambda_{rs})$. Let $\rho' \in F(\Gamma_{rs})$ and $\rho : \overline{I\!R^n} \to [0, \infty]$ be defined by $\rho(x) = \rho'(g(x)) \cdot |g'(x)|$ if $x \in D$, $\rho(x) = 0$ if $x \notin D$. Then $\rho \in \Gamma(\Lambda_{rs})$ and $M_{K^{n-1}og}(\Lambda_{rs}) \le \int\limits_{I\!R^n} \rho^n(x) \cdot K^{n-1}(g(x))\mathrm{d}x = \int\limits_D \rho'^n(g(x))|g'(x)|^n \cdot K^{n-1}(g(x))\mathrm{d}x = \int\limits_D \rho'^n(g(x)) \cdot K^{n-1}(g(x)) \cdot J_g(x)\mathrm{d}x = \int\limits_{g(D)} \rho'^n(y) \cdot K^{n-1}(y)\mathrm{d}y \le \int\limits_{I\!R^n} \rho'^n(y) \cdot K^{n-1}(y)\mathrm{d}y$. This implies that $M_{K^{n-1}og}(g(\Gamma_{rs})) = M_{K^{n-1}og}(\Lambda_{rs}) \le M_{K^{n-1}}(\Gamma_{rs})$. By the same argument, we have $M_{K^{n-1}}(\Gamma_{rs}) = M_{K^{n-1}ogog}(g(\Lambda_{rs})) \le M_{K^{n-1}og}(\Lambda_{rs})$ and we have

$$M_{K^{n-1}}(\Gamma_{rs}) = M_{K^{n-1}og}(\Lambda_{rs}). \tag{1}$$

We find

$$\int\limits_{g(D)\cap B(0,\frac{1}{r})} \exp(\mathcal{A}(K(g(x))))\mathrm{d}x = \int\limits_{g(D)\cap B(0,\frac{1}{r})} \exp(\mathcal{A}(K(g(x))))\cdot J_{g^{-1}}(g(x))\cdot J_g(x)\mathrm{d}x =$$

$$\int\limits_{D\cap CB(0,r)} \exp(A(K(y)))\cdot J_{g-1}(y)\,\mathrm{d}y = \int\limits_{D\cap CB(0,r)} \exp(\mathcal{A}(K(y)))\cdot \frac{1}{|y|^{2n}}\mathrm{d}y < \infty.$$

Let now $\Gamma'_{rs} = \Delta\left(\overline{B}\left(0, \frac{1}{s}\right), CB\left(0, \frac{1}{r}\right), B\left(0, \frac{1}{r}\right) \setminus \overline{B}\left(0, \frac{1}{s}\right)\right)$ for $1 < r < s$ and let $Q : I\!R^n \to [0, \infty]$ be defined by $Q(x) = K(g(x))$ if $x \in g(D), Q(x) = 1$ if $x \notin g(D)$. Since $\int\limits_{B(0,\frac{1}{r})} \exp(\mathcal{A}(Q(x)))\,\mathrm{d}x < \infty$, we see from Theorem

5.3, page 24, [7] that $\mathrm{cap}_{Q^{n-1}}\left(B\left(0,\frac{1}{r}\right),\overline{B}\left(0,\frac{1}{s}\right)\right)\to 0$ when $s\to\infty$ and $r>0$ is keeped fixed. We have $M_{K^{n-1}og}(\Lambda_{rs})\leq M_{Q^{n-1}}(\Lambda_{rs})\leq M_{Q^{n-1}}(\Gamma'_{rs})\leq \mathrm{cap}_{Q^{n-1}}(B(0,\frac{1}{r}),\overline{B}(0,\frac{1}{s}))$ and this implies that

$$(2).\qquad M_{K^{n-1}og}(\Lambda_{rs})\to 0 whens \to\infty\ andr>0iskeepedfixed$$

Using (1) and (2), the proof is finished.

Theorem 1. Let $D\subset I\!R^n$ be open, $f:D\to I\!R^n$ be a map of finite distortion, satisfying condition $(\mathcal{A})$ and let $K:D\to[0,\infty]$ be the dilatation map of f. Then $K\in L^p_{\mathrm{loc}}(D)$ for every $p>0$ and $f\in W^{1,p}_{\mathrm{loc}}(D,I\!R^n)$ for every $1\leq p<n$.

Proof. Suppose first that $p\geq 1$ and let $g:(0,\infty)\to I\!R_+$ be defined by $g(t)=\exp\left(\mathcal{A}\left(t^{\frac{1}{p}}\right)\right)$ for $t>0$. Then g is strictly increasing on $(0,\infty)$ and from Lemma 2.4 [8] there exists $b>1$ such that g is convex on (b,∞). Let $F:D\to[0,\infty]$, $F(x)=K(x)$ if $K(x)>b$, $F(x)=b$ if $K(x)\leq b$. Then F is measurable and we have, using Jensen's inequality, that $g(\int_B K^p(x)\mathrm{d}x)\leq g(\int_B F^p(x)\mathrm{d}x)\leq g(\int_B F^p(x)\mathrm{d}x/\mu_n(B))\leq \int_B g(F^p(x))\mathrm{d}x/\mu_n(B)$, where $B\subset D$ is compact and such that $\mu_n(B)<1$.

It results that

$$\int_B K^p(x)\,\mathrm{d}x\leq\left[\mathcal{A}^{-1}\left(\log\left(\max\left\{1,\int_B\exp\left(\mathcal{A}\left(F(x)\right)\right)\mathrm{d}x/\mu_n(B)\right\}\right)\right)\right]^p<\infty.$$

If $0<p<1$, then $\int_B K^p(x)\,\mathrm{d}x\leq\left(\int_B K(x)\,\mathrm{d}x\right)^p<\infty$. We therefore proved that $\int_B K^p(x)\,\mathrm{d}x<\infty$ if $B\subset D$ is compact with $\mu_n(B)<1$ for every $p>0$, hence $K\in L^p_{\mathrm{loc}}(D)$ for every $p>0$.

Let now $1\leq p<n$. Then we have from [7], Corollary 5.5, [5] Prop. 2.5 and [6] Theorem 1.1. that f is continuous and either constant on D, or open and discrete. We can suppose that f is open, discrete on D and let B a ball with $\overline{B}\subset D$. Then $N(f,B)=\sup_{y\in I\!R^n}\mathrm{Card}\,f^{-1}(y)\cap B<\infty$ and from Theorem 6.3.2, page 107 [3], we see that $\int_B J_f(x)\,\mathrm{d}x<\infty$. We have, using Hölder's inequality, that

$$\int_B|f'(x)|^p\,\mathrm{d}x\leq\left(\int_B K(x)^{\frac{p}{n-p}}\,\mathrm{d}x\right)^{\frac{n-p}{n}}\cdot\left(\int_B J_f(x)\,\mathrm{d}x\right)^{\frac{p}{n}}<\infty.$$

Since f is continuous and $f \in W^{1,1}_{\mathrm{loc}}(D, I\!R^n)$, we see that f is ACL^1 on B, hence f is ACL, and from Prop.1.2, page 6, [11], we see that $f \in W^{1,p}_{\mathrm{loc}}(D, I\!R^n)$ for $1 \le p < n$.

3. Proofs of the main results.

Theorem 2. *Let* $n \ge 3$, $B = \{a_1, \dots, a_j, \infty\}$, $K \subset I\!R^n \setminus B$ *closed in* $I\!R^n \setminus B$ *with* $\operatorname{int} K = \phi$, $f : I\!R^n \setminus (K \cup B) \to I\!R^n$ *be a map of finite distortion, of* $K(x)$ *dilatation, satisfying condition* $(\mathcal{A})$ *and a local homeomorphism such that* $\operatorname{int} C(f,K) = \phi$, $C(f,K)$ *is closed and* $I\!R^n \setminus C(f,K)$ *is connected and let* $E = K \cup B \cup f^{-1}(C(f,K))$. *Suppose that there exists* $\delta_p > 0$ *such that* $\int_{B(a_p,\delta_p)} \exp(\mathcal{A}(K(x)))\,\mathrm{d}x < \infty$, $p = 1, \dots, j$ *and that there exists* $\delta_0 > 0$ *such that* $\int_{CB(0,\delta_0)\cap(I\!R^n\setminus(K\cup B))} \exp(\mathcal{A}(K(x))) \cdot \frac{1}{|x|^{2n}} < \infty$. *Then*

1) *either we can lift any path* $p : [0,1] \to I\!R^n \setminus C(f,K)$ *from every point* x *with* $f(x) = p(0)$, *or we can eliminate some set* $A \subset B$, *extending* f *by continuity to a local homeomorphism on* A, *such that the extended map (also denoted by* f*) lifts any path* $p : [0,1] \to I\!R^n \setminus C(f,K)$ *from every point* x *with* $f(x) = p(0)$, *and if* $A \ne \phi$ *and* $\infty \in A$, *some lifted path can go through* ∞.

2) *if* $C(f,K)$ *is compact, then* $\operatorname{int} E = \phi$ *and* f *is injective on* $I\!R^n \setminus E$ *and if* f *can be continuously extended on* K, *then* $f|\left(\overline{I\!R^n}\setminus K\right)\setminus(B\setminus A) : (\overline{I\!R^n}\setminus K)\setminus(B\setminus A) \to I\!R^n \setminus C(f,K)$ *is a homeomorphism. Finally if* f *can be continuously extended to an open map on* K, *then* $f : \overline{I\!R^n} \to \overline{I\!R^n}$ *is a homeomorphism.*

Proof. Let $\Gamma = \{\gamma : [0,1] \to I\!R^n \setminus (K \cup B) \text{ path} | \ \gamma \text{ has some limit point in } B\}$. We see from Lemma 1 and Theorem 5.3, page 24, [7] that $M_{K^{n-1}}(B) = 0$, hence $M_{K^{n-1}}(\Gamma) = 0$. Using (i), we see that $M(f(\Gamma)) \le M_{K^{n-1}}(\Gamma) = 0$, hence $M(f(\Gamma)) = 0$ and from Theorem 1 [2], 1) is proved.

Suppose now that $C(f,K)$ is compact and let $r > 0$ be such that $C(f,K) \subset B(0,r)$, let $0 < r < s$ and let $x \in I\!R^n \setminus (K \cup (B \setminus A))$ be such that $f(x) \in CB(0,s)$. Let Q be the component of $f^{-1}(CB(0,r))$ which contains x and let U be the component of $f^{-1}(CB(0,s))$ which contains x. Then $\overline{U} \subset Q$ and since $CB(0,r)$ is simply connected and $f|Q : Q \to CB(0,r)$ is a local homeomorphism which lifts the paths, we see that $f|Q : Q \to CB(0,r)$ is a homeomorphism, and hence $f|\overline{U} : \overline{U} \to \overline{CB(0,s)}$ is also a homeomorphism. Since $f|\partial U : \partial U \to S(0,s)$ is a homeomorphism, we use Jordan's theorem to see that $I\!R^n \setminus \partial U$ has exactly two components, one of them being U, and since $\overline{U}$ is homeomorphic to $\overline{CB(0,s)}$, we see that U is unbounded. If U_0 is any other component of $f^{-1}(CB(0,s))$, we see by the same argument that $f|\overline{U}_0 : \overline{U}_o \to \overline{CB(0,s)}$ is a homeomorphism, that ∂U_0 bounds a Jordan domain and U_0 is the unbounded component of $I\!R^n \setminus \partial U_0$. Let $\alpha > 0$ be such that $\partial U \cup \partial U_0 \subset B(0,\alpha)$, Then $CB(0,\alpha) \subset U \cap U_0$, hence $U = U_0$ and we obtain that $f^{-1}(CB(0,s))$ has a single component U on which f is injective, $K \subset C\overline{U}$ and $U = f^{-1}CB(0,s)$.

Since f is a local homeomorphism on $\mathbb{R}^n \setminus (K \cup B)$ and $\operatorname{int} C(f,K) = \phi$, we see that $\operatorname{int} E = \phi$. Let $x_1, x_2 \in \mathbb{R}^n \setminus E$ be such that $f(x_1) = f(x_2) = y$ and let $z \in CB(0,s)$ and $p : [0,1] \to \mathbb{R}^n \setminus C(f,K)$ be a path such that $p(0) = y$ and $p(1) = z$. We can lift p from x_1 and x_2 and find some path $q_i : [0,1] \to \mathbb{R}^n$ such that $q_i(0) = x_i$, $f \circ q_i = p$, $i = 1,2$. Since $q_i(1) \in f^{-1}(CB(0,s)) = U$, $f(q_i(1)) = p(1)$, $i = 1,2$ and we proved that f is injective on U, we obtain that $q_1(1) = q_2(1)$. We use now the property of the uniqueness of path lifting for local homeomorphisms to conclude that $x_1 = x_2$. We therefore proved that f is injective on $\mathbb{R}^n \setminus E$.

Suppose now that f can be continuously extended on K and let $a \in \overline{\mathbb{R}^n} \setminus (B \setminus A)$ be a point such that f is open at a. We show that $\{a\} = f^{-1}(f(a))$. Indeed, if there exists $b \neq a, b \in \overline{\mathbb{R}^n} \setminus (B \setminus A)$ such that $f(a) = f(b)$, let $U_1 \in \mathcal{V}(a)$ and $V \in \mathcal{V}(f(a))$ be such that $f(U_1) = V$. Since f is continuous at b, we can find $U_2 \in \mathcal{V}(b)$ with $\overline{U_1} \cap \overline{U_2} = \phi$ such that $f(U_2) \subset V$ and since $\operatorname{int} E = \phi$, we can find $\alpha \in U_2 \setminus E$. Then we can find $\beta \in U_1 \setminus E$ with $f(\alpha) = f(\beta)$, which represents a contradiction, since we proved that f is injective on $\mathbb{R}^n \setminus E$. It results now immediately that $f| \left(\overline{\mathbb{R}^n} \setminus K\right) \setminus (B \setminus A) : \left(\overline{\mathbb{R}^n} \setminus K\right) \setminus (B \setminus A) \to \mathbb{R}^n \setminus f(K)$ is a homeomorphism, and if we additionally suppose that f is also an open map on K, we find that $f|\overline{\mathbb{R}^n} \setminus (B \setminus A) : \overline{\mathbb{R}^n} (B \setminus A) \to \mathbb{R}^n$ is a homeomorphism. In the last case we see by topological reasons that $B \setminus A = \{a\}$, with $a \in B$ and that $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ can be extended to a homeomorphism.

Proof of Theorem 3. We take $K = \phi$ and $B = \{\infty\}$ in Theorem 2.

Proof of Theorem 4. Since $M_{K^{n-1}}(F) = 0$, we see that cap $F = 0$ and hence that $\mathbb{R}^n \setminus F$ is a domain, and since f is nonconstant on $\mathbb{R}^n \setminus F$, we see that f is open, discrete on $\mathbb{R}^n \setminus F$. Suppose that cap $Cf(\mathbb{R}^n \setminus F) > 0$ and let $K \subset \mathbb{R}^n \setminus F$ be compact, connected, with Card $K > 1$. Then $E = (f(\mathbb{R}^n \setminus F), f(K))$ is a capacitor and $q(f(K)) > 0$, and from Lemma 2.6, page 65, [11], there exists $\delta > 0$ such that $\delta <$ cap E. Let $\Gamma' = \Delta(f(K), Cf(\mathbb{R}^n \setminus F), \mathbb{R}^n)$ and let Γ be the family of all maximal liftings of some path from Γ' starting from some point of K. Then $\Gamma' < f(\Gamma)$ and if $\gamma \in \Gamma$, then γ has at least a limit point in $F \cup \{\infty\}$. Since $\displaystyle\int_{CB(0,r)} \exp(\mathcal{A}(K(x))) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x < \infty$, we see from Lemma 1 that $M_{K^{n-1}}(\infty) = 0$, hence $M_{K^{n-1}}(F \cup \{\infty\}) = 0$ and this implies that $M_{K^{n-1}}(\Gamma) = 0$. We use now (i) and we obtain that $\delta <$ cap $(E) = M(\Gamma') \leq M(f(\Gamma)) \leq M_{K^{n-1}}(\Gamma) = 0$, which represents a contradiction. We therefore proved that cap $Cf(\mathbb{R}^n \setminus F) = 0$.

Remark 1. The condition $\displaystyle\int_{CB(0,r)} \exp(\mathcal{A}(K(x))) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x < \infty$, which ensures that $M_{K^{n-1}}(\infty) = 0$, and used in our generalizations given to Zoric's theorem and Picard's theorem can be realized for instance if $K(x) \leq K_p$ for $x \in B(0,p)$, $p \geq p_0$ and $\limsup\limits_{p\to\infty} \frac{K_p}{\ln p} = \alpha < n$ (i.e. it holds for a locally quasiregular map having some logarithmic growth of the constant of quasiregularity near ∞). Indeed, we take $\mathcal{A}(t) = t$ and if $p_1 \geq p_0$ is such that $K_p < \alpha \cdot \ln p$

for $p \geq p_1$, we have

$$\int\limits_{CB(0,p_1)} \exp\left(\mathcal{A}\left(K\left(x\right)\right)\right) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x = \sum_{p \geq p_1} \int\limits_{B(0,p+1) \setminus B(0,p)} \exp\left(\mathcal{A}\left(K\left(x\right)\right)\right) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x \leq$$

$$\sum_{p \geq p_1} \int\limits_{B(0,p+1) \setminus B(0,p)} \exp\left(K_{p+1}\right) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x \leq C_0 \cdot \sum_{p \geq p_1} \frac{(p+1)^{\alpha}\left((p+1)^n - p^n\right)}{p^{2n}} \leq$$

$$C_1 \cdot \sum_{p \geq p_1} \frac{1}{p^{n-\alpha+1}} < \infty.$$

Here C_0 and C_1 are some constants.

We can generalize in this way Theorem 8 and Theorem 9 from [2], obtaining a generalization of Zoric's theorem for locally quasiregular mappings having a logarithmic growth of the constant of quasiregularity near ∞.

In the same manner, the condition $\int\limits_{B(b,\rho)} \exp\left(\mathcal{A}\left(K\left(x\right)\right)\right) \mathrm{d}x < \infty$, which ensures that $M_{K^{n-1}}\left(b\right) = 0$ is realized for instance if $K\left(x\right) \leq K_p$ on $CB(b, \frac{1}{p})$ for $p \geq p_0$ and $\limsup\limits_{p \to \infty} \frac{K_p}{\ln p} = \alpha < n$. This condition holds for locally quasiregular mappings having some logarithmic growth of the constant of quasiregularity near the critical point $b \in \partial D$ and it can be used in Theorem 2 and Theorem 4.

Remark 2. The condition $\int\limits_{CB(0,r)} \exp\left(\mathcal{A}\left(K\left(x\right)\right)\right) \cdot \frac{1}{|x|^{2n}} \mathrm{d}x < \infty$ used in Theorem 4 is necessary, since we can find bounded homeomorphisms $f : I\!R^n \to B\left(0,1\right)$ of finite distortion and satisfying condition $(\mathcal{A})$ (see [7], page 28, or [6]).

Proof of Theorem 5. Let $x \in D$ be fixed, and $\varepsilon > 0$ with $\overline{B}\left(x, \varepsilon\right) \subset D$. We fix $\varepsilon > 0$ and suppose that there exists $\rho > 0, r_p \to 0$ and $f_p \in W$ such that $q\left(f_p\left(\overline{B}\left(x, r_p\right)\right)\right) \geq \rho$ for every $p \in I\!N$. Then f_p are open, discrete maps for every $p \in I\!N$ and we can find $\delta > 0$ such that $\delta \leq \mathrm{cap}\ \left(CM, f_p\left(\overline{B}\left(x, r_p\right)\right)\right)$ for every $p \in I\!N$ (see Lemma 2.6, page 65, [11]). We have, using (ii) that $\delta \leq \mathrm{cap}(CM, f_p(\overline{B}(x, r_p))) \leq \mathrm{cap}(f_p(B(x, \varepsilon)), f_p(\overline{B}(x, r_p))) \leq \mathrm{cap}_{K^{n-1}}(B(x,\varepsilon), \overline{B}(x,r_p)) \to 0$ if $r_p \to 0$ and $\varepsilon > 0$ is keeped fixed, which represents a contradiction. We therefore proved that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $f\left(B\left(x, \delta_\varepsilon\right)\right) \subset B_q\left(f\left(x\right), \varepsilon\right)$ for every $f \in W$, i.e. W is equicontinuous at x.

Proof of Theorem 6. Let $x \in D, \delta > 0$ such that $\overline{B}\left(x, \delta\right) \subset D$ and $M > 0$ be such that $\left|f\left(y\right)\right| \leq M$ for every $y \in \overline{B}\left(x, \delta\right)$ and every $f \in W$ and let $W_0 = \left\{f \left|\overline{B}\left(x, \delta\right)\right| f \in W\right\}$. Using Theorem 5, we see that W_0 is equicontinuous at x, hence W is equicontinuous at x. We proved that W is equicontinuous, and we take on D the euclidean metric and on $\overline{I\!R^n}$ the chordal metric. Using Theorem 20.4, page 68 [12], if $\left(f_p\right)_{p \in I\!N}$ is a sequence of mappings from W, we can find a subsequence $\left(f_{p_k}\right)_{k \in I\!N}$ and a map $f : D \to \overline{I\!R^n}$ such that $f_{p_k} \to f$ uniformly on the compact subsets from D. If $x \in D$ is fixed and $M_x > 0$ is such that $|f_{p_k}(x)| \leq M_x$ for every $k \in \mathbb{N}$, we see that $|f(x)| \leq M_x$, hence f takes finite values and it results that W is a normal family.

Theorem 7. *Let $D \subset \mathbb{R}^n$ be open, W be a family of mappings $f : D \to \mathbb{R}^n$ of finite distortion, having the same distortion map K, satisfying condition $(\mathcal{A})$ and let $x \in D$ be such that there exists $r, \delta > 0$ such that $\overline{B}(x,\delta) \subset D$ and $f(B(x,\delta)) \subset B(f(x), r)$ for every $f \in W$. Then W is equicontinuous at x, and we take on D and on $\mathbb{R}^n$ the euclidean distance.*

Proof. Let $f \in W$. If f is constant on $B(x,\delta)$, then, $f(B(x,\rho)) = \{f(x)\} \subset B(f(x), \varepsilon)$ for every $0 < \rho < \delta$, $0 < \varepsilon < r$. Suppose that f is open, discrete on $B(x,\delta)$ and let $0 < \varepsilon < r$ and $0 < \rho < \delta$ and $E = \left(B(x,\delta), \overline{B}(x,\rho)\right)$. Then $f(E) = (f\left(B(x,\delta), f(\overline{B}(x,\rho)\right)$ is a capacitor and let ν_n be the function defined in [11], page 60.

We keep $\delta > 0$ fixed and we let $\rho \to 0$ such that $\operatorname{cap}_{K^{n-1}}(E) \leq \nu_n\left(\frac{\varepsilon}{r}\right)$ for $0 < \rho < \delta_\varepsilon$. Then $\nu_n\left(\frac{|f(y)-f(x)|}{r}\right) \leq \operatorname{cap} f(E) \leq \operatorname{cap}_{K^{n-1}}(E) \leq \nu_n\left(\frac{\varepsilon}{r}\right)$ for every $y \in B(x,\rho)$, and since ν_n is strictly increasing, we find that $|f(y) - f(x)| \leq \varepsilon$ for every $y \in B(x, \delta_\varepsilon)$ and every $f \in W$, i.e. W is equicontinuous at x.

Remark 3. The last theorem shows the interesting thing thet if W is a family of mappings $f : D \to \mathbb{R}^n$ of finite distortion, having the same distortion map K and satisfying condition $(\mathcal{A})$ and x is a point from D, then it is sufficient to exist a single $r > 0$ and a single $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x), r)$ for every $f \in W$ to obtain the equicontinuity of the family W in the point x. Taking $f : D \to \mathbb{R}^n$ a map of finite distortion and satisfying condition $(\mathcal{A})$, $f_\lambda = f + \lambda$, $W = (f_\lambda)_{\lambda \in \mathbb{R}^n}$, we see immediately that W is equicontinuous. Since $\bigsqcup_{g \in W} g(x) = \mathbb{R}^n$ for every $x \in D$, we cannot apply Theorem 5 to deduce the equicontinuity of the family W, but we can use Theorem 7.

Theorem 8. *Let $D \subset \mathbb{R}^n$ be a domain, W be a family of homeomorphisms $f : D \to f(D)$ of finite distortion, having the same dilatation* map *K, satisfying condition $(\mathcal{A})$ such that there exists $r > 0$ such that for every $f \in W$, there exists $a_f, b_f \notin \operatorname{Im} f$ with $q(a_f, b_f) \geq r$. Then W is equicontinuous, and we take the euclidean distance on D and the chordal distance on $\mathbb{R}^n$.*

Proof. We follow the proof from Theorem 19.2, page 65. Let λ_n be the function defined in [12], 12.4, page 38 and let $x_0 \in D$ and $0 < \varepsilon < r$. Let $Q_0 = B(x_0, \alpha)$, $Q_1 = B(x_0, \beta)$ be such that $0 < \alpha < \beta$ and $\overline{B}(x_0, \beta) \subset D$ and let $A = R(\overline{Q}_0, CQ_1)$. Then $f(A) = R(f(\overline{Q}_0), Cf(Q_1))$ and $q(Cf(Q_1)) \geq q(a_f, b_f) \geq r, q(f(\overline{Q}_0)) \geq q(f(x), f(x_0))$ for every $x \in \overline{Q}_0$. Let $x \in \overline{Q}_0$ be fixed and $t = \min\{q(f(x_0), f(x)), r\}$. We keep $\beta > 0$ fixed and we choose α small enouh such that $M_{K^{n-1}}(\Gamma_A) \leq \lambda_n(\varepsilon)$. Then $\lambda_n(t) \leq M(f(\Gamma_A)) \leq M_{K^{n-1}}(\Gamma_A) \leq \lambda_n(\varepsilon)$ and since λ_n is increasing, we see that $t \leq \varepsilon$, and since $\varepsilon < r$, we obtain that $t = q(f(x), f(x_0))$. We therefore proved that $q(f(x), f(x_0)) \leq \varepsilon$ for every $x \in U_0$ and every $f \in W$.

Theorem 9. *Let $D \subset \mathbb{R}^n$ be a domain, W be a family of homeomorphisms $f : D \to f(D)$ of finite distortion, having the same dilatation* map *K, satisfying condition $(\mathcal{A})$ and such that one of the following condition is satisfied:*

1) *there exists $x_1, x_2 \in D$ and $r > 0$ such that each $f \in W$ omits a point a_f*

with $q(a_f, f(x_i)) \geq r$ *for* $i = 1, 2$.

2) *there exists* $x_i \in D$ *and* $r > 0$ *such that* $q(f(x_i), f(x_j)) \geq r$ *for* $i \neq i, i, j = 1, 2, 3$ *and every* $f \in W$.

Then W *is equicontinuous.*

Proof. Suppose that condiotion 1) is satisfied and let $D_k = D \setminus \{x_k\}$ for $k = 1, 2$. Using Theorem 8, we see that the families $W_k = \{f \,|D_k|\, f \in W\}$ are equicontinuous on D_k for $k = 1, 2$, hence W is equicontinuous. Suppose now that condition 2) is satisfied and let $D_{ij} = D \setminus \{x_i, x_j\}$ and $W_{ij} = \{f \,|D_{ij}|\, f \in W\}$ for $i, j = 1, 2, 3$. We see from Theorem 8 that the families W_{ij} are equicontinuous on D_{ij} for $i, j = 1, 2, 3$, hence W is equicontiuous.

Corollary 1. *Let* $D \subset I\!R^n$ *be a domain,* W *be a family of homeomorphisms* $f : D \to f(D)$ *of finite distortion, having the same dilatation* map K, *satisfying condition* $(\mathcal{A})$ *such that* $f(a_i) = b_i, i = 1, 2, 3$ *for every* $f \in W$, *where* a_1, a_2, a_3 *are three different points from* D *and* b_1, b_2, b_3 *are three different points from* $I\!R^n$. *Then* W *is equicontinuous.*

Theorem 10. *Let* D, D_j *be domains in* $I\!R^n$, $f_j : D \to D_j$ *be homeomorphisms of finite distortion, having the same dilatation* map K, *satisfying condition* $(\mathcal{A})$ *and such that* $f_j \to f$. *Then, if* Card Im $f \geq 3$, *it results that* $f : D \to D'$ *is a homeomorphism onto a domain* D' *from* $\overline{I\!R^n}$, *and if* $f_j \to f$ *uniformly on the compact subsets from* D, *then* f *is either constant, or it is a homeomorphism onto a domain from* $\overline{I\!R^n}$.

Proof. We follow the proof fromTheorem 21.1, page 9, [12]. Let b_1, b_2, b_3 be three different points from Im f, $a_k \in D$ such that $f(a_k) = b_k$ for $k = 1, 2, 3$, and let $r > 0$ be such that $q(f(a_i), f(a_j)) \geq r$ for $i \neq j$, $i, j = 1, 2, 3$. Then there exists $j_0 \in I\!N$ such that $q(f_j(a_i), f_j(a_k)) \geq \frac{r}{2}$ for $i \neq k$, $i, k = 1, 2, 3$ and $j \geq j_0$, and from Theorem 9, condition 2), it results that the family $W = (f_j)_{j \geq j_0}$ is equicontinuous. Using Theorem 20.3, page 68, [12], we see that $f_j \to f$ uniformly on the compact subsets from D and hence f is continuous on D. Using Brouwer's theorem, it is enough to prove that f is injective on D.

Suppose that there exists $z_1, z_2 \in D, z_1 \neq z_2$ such that $f(z_1) = f(z_2)$ and let $r > 0$ be such that $z_2 \notin \overline{B}(z_1, r)$. Then $f_j(S(z_1, r))$ separates the points $f_j(z_1)$ and $f_j(z_2)$, hence we can find $x_j \in S(z_1, r)$ such that

$$q(f_j(x_j), f_j(z_1)) \leq q(f_j(z_1), f_j(z_2)) \; foreveryj \in I\!N. \tag{1}$$

Taking a subsequence, we can suppose that $x_j \to x \in S(z_1, r)$ and since W is equicontinuous at x, we have $q(f_j(x_j), f(x)) \leq q(f_j(x_j), f_j(x)) + q(f_j(x), f(x)) \to 0$. Letting $j \to \infty$ in (1), we find that $q(f(x), f(z_1)) \leq q(f(z_1), f(z_2)) = 0$, hence $f(z_1) = f(x)$.We proved that f is not injective in any neighbourhood of z_1.

We prove now that every point $x \in D$ has a neighbourhood U such that f is either injective on U, or it is constant on U. Indeed, suppose that this thing is fakse. Then we can find U a ball centered at x and u_1, u_2, u_3 distinct points in U with $f(u_1) \neq f(u_2), f(u_2) = f(u_3)$ and since W is equicontinuous at x, we can take U such that $qC(f_j(U)) \geq 1$ for every $j \in I\!N$, and we also take U such that $\int_U K^{n-1}(z)\,dz < \infty$. We join u_1 and u_2 by an arc J_0 from U and we choose J_1

an arc joining u_3 with a point $u_4 \in \partial U$ in U. Then $A = R(\operatorname{Im} J_0, C(U \setminus \operatorname{Im} J_1))$ is a ring and $A_j = f_j(A)$ is a ring $R(C_{oj}, C_{1j})$, where $C_{oj} = f_j(J_0), C_{1j} = Cf_j(U \setminus \operatorname{Im} J_1)$ for $j \in I\!N$. Let $\lambda_n(r,t)$ be the function defined in [12], 12.6, page 39 and let $r_j = q(f_j(u_1), f_j(u_2)), t_j = q(f_j(u_2), f_j(u_3))$ for $j \in I\!N$. Then $q(C_{oj}) \geq r_j$, $q(C_{oj}, C_{1j}) \leq t_j$ for $j \in I\!N, t_j \to 0$, and taking a subsequence, we can suppose that $r_j \geq r$ for every $j \in I\!N$. It results that $M(\Gamma_{A_j}) \geq \lambda_n(r_j, t_j) \to \infty$ for $j \to \infty$. Let now $\delta = d(\operatorname{Im} J_0, C(U \setminus \operatorname{Im} J_1)) > 0$ and let $\rho : \overline{I\!R^n} \to [0, \infty]$ be defined by $\rho(z) = \frac{1}{\delta}$ if $z \in U, \rho(z) = 0$ if $z \notin U$. Then $\rho \in F(\Gamma_A)$, hence $M(\Gamma_A) \leq \int\limits_{I\!R^n} \rho^n(z) \cdot K^{n-1}(z)dz = \frac{1}{\delta^n} \cdot \int\limits_U K^{n-1}(z)dz < \infty$. Using (i), we obtain that $\infty > M_{k^{n-1}}(\Gamma_A) \geq M(\Gamma_{A_j}) \to \infty$, which represents a contradiction.

Let now $Q_1 = \{z \in D |$ there exists $V \in \mathcal{V}(z)$ such that f is injective on $V\}$, and let $Q_2 = \{x \in D |$ there exists $V \in \mathcal{V}(z)$ such that f is constant on $V\}$. Then $D = Q_1 \cup Q_2$ and since $z_1 \notin Q_1$ it results that $z_1 \in Q_2$, hence $Q_2 \neq \phi$. Since D is connected, it results that $D = Q_2$, i.e. f is locally constant on D and hence f is constant on D. We obtained a contradiction, because $\operatorname{Card} \operatorname{Im} f \geq 3$. We therefore proved that $f : D \to D'$ is a homeomorphism onto a domain D' from $\overline{I\!R^n}$.

Theorem 11. *Let D, D' be domains in $I\!R^n$ with* Card $\partial D' \geq 2, F \subset D$ *be compact and let W be a family of homeomorphisms $f : D \to D'$ of finite distortion, having the same dilatation* map K *and satisfying condition* $(\mathcal{A})$. *Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $f \in W$ and $q(f(F), \partial D') < \delta$, it results that $q(f(F)) < \varepsilon$.*

Proof. Suppose that the theorem is not true. Then there exists $\varepsilon > 0$ and a sequence $(f_j)_{j \in I\!N}$ from W such that $q(f_j(F), \partial D') < \frac{1}{j}$ and $q(f_j(F)) \geq \varepsilon$ for every $j \in I\!N$. Since Card $\partial D' \geq 2$, there exists at least two points $y_k \notin \operatorname{Im} f_j$ for every $j \in I\!N$ and $k = 1, 2$ and from Theorem 8 we see that the family $(f_j)_{j \in I\!N}$ is equicontinuous. Taking a subsequence, we can find a map $f : D \to \overline{I\!R^n}$ such that $f_k \to f$ uniformly on the compact subsets from D. Using Theorem 10, we see that either f is a constant map on D, or $f : D \to G$ is a homeomorphism onto a domain G from $\overline{I\!R^n}$. Since $q(f_j(F)) \geq \varepsilon > 0$ for $j \in I\!N$, it results that f cannot be constant on D, hence $f : D \to G$ is a homeomorphism.

We show that $G \subset D'$. Indeed, let $y \in G$ and $x \in D$ be such that $y = f(x)$ and let $U \in \mathcal{V}(x)$ be such that $\overline{U} \subset D, f(x) \notin f(\partial U)$ and let $r = q(f(x), f(\partial U))$. Since $f_j \to f$ uniformly on $\overline{U}$, there exists $j_0 \in I\!N$ such that $q(f_j(z), f(z) < \frac{r}{3}$ for every $z \in \overline{U}$ and every $j \geq j_0$. Then if $V = B_q(y, \frac{r}{3})$, we see that $V \cap f_j(U) \neq \phi, \partial f_j(U) = f_j(\partial U)$ and $f_j(\partial U) \cap V = \phi$ for $j \geq j_0$. We have $V = (V \cap f_j(U)) \cup (V \cap \partial f_j(U)) \cup (V \cap C\overline{f_j(U)}) = (V \cap f_j(U)) \cup (V \cap C\overline{f_j(U)})$ for $j \geq j_0$, and since V is connected, we see that $V = V \cap f_j(U)$ and hence $V \subset f_j(U)$ for $j \geq j_0$. It results that $y \in V \subset f_j(U) \subset D'$. We therefore proved that $G \subset D'$, hence $\delta = q(f(F), \partial D')) > 0$. Since $f_j \to f$ uniformly on F and $q(f_j(F), \partial D')) \to 0$, we obtained a contradiction. The theorem is now proved.

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