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## Mappings of finite distortion: boundary extension

Mihai Cristea

Abstract: We generalize some results of O.Martio and S.Rickman [10] concerning the boundary extension and the asymptotic values of quasiregular mappings. This results are known for bounded analytic functions as Lindelöf's theorem, Iversen's theorem, Iversen-Tsuji's theorem, Noshiro's theorem and Cartwrigtht's theorem. We also generalize a result of M.Vuorinen concerning the boundary extension of closed quasiregular mappings. We also prove some maximum principles, some eliminability theorems and a theorem concerning the density of the points $b \in \partial B^n$ at which a map $f : B^n \to I\!R^n$ of finite distortion has some asymptotic values. Our extensions holds for mappings of finite distortion satisfying condition $(\mathcal{A})$, a new class of mappings recently studied in [7], [8], which generalizes the known class of quasiregular mappings.

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### 1. Introduction

A mapping $f : D \to I\!R^n$, where $D \subset I\!R^n$ is a domain, is said to have finite distortion if the following conditions are satisfied:

1) $f \in W^{1,1}_{\text{loc}}(D, I\!R^n)$.

2) There exists a measurable function $K : D \to [0,\infty]$ finite a.e. such that $|f'(x)|^n \leq K(x) \cdot J_f(x)$ a.e.

3) The Jacobian determinant is locally integrable.

If $K \in L^\infty(D)$ we obtain the known class of quasiregular mappings and we refer the reader to [12], [13] for the monographs dedicated to this subject. Quasiregular mappings satisfies an important topological property, they are either constant, or open, discrete. If the distortion map $K \in L^p_{\text{loc}}(D)$, with $p > n-1$, and $f \in W^{1,n}_{\text{loc}}(D, I\!R^n)$, then it is shown in [9] that a map with finite distortion is open, discrete.

Recently were considered in [7], [8] mappings $f : D \to I\!R^n$ of finite distortion for which there exists $\mathcal{A} : [0,\infty) \to [0,\infty)$ smooth, strictly increasing, with $\mathcal{A}(0) = 0$, $\lim_{t\to\infty} \mathcal{A}(t) = \infty$ and satisfying the conditions:

$(\mathcal{A}_0)$ $\exp(\mathcal{A}(K)) \in L^1_{\text{loc}}(D)$.

$(\mathcal{A}_1)$ $\int_1^\infty \frac{\mathcal{A}'(t)}{t} \mathrm{d}t = \infty$.

$(\mathcal{A}_2)$ there exists $t_0 \in (0,\infty)$ such that $\mathcal{A}'(t) \cdot t$ increases to infinity for $t \geq t_0$.

As in [7], we shall say that a map of finite distortion satisfies condition $(\mathcal{A})$ if satisfies conditions $(\mathcal{A}_0)$, $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. In [4], Theorem 1.3 it is shown that such nonconstant mappings are open, discrete if $\mathcal{A}(t) = \lambda t$ for some $\lambda > 0$. We shall show in Lemma 1 that mappings $f : D \to I\!R^n$ with finite distortion satisfying condition $(\mathcal{A})$ are such that the dilatation map $K$ is in $L^p_{\text{loc}}(D)$ for every $p > 0$ and this ensures (see [5], Prop.2.5, [6] Theorem 1.1 or Theorem 2.1 [7]) that such mappings are continuous and either constant, or open and discrete. Our proofs are based on the following basic ingredients, valid for mappings $f : D \to I\!R^n$ of finite distortion and satisfying condition $(\mathcal{A})$

(i) $M(f(\Gamma)) \leq M_{K^{n-1}}(\Gamma)$ for every path family $\Gamma$ from $D$.

(ii) $\operatorname{cap}_{K^{n-1}}(B(x,R)), \overline{B}(x,r)) \to 0$ when $r \to 0$ and $R > 0$ is keept fixed established in [7], Corollary 4.2 and Theorem 5.3.

We shall prove first an eliminability result for mappings of finite distortion satisfying condition $(\mathcal{A})$, generalizing some known facts from the theory of quasiregular mappings (Theorem 2.8 and Theorem 2.9, page 64, [13]).

**Theorem 1.** *Let $n \geq 2$, $D \subset \mathbb{R}^n$ be a domain, $F \subset D$ closed in $D, f : D \backslash F \to \mathbb{R}^n$ a map of finite distortion, of $K(x)$ dilatation, satisfying condition $(\mathcal{A})$ such that $M_{K^{n-1}}(F) = 0$. Let $x \in F$ and suppose that*

a) *there exists $\rho_x > 0$ such that $\overline{B}(x,\rho_x) \subset D$ and $\int_{B(x,\rho_x)} \exp(\mathcal{A}(K(y)))\,dy < \infty$.*

b) $\operatorname{cap} Cf(B(x,\rho_x)\backslash F) > 0$.

*Then $f$ extends by continuity in $x$ and if $x$ is an isolated point of $F$, then $x$ is eliminable for $f$. If $f$ satisfies conditions* a) *and* b) *in every point from $F$ and we also denote by $f$ the extended* map *and* $\operatorname{int} f(F) = \phi$, *then $F$ is eliminable.*

We generalize now some results which for meromorphic functions in the plane are known as Iversen's theorem and Cartwright's theorem. Our result also generalizes some extensions of the previous theorems established for quasiregular mapping in Theorem 2.6, page 170, [13] and Corollary 4.4, [10].

**Theorem 3.** *Let $n \geq 2$, $D \subset \mathbb{R}^n$ be a domain, $F \subset D$ closed in $D, f : D\backslash F \to \mathbb{R}^n$ a nonconstant map of finite distortion, of $K(x)$ dilatation, satisfying condition $(\mathcal{A})$ such that $M_{K^{n-1}}(F) = 0$ and let $x \in F$ be an essential singularity of $f$ such that there exists $\rho_x > 0$ such that $\overline{B}(x,\rho_x) \subset D$ and $\int\limits_{B(x,\rho_x)} \exp(\mathcal{A}(K(y)))\,dy < \infty$. Then, if $x$ is an isolated point of $F$, it results that $\overline{\mathbb{R}^n}\backslash f(B(x.r)\backslash F) \subset A(f,x)$ for every $0 < r < \rho_x$ and in the general case, there exists $x_k \in F, x_k \to x$ such that $\overline{\mathbb{R}^n}\backslash f(B(x,r)\backslash F) \subset A(f,x_k)$ for every $k \in \mathbb{N}$ and every $0 < r < \rho_x$.*

The following result was proved for meromorphic functions in the plane by K.Noshiro [11] and for $K$-quasiregular mappings by 0. Martio and S.Rickman in [10], Theorem 4.2.

**Theorem 4.** *Let $n \geq 2, D \subset \mathbb{R}^n$ be a domain, $E \subset \partial D$, $f : D \to \mathbb{R}^n$ a map of finite distortion, of $K(x)$ dilatation satisfying condition $(\mathcal{A})$ such that $M_{K^{n-1}}(E) = 0$. Let $x \in \partial D$ and $z \in \mathbb{R}^n$ be such that there exists $\rho_x > 0$ such that $z \in (C(f,x)\backslash C(f,x,\partial D\backslash E)) \cap Cf(B(x,\rho_x) \cap D)$. Then either $z \in A(f,x)$,* or there *exists $x_k \in E, x_k \to x$ such that $z \in A(f,x_k)$ for every $k \in \mathbb{N}$.*

The following theorem extends a result of O.Martio and S.Rickman [10] concerning the density of the points $b \in \partial B^n$ at which a $K$-quasiregular map $f : B^n \to \mathbb{R}^n$ with $\operatorname{cap} Cf(B^n) > 0$ has some asymptotic values (see also [13], Theorem 2.4, page 170).

**Theorem 5.** *Let $n \geq 2, D \subset \mathbb{R}^n$ be a domain, $B = \{b \in \partial D|$ there exists $\gamma : [0,1) \to D$ a path such that $\lim\limits_{t\to 1}\gamma(t) = b\}$ and let $E = \{b \in B|$ there exists $\gamma : [0,1) \to D$ a path with $\lim\limits_{t\to 1}\gamma(t) = b$* and there exists $\lim\limits_{t\to 1} f(\gamma(t))\}$.

*Let $f : D \to \mathbb{R}^n$ be a* map *of finite distortion, of $K(x)$ dilatation, satisfying* *condition* $(\mathcal{A})$ *and* $\exp(\mathcal{A}(K)) \in L^1_{\text{loc}}(D \cup B)$ *and suppose that* $M_{K^{n-1}}(B \cap B(b,\varepsilon)) > 0$ *and* $\operatorname{cap} Cf(B(b,\varepsilon) \cap D) > 0$ *for every* $b \in B$ *and every* $\varepsilon > 0$. *Then* $M_{K^{n-1}}(E \cap B(b,\varepsilon)) > 0$ *for every* $b \in B$ *and every* $\varepsilon > 0$, *hence* $E$ *is densely in* $B$.

We shall prove that if the boundary cluster set $C(f, b, \partial G \backslash E)$ is small enough, then we have the equality $C(f, b, G) = C(f, b, \partial G \backslash E)$ and the result seems to be new even for quasiregular mappings.

**Theorem 6.** *Let* $n \geq 2, D \subset I\!R^n$ *be a domain,* $E \subset \partial D, f : D \to I\!R^n$ *a* map *of finite distortion, of* $K(x)$ *dilatation, satisfying condition* $(\mathcal{A})$ *such that* $M_{K^{n-1}}(E) = 0$. *Let* $G \subset D$ *be a domain,* $b \in \partial D \cap (K \backslash E)'$ *such that* $\partial G = K \cup C$, *with* $C$ *closed,* $b \in K \backslash C$, *and suppose that there exists* $\rho > 0$ *such that* $\int_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))dx < \infty$ *and* $\operatorname{cap} Cf(B(b,\rho)) > 0$ *and* $m_{n-1}(C(f, b, K \backslash E)) = 0$. *Then, if we let* $M = Cf(B(b,\rho))$, *it results that either* $\operatorname{cap}(M \backslash C(f, b, K \backslash E)) = 0$, *or that* $C(f, b, K \backslash E) = C(f, b, G)$.

A theorem of Lindelöf says that if $f : B^2 \to \mathbb{C}$ is meromorphic and $f$ admits two distinct asymptotic values at some point $b \in \partial B^2$, then $f$ assumes infinitely often in any neighbourhood of $b$ all values of the extended complex plane, with at most two possible exceptions. We shall use the preceeding theorem to prove the following generalization of Lindelöf's theorem for plane mappings of finite distortion, and the result seems to be new even for quasiregular mappings.

**Theorem 8 .** *Let* $D \subset I\!R^2$ *be a domain,* $f : D \to I\!R^2$ *be a map of finite distortion, of* $K(x)$ *dilatation, satisfying condition* $(\mathcal{A})$, *and let* $b \in \partial D$ *be such that* $f$ *admits two distinct asymptotic values at* $b$ *and there exists* $\rho > 0$ *such that* $\int_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))\,dx < \infty$. *Then there exists* $M \subset I\!R^2$ *with* cap $M = 0$ *and such that* $I\!R^2 \backslash M \subset f(U \cap D)$ *for every* $U \in \mathcal{V}(b)$.

We prove the following maximum principle.

**Theorem 9.** *Let* $n \geq 2$, $D \subset I\!R^n$ *be a domain,* $E \subset \partial D$, $f : D \to I\!R^n$ *be a* map *of finite distortion, of* $K(x)$ *dilatation, satisfying condition* $(\mathcal{A})$ *such that* $M_{K^{n-1}}(E) = 0$. *Let* $G \subset D$ *be a domain with* $\partial G = K \cup C, C$ *closed,* $b \in \partial D \cap (K \backslash C) \cap (K \backslash E)'$ *and suppose that there exists* $\rho > 0$ *such that* $\int_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))dx < \infty$ *and let* $M = Cf(B(b,\rho) \cap D)$. *Suppose that* $\operatorname{cap}(M \cap CB(x,a)) > 0$ *and that* $C(f, b, K \backslash E) \subset B(x,a)$. *Then* $C(f, b, G) \subset B(x,a)$.

We shall use this maximum principle to generalize a maximum principle established in Corollary 3.9, [10] for quasiregular mappings.

**Theorem 10.** *Let* $n \geq 2$, $D \subset I\!R^n$ *be a bounded domain,* $E \subset \partial D$ *such that* $\partial D \backslash E$ *is densely in* $\partial D, f : D \to I\!R^n$ *a* map *of finite distortion of* $K(x)$ *dilatation, satisfying condition* $(\mathcal{A})$ *such that* $M_{K^{n-1}}(E) = 0$. *Suppose that there exists* $M > 0$ *such that* $\limsup_{y\to x} |f(y)| \leq M$ *for every* $x \in \partial D \backslash E$, *and for every* $x \in E$ *there exists* $\rho_x > 0$ *such that* $\int_{B(x,\rho_x)\cap D} \exp(\mathcal{A}(K(y)))dx < \infty$ *and* $\operatorname{cap}(Cf(B(x,\rho_x) \cap D) \cap CB(0,M) > 0$. *Then* $|f(x)| \leq M$ *for every* $x \in D$.

We shall also use Theorem 9 to prove a theorem which is known for meromorphic functions in the plane as Iversen-Tsuji's theorem. A version for quasimeromorphic mappings was given by O.Martio and S.Rickman in [10], Theorem 3.3.

**Theorem 11.** *Let $n \geq 2$, $D \subset I\!R^n$ be a domain, $E \subset \partial D$, $f : D \to I\!R^n$ be a map of finite distortion, of $K(x)$ dilatation, satisfying condition $(\mathcal{A})$ such that $M_{K^{n-1}}(E) = 0$. Let $b \in (\partial D \backslash E)'$ be such that there exists $\rho > 0$ such that $\int\limits_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))\mathrm{d}x < \infty$, let $M = Cf(B(b,\rho) \cap D)$ and suppose that $\operatorname{cap}(M \cap CB(0,r)) > 0$ for every $r > 0$. Then $\limsup\limits_{x\to b} |f(x)| = \limsup\limits_{\substack{z\to b \\ z\in\partial D\backslash E}} \left( \limsup\limits_{x\to z} |f(x)| \right)$.*

Our generalizations given in Theorem 4, Theorem 10 and Theorem 11 to some results from [10] bring something new even if $f$ is supposed to be a quasiregular map, since the singular set $E \subset \partial D$ is not supposed to be compact in $\partial D$.

We finally generalize Theorem 4.10 from [16], extendings result of M.Vuorinen concerning the boundary extension of closed quasiregular mappings.

**Theorem 12.** *Let $n \geq 2$, $D, D'$ be domains in $I\!R^n$, $f : D \to D'$ a map of finite distortion, satisfying condition $(\mathcal{A})$, of $K(x)$ dilatation, $b \in \partial D$ such that $D$ is locally connected at $b$ and there exists $\rho > 0$ such that $\int\limits_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))\mathrm{d}x < \infty$. Suppose that $C(f,b) \subset \partial D'$ and $A(f,x) \subset \partial D'$ for every $x \in B(b,\rho) \cap \partial D$ and that $D'$ has property $\mathrm{P}_2$ in some point from $C(f,b)$. Then $f$ can be continuously extended at $b$.*

## 2. Preliminaries

We call a path $q : [0,1) \to I\!R^n$an open path,and a point $x \in \overline{I\!R^n}$ will be called a limit point of $q$ if there exists $t_p \nearrow 1$ such that $q(t_p) \to x$. If $\Gamma$ is a path family in $I\!R^n$, we define $F(\Gamma) = \{\rho : \overline{I\!R^n} \to [0,\infty]$ Borel maps $| \int\limits_\gamma \rho \mathrm{d}s \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. If $D \subset I\!R^n$ is open, $\omega : D \to [0,\infty]$ is measurable and finite a.e. we define $\widetilde{\omega} : I\!R^n \to [0,\infty]$ by $\widetilde{\omega}(x) = \omega(x)$ if $x \in D$, $\widetilde{\omega}(x) = 0$ if $x \notin D$ and we define the $\omega$modulus of $\Gamma$ by $M_\omega(\Gamma) = \inf\limits_{\rho\in F(\Gamma)} \int\limits_{I\!R^n} \rho^n(x) \cdot \widetilde{\omega}(x)\,\mathrm{d}x$, and for $\omega \equiv 1$ we obtain the usual modulus $M(\Gamma)$. If $\Gamma_1, \Gamma_2$ are paths families in $I\!R^n$, we say that $\Gamma_1 > \Gamma_2$ if every path $\gamma \in \Gamma_1$ has a subpath in $\Gamma_2$. As in the classical case, we prove that if $\Gamma_1 > \Gamma_2$, then $M_\omega(\Gamma_1) \leq M_\omega(\Gamma_2)$ and that $M_\omega\left(\bigsqcup\limits_{i=1}^{\infty} \Gamma_i\right) \leq \sum\limits_{i=1}^{\infty} M_\omega(\Gamma_i)$. Also, if $\omega_1 \leq \omega_2$, then $M_{\omega_1}(\Gamma) \leq M_{\omega_2}(\Gamma)$. We define, for $D \subset I\!R^n$ open and $E, F \subset \overline{D}$ by $\Delta(E,F,D)$ the family of all paths, open or not, which joins $E$ with $F$ in $D$.

We say that $E = (D,C)$ is a condenser if $C$ is compact, $D$ is open, $C \subset D \subset I\!R^n$, and we define the capacity of $E$, $\operatorname{cap} E = \inf \int\limits_{I\!R^n} |\nabla u|^n(x)\,\mathrm{d}x$, where $u \in C_0^\infty(D)$ and $u \geq 1$ on $C$ and we let $\Gamma_E = \{\gamma : [a,b) \to D$ paths $|\gamma(a) \in C, \gamma$ has a limit point in $\partial D\}$. We know from Prop.10.2, [13], page 54 that cap $E = M(\Gamma_E)$:

If $E=(D,C)$ is a condenser, $\omega : D \to [0,\infty]$ is measurable and finite a.e., we define the $\omega$capacity of $E$, $\text{cap}_\omega E = \inf \int_{\mathbb{R}^n} |\nabla u|^n (x) dx$, where $u \in C_0^\infty(D)$ and $u \geq 1$ on $C$ and we see that if $u$ is a test function for $\text{cap}_\omega E$, then $\rho = |\nabla u| \in F(\Gamma_E)$, and this implies that $M_\omega(\Gamma_E) \leq \text{cap}_\omega(E)$.

If $C \subset \mathbb{R}^n$ is compact, we say that cap $C=0$ if cap $(A,C)=0$ for some open set $A$ from $\mathbb{R}^n$ and from Lemma 2.2, [13], page 64, the definition is independent on the open set $A$ with $C \subset A \subset \mathbb{R}^n$. If $C \subset \mathbb{R}^n$ is arbitrary, we say that $\exp C = 0$ if cap $K=0$ for every $K \subset C$ compact.

If $D \subset \mathbb{R}^n$ is open, $\omega : D \to [0,\infty]$ is measurable and finite a.e., $A \subset D$ is a set, we say that $A$ is of zero $\omega$-modulus (and we write $M_\omega(A)=0$) if the $\omega$-modulus of all paths having some limit point in $A$ is zero. We write $M_\omega(A)>0$ if $A$ is not of zero $\omega$-modulus. Since $M(\Gamma) \leq M_\omega(\Gamma)$ if $\omega \geq 1$, we see that if $\omega \geq 1$ and $A \subset D$ is such that $M_\omega(A)=0$, it results that cap $A=0$.

If $A$ is countable, $A \subset D$ and $\lim_{r\to 0} M_\omega\left(\Delta\left(\overline{B}(x,r), CB(x,R), D\right)\right) = 0$ for every $x \in A$ and every $0<R$ with $\overline{B}(x,R) \subset D$, we prove as in the classical case that $M_\omega(A)=0$. Using Lemma 2, we see that this thing holds for instance if $K : D \to [0,\infty]$ is measurable and finite a.e. and for every $x \in A$ there exists $\rho_x > 0$ such that $\overline{B}(x,\rho_x) \subset D$ and $\int_{B(x,\rho_x)} \exp(\mathcal{A}(K(y)))dy < \infty$ for some map $\mathcal{A}$ satisfying conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$and we take $\omega = K^{n-1}$.

If $D \subset \mathbb{R}^n$ is open, $b \in \partial D$ and $f : D \to \mathbb{R}^n$ is a map, we let $C(f,b) = \{w \in \overline{\mathbb{R}^n} \mid \text{there exists } b_p \in D,\ b_p \neq b,\ b_p \to b \text{ such that } f(b_p) \to w\}$, if $B \subset \partial D$ we let $C(f,B) = \bigcup_{b\in B} C(f,b)$ and if $K \subset D$ is such that $b \in K'$, we let $C(f,b,K) = \{w \in \overline{\mathbb{R}^n} \mid \text{there exists } b_p \in K, b_p \neq b, b_p \to b \text{ such that } f(b_p) \to w\}$. If $(U_m)_{m\in\mathbb{N}}$ is a fundamental system of neighbourhoods of $b$ , then $C(f,b) = \bigcap_{m=1}^{\infty} \overline{f(U_m \cap D)}, C(f,b,K) = \bigcap_{m=1}^{\infty} \overline{f(U_m \cap K)}$ and we usually take $U_{m+1} \subset U_m$ for every $m \in \mathbb{N}$. Suppose now that $b \in K', K \subset \overline{D}$ and take $F : \overline{D} \to \mathcal{P}\left(\overline{\mathbb{R}^n}\right), F(x) = f(x)$ if $x \in D, F(x) = C(f,x)$ if $x \in \partial D$. We define in this case $C(f,b,K) = \bigcap_{m=1}^{\infty} \overline{F(U_m \cap (K \setminus \{b\}))}$. We see that if $f$ is continuous, $K \subset D, b \in \partial D$ and $(U_m)_{m\in\mathbb{N}}$ is a fundamental system of neighbourhoods of $b$ such that $U_{m+1} \subset U_m$ and $U_m \cap K$ is connected for every $m \in \mathbb{N}$, then it results that $C(f,b,K)$ is a connected set from $\overline{\mathbb{R}^n}$. If $K \subset \partial D$, then $C(f,b,K)$ is called the boundary cluster set of $f$ at $b$ and $C(f,b)$ is called the cluster set of $f$ at $b$.

If $D \subset \mathbb{R}^n$ is a domain and $b \in \partial D$, we say that $D$ is locally connected at $b$ if there exists $\mathcal{U}$ a fundamental system of neighbourhoods of $b$ such that $U \cap D$ is connected for every $U \in \mathcal{U}$. We say that $D$ has property $\text{P}_2$ at $b$ (following [14], page 54) if for every point $b_1 \neq b, b_1 \in \partial D$, there exists $F \subset D$ compact and $\delta > 0$ such that $M(\Delta(E,F,D)) \geq \delta$ for every $E \subset D$ connected such that $\overline{E}$ contains $b$ and $b_1$.

If $E, F$ are Hausdorff spaces and $f : E \to F$ is a map, we say that $f$ is open if $f$ carries open sets into open sets, we say that $f$ is discrete if $f^{-1}(y)$ is empty or a discrete set for every $y \in F$ and we say that $f$ is a light map if for every $x \in E$ and $V \in \mathcal{V}(x)$, there exists $U \in \mathcal{V}(x)$ with $\partial U \cap \left(f^{-1}(f(x))\right) = \phi$ and $U \subset V$. We denoted here by $\mathcal{V}(x) = \{U \subset Eopen | x \in U\}$. If $p : [0,1] \to F$ is a path, $x \in E$ is such that $f(x) = p(0)$, we say that $q : [0,a) \to E$ is a maximal lifting of $p$ from $x$ if $q(0 = x, 0 < a \leq 1\ ,\ f \circ q = p|[0,a)$ and $q$ is maximal with this property. If $q$ is defined on $[0,1]$, we say that $q$ is a lifting of $p$ from $x$. If $E, F$ are domains from $I\!R^n$and $f$ is continuous, open and discrete, there exists always a maximal lifting.

Let $X$ be a separable metric space and $\mathcal{A} = (A_i)_{i \in I}$ be a collection of sets. We define the superior limit of the collection $(\mathcal{A})$ to be $\limsup\ A_i = \{x \in X|$ every neighbourhood of $x$ contains points from infinitely many sets $A_i\}$ and we define the inferior limit of the collection $\mathcal{A}$ to be $\liminf A_i = \{x \in X|$ every neighbourhood of $x$ contains points of all but a finite number of the sets $A_i\}$. If for a collection $\mathcal{A}$ $\limsup A_i = \liminf A_i$, we say that $\mathcal{A}$ is convergent and we write $B = \lim A_i$, where $B$ is the common value of $\limsup A_i$ and $\liminf A_i$. We know from [17], Theorem 7.1, page 11, that every infinite sequence of sets $(A_i)_{i \in I}$ contains a convergent subsequence and from [17], page 15, we see that if $(A_i)_{i \in I}$ is a convergent sequence of compact connected sets such that $\overline{\bigsqcup_{i \in I} A_i}$ is compact, then it results that $\lim A_i$ is compact and connected.

If $D \subset I\!R^n$ is open and $f : D \to I\!R^n$ is a map, $A \subset D$, we let $N(f, A) = \sup_{y \in I\!R^n} \text{Card}\ \left(f^{-1}(y) \cap A\right)$. If $b \in \partial D, \gamma : [0,1) \to D$ is a path such that $\lim_{t \to 1} \gamma(t) = b$ and $w \in \overline{I\!R^n}$ is such that $w = \lim_{t \to 1} f(\gamma(t))$, we say that $w$ is an asymptotic value of $f$ at $b$, and we denote by $A(f, b)$ the set of all asymptotic values of $f$ at $b$. We let $B^n = \{x \in I\!R^n|\ |x| < 1\}$ and we let $\mu_n$ the Lebesgue measure from $I\!R^n$. We shall denote by $q$ the chordal metric in $\overline{I\!R^n}$ given by $q(a, b) = |a - b| \cdot \left(1 + |a|^2\right)^{-\frac{1}{2}} \cdot (1 + |b|^2)^{-\frac{1}{2}}$ if $a \neq b, a, b \in I\!R^n, q(a, \infty) = \left(1 + |a|^2\right)^{-\frac{1}{2}}$ if $a \in I\!R^n$ and we denote by $d$ the euclidean metric in $I\!R^n$ and by $|a - b|$ the euclidean distance between $a$ and $b$ in $I\!R^n$. We denote by $B_q(x, r)$, respectively $B(x, r)$ the ball of center $x$ and radius $r$ if we consider on $I\!R^n$ the chordal metric, respectively the euclidean metric and in the same way we denote for a set $A \subset I\!R^n$ by $q(A)$, respectively $d(A)$ the diameter of $A$. We denote by $m_p$ the $p$-Hausdorff measure in $I\!R^n$, and if $A \subset I\!R^n, p, t > 0$, we let $m_p^t(A) = \inf \sum_{i=1}^{\infty} d(A_i)^p$, where $A \subset \bigcup_{i=1}^{\infty} A_i$ and $d(A_i) < t$ for $i \in I\!N$ and $m_p^*(A) = \lim_{t \to 0} m_p^t(A)$.

We denote by $W^{1,p}_{\text{loc}}(D, I\!R^n)$ the Sobolev space of all functions $f : D \to I\!R^n$ which are locally in $L^p(D)$ together with their first order weak partial derivatives. Let $D \subset I\!R^n$ be open, $F \subset D$ closed in $D$ with $\mu_n(F) = 0$, $f : D \backslash F \to I\!R^n$ a map of finite distortion, of $K(x)$ dilatation, satisfying condition $(\mathcal{A})$. We say that the set $F$ is eliminable for $f$ if $f$ extends to a map of finite distortion, of $K(x)$ dilatation, satisfying condition $(\mathcal{A})$ on $D$. A point $b \in F$ will be called an essential singularity of $f$ if the limit $\lim_{x \to b} f(x)$ does not exist. If $\gamma : [a, b] \to I\!R^n$ is a path, we let $\gamma^- : [a, b] \to I\!R^n$ defined by $\gamma^-(t) = \gamma(a + b - t)$

for $t \in [a,b]$ and if $\gamma_1 : [a,b] \to \mathbb{R}^n, \gamma_2 : [c,d] \to \mathbb{R}^n$ are paths, with $\gamma_1(b) = \gamma_2(c)$ we let $\gamma_1 \vee \gamma_2 : [a, b+d-c] \to \mathbb{R}^n$ defined by $(\gamma_1 \vee \gamma_2)(t) = \gamma_1(t)$ if $t \in [a,b]$, $(\gamma_1 \vee \gamma_2)(t) = \gamma_2(t+c-b)$ if $t \in [b, b+d-c]$.

The following Lemma is essential for the properties of mappings with finite distortion, of $K(x)$ dilatation and satisfying condition $(\mathcal{A})$.

**Lemma 1.** Let $D \subset \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^n$ a map of finite distortion, of $K(x)$ dilatation and satisfying condition $(\mathcal{A})$. Then the dilatation map $K$ is in $L^p_{\text{loc}}(D)$ for every $p > 0$.

The proof of this lemma is given by the following:

**Proposition 1.** Let $\mathcal{A} : [0,\infty) \to [0,\infty)$ be smooth, strictly increasing, with $\mathcal{A}(0) = 0$, $\lim\limits_{t\to\infty} \mathcal{A}(t) = \infty$, satisfying condition $(\mathcal{A}_2)$, $D \subset \mathbb{R}^n$ be open, $K : D \to [0,\infty]$ measurable and finite a.e. such that there exists $B \subset D$ measurable with $\mu_n(B) < 1$ and $\int\limits_B \exp(\mathcal{A}(K(x)))\,dx < \infty$. Then $\int\limits_B K^p(x)\,dx < \infty$ for every $p > 0$.

*Proof.* Suppose first that $p \geq 1$ and let $f : (0,\infty) \to \mathbb{R}_+$ be defined by $f(t) = \exp\left(\mathcal{A}\left(t^{\frac{1}{p}}\right)\right)$ for $t > 0$. Then $f$ is strictly increasing on $(0,\infty)$ and from Lemma 2.4, [8], there exists $b > 1$ such that $f$ is convex on $(b,\infty)$. Let $Q : D \to [0,\infty]$, $Q(x) = K(x)$ if $K(x) > b$, $Q(x) = b$ if $K(x) \leq b$. Then $Q$ is measurable and we have, using Jensen's inequality, that $f\left(\int\limits_B K^p(x)\,dx\right) \leq$

$f\left(\int\limits_B Q^p(x)\,dx\right) \leq f\left(\int\limits_B Q^p(x)\,dx/\mu_n(B)\right) \leq \int\limits_B f(Q^p(x))dx/\mu_n(B)$.

It results that

$$\int\limits_B K^p(x)\,dx \leq \left[\mathcal{A}^{-1}\left(\log\left(\max\left\{1, \int\limits_B \exp(\mathcal{A}(Q(x)))/\mu_n(B)\right\}\right)\right)\right]^p < \infty.$$

If $0 < p < 1$, then $\int\limits_B K^p(x)\,dx \leq \left(\int\limits_B K(x)\,dx\right)^p < \infty$.

Another useful lemma is the following:

**Lemma 2.** Let $D \subset \mathbb{R}^n$ be a domain, $b \in \partial D$, $K : D \to [0,\infty]$ measurable and finite a.e., $\mathcal{A} : [0,\infty) \to [0,\infty)$ be smooth, strictly increasing, with $\mathcal{A}(0) = 0$, $\lim\limits_{t\to\infty} \mathcal{A}(t) = \infty$, satisfying conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ and suppose that there exists $\delta > 0$ such that $\int\limits_{D\cap B(b,\delta)} \exp(\mathcal{A}(K(x)))\,dx < \infty$. Then $M_{K^{n-1}}(\Delta(\overline{B}(b,r) \cap D, CB(b,\rho) \cap D, D) \to 0$ when $\delta > 0$ is keept fixed and $r \to 0$.

*Proof.* Let $Q : \overline{\mathbb{R}^n} \to [0,\infty]$, $Q(x) = K(x)$ if $x \in D$, $Q(x) = 1$ if $x \notin D$, $0 < r < \delta$ and let $\Gamma_{r\delta} = \Delta\left(\overline{B}(b,r) \cap D, CB(b,\delta) \cap D, D\right)$, $\Lambda_{r\delta} = \Delta\left(\overline{B}(r,\delta), CB(b,\delta), B(b,\delta) \setminus \overline{B}(b,r)\right)$. Since $\int\limits_{B(b,\delta)} \exp(\mathcal{A}(Q(x)))\,dx < \infty$, we see from Theorem 5.3, page 24, [7], that $\text{cap}_{Q^{n-1}}\left(B(b,\delta), \overline{B}(b,r)\right) \to 0$ when $r \to 0$ and $\delta > 0$ is keept fixed. We have

$$M_{K^{n-1}}(\Gamma_{r\delta}) \leq M_{Q^{n-1}}(\Gamma_{r\delta}) \leq M_{Q^{n-1}}(\Lambda_{r\delta}) \leq \text{cap}_{Q^{n-1}}\left(B(b,\delta), \overline{B}(b,r)\right)$$

and the theorem is proved.

**Lemma 3.** let $n \geq 2, G, D$ open sets in $\mathbb{R}^n$ with $\mu_n(D \cap G) < \infty, \omega : D \to [0, \infty]$ measurable and finite a.e., $\Gamma$ a path family in $G$ and $\Gamma_r = \{\gamma \in \Gamma | \gamma$ is rectifiable$\}$ and suppose that there exists $p > 1$ such that $\int\limits_{G \cap D} \omega^p(x) \mathrm{d}x < \infty$. Then $M_\omega(\Gamma) = M_\omega(\Gamma_r)$.

*Proof.* We shall use some arguments from Theorem 6.9, page 19, [14]. Since $\Gamma_r \subset \Gamma$, we see that $M_\omega(\Gamma_r) \leq M_\omega(\Gamma)$. We show that $M_\omega(\Gamma) \leq M_\omega(\Gamma_r)$. Let $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\rho_0 : \overline{\mathbb{R}^n} \to [0, \infty)$ be defined by $\rho_0(x) = \frac{1}{|x| \ln |x|}$ if $x \in G \cap CB(0,2)$, $\rho_0(x) = 1$ if $x \in G \cap B(0,2)$, $\rho_0(x) = 0$ if $x \notin G$. Then, using Hölder's inequality, we have

$$\int\limits_{\mathbb{R}^n} \rho_0^n(x) \cdot \omega(x) \,\mathrm{d}x = \int\limits_{D \cap G} \rho_0^n(x) \cdot \omega(x) \,\mathrm{d}x =$$

$$\int\limits_{D \cap G \cap B(0,2)} \omega(x) \,\mathrm{d}x + \int\limits_{D \cap G \cap CB(0,2)} \rho_0^n(x) \cdot \omega(x) \,\mathrm{d}x \leq$$

$$\int\limits_{D \cap G} \omega(x) \,\mathrm{d}x + \left( \int\limits_{CB(0,2)} \rho_0^{nq}(x) \,\mathrm{d}x \right)^{\frac{1}{q}} \cdot \left( \int\limits_{D \cap G} \omega^p(x) \,\mathrm{d}x \right)^{\frac{1}{p}} \leq$$

$$\left[ C_1 + C_2 \left( \int\limits_2^\infty \frac{r^{n-1}}{r^{nq} \cdot (\ln r)^{nq}} \mathrm{d}r \right)^{\frac{1}{q}} \right] \cdot \left( \int\limits_{D \cap G} \omega^p(x) \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

Here $C_1$ and $C_2$ are some constants .

Let now $\gamma \in \Gamma \backslash \Gamma_r$. If $\gamma$ is bounded, there exists $a > 0$ such that $\rho_0(x) > a$ on $\operatorname{Im} \gamma$, hence $1 \leq \infty = \int\limits_\gamma \rho_0(x) \,\mathrm{d}s$. If $\gamma \in \Gamma \backslash \Gamma_r$ is unbounded, we choose $x \in \operatorname{Im} \gamma$ with $|x| > 2$ and we see that $\infty = \int\limits_{|x|}^\infty \frac{\mathrm{d}r}{r \ln r} \leq \int\limits_\gamma \rho_0 \mathrm{d}s$. It results that $\int\limits_\gamma \rho_0 \mathrm{d}s = \infty$ for every $\gamma \in \Gamma \backslash \Gamma_r$.

Let $\rho \in F(\Gamma_r)$, $\varepsilon > 0$ and let $\rho_\varepsilon = (\rho^n + \varepsilon^n \cdot \rho_0^n)^{\frac{1}{n}}$. Then, if $\gamma \in \Gamma_r$, we have $1 \leq \int\limits_\gamma \rho \mathrm{d}s \leq \int\limits_\gamma \rho_\varepsilon \mathrm{d}s$, and if $\gamma \in \Gamma \backslash \Gamma_r$, we have $1 \leq \infty = \varepsilon \cdot \int\limits_\gamma \rho_0 \mathrm{d}s \leq \int\limits_\gamma \rho_E \mathrm{d}s$ and this implies that $\rho_\varepsilon \in F(\Gamma)$. We obtain that $M_\omega(\Gamma) \leq \int\limits_{\mathbb{R}^n} \rho_\varepsilon^n(x) \cdot \omega(x) \,\mathrm{d}x = \int\limits_{\mathbb{R}^n} \rho^n(x) \cdot \omega(x) \,\mathrm{d}x + \varepsilon^n \cdot \int\limits_{\mathbb{R}^n} \rho_0^n(x) \cdot \omega(x) \,\mathrm{d}x$ and letting $\varepsilon \to 0$, we see that $M_\omega(\Gamma) \leq \int\limits_{\mathbb{R}^n} \rho^n(x) \cdot \omega(x) \,\mathrm{d}x$ for every $\rho \in F(\Gamma_r)$, hence $M_\omega(\Gamma) \leq M_\omega(\Gamma_r)$. We therefore proved that $M_\omega(\Gamma) = M_\omega(\Gamma_r)$.

**Remark 1.** The theorem remains true if we replace the condition "$\int\limits_{D \cap G} \omega^p(x) \,\mathrm{d}x < \infty$ for some $p > 1$" by "$\int\limits_D \rho_0^n(x) \cdot \omega(x) \,\mathrm{d}x < \infty$".

**Lemma 4.** Let $r_1, \rho, R > 0$ and $x \in \mathbb{R}^n$ be fixed with $0 < \rho + r_1 < R$ and let $E \subset C\overline{B}(x, \rho + r_1)$ be such that cap $E > 0$. Then, for every $r > 0$, there exists $\delta > 0$ such that $M(\Delta(E, C, \overline{B}(x, \rho)) \geq \delta$ for every continuum $C \subset B(x, R) \backslash \overline{B}(x, \rho + r_1)$ with $d(C) \geq r$ and $C \cap E = \phi$.

*Proof.* Let $C \subset B(x,R)\backslash\overline{B}(x,\rho+r_1)$ be a continuum with $\mathrm{d}(C) \geq r$ and $C \cap E = \phi$, let $g$ be the inversion of center $x$ and radius $\rho$ and let $\Gamma = \Delta\left(E, C, C\overline{B}(x,\rho)\right)$ and $\widetilde{\Gamma} = \Delta(g(E), g(C), B(x,\rho))$. Then $M(\Gamma) = M\left(\widetilde{\Gamma}\right)$, $\operatorname{cap} g(E) > 0$, $\mathrm{d}(g(C)) \geq r'$, $g(C) \cup g(E) \subset B(x,\rho')$, with $0 < \rho' < \rho$ such that $r'$ and $\rho'$ depends on $r, r_1, \rho, R$. Let $\Gamma_1 = \Delta(g(C), S(x,\rho), B(x,\rho))$, $\Gamma_2 = \Delta(g(E), S(x,\rho), B(x,\rho))$. Then $M(\Gamma_k) = \delta_k > 0$ for $k = 1,2$, and let $\rho \in F\left(\widetilde{\Gamma}\right)$. If $3\rho \notin F(\Gamma_1) \cup F(\Gamma_2)$, there exists $\gamma_k \in \Gamma_k$ with $\int\limits_{\gamma_k} \rho \mathrm{d}s < \frac{1}{3}$ for $k = 1,2$, and let $\Gamma_3 = \Delta\left(\operatorname{Im}\gamma_1, \operatorname{Im}\gamma_2, B(x,\rho)\backslash\overline{B}(x,\rho')\right)$. Then $M(\Gamma_3) = \delta_3 \geq c_n \ln\frac{\rho}{\rho'} > 0$ and $3\rho \in F(\Gamma_3)$. It results that $3\rho \in F(\Gamma_1) \cup F(\Gamma_2) \cup F(\Gamma_3)$, hence $M(\Gamma) = M\left(\widetilde{\Gamma}\right) \geq \delta = \frac{1}{3^n}\min\{\delta_1, \delta_2, \delta_3\} > 0$.

**Lemma 5.** Let $C_0 \subset I\!\!R^n$ be compact, $C_1 \subset I\!\!R^n$ closed with $C_0 \cap C_1 = \phi$, $F \subset I\!\!R^n$ such that $F \cap (C_0 \cup C_1) = \phi$, and let $\Gamma = \Delta(C_0, C_1, I\!\!R^n \backslash F)$. Then $M(\Gamma) < \infty$.

*Proof.* Let $r = d(C_0, C_1) > 0$ and $\rho : \overline{I\!\!R^n} \to [0,\infty]$ be defined by $\rho(x) = \frac{1}{r}$ if $x \in B(C_0, r)$, $\rho(x) = 0$ if $x \notin B(C_0, r)$. Then $\rho \in F(\Gamma)$, hence $M(\Gamma) \leq \int\limits_{I\!\!R^n} \rho^n(x)\,\mathrm{d}x \leq \mu_n(B(C_0,r))/r^n < \infty$.

**Lemma 6.** Let $M, F$ be closed subsets of $\overline{I\!\!R^n}$, $C_m, C$ compact, connected subsets from $I\!\!R^n$ with $C = \lim\limits_{m\to\infty} C_m$, $\operatorname{Card} C > 1$ and $M \cap C = \phi$, $F \cap (M \cup C) = \phi$. Then, if $\Gamma = \Delta(C, M, I\!\!R^n \backslash F)$, $\Gamma_m = \Delta(C_m, M, I\!\!R^n \backslash F)$, it results that $\lim\limits_{m\to\infty} M(\Gamma_m) = M(\Gamma)$.

*Proof.* We shall use in the proof some arguments from Theorem 37.1, [14]. We see from Lemma 5 that $M(\Gamma) < \infty$ and using a Möbius transformation, we can suppose that $M \cup F \subset I\!\!R^n$. Taking a subsequence, we can presume that there exists $r > 0$ such that $d(C_m) \geq r$, $d(C) \geq r$ for every $m \in I\!\!N$. Let $\delta = \min\{d(C,F), d(C,M), \frac{r}{2}\}$. Let $\rho \in F(\Gamma)$ with $\rho \in L^n(I\!\!R^n)$ and let $q > 1$. We show that there exists $m_0 \in I\!\!N$ such that $q\rho \in F(\Gamma_m)$ for every $m \geq m_0$. Indeed, otherwise, taking a subsequence, we can find $\gamma_m : [a_m, b_m] \to I\!\!R^n \backslash F$ a path with $\gamma_m(a_m) \in C_m$, $\gamma_m(b_m) \in M$ such that $\int\limits_{\gamma_m} q\rho \mathrm{d}s < 1$ for every $m \in I\!\!N$. Let $r_m = d(\gamma_m(a_m), C)$ and $x_m \in C$ be such that $r_m = d(x_m, \gamma_m(a_m))$ for $m \in I\!\!N$. We show that $r_m \to 0$.

Indeed, otherwise there exists $\lambda > 0$ and a subsequence $(r_{m_k})_{k\in I\!\!N}$ such that $r_{m_k} > \lambda$ for every $k \in I\!\!N$. Taking some subsequence, we can find $y \in I\!\!R^n$ such that $\gamma_{m_k}(a_{m_k}) \to y$ and using the definition of the set $C$, we see that $y \in C$. On the other side, we have $\lambda < r_{m_k} = d\left(\gamma_{m_k}(a_{m_k}), C\right) \leq d\left(\gamma_{m_k}(a_{m_k}), y\right) \to 0$, which represents a contradiction. It results that $r_m \to 0$, and taking a subsequence, we can suppose that $r_m < \delta$ for $m \in I\!\!N$.

Let $\Delta_m = \Delta(C, \operatorname{Im}\gamma_m, ((B(x_m,\delta)\backslash\overline{B}(x_m, r_m)\backslash F))$ for $m \in I\!\!N$. Then $B(x_m,\delta) \cap F = \phi$, $S(x_m, t) \cap C \neq \phi$, $S(x_m,t) \cap \operatorname{Im}\gamma_m \neq \phi$ for every $r_m < t < \delta$ and every $m \in I\!\!N$, hence $M(\Delta_m) \geq c_n \cdot \ln\frac{\delta}{r_m}$ for $m \in I\!\!N$.

Let now $m \in I\!\!N$ be fixed and $\alpha : [a,b] \to I\!\!R^n$, $\alpha \in \Delta_m$, $\alpha(a) \in C$, $\alpha(b) \in \operatorname{Im}\gamma_m$ and let $c = \inf\{t \in [a,b]\,|\alpha(t) \in \operatorname{Im}\gamma_m\}$ and let $c_m \in [a_m, b_m]$ be such that $\gamma(c) = \gamma_m(c_m)$. We take $\beta = \alpha| [a, c] \vee \gamma_m| [c_m, b_m]$ and we see that $\beta \in \Gamma$. Then $1 \leq \int\limits_{\beta} \rho ds \leq \int\limits_{\alpha} \rho ds + \int\limits_{\gamma_m} \rho ds \leq \int\limits_{\alpha} \rho ds + \frac{1}{q}$, hence $\frac{q}{q-1} \cdot \rho \in F(\Delta_m)$. We have $\infty > \int\limits_{I\!R^n} \rho^n(x)\,dx \cdot \left(\frac{q}{q-1}\right)^n \geq M(\Delta_m) \geq c_n \cdot \ln \frac{\delta}{r_m} \to \infty$ if $m \to \infty$, which represents a contradiction. We therefore proved that there exists $m_0 \in I\!N$ such that $q\rho \in F(\Gamma_m)$ for $m \geq m_0$.

Let now $\varepsilon > 0$ and $q = 1 + \varepsilon$ and let $\rho_\varepsilon \in F(\Gamma)$ be such that $M(\Gamma) + \varepsilon > \int\limits_{I\!R^n} \rho_\varepsilon^n(x)\,dx$. Then there exists $n_\varepsilon \in I\!N$ such that $(1+\varepsilon)\rho_\varepsilon \in F(\Gamma_m)$ for $m \geq n_\varepsilon$, hence we have

(1) $$M(\Gamma_m) \leq (1+\varepsilon)^n \cdot \int\limits_{I\!R^n} \rho_\varepsilon^n(x)\,dx \leq (1+\varepsilon)^n \cdot (M(\Gamma) + \varepsilon)\, form \geq n_\varepsilon.$$

We can also suppose that $M(\Gamma_m) \leq 2M(\Gamma)$ for $m \geq n_\varepsilon$. Let $q > 1$ and $\rho_m \in F(\Gamma_m)$ be such that $\int\limits_{I\!R^n} \rho_m^n(x)\,dx < M(\Gamma_m) + \varepsilon$ for $m \geq n_\varepsilon$. We show that there exists $q_\varepsilon \geq n_\varepsilon$ such that $q \cdot \rho_m \in F(\Gamma)$ for every $m \geq q_\varepsilon$. Indeed, otherwise, taking a subsequence, we can presume that we can find $\gamma_m : [a_m, b_m] \to I\!R^n, \gamma_m \in \Gamma$, with $\gamma_m(a_m) \in C, \gamma_m(b_m) \in M$ and $\int\limits_{\gamma_m} q \cdot \rho_m ds < 1$ for every $m \in I\!N$, and let $x_m \in C_m$be such that $r'_m = d(\gamma_m(a_m), C_m) = d(x_m, \gamma_m(a_m))$ for every $m \in I\!N$. We show that $r'_m \to 0$.

Indeed, otherwise we can find $\lambda > 0$ and $\left(r'_{m_k}\right)_{k \in I\!N}$ with $r'_{m_k} > \lambda$ for $k \in I\!N$ and let $x \in C$ such that there exists $\gamma_{m_{k_p}}\left(a_{m_{k_p}}\right) \to x$. Then $B\left(x, \frac{\lambda}{2}\right) \cap C_{m_{k_p}} = \phi$ for $p \in I\!N$ great enough, which represents a contradiction, since $x \in C = \lim C_m$. It results that $r'_m \to 0$ and taking a subsequence, we can suppose that $r'_m < \frac{\delta}{2}$ for $m \in I\!N$, hence $\frac{\delta}{2} < \min\{d(x_m, F), d(x_m, M)\}$ for $m \in I\!N$.

Let $\Delta'_m = \Delta(C_m, \operatorname{Im}\gamma_m, ((B\left(x_m, \frac{\delta}{2}\right) \setminus B(x_m, r'_m) \setminus F))$ for $m \in I\!N$. Then $B\left(x_m, \frac{\delta}{2}\right) \cap F = \phi, S(x_m, t) \cap C_m \neq \phi, S(x_m, t) \cap \operatorname{Im}\gamma_m \neq \phi$ for every $r'_m < t < \frac{\delta}{2}$ and every $m \in I\!N$, hence $c_n \cdot \ln \frac{\delta}{2r'_m} \leq M(\Delta'_m)$ for $m \in I\!N$.

Let now $m \in I\!N$ be fixed and let $\alpha : [a, b] \to I\!R^n, \alpha \in \Delta'_m, \alpha(a) \in C_m, \alpha(b) \in M$ and $c = \inf\{t \in [a, b] \,|\alpha(t) \in \operatorname{Im}\gamma_m\}$ and let $c_m \in [a_m, b_m]$ be such that $\alpha(c) = \gamma_m(c_m)$. We take $\beta = \alpha|[a, c] \vee \gamma_m| [c_m, b_m]$ and we see that $\beta \in \Gamma_m$, hence $1 \leq \int\limits_{\beta} \rho_m ds \leq \int\limits_{\alpha} \rho_m ds + \int\limits_{\gamma_m} \rho_m ds \leq \int\limits_{\alpha} \rho_m ds + \frac{1}{q}$. Then $\frac{q}{q-1} \cdot \rho_m \in F(\Delta'_m)$, hence $\left(\frac{q}{q-1}\right)^n \cdot (2M(\Gamma) + \varepsilon) \geq \left(\frac{q}{q-1}\right)^n \cdot \int\limits_{I\!R^n} \rho_m^n(x)\,dx \geq M(\Delta'_m) \geq c_n \cdot \ln \frac{\delta}{2r'_m} \to \infty$ if $m \to \infty$, which represents a contradiction. It results that we can find $m_\varepsilon \geq n_\varepsilon$ such that $(1+\varepsilon) \cdot \rho_m \in F(\Gamma)$ for $m \geq m_\varepsilon$, hence

(2) $$M(\Gamma) \leq (1+\varepsilon)^n \cdot \int\limits_{I\!R^n} \rho_m^n(x)\,dx \leq (1+\varepsilon)^n \cdot (M(\Gamma_m) + \varepsilon)\, form \geq m_\varepsilon.$$

Using (1) and (2), we see that $\lim\limits_{m \to \infty} M(\Gamma_m) = M(\Gamma)$, q.e.d.

**Lemma 7.** Let $K \subset I\!R^n, y \in I\!R^n, p > 0$ such that $m_p(K) = 0$ and let $M(K, y) = \{z \in I\!R^n |$ there exists $w \in K$ and $t \geq 0$ such that $z = (1-t)y + tw\}$. Then $m_{p+1}(M(K, y)) = 0$.

*Proof.* We show that $m_{p+1}(K \times [m, m+1]) = 0$ for every $m \in I\!N$. We fix $m \in I\!N$ and let $0 < \varepsilon < 1, 0 < t < 1$ such that $m_p^t(K) < \varepsilon$ and take a covering $K \subset \bigcup_{i=1}^{\infty} A_i$ of $K$ such that $d(A_i) < t$ for every $i \in I\!N$ and $\sum_{i=1}^{\infty} d(A_i)^p < \varepsilon$. Let $\ell_i = d(A_i)$ and $\alpha_i = \left[\frac{1}{\ell_i}\right] + 1$ for $i \in I\!N$. Then $1 \leq \alpha_i \cdot \ell_i \leq 2$ for every $i \in I\!N, A_i \times [m, m+1] \subset \bigsqcup_{j=1}^{\alpha_i} A_i \times \left[m + \frac{j-1}{\alpha_i}, m + \frac{j}{\alpha_i}\right]$ and there exists a constant $C(n)$ depending only on $n$ such that $d\left(A_i \times \left[m + \frac{j-1}{\alpha_i}, m + \frac{j}{\alpha_i}\right]\right) \leq C(n) \cdot d(A_i)$ for every $i \in I\!N$ and $j = 1, \ldots, \alpha_i$ and let $r = C(n) \cdot t$. We have $K \times [m, m+1] \subset \bigsqcup_{i=1}^{\infty} \bigsqcup_{j=1}^{\alpha_i} A_i \times \left[m + \frac{j-1}{\alpha_i}, m + \frac{j}{\alpha_i}\right]$ and $d\left(A_i \times \left[m + \frac{j-1}{\alpha_i}, m + \frac{j}{\alpha_i}\right]\right) \leq r$ for $i \in I\!N$ and $j = 1, \ldots, \alpha_i$. This implies

$$m_{p+1}^r(K \times [m, m+1]) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\alpha_i} d\left(A_i \times \left[m + \frac{j-1}{\alpha_i}, m + \frac{j}{\alpha_i}\right]\right)^{p+1} \leq$$

$$C(n)^{p+1} \cdot \sum_{i=1}^{\infty} \alpha_i \cdot d(A_i)^{p+1} \leq C(n)^{p+1} \cdot \sum_{i=1}^{\infty} \alpha_i \cdot \ell_i \cdot d(A_i)^p \leq 2 \cdot C(n)^{p+1} \cdot \sum_{i=1}^{\infty} d(A_i)^p$$

$$\leq 2 \cdot C(n)^{p+1} \cdot \varepsilon.$$

Letting first $t \to 0$, then $\varepsilon \to 0$, we obtain that $m_{p+1}(K \times [m, m+1]) = 0$ for every $m \in I\!N$ and hence that $m_{p+1}(K \times [0, \infty])) = 0$.

Let now $H : I\!R^n \times [0, \infty) \to I\!R^n$ be defined by $H(x, t) = (1-t)y + tx$ for $x \in I\!R^n$ and $t \geq 0$. Then $H$ is a $C^\infty$ map and $M(K, y) = H(K \times [0, \infty))$, and this implies that $m_{p+1}(M(K, y)) = 0$.

## 3. Proofs of the main results

*Proof of Theorem* 1. Suppose that $f$ is not continuous at $x$. Then there exists $x_j \to x, y_j \to x$ such that $f(x_j) \to b_1, f(y_j) \to b_2$, with $b_1 \neq b_2$ and let $r_j = \max\{2|x_j|, 2|y_j|\}$ for $j \in I\!N$. Since cap $F = 0$, it results that $F$ is nowhere disconnecting and let $C_j$ be a connected set joining $x_j$ with $y_j$ in $B(x, r_j) \setminus F$. Let $\rho > 0$ be such that $q(f(C_j)) \geq \rho$ for every $j \in I\!N$. Using Lemma 2.6, page 65, [13], there exists $\delta > 0$ such that $\delta <$ cap$(f(B(x, \delta_x) \setminus F, f(C_j))$ for every $j \in I\!N$.

Let $\Gamma_j' = \{\gamma : [0, 1] \to I\!R^n$ path $|\gamma(0) \in f(C_j), \gamma([0, 1])) \subset f(B(x, \rho_x) \setminus F)$, $\gamma(1) \in \partial f(B(x, \rho_x) \setminus F)\}$ and let $\Gamma_j$ be the family of all maximal liftings of some paths from $\Gamma_j'$ starting from some point from $C_j$ for $j \in I\!N$. Let $\Gamma_{1j} = \{\gamma \in \Gamma_j | \gamma$ has a limit point in $F\}$ and $\Gamma_{2j} = \{\gamma \in \Gamma_j | \gamma$ has a limit point outside $B(x, \rho_x)\}$ for $j \in I\!N$. Then $\Gamma_j' < f(\Gamma_j), \Gamma_j = \Gamma_{1j} \cup \Gamma_{2j}, M_{K^{n-1}}(\Gamma_{1j}) = 0$ and $\Gamma_{2j} < \Delta(\overline{B}(x, r_j) \setminus F, C(B(x, \rho_x) \setminus F), I\!R^n \setminus F)$ for $j \in I\!N$. Using (i) and Lemma 2, we see that

$$\delta < \text{cap } (f(B(x, \rho_x) \setminus F), f(C_j)) = M(\Gamma_j') \leq M(f(\Gamma_j)) \leq M_{K^{n-1}}(\Gamma_j)$$

$\le M_{K^{n-1}}(\Gamma_{1j}) + M_{K^{n-1}}(\Gamma_{2j}) = M_{K^{n-1}}(\Gamma_{2j}) \le M_{K^{n-1}}(\Delta(\overline{B}(x,r_j)\backslash F, C(B(x,\rho_x)\backslash F), I\!R^n\backslash F) \to 0$ if $j \to \infty$. We obtained a contradiction, hence we proved that $f$ is continuous at $x$.

Suppose now that for every $x \in F$ there exists $\rho_x > 0$ such that cap $C\left(f\left(B,\rho_x\right)\backslash F\right) > 0$ and $\int\limits_{B(x,\rho_x)} \exp\left(\mathcal{A}\left(K\left(y\right)\right)\right) \mathrm{d}y < \infty$. Using the first part of the proof, we see that $f$ extends continuously on $F$, and we also denote by $f$ the extended map on $D$. Then $f$ is continuous on $D$, and since the theorem is immediately if $f$ is constant on $D$, we can suppose that $f$ is open, discrete on $D\backslash F$. Using the openness and the discreetness of $f$ on $D\backslash F$, we see that $i\left(f,x\right)$ has a constant, nonvanishing sign on $D\backslash F$, and we easy see that $f$ is a light map on $D$. We use now Theorem 1, [2] to see that $f$ is an open, discrete map on $D$.

Let now $x \in F$ be fixed. We can find $0 < \alpha_x < \rho_x$ such that $N\left(f, B\left(x,\alpha_x\right)\right) < \infty$ and let $I\!N = N\left(f, B\left(x,\alpha_x\right)\right)$. Since $f \in W^{1,1}_{\text{loc}}\left(D\backslash F\right)$, we use Theorem 6.3.2, [3] page 107 to see that $\int\limits_B J_f\left(x\right)\mathrm{d}x \le N \cdot \mu_n\left(f\left(B\right)\right)$ for every ball $B$ from $D\backslash F$. We cover now $B\left(x,\alpha_x\right)\backslash F$ with some balls $B_i, i \in I\!N$ such that there exists a number $L$ depending only on $n$ such that every point from $B\left(x,\alpha_x\right)\backslash F$ belongs to at most $L$ balls $B_i$. Then

$$\int\limits_{B(x,\alpha_x)} J_f\left(z\right)\mathrm{d}z = \int\limits_{B(x,\alpha_x)\backslash F} J_f\left(z\right)\mathrm{d}z \le \sum_{i=1}^{\infty} \int\limits_{B_i} J_f\left(z\right)\mathrm{d}z \le N \cdot \sum_{i=1}^{\infty} \mu_n\left(f\left(B_i\right)\right)$$

$\le N^2 \cdot L \cdot \mu_n\left(f\left(B\left(x,\alpha_x\right)\right)\right) < \infty$.

Since $\int\limits_{B(x,\alpha_x)} \exp\left(\mathcal{A}\left(K\left(y\right)\right)\right)\mathrm{d}y < \infty$, we see from Proposition 1 that $\int\limits_{B(x,\alpha_x)} K^{\frac{1}{n-1}}\left(z\right)\mathrm{d}z < \infty$ and using Hölder's inequality, we obtain that

$$\int\limits_{B(x,\alpha_x)} \left|f'\left(y\right)\right|\mathrm{d}y \le \left(\int\limits_{B(x,\alpha_x)} K^{\frac{1}{n-1}}\left(y\right)\mathrm{d}y\right)^{\frac{n-1}{n}} \cdot \left(\int\limits_{B(x,\alpha_x)} J_f\left(y\right)\mathrm{d}y\right)^{\frac{1}{n}} < \infty.$$

Now $f$ is obviously an ACL map on $B\left(x,\alpha_x\right)$, hence $f$ is $\mathrm{ACL}^1$ on $B\left(x,\alpha_x\right)$ and from Proposition 1.2, [13], page 6, we see that $f \in W^{1,1}_{\text{loc}}\left(B\left(x,\alpha_x\right)\right)$.

We also obtain

**Theorem 2.** *Let $n \ge 2, D \subset I\!R^n$ be a domain, $F \subset D$ be closed in $D, f : D\backslash F \to I\!R^n$ be a nonconstant map of finite distortion, of $K\left(x\right)$ dilatation, satisfying condition $(\mathcal{A})$ such that $M_{K^{n-1}}\left(F\right) = 0$ and let $x \in F$ be an essential singularity of $f$ such that there exists $\rho_x > 0$ such that $\overline{B}\left(x,\rho_x\right) \subset D$* and $\int\limits_{B(x,\rho_x)} \exp\left(\mathcal{A}\left(K\left(y\right)\right)\right)\mathrm{d}y < \infty$. *Then it results that* cap $C\left(f\left(B\left(x,r\right)\backslash F\right)\right) = 0$ *for every $0 < r < \rho_x$.*

*Proof of Theorem 3.* Let $0 < r < \rho_x$ and $x \in \overline{I\!R^n}\backslash f\left(B\left(x,r\right)\backslash F\right)$, and we can suppose that $z \neq \infty$. Since cap $F = 0$, we can also take $0 < r < \rho_x$

such that $S(x,r) \cap F = \phi$, and since $z \notin f(B(x,r) \setminus F)$, it results that $\alpha = d(z, f(S(x,r))) > 0$. We see from Theorem 2 that cap $Cf(B(x,r) \setminus F) = 0$ and we use the fact that $f(B(x,r) \setminus F)$ is an open, nonempty set to find $0 < r' < \alpha$, a connected set $Q \subset B(x,r)$ and a cap $C$ of the sphere $S(z,r')$ such that $f(Q) = C$. We denote for $y \in C$ and $i \in I\!N$ by $\gamma_{yi}$the path $\gamma_{yi} : \left[0, 1 - \frac{1}{i}\right] \to B(z,r')$ defined by $\gamma_{yi}(t) = (1-t)y + tz$ for $t \in \left[0, 1 - \frac{1}{i}\right]$ and let $A_i = \{y \in C|$ the path $\gamma_{yi}$ cannot be lifted from every point from $Q\}$ for $i \in I\!N$.

Suppose that $m_{n-1}(A_i) > 0$ for some $i \in I\!N$. Let $\Gamma_i' = \{\gamma_{yi} | y \in A_i\}$ and let $\Gamma_i$ be the family of all maximal liftings of the paths from $\Gamma_i'$ starting from some point from $Q$. Then $f(\Gamma_i) < \Gamma_i'$, every path from $\Gamma_i$ is contained in $B(x,r) \setminus F$ and has some limit point in $F$, hence $M_{K^{n-1}}(\Gamma_i) = 0$. Since $m_{n-1}(A_i) > 0$, we see that $M(\Gamma_i') > 0$ and we use now (i) to see that $0 < M(\Gamma_i') \leq M(f(\Gamma_i)) \leq M_{K^{n-1}}(\Gamma_i) = 0$, which represents a contradiction. It results that $m_{n-1}(A_i) = 0$ for every $i \in I\!N$.

Let now $y \in C \setminus \bigcup\limits_{i=1}^{\infty} A_i$ and $\gamma_y : [0,1] \to \overline{B}(z,r')$ be defined by $\gamma_y(t) = (1-t)y + tz$ for $t \in [0,1]$. It results that if $b \in Q$ is such that $f(b) = y$, we can lift $\gamma_y|[0,1)$ from $b$ and let $q : [0,1) \to I\!R^n$ be a path such that $q(0) = b$ and $f \circ q = \gamma_y|[0,1)$. Then $\operatorname{Im} q \subset B(x,r)$ and let $B_i = q\left(\left[1 - \frac{1}{i}, 1\right)\right)$ for $i \in I\!N$ and $B$ the set of all limit points of $q$. Then $B = \limsup B_i$, hence $B$ is a connected set from $B(x,r)$. Suppose that Card $B > 1$. Then $\operatorname{cap}(B \setminus F) > 0$ and $f(B \setminus F) \subset \{z\}$, which contradicts the discreteness of $f$ on $D \setminus F$. It results that Card $B = 1$, hence there exists $c = \lim\limits_{t \to 1} q(t)$ and $c \in F$ and hence $z \in A(f,c)$. If $x$ is an isolated point of $F$, then $c = x$, hence $z \in A(f,x)$. If $x \in F'$ we can find $r_k \to 0, r_k < r$ with $S(x,r_k) \cap F = \phi$, hence $z \in I\!R^n \setminus f(B(x,r) \setminus F) \subset I\!R^n \setminus f(B(x,r_k) \setminus F)$ and as before we find $x_k \in B(x;r_k) \cap F$ such that $z \in A(f,x_k)$ for every $k \in I\!N$ and the theorem is proved.

*Proof of Theorem* 4. Let $o < r_k < \rho_k, r_k \searrow 0$ such that $S(x,r_k) \cap E = \phi$ and let $F_k = C\left(f, \overline{B}(x,r_k) \cap ((\partial D \setminus E) \setminus \{x\})\right)$ for $k \in I\!N$. Then $C(f,x,\partial D \setminus E) = \bigcap\limits_{k=1}^{\infty} \overline{F}_k$ and $F_{k+1} \subset F_k$ for $k \in I\!N$ and since $z \notin C(f,x,\partial D \setminus E)$, we can presume that $z \notin \overline{F}_k$ for every $k \in I\!N$. Let $\rho_k > 0$ be such that $B(z,\rho_k) \cap \overline{F}_k = \phi$ and let us prove that there exists $\rho_k' < \rho_k$ such that $B(z,\rho_k') \cap f(D \cap S(x,r_k)) = \phi$. Indeed, otherwise we can find $a_i \in D \cap S(x,r_k)$ with $f(a_i) \to z$. Taking a subsequence, we can suppose that $a_i \to a_0 \in S(x,r_k)$. Then $a_0 \notin D$, since in this case we obtain that $z = f(a_0) \in f(D \cap B(x,\rho_x))$, which represents a contradiction. Also, $a_0 \notin E$, since $a_0 \in S(z,r_k)$ and $S(z,r_k) \cap E = \phi$. If $a_0 \in \partial D \setminus E$, then $z \in C(f,a_0) \subset F_K$, which represents a contradiction. It results that for every $k \in I\!N$, we can find $\rho_k' > 0$ such that $B(z,\rho_k') \cap (F_k \cup f(D \cap S(x,r_k))) = \phi$. Let now $k \in I\!N$ be fixed.

Since $z \in C(f,x)$, we can find a point $a_k \in D \cap B(x,r_k)$ such that $f(a_k) \in B(z,\rho_k')$. We can presume that $f$ is not constant on $D$, hence $f$ is open, discrete on $D$ and this implies that we can find $0 < r_k' < \rho_k'$, a connected set $Q_k \subset B(x,r_k) \cap D$ and $C_k$ a cap of the sphere $S(z,r_k')$ such that $f(Q_k) = C_k$. We denote for $y \in C_k$ and $i \in I\!N$ by $\gamma_{yi} : \left[0, 1 - \frac{1}{i}\right] \to B(z,r_k')$ the path defined by $\gamma_{yi}(t) = (1-t)y + tz$ for $t \in \left[0, 1-\frac{1}{i}\right]$ and let $A_i = \{y \in C_k | \gamma_{yi}$ can not be lifted from any point from $Q_k\}$ for $i \in I\!N$. Suppose that $m_{n-1}(A_i) > 0$ and let $\Gamma_i' = \{\gamma_{yi} | y \in A_i\}$ and let $\Gamma_i$ be the family of all maximal liftings of the paths from $\Gamma_i'$ starting from some point from $Q_k$. Then every path $\gamma \in \Gamma_i$ is contained in $B(x, r_k) \cap D$ and cannot have some limit point in $D \cap S(x, r_k)$ or in $B(x, r_k) \cap (\partial D \backslash E)$, hence $\gamma$ can have only some limit points in $E$. Since $M_{K^{n-1}}(E) = 0$, we see that $M_{K^{n-1}}(\Gamma_i) = 0$ and since $m_{n-1}(A_i) > 0$, it results that $M(\Gamma_i') > 0$, Since $f(\Gamma_i) < \Gamma_i'$ and using (i), we obtain that $0 < M(\Gamma_i') \leq M(f(\Gamma_i)) \leq M_{K^{n-1}}(\Gamma_i) = 0$, which represents a contradiction. It results that $m_{n-1}(A_i) = 0$ for every $i \in I\!N$.

Let now $y \in C_k \backslash \bigsqcup_{i=1}^{\infty} A_i$ and let $\gamma_y : [0,1] \to \overline{B}(z, r_k')$ be defined by $\gamma_y(t) = (1-t)y + tz$ for $t \in [0,1]$. Then we can find $q : [0,1) \to D \cap B(x, r_k)$ a path such that $q(0) \in Q_k$, $f \circ q = \gamma_y | [0,1)$ and we show that there exists $x_k = \lim_{t \to 1} q(t)$. Indeed, let $x_k$ be a limit point of $q$ and suppose that we can find another limit point $y_k$ of $q$ such that $y_k \neq x_k$. Let $0 < t_p < t_{p+1} <, \ldots, < 1$ with $t_p \nearrow 1$ and $|q(t_{2p}) - x_k| < \frac{1}{2p}, |q(t_{2p+1}) - y_k| < \frac{1}{2p+1}$ and let $A_p = q([t_{2p}, t_{2p+1}])$ for $p \in I\!N$. Then $A = \limsup A_p$ is a connected set from $\partial D \cap \overline{B}(x, r_k)$, $x_k, y_k \in A$, and since cap $E = 0$, we can find a point $u \in A \backslash (E \cup \{x\})$. Then we can find $s_p \nearrow 1$ such that $q(s_p) \to u$ and we see that $f(q(s_p)) \to z$. It results that $z \in C(f, u) \subset F_k$, which represents a contradiction. We found that $\lim_{t \to 1} q(t) = x_k \in \partial D \cap \overline{B}(x, r_k)$, hence $z \in A(f, x_k)$ for $k \in N$. Then either $x_k = x$ for some $k \in I\!N$, hence $z \in A(f, x)$, or $x_k \neq x$ for $k \in I\!N$ and then $x_k \in E, x_k \to x$.

*Proof of Theorem* 5. Suppose that there exists $b \in B$ and $\varepsilon > 0$ such that $M_{K^{n-1}}(E \cap B(b, \varepsilon)) = 0$. Since $M_{K^{n-1}}\left(B \cap B\left(b, \frac{\varepsilon}{2}\right)\right) > 0$, we can find $y \in (B \backslash E) \cap B(b, \frac{\varepsilon}{2})$ and let $\beta : [0,1) \to D \cap B(y, \frac{\varepsilon}{2})$ be a path such that $\lim_{t \to 1} \beta(t) = y$ and $\lim_{t \to 1} f(\beta(t))$ does not exists. Then we can find $s_m \nearrow 1$ with $\lim_{m \to \infty} f(\beta(s_{2m})) = u \neq v = \lim_{m \to \infty} f(\beta(s_{2m+1}))$ and let $F_m = f(\beta([s_{2m}, s_{2m+1}]))$ for $m \in I\!N$ and let $r > 0$ be such that $q(F_m) \geq r$ for every $m \in I\!N$. Let $r_m \to 0$ be such that $\beta([s_{2m}, s_{2m+1}]) \subset B(y, r_m)$, and we can suppose that $r_m < \frac{\varepsilon}{2}$ for every $m \in I\!N$.

Let $\Gamma_m' = \{\gamma : [0,1] \to I\!R^m$ path $| \gamma(0) \in F_m, \gamma(1) \in Cf(B(b, \varepsilon) \cap D)\}$ and let $\Gamma_m$ be the family of all maximal liftings of some paths from $\Gamma_m'$ starting from some point from $\beta | [s_{2m}, s_{2m+1}]$ for $m \in I\!N$. Let $\Gamma_{m_1} = \{\gamma : [0, c] \to D$ path$|$ either $\gamma \in \Gamma_m$ and $\operatorname{Im} \gamma \subset B\left(y, \frac{\varepsilon}{2}\right)$, or there exists $c \leq d$ and $\alpha : [0, d] \to D, \alpha \in \Gamma_m$ such that $\gamma = \alpha | [0, c], \alpha(0) \in \beta | [s_{2m}, s_{2m+1}]$ and $c = \inf\{t \in [0, d] \ | \alpha(t) \in S\left(y, \frac{\varepsilon}{2}\right)\}\}$ and let $\Gamma_{m2} = \{\gamma \in \Gamma_{m1} | \gamma$ is rectifiable$\}$.

Since we can suppose that $\varepsilon > 0$ is chosen such that $\int_{D \cap B(b, \varepsilon)} \exp(\mathcal{A}(K(x))) dx < \infty$, we use Lemma 1 to see that $\int_{D \cap B\left(y, \frac{\varepsilon}{2}\right)} K^p(x) dx < \infty$ for every $p > 0$ and from Lemma 3 we see that $M_{K^{n-1}}(\Gamma_{m1}) = M_{K^{n-1}}(\Gamma_{m2})$ for every $m \in I\!N$. We see that $f(\Gamma_{m1}) < \Gamma_m'$ and since cap $Cf(B(b, \varepsilon) \cap D) > 0$ and $q(F_m) \geq r$ for every $m \in I\!N$, we use Lemma 2.6, [13], page 65 to find $\delta > 0$ such that $\delta <$ cap $(f(D \cap B(b, \varepsilon)), F_m))$ for every $m \in I\!N$. Let $\Gamma_{m3} = \{\gamma \in \Gamma_{m2} | \operatorname{Im} \gamma \cap$

$S(y, \frac{\varepsilon}{2}) = \phi\}$ for $m \in I\!N$. Then, if $\gamma \in \Gamma_{m3}, \gamma : [0,1) \to D \cap B\left(y, \frac{\varepsilon}{2}\right)$, the rectifiabilities of $\gamma$ implies that there exists $\beta_\gamma = \lim_{t \to 1} \gamma(t) \in \partial D$, and of course, $f$ has some asymptotic limit at $\beta_\gamma$ and hence $\beta_\gamma \in E$. Since $M_{K^{n-1}}(E \cap B(b, \varepsilon)) = 0$, it results that $M_{K^{n-1}}(\Gamma_{m3}) = 0$, hence $M_{K^{n-1}}(\Gamma_{m2}) = M_{K^{n-1}}(\Gamma_{m2} \backslash \Gamma_{m3})$ for $m \in I\!N$.

Then $\Gamma_{m2} \backslash \Gamma_{m3} \subset \Delta\left(D \cap \overline{B}(y, r_m), D \cap CB\left(y, \frac{\varepsilon}{2}\right), D\right)$ for $m \in I\!N$. Using (i) we have

$$\delta < \text{cap } (f(D \cap B(b,\varepsilon)), F_m) = M(\Gamma'_m) \leq M(f(\Gamma_{m1})) \leq M_{K^{n-1}}(\Gamma_{m1}) =$$

$$M_{K^{n-1}}(\Gamma_{m2}) = M_{K^{n-1}}(\Gamma_{m2} \backslash \Gamma_{m3}) \leq M_{K^{n-1}}(\Delta(\overline{B}(y, r_m) \cap D, D \cap CB(y, \frac{\varepsilon}{2}), D) \to 0$$

if $m \to \infty$. We obtained a contradiction, and it results that $M_{K^{n-1}}(E \cap B(b, \varepsilon)) > 0$ for every $b \in B$ and every $\varepsilon > 0$.

*Proof of Theorem 6.* Suppose that cap $(M \backslash C(f, b, K \backslash E)) > 0$. Since $I\!R^n \backslash C(f, b, K \backslash E)$ is open, it is an union of closed balls $\overline{B}_i$, hence $M \backslash C(f, b, K \backslash E) = \bigsqcup_{i=1}^{\infty} M \cap \overline{B}_i$ and this implies that there exists $i \in I\!N$ such that cap $M \cap \overline{B}_i > 0$. We can therefore presume that there exists $M_1 \subset M$ compact in $I\!R^n$ with cap $M_1 > 0$ and $M_1 \cap C(f, b, K \backslash E) = \phi$. We can also easy see that $C(f, b, K \backslash E) \subset C(f, b, G)$.

Suppose that there exists a point $y_1 \in C(f, b, G) \backslash C(f, b, K \backslash E)$, and we can take $y_1 \notin M_1, y_1 \neq \infty$. Taking a subsequence, we can find $x_m \in B\left(b, \frac{1}{m}\right) \cap G$ such that $f(x_m) \to y_1$ and we can suppose that $B\left(b, \frac{1}{m}\right) \cap \partial G \subset K$ for every $m \in I\!N$. Let $G_m = B\left(b, \frac{1}{m}\right) \cap G$, $K_m = B\left(b, \frac{1}{m}\right) \cap K$ for $m \in I\!N$ and let $z_m \in (K \backslash E) \cap B\left(b, \frac{1}{m}\right)$ for $m \in I\!N$. Since cap $E = 0$, we see that $m_\alpha(E) = 0$ for every $\alpha > 0$ and from Lemma 7 we see that $m_p(M(E, x_m) \cup M(E, z_m)) = 0$ for every $1 < p < n$, and this implies that we can find a point $w_m \in G_m \backslash (M(E, z_m) \cup M(E, x_m))$ for $m \in I\!N$. Let $q_m : [0,1] \to B\left(b, \frac{1}{m}\right)$ be a path such that $q_m(0) = x_m, q_m(1) = z_m$ and $\text{Im}\, q_m = [x_m, w_m] \cup [w_m, z_m]$. Then $\text{Im}\, q_m \cap E = \phi$ and let $t_m = \inf\{t \in [0,1] \,|\, q_m(t) \in \partial G_m\}$ and $\lambda_m = q_m | [0, t_m]$ for $m \in I\!N$. Then $\lambda_m(0) = x_m, \lambda_m([0, t_m)) \subset G_m, \lambda_m(t_m) \in \partial G_m$ and we see that $\lambda_m(t_m) \in K_m \backslash E$ for every $m \in I\!N$.

Let $r' = \min\{q(y_1, M_1), q(y_1, C(f, b, K \backslash E)), q(M_1, C(f, b, K \backslash E))\}$. Then $r' > 0$ and we also suppose that $\frac{1}{m} < \rho$ for $m \in I\!N$. Let $F : \overline{D} \to \mathcal{P}\left(\overline{I\!R^n}\right)$ be defined by $F(x) = f(x)$ if $x \in D, F(x) = C(f, x)$ if $x \in \partial D$. Then $C(f, b, K \backslash E) = \bigsqcap_{m=1}^{\infty} \overline{F(K_m \backslash (E \cup \{b\}))}$, and taking a subsequence, we can presume that $F(K_m \backslash (E \cup \{b\})) \subset B_q\left(C(f, b, K \backslash E), \frac{r'}{4}\right)$ for $m \in I\!N$. Then $C(f, \lambda_m(t_m), \text{Im}\, \lambda_m) \subset B_q\left(C(f, b, K \backslash E), \frac{r'}{4}\right), f(\lambda_m(0)) = f(x_m) \to y_1$, hence $\text{Im}\, \lambda_m \cap B_q\left(C(f, b, K \backslash E), \frac{r'}{4}\right) \neq \phi$ and $\text{Im}\, \lambda_m \cap B_q\left(y_1, \frac{r'}{4}\right) \neq \phi$ for every $m \in I\!N$. Taking $r = \frac{r'}{4}$, we can find a subpath $\alpha_m$ of $\lambda_m$ such that if $H_m = \text{Im}\, \alpha_m, Q_m = \text{Im}\, f \circ \alpha_m$, to have $f(H_m) = Q_m$, $Q_m \subset B_q(y_1, 3r), Q_m \cap B_q(y_1, r) \neq \phi, Q_m \cap CB_q(y_1, 2r) \neq \phi$ for $m \in I\!N$. We see that $q(Q_m) \geq$

$r, q(Q_m, C(f,b,K\backslash E)) \geq r, q(Q_m, M_1) \geq r$ and $Q_m$ are compact, connected sets for every $m \in I\!N$.

Using Theorem 7.1, page 11, [17] and taking a subsequence we can suppose that $\lim Q_m = Q$, and we see that $Q$ is compact connected, $Q \subset B_q(y_1, 3r)$, $q(Q) \geq r, Q \cap (M_1 \cup B_q(C(f,b,K\backslash E), r) = \phi$. Let $R_2 = \Delta(Q, M_1, I\!R^n), R_1 = \Delta(Q, M_1, CC(f,b,K\backslash E))$. Using Lemma 2.6, [13], page 65, we see that there exists $\delta > 0$ such that $M(R_2) > \delta$. Since $C(f,b,K\backslash E)$ is closed, $(C \cup M_1) \cap (C(f,b,K\backslash E)) = \phi$ and $m_{n-1}(C(f,b,K\backslash E)) = 0$, we see from [15] that $M(R_2) = M(R_1)$.

Let $\Delta_j = \Delta(Q, M_1, C\overline{B}_q(C(f,b,K\backslash E), \frac{1}{j}))$ for $j \in I\!N$. Then $\Delta_j \nearrow R_1$ and using a result of Ziemer [18], we see that $M(\Delta_j) \nearrow M(R_1)$. We can therefore find $r_0 < \frac{r}{2}$ such that if $U = B_q(C(f,b,K\backslash E), r_0)$ and $\Gamma' = \Delta(Q, M_1, C\overline{U})$, to have that $M(\Gamma') > \frac{\delta}{2}$. Let $\Gamma'_m = \Delta(Q_m, M_1, C\overline{U})$ for $m \in I\!N$. Using Lemma 6, we see that $\lim\limits_{m\to\infty} M(\Gamma'_m) = M(\Gamma')$ and taking again a subsequence, we can presume that $M(\Gamma'_m) \geq \frac{\delta}{4}$ and $F(K_m \backslash (E \cup \{b\})) \subset U$ for every $m \in I\!N$.

Let $\Gamma_m$ be the family of all maximal liftings of the paths from $\Gamma'_m$ starting from some point from $H_m$ and let $\Gamma_{1m} = \{\gamma : [0,d] \to D \text{ path} |$ either $\gamma \in \Gamma_m$ and $\operatorname{Im}\gamma \subset G_1$, or there exists $c \geq d$ and $\beta : [0,c] \to D, \beta \in \Gamma_m$ such that $\gamma = \beta | [0,d], \beta(0) \in H_m$ and $d = \inf\{t \in [0,c] | \beta(t) \in G_1\}\}$ for $m \in I\!N$. Let $\Gamma_{2m} = \{\gamma \in \Gamma_{1m} | \gamma$ is rectifiable$\}$ for $m \in I\!N$. Since every path $\gamma$ from $\Gamma_{1m}$ is included in $G_1, G_1 \subset B(b,\rho)$ and $\int\limits_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))\,dx < \infty$, we apply Lemma 1 to see that $\int\limits_{B(b,\rho)\cap D} K^p(x)\,dx < \infty$ for every $p > 0$. We use now Lemma 3 to see that $M_{K^{n-1}}(\Gamma_{1m}) = M_{K^{n-1}}(\Gamma_{2m})$ for every $m \in I\!N$.

Let $\Gamma_{3m} = \{\gamma \in \Gamma_{2m} | \gamma$ ends in a point from $K_1 \backslash E\}, \Gamma_{4m} = \{\gamma \in \Gamma_{2m} | \gamma$ ends in a point from $E\}, \Gamma_{5m} = \{\gamma \in \Gamma_{2m} | \gamma$ ends in a point from $S(b,1)\}$ for $m \in I\!N$. We see that $f(\Gamma_{1m}) < \Gamma'_m$ for $m \in I\!N$. If $\gamma : [c,d] \to \overline{G}_1$ is a path from $\Gamma_{3m}$, with $\gamma(c) \in H_m, \gamma(d) \in K_1 \backslash (E \cup \{b\})$, then there exists $\lim\limits_{t\to d} f(\gamma(t)) \notin \overline{U}$ and on the other side, $\lim\limits_{t\to d} f(\gamma(t)) \in C(f, \gamma(d), \operatorname{Im}\gamma) \subset F(K_1 \backslash (E \cup \{b\})) \subset U$, which represents a contradiction. It results that $\Gamma_{3m} = \phi$ for $m \in I\!N$. Since $M_{K^{n-1}}(E) = 0$, we see that $M_{K^{n-1}}(\Gamma_{4m}) = 0$, and we also see that $\Gamma_{5m} \subset \Delta\left(\overline{B}(b, \frac{1}{m}) \cap G_1, CB(b,1) \cap G_1, G_1\right)$ for $m \in I\!N$. We see, using (i), that

$$\frac{\delta}{4} \leq M(\Gamma'_m) \leq M(f(\Gamma_{1m})) \leq M_{K^{n-1}}(\Gamma_{1m}) = M_{K^{n-1}}(\Gamma_{2m}) \leq$$

$M_{K^{n-1}}(\Gamma_{3m}) + M_{K^{n-1}}(\Gamma_{4m}) + M_{K^{n-1}}(\Gamma_{5m}) = M_{K^{n-1}}(\Gamma_{5m}) \leq M_{K^{n-1}}(\Delta(G_1 \cap \overline{B}(b, \frac{1}{m}), CB(b,1) \cap G_1, G_1) \to 0$ if $m \to \infty$, which represents a contradiction.

We proved in this way that if cap $(M \backslash C(f,b,K\backslash E)) > 0$, then it results that $C(f,b,K\backslash E) = C(f,b,G)$.

**Remark 2.** If we additionally suppose in the preceeding theorem that $m_{n-1}(C(f(B(b,\rho))) > 0$, then the condition cap $(M \backslash C(f,b,K\backslash E)) > 0$ is satisfied and hence in this case we have $C(f,b,K\backslash E) = C(f,b,G)$. An important such case holds of course when we suppose $f$ to be bounded near the point $b$.

**Remark 3.** We can take for instance $G$ to be a cone centered in the point $b$ and of some angle $0 < \varphi < \frac{\pi}{2}$ and $K$ to be the border of this cone.We can also take $G = D$ and in this case $K \subset \partial D$. We can also meet the situation when $K \cap \partial D \cap B\left(b, \frac{1}{m}\right) \neq \phi, K \cap D \cap B\left(b, \frac{1}{m}\right) \neq \phi$, for every $m \in I\!N$.

In the case $n = 2$ we obtain

**Theorem 7.** Let $D \subset I\!R^2$ be a domain, $f : D \to I\!R^2$ be a map of finite distortion, of $K(x)$ dilatation, satisfying condition $(\mathcal{A})$. Let $b \in \partial D$ be such that there exists $\rho > 0$ such that $\int\limits_{B(b,\rho)\cap D} \exp\left(\mathcal{A}\left(K(x)\right)\right) \mathrm{d}x < \infty$ and cap $C\left(f(B(b,\rho)\right) > 0$ and let $G \subset D$ be a Jordan domain such that $\partial G = \operatorname{Im}\left(\gamma_1 \vee \gamma_2^- \vee \gamma_3\right)$, where $\gamma_k : [0,1] \to \overline{D}$ are some arcs, $k = 1,2,3, \gamma_k\left([0,1)\right) \subset D$, $\lim\limits_{t\to 1}\gamma_k(t) = b$, $\lim\limits_{t\to 1} f\left(\gamma_k(t)\right) = c, k = 1,2$ and $\operatorname{Im}\gamma_3 \subset D$. Then $\lim\limits_{\substack{z\to b\\ z\in G}} f(z) = c$.

*Proof.* We take $K = \operatorname{Im}\gamma_1 \cup \operatorname{Im}\gamma_2, C = \operatorname{Im}\gamma_3$ and we see that $C(f,b,K) = \{c\}$, so that cap $C(f,b,K) = 0$. Taking $M = Cf\left(B(b,\rho)\right)$, we see that cap $\left(M \backslash C(f,b,K)\right) > 0$ and we apply now Theorem 6 to see that $C(f,b,K) = C(f,b,G)$, hence $\lim\limits_{\substack{z\to b\\ z\in G}} f(z) = c$.

*Proof of Theorem 8.* Since the locus of a path is also the locus of an arc, we can suppose that there exists $\gamma_k : [0,1] \to \overline{D}$ arcs such that $\gamma_k\left([0,1)\right) \subset D$, $\lim\limits_{t\to 1}\gamma_k(t) = b$ and $\lim\limits_{t\to 1} f\left(\gamma_k(t)\right) = b_k, k = 1,2$, with $b_1 \neq b_2$. The last condition allows us to take $\gamma_1$ and $\gamma_2$ such that $\operatorname{Im}\gamma_1 \cap \operatorname{Im}\gamma_2 = \{b\}$, and let $\gamma_3 : [0,1] \to D$ be an arc such that there exists a Jordan domain $G \subset D$ such that $\partial G = \operatorname{Im}(\gamma_1 \vee \gamma_2^- \vee \gamma_3)$. Let $M_1 = Cf\left(B(b,\rho)\right)$. If cap $M_1 > 0$, then, taking $K = \operatorname{Im}\gamma_1 \cup \operatorname{Im}\gamma_2$ and $C = \operatorname{Im}\gamma_3$, we see that $C(f,b,K) = \{b_1, b_2\}$, hence cap $C(f,b,K) = 0$ and cap$(M_1 \backslash C(f,b,K)) > 0$. We apply Theorem 6 to see that $C(f,b,G) = C(f,b,K) = \{b_1,b_2\}$ and since $C(f,b,G)$ is connected, we obtained a contradiction. It results that cap $M_1 = 0$ and taking $M_m = Cf\left(B\left(b, \frac{1}{m}\right)\right)$ for $m \in I\!N$, we prove in the same way that cap $M_m = 0$ for every $m \in I\!N$ and the theorem is proved.

**Remark 4.** As in Theorem 6, we can take the asymptotic limits in Theorem 7 and Theorem 8 over some paths $\gamma : [0,1) \to D$ ending in $b \in \partial D$ and avoiding some sets $E \subset \operatorname{Im}\gamma$ with $M_{K^{n-1}}(E) = 0$.

*Proof of Theorem 9.* Since $C(f,b,K\backslash E)$ is a compact subset from $I\!R^n$, there exists $a' < a$ such that $C(f,b,K\backslash E) \subset B(x,a')$. We can find $M_1 \subset M$ a compact set from $I\!R^n$ with cap $M_1 > 0$ and $M_1 \cap \overline{B}\left(x, \frac{a+a'}{2}\right) = \phi$. Suppose that there exists $y_1 \in C(f,b,G) \backslash B(x,a)$, and we can take $y_1 \notin M_1, y_1 \neq \infty$.

Let $G_m = B\left(b, \frac{1}{m}\right) \cap G, K_m = B\left(b, \frac{1}{m}\right) \cap K$ for $m \in I\!N$. Taking a subsequence, we can find $x_m \in G_m$ such that $f(x_m) \to y_1$ and we can suppose that $B\left(b, \frac{1}{m}\right) \cap \partial G \subset K$ and $\frac{1}{m} < \rho$ for every $m \in I\!N$. As in Theorem 6, we can find some paths $\lambda_m : [0,1] \to B\left(b, \frac{1}{m}\right)$ such that $\lambda_m(0) = x_m, \lambda_m\left([0,1)\right) \subset G_m, \lambda_m(1) \in K_m \backslash E$ for $m \in I\!N$. Let $r' = \min\{d(y_1, M_1), d\left(M_1, \overline{B}(x,a')\right)$, $d\left(y_1, \overline{B}(x,a')\right)\}$. Then $r' > 0$ and let $F : \overline{D} \to \mathcal{P}\left(\overline{I\!R^n}\right)$ be defined by $F(x) = f(x)$ if $x \in D, F(x) = C(f,x)$ if $x \in \partial D$. Then $C(f,b,K\backslash E) = \bigcap\limits_{m=1}^{\infty} F(K_m \backslash$

$\setminus (E \cup \{b\}))$ and taking a subsequence, we can presume that $F(K_m \setminus (E \cup \{b\}))$ $\subset B(x, a')$ for every $m \in I\!N$.

Then $C(f, \lambda_m(1), \operatorname{Im} \lambda_m) \subset B(x, a'), f(\lambda_m(0)) = f(x_m) \to y_1$, hence $\operatorname{Im} \lambda_m \cap B(x, a') \neq \phi, \operatorname{Im} \lambda_m \cap B\left(y_1, \frac{r'}{4}\right) \neq \phi$ for every $m \in I\!N$. Taking $r = \frac{r'}{4}$, we can find a subpath $\alpha_m$ of $\lambda_m$ such that if $H_m = \operatorname{Im} \alpha_m, Q_m = \operatorname{Im} f \circ \alpha_m$, we have $f(H_m) = Q_m, Q_m \subset B(y_1, 3r), Q_m \cap B(y_1, r) \neq \phi, Q_m \cap CB(y_1, 2r) \neq \phi$ for $m \in I\!N$. Then $d(Q_m) \geq r, d(Q_m, M_1) \geq r$ and $Q_m$ are compact, connected in $I\!R^n$ for every $m \in I\!N$. Using Theorem 7.1, page 11, [17] and taking a subsequence, we can suppose that $\lim Q_m = Q$, with $Q$ compact, connected in $I\!R^n, Q \subset B(y_1, 3r), d(Q) \geq r, Q \cap M_1 = \phi, Q \cap \overline{B}(x, a' + r) = \phi$ and let $R > 0$ be such that $B(y_1, 3r) \subset B(x, R)$ and $\Gamma' = \Delta\left(Q, M_1, C\overline{B}(x, a')\right)$. We see that $Q \subset B(x, R) \setminus \overline{B}(x, a' + r)$ and $M_1 \cap \overline{B}(x, a' + r) = \phi$, cap $M_1 > 0$ and from Lemma 4 we see that there exists $\delta > 0$ such that $M(\Gamma') > \delta$.

Let $\Gamma'_m = \Delta\left(Q_m, M_1, C\overline{B}(x, a')\right)$ for $m \in I\!N$. Using Lemma 6, we see that $\lim\limits_{m \to \infty} M(\Gamma'_m) = M(\Gamma')$ and taking a subsequence, we can presume that $M(\Gamma'_m) \geq \frac{\delta}{2}$ for every $m \in I\!N$. Let now $\Gamma_m$ be the family of all maximal liftings of some paths from $\Gamma'_m$ starting from some point of $H_m$ for $m \in I\!N$. Let $\Gamma_{1m} = \{\gamma : [0, c] \to D$ paths $|$ either $\gamma \in \Gamma_m$ and $\operatorname{Im} \gamma \subset G_1$, or there exists $c \leq d$ and $\beta : [0, d] \to D, \beta \in \Gamma_m$ with $\beta(0) \in H_m, \beta[0, c] = \gamma$ and $c = \inf\{t \in [0, d] \mid \beta(t) \notin G_1\}\}$ and let $\Gamma_{2m} = \{\gamma \in \Gamma_{1m} \mid \gamma$ is rectifiable$\}$ for $m \in I\!N$. Since every path $\gamma \in \Gamma_{1m}$ is contained in $\overline{G}_1, \overline{G}_1 \subset B(b, \rho)$ and $\int\limits_{B(b,\rho) \cap D} \exp(\mathcal{A}(K(x)))\,dx < \infty$, we see from Lemma 1 and Lemma 3 that $M_{K^{n-1}}(\Gamma_{1m}) = M_{K^{n-1}}(\Gamma_{2m})$ for every $m \in I\!N$.

Let $\Gamma_{3m} = \{\gamma \in \Gamma_{2m} | \gamma$ ends in a point from $K_1 \setminus E\}, \Gamma_{4m} = \{\gamma \in \Gamma_{2m} | \gamma$ ends in a point from $E\}$ and $\Gamma_{5m} = \{\gamma \in \Gamma_{2m} | \gamma$ ends in a point from $S(b, 1)\}$ for $m \in I\!N$. Let $\gamma : [c, d] \to \overline{G}_1$ be a path from $\Gamma_{3m}$, with $\gamma(c) \in H_m, \gamma(d) \in K_1 \setminus (E \cup \{b\})$. Then there exists $\lim\limits_{t \to d} f(\gamma(t)) \notin \overline{B}(x, a')$ and on the other side, we have $\lim\limits_{t \to d} f(\gamma(t)) \in C(f, \gamma(d), \operatorname{Im} \gamma) \subset F(K_1 \setminus (E \cup \{b\})) \subset B(x, a')$, which represents a contradiction. It results that $\Gamma_{3m} = \phi$ for every $m \in I\!N$.

We see that $f(\Gamma_{1m}) < \Gamma'_m$, that $\Gamma_{2m} \subset \Gamma_{3m} \cup \Gamma_{4m} \cup \Gamma_{5m}$, that $M_{K^{n-1}}(\Gamma_{4m}) = 0$ and that $\Gamma_{5m} \subset \Delta\left(\overline{B}\left(b, \frac{1}{m}\right) \cap G_1, CB(b, 1) \cap G_1, G_1\right)$ for every $m \in I\!N$. Using (i) we have

$$\frac{\delta}{2} \leq M(\Gamma'_m) \leq M(f(\Gamma_{1m})) \leq M_{K^{n-1}}(\Gamma_{1m}) = M_{K^{n-1}}(\Gamma_{2m}) \leq M_{K^{n-1}}(\Gamma_{3m})$$

$$+ M_{K^{n-1}}(\Gamma_{4m}) + M_{K^{n-1}}(\Gamma_{5m}) = M_{K^{n-1}}(\Gamma_{5m}) \leq M_{K^{n-1}}(\overline{B}(b, \frac{1}{m}) \cap G_1,$$

$CB(b, 1) \cap G_1, G_1) \to 0$ if $m \to \infty$.

We obtained in this way a contradiction. We therefore proved that $C(f, b, G)$ $\subset B(x, a)$.

**Remark 5.** We see that is we additionally suppose in the preceeding theorem that $f$ is bounded in a neighbourhood of $b$ and $C(f, b, K \setminus E) \subset B(x, a)$, then it results that $C(f, b, G) \subset B(x, a)$. We can also prove Theorem 7 using Theorem 9 instead of Theorem 6.

*Proof of Theorem* 10. We can suppose that $f$ is not constant on $D$. We see that $C(f, x) \subset \overline{B}(0, M)$ for every $x \in \partial D \backslash E$ and this implies that $C(f, b, \partial D \backslash E) \subset \overline{B}(0, M)$ for every $b \in (\partial D \backslash E)'$ and since $E \subset (\partial D \backslash E)'$, we use Theorem 9 to see that $C(f, b) \subset \overline{B}(0, M)$ for every $b \in \partial D$. Let $L = \sup_{x \in D} |f(x)|$ and suppose that $L > M$. Let $x_j \in D$ be such that $|f(x_j)| \to L$. Since $\overline{D}$ is compact and taking a subsequence, we can presume that $x_j \to x_0 \in \overline{D}$. If $x_0 \in \partial D$, then, taking again a subsequence, we can suppose that there exists $w \in \overline{I\!R^n}$ such that $f(x_j) \to w$, and in this case we obtain that $w \in C(f, x_0)$ and $|w| = L > M$ which contradicts the fact proved before that $C(f, b) \subset \overline{B}(0, M)$ for every $b \in \partial D$. If $x_0 \in D$, then $f(x_j) \to f(x_0)$, hence $|f(x_0)| = L$ and since $f$ is open at $x_0$, this contradicts the definition of $L$. It results that $L = M$, hence $|f(x)| \leq M$ for every $x \in D$.

*Proof of Theorem* 11. Let $\alpha = \limsup\limits_{x \to b} |f(x)|$ and $\beta = \limsup\limits_{\substack{z \to b \\ z \in \partial D \backslash E}} (\limsup\limits_{x \to z} |f(x)|)$. Then $\beta \leq \alpha$ and we can suppose that $\beta < \infty$ and let $\varepsilon > 0$ and define $\psi(z) = \limsup\limits_{x \to z} |f(x)|$ for $z \in \partial D \backslash E$. Then there exists $\delta_\varepsilon > 0$ such that $0 \leq \psi(z) < \beta + \varepsilon$ for $z \in B(b, \delta_\varepsilon) \cap (\partial D \backslash E)$ and hence $C(f, z) \subset \overline{B}(0, \beta + \varepsilon)$ for every $z \in B(b, \delta_\varepsilon) \cap (\partial D \backslash E)$. This implies that $C(f, b, \partial D \backslash E) \subset B(0, \beta + 2\varepsilon)$ and since cap $(M \cap CB(0, \beta + 2\varepsilon)) > 0$, we apply Theorem 9 to see that $C(f, b) \subset B(0, \beta + 2\varepsilon)$. Letting $\varepsilon \to 0$, we see that $C(f, b) \subset \overline{B}(0, \beta)$, hence $\alpha \leq \beta$ and hence $\alpha = \beta$.

*Proof of Theorem* 12. Suppose that there exists $b_1, b_2 \in C(f, b), b_1 \neq b_2$ and $D'$ has property $P_2$ at $b_1$. Let $r_j \to 0, U_j \in \mathcal{V}(b)$ be such that $U_{j+1} \subset U_j, U_j \subset B(b, r_j)$ and $U_j \cap D$ is connected for every $j \in I\!N$ and let $F \subset D'$ be compact. Since $C(f, b) \subset \partial D'$, we can suppose, taking a subsequence, that there exists $0 < \rho_0 < \rho$ such that $\overline{f(U_j \cap D)} \cap F = \phi$ and $f^{-1}(F) \cap B(b, \rho_0) = \phi$ for every $j \in I\!N$, and we can suppose that $r_j < \rho_0$ and let $\Gamma_j' = \Delta(f(U_j \cap D), F, D')$ for $j \in I\!N$. Since $f(U_j \cap D)$ is connected, $b_1, b_2 \in \overline{f(U_j \cap D)}$ and $D'$ has property $P_2$ at $b_1$, there exists $\delta > 0$ such that $\delta \leq M(\Gamma_j')$ for every $j \in I\!N$, and let $\Gamma_j$ be the family of all maximal liftings of the paths from $\Gamma_j'$ starting from some point from $U_j \cap D$ for $j \in I\!N$. Let $\Gamma_{1j} = \{\gamma : [0, c] \to D$ path $|$ either $\gamma \in \Gamma_j$ and $\operatorname{Im} \gamma \subset B(b, \rho_0)$, or there exists $c \leq d$ and $\beta : [0, d] \to D, \beta \in \Gamma_j$ such that $\gamma = \beta|[0, c], \gamma(0) \in U_j \cap D$ and $c = \inf\{t \in [0, d] | \beta(t) \in B(b, \rho_0)\}\}$ for $j \in I\!N$. Let $\Gamma_{2j} = \{\gamma \in \Gamma_{1j} | \gamma$ is rectifiable$\}$ for $j \in I\!N$. Using Lemma 1 and Lemma 3, we see that $M_{K^{n-1}}(\Gamma_{1j}) = M_{K^{n-1}}(\Gamma_{2j})$ for $j \in I\!N$. Also, if $\gamma \in \Gamma_{2j}, \gamma(0) \in U_j \cap D$, $\gamma : [0, 1] \to D$, then we cannot have $\gamma(1) \in B(b, \rho_0) \cap \partial D$, since the hypothesis implies that $A(f, \gamma(1)) \subset \partial D'$ and on the other side $\operatorname{Im} f \circ \gamma \subset D'$. This implies that $\Gamma_{2j} \subset \Delta\left(\overline{B}(b, r_j) \cap D, CB(b, \rho_0) \cap D, D\right)$ for every $j \in I\!N$ and using (i) we obtain

$$\delta \leq M(\Gamma_j') \leq M(f(\Gamma_{1j})) \leq M_{K^{n-1}}(\Gamma_{1j}) = M_{K^{n-1}}(\Gamma_{2j}) \leq M_{K^{n-1}}(\Delta(\overline{B}(b, r_j)$$

$\cap D, CB(b, \rho_0) \cap D, D) \to 0$ if $j \to \infty$, which represents a contradiction. It results that Card $C(f,b)=1$, hence $f$ can be continuously extended at $b$.

**Remark 6.** The important condition $\int\limits_{B(b,\rho)\cap D} \exp(\mathcal{A}(K(x)))\,\mathrm{d}x < \infty$ used in the preceedings theorems, which ensures that $M_{K^{n-1}}(\Delta(\overline{B}(b,r_j)\cap D, CB(b,\rho)\cap D, D) \to 0$ if $r_j \to 0$ and $\rho > 0$ is keept fixed, can be realized for instance if $K(x) \leq K_p$ on $D \cap CB\left(b,\frac{1}{p}\right)$ for $p \in I\!N$ and $\limsup\limits_{p\to\infty} \frac{K_p}{\ln p} = \alpha < n$. Indeed, we take $\mathcal{A}(t) = t$ and if $p_0 \in I\!N$ is such that $K_p < \ln p^{\alpha}$ for $p \geq p_0$, we have

$$\int\limits_{B\left(b,\frac{1}{p_0}\right)\cap D} \exp(\mathcal{A}(K(x)))\,\mathrm{d}x = \sum_{p\geq p_0+1} \int\limits_{\left(B\left(b,\frac{1}{p-1}\right)\setminus B\left(b,\frac{1}{p}\right)\right)\cap D} \exp(\mathcal{A}(K(x)))\,\mathrm{d}x \leq$$

$$\sum_{p\geq p_0+1} \int\limits_{\left(B\left(b,\frac{1}{p-1}\right)\setminus B\left(b,\frac{1}{p}\right)\right)\cap D} \exp(K_p)\,\mathrm{d}x \leq C_0 \cdot \sum_{p\geq p_0+1} p^{\alpha} \cdot \left(\frac{1}{(p-1)^n} - \frac{1}{p^n}\right) \leq$$

$C_1 \cdot \sum\limits_{p\geq p_0+1} \frac{1}{p^{n-\alpha+1}} < \infty$. Here $C_0$ and $C_1$ are some constants depending only on $n$.

This shows that the preceedings theorems holds for instance for locally quasiregular maps having some logarithmic growth of the constant of quasiregularity near the critical point $b \in \partial D$.

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