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by

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Classes of BCK algebras - Part I

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Dedicated to Grigore C. Moisil (1905-1973)
(9 December 2003)

Abstract

In this paper we study the BCK algebras and their particular classes: the BCK(P) (residuated) lattices, the Hájek(P) (BL) algebras and the Wajsberg (MV) algebras, we introduce new classes of BCK(P) lattices, we establish hierarchies and we give many examples. The paper has five parts.

In the first part, the most important part, we decompose the divisibility and the pre-linearity conditions from the definition of a BL algebra into four new conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). We study the additional conditions (WNM) (weak nilpotent minimum) and (DN) (double negation) on a BCK(P) lattice. We introduce the ordinal sum of two BCK(P) lattices and prove in what conditions we get BL algebras or other structures, more general, or more particular than BL algebras.

In part II, we give examples of some finite bounded BCK algebras. We introduce new generalizations of BL algebras, named α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). By adding the conditions (WNM) and (DN) to these classes, we get more classes; among them, we get many generalizations of Wajsberg (MV) algebras and of R_0 (NM) algebras. The subclasses of $_{(WNM)}$ Wajsberg algebras ($_{(WNM)}$MV algebras) and of $_{(WNM)}$Hájek algebras ($_{(WNM)}$BL algebras) are introduced. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I.

In part III, we give examples of finite MV and $_{(WNM)}$MV algebras, of Hájek(P) (i.e. BL) algebras and $_{(WNM)}$BL algebras and of $\alpha\gamma\delta$ (i.e. divisible BCK(P) lattices (divisible residuated lattices or divisible integral, residuated, commutative l-monoids)) and of divisible $_{(WNM)}$BCK(P) lattices.

In part IV, we stress the importance of $\alpha\beta\gamma$ algebras versus $\alpha\beta$ (i.e. MTL) algebras algebras and of R_0 (i.e. NM) algebras versus Wajsberg (i.e. MV) algebras and of $_{(WNM)}\alpha\beta\gamma$ algebras versus BL algebras and of $\alpha\gamma$ versus $\alpha\gamma\delta$ algebras. We give examples of finite IMTL algebras and of $_{(WNM)}$IMTL (i.e. NM) algebras), of $\alpha\beta\gamma$ algebras and of $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras and finally of $\alpha\gamma$ algebras.

In part V, we give other examples of finite BCK(P) lattices, finding examples for the others remaining an open problem. We make final remarks and formulate final open problems.

Keywords MV algebra, Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, divisible BCK(P) lattice, α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, $_{(WNM)}$MV, $_{(WNM)}$BL, $_{(WNM)}$ $\alpha\beta\gamma$, Roman algebra

1 Introduction

The results of this paper in three parts concern more related algebras and they were found because we have worked only with "left" algebras, not with some "left" algebras and other "right" algebras, and because we have worked only with the implication ($\to$) as primitive operation, not with both implication and the t-norm ($\odot$). To be more explicite, we shall develop the two ideas.

1.1 The first idea: to work only with "left" algebras

The first idea is related to the two diferent possible definitions of some algebras, as "left" or as "right" algebras. For instance, MV algebras (1958) and BCK algebras (1966) were (initially) defined as "right" algebras, while Wajsberg algebras (1984), residuated lattices (1924) and BL algebras (1994) were (initially) defined as "left" algebras.

When working simultaneously with different algebras, we claim that is better to choose: "left", or "right", and then use the appropiate definitions. Otherwise, it is difficult to "see" the connections between the algebras and to build examples - of "ordinal sums", for examples (see Part II). In this paper, we shall work only with "left" algebras.

The notions of "left" and "right" algebras are connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm on $[0,1]$, respectively, and are discussed in detail in [20]. We can also say that they are connected with the "negative (left)" cone and with the "positive (right)" cone, respectively, of an l-group (lattice-ordered group).

Recall that at the beginning, t-norms (triangular norms) and t-conorms were defined on the real interval $[0,1]$, namely:

A binary operation $\odot$ on the real interval $[0,1]$ is a *t-norm* iff it is commutative, associative, non-decreasing (isotone) in the first argument (i.e. if $x \leq y$, then $x \odot z \leq y \odot z$, for every $x, y, z \in [0,1]$), and hence in the second argument too, and it has 1 as neutral element (i.e. $x \odot 1 = x$ (and consequently, $x \odot 0 = 0$), for every $x \in [0,1]$).

A binary operation $\oplus$ on the real interval $[0,1]$ is a *t-conorm* iff it is commutative, associative, non-decreasing in the first argument and hence in the second argument too, and it has 0 as neutral element.

We have defined in a natural way, in [38], a *t-norm* $\odot$ *on a poset* $(A, \geq, 1)$ *with greatest element* 1 iff the above mentioned axioms are fulfilled and a *t-conorm* $\oplus$ *on a poset* $(A, \leq, 0)$ *with smallest element* 0 iff the above mentioned corresponding axioms are fulfilled.

Recall also the following definition: a *partially ordered, abelian (i.e. commutative), integral monoid* or a *pocim* for short is an algebra $(A, \geq, \odot, 1)$ such that: $(A, \geq, 1)$ is a poset with greatest element 1, $(A, \odot, 1)$ is an abelian monoid (i.e. $\odot$ is commutative, associative and has 1 as neutral element) and $\odot$ is non-decreasing in the first argument (or, $\odot$ is compatible with $\geq$) and hence in the second argument too; *integral* means that the greatest element of the poset $(A, \geq)$ coincides with the neutral element of the abelian monoid.

Recal also [38] that the statement: "$\odot$ is a t-norm on the poset $(A, \geq, 1)$ with greatest element 1" is equivalent with the statement: "the algebra $(A, \geq, \odot, 1)$ is a pocim".

The passage from the (definition of) "right" algebra to its inverse, the "left" algebra, is made by replacing everywhere the t-conorm $\oplus$ by the t-norm $\odot$, the co-residuum $\to_R$ by the residuum $\to = \to_L$ ("R" comes from "right", "L" comes from "left"), by replacing 0 by 1 (and 1 by 0), by replacing the binary relation $\leq$ by its inverse relation, $\geq$.

The passage from the "left" algebra to its inverse, the "right" algebra, is made by replacing everywhere the t-norm $\odot$ by the t-conorm $\oplus$, the residuum $\to = \to_L$ by the co-residuum $\to_R$, by replacing 1 by 0 (and 0 by 1), by replacing the binary relation $\geq$ by its inverse relation, $\leq$.

We shall denote by a bolded name the class of corresponding algebras.

MV algebras were introduced in 1958, by C. C. Chang [10], as "right" algebras, as a model of $\aleph_0$-valued Lukasiewicz logic.

A *(right-) MV algebra* is an algebra $(A, \oplus, ^-, 0)$ of type (2,1,0) satisfying the following axioms (see [11]): for all $x, y, z \in A$,

(MV1-R) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV2-R) $x \oplus y = y \oplus x$,

(MV3-R) $x \oplus 0 = x$,

(MV4-R) $(x^-)^- = x$,
(MV5-R) $x \oplus 0^- = 0^-$,
(MV6-R) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

Definition 1.1 A *left-MV algebra* is an algebra $(A, \odot, ^-, 1)$, of type (2,1,0), satisfying, for all $x, y, z \in A$ [20], [38]:
(MV1-L) $x \odot (y \odot z) = (x \odot y) \odot z$,
(MV2-L) $x \odot y = y \odot x$,
(MV3-L) $x \odot 1 = x$,
(MV4-L) $(x^-)^- = x$,
(MV5-L) $x \odot 1^- = 1^-$,
(MV6-L) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.

Let **MV** denote the class of left-MV algebras.

Remarks 1.2 Recall that in a left-MV algebra $\mathcal{A} = (A, \odot, ^-, 1)$ we have the following properties:
0) $x \oplus y \stackrel{def}{=} (x^- \odot y^-)^-$, $0 \stackrel{def}{=} 1^-$, $0^- = 1$.
1) $(A, \wedge, \vee, 0, 1)$ is a bounded, distributive lattice, where for all $x, y \in A$:
$x \wedge y = x \odot (x \odot y^-)^- = x \odot (x^- \oplus y) = y \odot (y \odot x^-)^- = y \odot (y^- \oplus x)$,
$x \vee y = [x^- \odot (x^- \odot y)^-]^- = x \oplus (x^- \odot y) = y \oplus (y^- \odot x) = [y^- \odot (y^- \odot x)^-]^-$.
2) The binary relation defined by: for all $x, y \in A$: $x \leq y \Leftrightarrow x \odot y^- = 0 \Leftrightarrow x^- \oplus y = 1$ is the partially ordered relation of the lattice.
3) For all $x, y, z \in A$: $x \leq y$ implies $x \odot z \leq y \odot z$.
4) For all $x \in A$, $x \odot x^- = 0$.
5) For all $x, y, z \in A$, $x \odot y \leq z \Leftrightarrow y \leq (x \odot z^-)^- \Leftrightarrow x \leq (y \odot z^-)^-$.
6) For all $y, z \in A$, $\max\{x \mid x \odot y \leq z\} = (y \odot z^-)^-$.
7) If we define the residuum $\rightarrow$ by:

$$x \rightarrow y \stackrel{def}{=} (x \odot y^-)^- = x^- \oplus y,$$

then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a (left-) residuated lattice verifying, for all $x, y \in A$:
$x \vee y = (x \rightarrow y) \rightarrow y = (x \odot y^-)^- \rightarrow y = ((x \odot y^-)^- \odot y^-)^- = y \oplus (x \odot y^-)$,
$x \wedge y = x \odot (x \odot y^-)^- = x \odot (x \rightarrow y) = (y^- \vee x^-)^-$ and
$(x \rightarrow y) \vee (y \rightarrow x) = 1$ (i.e. $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a (left-)BL algebra). Moreover, $x \rightarrow y = (x^- \rightarrow_R y^-)^-$ and $x \rightarrow_R y = (x^- \rightarrow y^-)^-$.

Wajsberg algebras were introduced in 1984, by Font, Rodriguez and Torrens [22], as left-algebras; they are a model of $\aleph_0$-valued Lukasiewicz logic too, studied by Wajsberg in 1935 [70].

Definition 1.3 A *(left-) Wajsberg algebra* is an algebra $(A, \rightarrow = \rightarrow_L, ^-, 1)$ of type $(2, 1, 0)$ such that, for all $x, y, z \in A$:
(W1) $1 \rightarrow x = x$,
(W2) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,
(W3) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
(W4) $(x^- \rightarrow y^-) \rightarrow (y \rightarrow x) = 1$.

Let **W** denote the class (or the category) of Wajsberg algebras.
MV algebras and Wajsberg algebras are categorically equivalent (see [22], Theorems 4 and 5).
Let $\equiv$ mean "is an equivalent definition", $\cong$ mean "are categorically equivalent" and $=$ mean "is a duplicate name" through this paper.
Then, we shall write: $\mathbf{W} \cong \mathbf{MV}$.

* * *

Residuated lattices, the algebraic counterpart of logics without contraction rule, have been investigated (cf. Kowalski-Ono [52]) by Krull [53], Dilworth [15], Ward and Dilworth [72], Ward [71], Balbes and Dwinger [4], Pavelka [62] and Idziak [34]. Residuated lattices have been known under many names; they have been called (cf. [52]) *BCK lattices* in [34], *full BCK-algebras* in [60], FL_{ew}*-algebras* in [61] and *integral, residuated, commutative l-monoids* in [33]; some of those definitions are free of 0. We shall use the following definition.

Definition 1.4 [52] (see [38])
A *(left-) residuated lattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ verifying:
(i) $(A, \wedge, \vee, 0, 1)$ is a lattice with first element 0 and last element 1 (under $\geq$),
(ii) $(A, \odot, 1)$ is an abelian (i.e. commutative) monoid,
(RP) for all $x, y, z \in A$, $x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z$.

Let **R-L** denote the class (or the category) of residuated lattices.

* * *

BL algebras were introduced in 1994 by Petr Hájek [29], [30], [31]. The starting point in defining and studying Basic Logic and BL algebras were the algebras of the form $([0,1], \min, \max, \odot, \rightarrow, 0, 1)$, where $\odot$ is a a continuous t-norm on $[0,1]$ and $\rightarrow$ is the associated residuum; these algebras are called *standard* BL algebras.

The most important continuous t-norms on $[0,1]$ are the following three: Lukasiewicz t-norm, Product t-norm, Gödel t-norm. These three t-norms have the following associated residua:

(1) Lukasiewicz:

$$x \odot_L y = \max(0, x+y-1), \quad x \rightarrow_L y = \begin{cases} 1, & \text{if } x \leq y \\ 1-x+y, & \text{if } x > y \end{cases} = \min(1, 1-x+y);$$

(2) Product (Gaines):

$$x \odot_P y = xy, \quad x \rightarrow_P y = \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{if } x > y, \end{cases} \quad \text{(Goguen implication)}$$

(3) Gödel (Brouwer):

$$x \odot_G y = \min(x, y), \quad x \rightarrow_G y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y, \end{cases} \quad \text{(Godel implication).}$$

The three t-norms and their associated residua correspond to the most significant fuzzy logics: Lukasiewicz logic, Product logic and Gödel logic, respectively. The MV algebras, the Product algebras and the Gödel algebras constitute the algebraic models for these three types of logics.

The class of BL algebras contains the MV algebras [10], [11], the Product algebras [32], [55], [30] and the Gödel algebras [30].

Definition 1.5 A *(left-) BL algebra* [30] is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ such that:
(B1) $\mathcal{A}$ is a residuated lattice,
(B2) $x \wedge y = x \odot (x \rightarrow y)$ (divisibility),
(B3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (pre-linearity).

Let **BL** denote the class (or the category) of BL algebras.

A (left-) BL algebra is a left-MV algebra iff it satisfies the condition (DN) (double negation): for all x,

$$(x^-)^- = x,$$

where $x^- = x \rightarrow 0$ [30].

The standard left-MV algebra is the (left-) BL algebra $([0,1], \min, \max, \odot_L, \rightarrow_L, 0, 1)$ determined by the above Lukasiewicz t-norm.

Definition 1.6

A *(left-) Product algebra* [30] is a BL algebra $\mathcal{A}$ which fulfills the following two conditions: for every $x, y, z \in A$:

(P1) $x \wedge x^{-} = 0$,

(P2) $(z^{-})^{-} \odot [(x \odot z) \to (y \odot z)] \leq x \to y$.

The standard Product algebra is the BL algebra $([0,1], \min, \max, \odot_P, \to_P, 0, 1)$ determined by the above Product t-norm.

A BL algebra which fulfills the condition (P1) is usually called a *SBL algebra.*

Let us name as *SSBL algebra* the BL algebra fulfilling the condition (P2).

Let **Product**, **SBL**, **SSBL** denote the classes of Product algebras, SBL and SSBL algebras, respectively.

Definition 1.7

A *(left-) Gödel algebra* [30] is a BL algebra $\mathcal{A}$ which fulfills the condition (G) (idempotent multiplication): for each $x \in A$,

(G) $x \odot x = x$.

Let **Godel** denote the class of Gödel algebras.

The standard Gödel algebra is the BL algebra $([0,1], \min, \max, \odot_G, \to_G, 0, 1)$ determined by the above Gödel t-norm.

Recall now the following definitions of two particular cases of residuated lattices and in the same time generalizations of BL algebras:

A weak-BL algebra [20] (or a weak-Hajek(P) algebra, more pedantically) is a residuated lattice satisfying the condition (B3) (a duplicate name in the literature for weak-BL algebras is "MTL (Monoidal t-norm based) algebras" [18]).

Let **MTL** denote the class of MTL algebras.

A *divisible* BCK(P) lattice (see [33]) is a BCK(P) lattice (residuated lattice) satisfying the condition (B2).

Let us recall (see [18]) that:

(1) a *WNM (Weak Nilpotent Minimum) algebra* is a MTL algebra satisfying the additional axiom:

(WNM) $$(x \odot y)^{-} \vee [(x \wedge y) \to (x \odot y)] = 1;$$

(2) an *IMTL algebra (Involutive Monoidal t-norm based Logic)* is a MTL algebra satisfying the condition (DN);

(3) a *NM (Nilpotent Minimum) algebra* is an IMTL algebra satisfying the axiom (WNM) (or a WNM algebra satisfying the condition (DN) (double negation)),

where, given a **weak negation** "n" (i.e. $x \leq n(n(x))$) on $[0,1]$ and the t-norm "$\odot_n$" and the implication "$\to_n$" defined as follows on $[0,1]$:

$$x \odot_n y = \begin{cases} 0, & \text{if } x \leq n(y) \\ \min(x,y), & \text{otherwise}, \end{cases} \qquad x \to_n y = \begin{cases} 1, & \text{if } x \leq y \\ \max(n(x),y), & \text{if } x > y, \end{cases}$$

then $\odot_n$ is a left-continuous t-norm, with $n(x) = x \to_n 0$, and $([0,1], \min, \max, \odot_n, \to_n, 0, 1)$ is a standard WNM algebra, for each weak negation n, while given a **strong (involutive) negation** "n" (i.e. $x = n(n(x))$) on $[0,1]$ and the Fodor's t-norm "$\odot_F$" and implication "$\to_F$"t-norm "$\odot_F$" defined as follows on $[0,1]$ [21]:

$$x \odot_F y = \begin{cases} 0, & \text{if } x \leq n(y), \\ \min(x,y), & \text{otherwise} \end{cases} \qquad x \to_F y = \begin{cases} 1, & \text{if } x \leq y \\ \max(n(x),y), & \text{if } x > y, \end{cases}$$

then $\odot_F$ is left-continuous also, with $n(x) = x \to 0$, and $([0,1], \min, \max, \odot_F, \to_F, 0, 1)$ is a standard NM algebra, for each strong negation n.

Remark 1.8 When working with BCK(P) algebras, residuated lattices, BL algebras etc. we start with the implication $\to$ and $0,1$, or with the implication $\to$, with the t-norm $\odot$ and $0,1$, and we define the negation $^-$ as $x^- = x \to 0$, which is **weak** (i.e. $x \leq (x^-)^-$), and we see what happens when the negation is **strong** (or **involutive**, or satisfies the **double negation** condition (DN)) (i.e. $x = (x^-)^-$). In [21], Fodor starts with a strong negation n and with the t-norm $\odot_F$ and defines the implication $\to_F$, which verifies $n(x) = x \to_F 0$. In [18], Fodor's implication is generalized, by starting with a weak negation.

Remark 1.9 We shall stress in this paper, especially in Part II and Part III, the importance of NM algebras; we shall prove that the class of Wajsberg (MV) algebras and the class of NM algebras are incomparable (with respect to set inclusion).

Let **WNM, IMTL, NM** denote the classes of WNM algebras, IMTL algebras, NM algebras, respectively. Then we have:

NM = **IMTL** + (WNM) = **WNM** + (DN).

Remark 1.10 We shall generalize, by following up the condition (WNM) in all BCK(P) lattices (residuated lattices). We shall call BCK(P) lattices satisfying the condition (WNM) as a "$_{(WNM)}$BCK(P)" lattices.

Recall also [63] that the IMTL algebras, introduced in 2001 by Esteva and Godo [18], are categorically equivalent with "weak-R_0" algebras, introduced in 1997 by G.J. Wang [73] and that NM algebras are categorically equivalent with R_0 algebras, introduced also in 1997 by G.J. Wang [73]:

Definition 1.11 [63]

(1) A *weak-R_0 algebra* or, a WR_0 for short is an algebra $\mathcal{M} = (M, \wedge, \vee, \to, ^-, 1)$ of order type $(2,2,2,1,0)$, such that:

- $(M, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, $\leq$ being the order relation,
- "$^-$" is an order reversing involution with respect to $\leq$,
- the following conditions hold: for all $x, y, z \in M$,
  (R1) $x^- \to y^- = y \to x$,
  (R2) $1 \to x = x$,
  (R3) $y \to z \leq (x \to y) \to (x \to z)$,
  (R4) $x \to (y \to z) = y \to (x \to z)$,
  (R5) $x \to (y \vee z) = (x \to y) \vee (x \to z)$.

(2) An *R_0-algebra*, or R_0 for short is a weak R_0-algebra verifying the additional condition (R6):
(R6) $(x \to y) \vee ((x \to y) \to (x^- \vee y)) = 1$.

Let **weak $\mathbf{R_0}$** and $\mathbf{R_0}$ denote the classes of weak R_0 algebras and R_0 algebras, respectively.

Remark 1.12
The conditions (R6) and (WNM) are not equivalent in an IMTL algebra $\mathcal{A}$ which is not an NM algebra, in the following sense: if there are $a, b \in A$ such that (R6) is not verified, it is possible that (WNM) be verified by those a, b, and vice-versa, as you can see in the examples of IMTL algebras from the Part IV.

1.2 The second idea: to work only with the implication

The second ideea is related to the similarity type of algebras. In many algebras connected with logics (residuated lattices, BL algebras, MV algebras, Wajsberg algebras, BCK algebras etc.) we have two adjoint operations: the implication (residuum) ($\rightarrow$) and the product (t-norm) ($\odot$). As it was largely developed in the survey-paper [38], there are two main ways of studying these algebras:

(1) either to start only with the residuum $\rightarrow$ as primitive operation (i.e. to start with the BCK algebra), and then its associated (derived) t-norm $\odot$ is defined, whenever it exists, by the condition:

(P) $x \odot y \overset{notation}{=} min\{z \mid x \leq y \rightarrow z\}$, for all x, y,

or, alternatively, to start with both $\rightarrow$ and $\odot$ (in this order), verifying then the condition:

(RP) $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$, for all x, y, z,

as very seldom is the case (see the definitions of BCK algebras, BCK lattices, Wajsberg algebras etc.), or

(2) either to start only with the t-norm $\odot$ as primitive operation (i.e. to start with the monoid), and then its asociated (derived) residuum $\rightarrow$ is defined, whenever it exists, by the condition:

(R) $y \rightarrow z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}$, for all y, z,

or, alternatively, to start with both $\odot$ and $\rightarrow$ (in this order), verifying then the condition:

(PR) $x \leq y \rightarrow z \Longleftrightarrow x \odot y \leq z$, for all x, y, z,

as very often is the case (see the definitions of monoids, pocrims, residuated lattices, BL algebras, MV algebras etc.).

We claim that it is better to start in the first way, namely to start with $\rightarrow$, alone, since $\rightarrow$ is more closed to logic than $\odot$ and the properties are more accessible. This implies to study "the deductive systems", not "the filters" of such defined algebras. Look for instance at the two conditions, divisibility ($x \wedge y = x \odot (x \rightarrow y)$) and pre-linearity ($(x \rightarrow y) \vee (y \rightarrow x) = 1$), which appear in the definition of a BL algebra; if the divisibility can be expressed in a nice way either only by means of $\odot$ (see [33] Lemma 2.5) or only by means of $\rightarrow$ (see Theorem 3.1), the pre-linearity cannot.

Thus, there are four (two plus two) different types of similarity for a "left" algebra of logic and usually one, maximum two, among the four different types are used in the literature, for each algebra.

For example, the five algebras: the reversed left-BCK(P) algebras (r-BCK(P)), the pocrims, the reversed left-BCK(P) lattices (r-BCK(P)-L), the residuated lattices (R-L) and the BL algebras determine a table (matrix) with 4 columns and 3 rows, where only five cells are filled. In [38] we have introduced the "missing" algebras, we have put them in the empty cells and it was proved that the algebras of the four different types of similarity (i.e. the algebras on the same row in the four colums) are categorically equivalent; the only problem is the problem of "names" for the four equivalent algebras. The complete table of all $12 = 4 \times 3$ algebras is presented in Figure 1, where the initial five algebras are marked by a bullet. Since the 12 algebras are (direct or indirect) generalizations (ascendents) of Wajsberg (MV) algebras, we have added a fourth row to the table, the row of Wajsberg and MV algebras, with completion of two columns - following the usual definitions - and whithout completion of two other columns.

Note that we shall use the following "signs" between categories of left algebras: the sign "=" will mean duplicate names, the sign " $\equiv$" will mean equivalent definitions, while the sign "$\cong$" will mean that the corresponding categories are equivalent. Thus, between categories of algebras of the same line in the table from Figure 1 we must use the sign $\cong$; for example, bounded, commutative reversed left-BCK lattices $\equiv$ Wajsberg algebras, while X-Hájek(RP) algebras = BL algebras.

Recall that the axioms appearing in the table from Figure 1 are the following [38]:

(A1) = (X1) $(A, \geq, 1)$ is a poset with greatest element 1,

(A2) $(A, \rightarrow, 1)$ verifies: for all x, y, z, (R1) $1 \rightarrow x = x$, (R2) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,

(A3) $x \rightarrow y = 1 \Longleftrightarrow x \leq y$, for all x, y,

(A4) $x \leq y \Longrightarrow z \rightarrow x \leq z \rightarrow y$, for all x, y, z,

(A5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for all x, y,

The world of left-algebras			
The general world of $\rightarrow$, 1		The general world of $\odot$, 1	
The world of $\rightarrow$, 1 (direct ascend. of W)	The world of $\rightarrow$, $\odot$, 1 (indirect ascend. of W)	The world of $\odot$, $\rightarrow$, 1 (indirect asc. of MV)	The world of $\odot$, 1 (direct asc. of MV)
• **r-BCK(P)** $(A, \geq, \rightarrow, 1)$ (A1),(A2),(A3),(A4), (P)	**r-BCK(RP)** $(A, \geq, \rightarrow, \odot, 1)$ (A1), (A2), (A3), (RP)	**X-BCK(RP)** $(A, \geq, \odot, \rightarrow, 1)$ (A1), (X2), (RP)	• **X-BCK(R)** = pocrim $(A, \geq, \odot, 1)$ (A1), (X2), (X3), (R)
• **r-BCK(P)-L** $(A, \wedge, \vee, \rightarrow, 0, 1)$ (B1),(A2),(A3),(A4), (P)	**r-BCK(RP)-L** $(A, \wedge, \vee, \rightarrow, \odot, 0, 1)$ (B1), (A2), (A3), (RP)	• **X-BCK(RP)-L** **= R-L** $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (B1), (X2), (RP)	**X-BCK(R)-L** **= X-R-L** $(A, \wedge, \vee, \odot, 0, 1)$ (B1), (X2), (X3), (R)
r-Ha(P) $(A, \wedge, \vee, \rightarrow, 0, 1)$ (B1),(A2),(A3),(A4), (P) (B2),(B3)	**r-Ha(RP)** $(A, \wedge, \vee, \rightarrow, \odot, 0, 1)$ (B1), (A2), (A3), (RP), (B2), (B3)	• **X-Ha(RP)= BL** $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (B1), (X2), (RP), (B2),(B3)	**X-Ha(R)= X-BL** $(A, \wedge, \vee, \odot, 0, 1)$ (B1), (X2), (X3), (R), (B2), (B3)
W $(A, \rightarrow, ^-, 1)$ (A2), (A5), (A6)			**MV** $(A, \odot, ^-, 1)$ (X2), (DN), (X4), (X6)

Figure 1: The table with four columns corresponding to the four different similarity types of algebras

(A6) $(x^- \to y^-) \to (y \to x) = 1$. (X2) $(A, \odot, 1)$ is an abelian (i.e. commutative) left-monoid,
(X3) $x \leq y \Rightarrow x \odot z \leq y \odot z$, for every x, y, z,
(X4) $x \odot 1^- = 1^-$, for all x,
(X5) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$, for all x, y,
(DN) $x^-)^- = x$, for all x,
(P) there exists $x \odot y \overset{notation}{=} min\{z \mid x \leq y \to z\}$, for all x, y,
(R) there exists $y \to z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}$, for all y, z,
(RP)= (PR) $x \odot y \leq z \Longleftrightarrow x \leq y \to z$, for all x, y, z,
(B1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
(B2) $x \wedge y = x \odot (x \to y)$, for all x, y,
(B3) $(x \to y) \vee (y \to x) = 1$, for all x, y.

Consequently, in this paper we shall start with $\to$, i.e. we shall work with reversed left-BCK algebras, reversed left-BCK(P) algebras, reversed left-BCK(P) lattices, reversed left-Hájek(P) algebras and Wajsberg algebras (see the algebras from the first column of the table from Figure 1). Note that we shall sometimes use the more used names, "MV algebras" and "BL algebras", rather than "Wajsberg algebras" and "reversed left-Hájek(P) algebras" respectively.

Following these comments, weak R_0-algebras and R_0-algebras "go with" Hájek(P) algebras and with Wajsberg algebras in column 1 of the table from Figure 1, while IMTL algebras and NM algebras "go with" BL algebras, in the 3^{rd} column of that table. Consequently, we should normally refer to "weak R_0-algebras, R_0-algebras, axiom (R6), Hájek(P) algebras and Wajsberg algebras", but sometimes we shall refer to "IMTL algebras, NM algebras, axiom (WNM), BL algebras and even MV algebras (from the 4^{th} column)" too.

* * *

The motivation of this paper was the following: trying to answer to the open problem 3.12 (3.35) from [38]: "find an example of reversed left-BCK(P) lattice with condition (DN) which is not with condition (C)", i.e. which is not a Wajsberg (MV) algebra, we found more examples; thus, a new problem arised: which is the difference between them? Thus, we arrived to decompose the divisibility and pre-linearity conditions in other conditions and so on.

The paper has five parts and twenty sections.

Part I is the main part of the whole paper and has 4 Sections . In Section 2, we recall the properties of reversed left-Hájek (BL) algebras by showing where they come from: most of the properties are coming from the basic BCK algebra and from the condition (P), some are coming from the lattice condition and very few from the two conditions, divisibility ((B2)) and pre-linearity ((B3)). In Section 3 we decompose the two conditions (B2) and (B3) into four conditions: $(C_{\to})$, $(C_{\vee})$, $(C_{\wedge})$, (C_X). This is the main result of this part. In Section 4 we define the ordinal sum of two BCK(P) lattices and prove that it is a BCK(P) lattice (Theorem 5.2).

Part II has two sections. In Section 5, we give examples of some finite bounded BCK algebras. In Section 6, we In Section 14, we introduce new generalizations of Hájek(P) (BL) algebras, named α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_{\to})$, $(C_{\vee})$, $(C_{\wedge})$, (C_X) found in Part I. We make the connections with MTL algebras [18] and with divisible integral, residuated, commutative l-monoids [33]. By adding the conditions (WNM) and (DN) to these classes, we get more classes: of ${}_{(WNM)}\alpha$ algebras, $\alpha_{(DN)}$ algebras, ${}_{(WNM)}\alpha_{(DN)}$ algebras etc. Thus, we get generalizations of BL and ${}_{(WNM)}$BL algebras, and of Wajsberg (MV) algebras and of NR_0 algebras. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I and Part II. We make the connections with MTL, WNM, IMTL and NM algebras [18], [21] and with R_0 [73], [63] and NR_0 algebras [54].

Part III has seven sections. In Section 7 we give examples of finite Wajsberg (MV) algebras, useful in the next sections. In Section 8 we give examples of finite linearly ordered reversed left-Hájek algebras (BL algebras) which are not Wajsberg (MV) algebras. In Section 9 we give examples of finite non-linearly ordered reversed left-Hájek algebras (BL) algebras which are not Wajsberg (MV) algebras. In Section 10 we give examples of infinite proper BL algebras, obtained as ordinal sums of two product algebras. In

Section 11 we give examples of finite divisible reversed left-BCK(P) lattices (divisible residuated lattices). In Section 12 we give an example of infinite proper divisible reversed left-BCK(P) lattice, as an ordinal sum of two product algebras. In Section 13 we present two open problems.

Part IV has four sections. In Section 14, we give examples of proper IMTL algebras and of NM algebras. In Section 15, we give examples of proper $\alpha\beta\gamma$ and of $_{(WNM)}\alpha\beta\gamma$ algebras. In Section 16, we give examples of proper $\alpha\gamma$ and of $_{(WNM)}\alpha\gamma$ algebras. In Section 17, we formulate some remarks and open problems.

Part V has three parts. In Section 18, we give other finite examples of generalizations of Wajsberg (MV) algebras and $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras. In Section 19, we give other finite examples of generalizations of Hájek(P) (BL) algebras and $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras. In Section 20, we give final remarks and open problems.

We assume the reader is familiar with [38], but the paper is self-contained as much as possible.

The old, already known results, are presented without proof.

2 Classes of BCK algebras

In this section we stress the fact that a Hájek (BL) algebra is a BCK algebra and we divide the properties of a BL algebra in three groups: those coming from the fact that it is a BCK algebra, those coming from the fact that it is a residuated lattices and finally those coming from the two conditions (B2) and (B3). Most of the results are old. The new most important results are Theorem 2.34, Propositions 2.39, 2.40, Theorems 2.41 and 2.42.

2.1 BCK algebras, reversed left-BCK(P) algebras (pocrims)

BCK algebras were introduced in 1966 by Kiyoshi Iséki as "right" algebras, starting from the systems of positive implicational calculus, weak positive implicational calculus by A. Church and BCI, BCK-systems by C.A. Meredith (cf. [49]):

A *(right-) BCK algebra* [49] is an algebra

$$\mathcal{A} = (A, \leq, \star, 0),$$

where $\leq$ is a binary relation on A, $\star$ is a binary operation on A and 0 is an element of A, verifying the following axioms: for all $x, y, z \in A$,

(I-R) $(x \star y) \star (x \star z) \leq z \star y$,

(II-R) $x \star (x \star y) \leq y$,

(III-R) $x \leq x$,

(IV-R) $0 \leq x$,

(V-R) $x \leq y,\ y \leq x \Longrightarrow x = y$,

(VI-R) $x \leq y \Longleftrightarrow x \star y = 0$,

or, equivalently, (see [28]) an algebra $(A, \star, 0)$ of type (2,0) satisfying the following axioms: for all $x, y, z \in A$,

(BCK-1-R) $[(x \star y) \star (x \star z)] \star (z \star y) = 0$,

(BCK-2-R) $x \star 0 = x$,

(BCK-3-R) $0 \star x = 0$,

(BCK-4-R) $x \star y = 0$ and $y \star x = 0$ imply $x = y$.

The left-BCK algebra is obtained by replacing the relation $\leq$ with the inverse relation, $\geq$, $\star$ with $\square$ and 0 with 1, as follows.

A *left-BCK algebra* is an algebra

$$\mathcal{A} = (A, \geq, \square, 1),$$

where $\geq$ is a binary relation on A, $\square$ is a binary operation on A and 1 is an element of A, verifying, for all $x, y, z \in A$, the axioms:

(I-L) $(x\Box y)\Box(x\Box z) \geq z\Box y$,
(II-L) $x\Box(x\Box y) \geq y$,
(III-L) $x \geq x$,
(IV-L) $1 \geq x$,
(V-L) $x \geq y,\ y \geq x \Longrightarrow x = y$,
(VI-L) $x \geq y \Longleftrightarrow x\Box y = 1$.
or, equivalently, is an algebra $(A, \Box, 1)$ of type (2,0) verifying the axioms corresponding to (BCK-1-R) - (BCK-4-R).

The *reversed* left-BCK algebra is obtained by reversing the operation $\Box$, i.e. by replacing $x\Box y$ by $y \to x = y \to_L x$, for all x, y. We need to reverse the left-BCK algebra in order to arrive to the implication $\to$, which appears in BL algebras.

Definition 2.1 A *reversed left-BCK algebra* is an algebra

$$\mathcal{A} = (A, \geq, \to, 1),$$

where $\geq$ is a binary relation on A, $\to$ is a binary operation on A and 1 is an element of A, verifying, the axioms: for all $x, y, z \in A$,
(I) $(z \to x) \to (y \to x) \geq y \to z$,
(II) $(y \to x) \to x \geq y$,
(III) $x \geq x$,
(IV) $1 \geq x$,
(V) $x \geq y,\ y \geq x \Longrightarrow x = y$,
(VI) $x \geq y \Longleftrightarrow y \to x = 1$,

or, equivalently,

Definition 2.2 A *reversed left-BCK algebra* is an algebra

$$(A, \to, 1)$$

of type (2,0) verifying the axioms: for all $x, y, z \in A$,
(BCK-1) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(BCK-2) $1 \to x = x$,
(BCK-3) $x \to 1 = 1$,
(BCK-4) $y \to x = 1$ and $x \to y = 1$ imply $x = y$.

We shall freely write $x \geq y$ or $y \leq x$ in the sequel.

Proposition 2.3 *(see [49])*
The following properties hold in a reversed left-BCK algebra:

$$x \leq y \Rightarrow y \to z \leq x \to z, \tag{1}$$

$$x \leq y,\ y \leq z \Rightarrow x \leq z, \tag{2}$$

$$z \to (y \to x) = y \to (z \to x), \tag{3}$$

$$z \leq y \to x \Leftrightarrow y \leq z \to x, \tag{4}$$

$$x \leq y \to x, \tag{5}$$

$$1 \to x = x, \tag{6}$$

$$x \leq y \Rightarrow z \to x \leq z \to y. \tag{7}$$

Recall that "$\geq$" is a partial order relation and that $(A, \geq, 1)$ is a poset (partial ordered set) with greatest element 1.

Theorem 2.4 *[38]*

i) Let $\mathcal{A} = (A, \geq, \rightarrow, 1)$ such that:

(A1) $(A, \geq, 1)$ *is a poset with greatest element* 1;

(A2) $(A, \rightarrow, 1)$ *verifies: for all* $x, y, z \in A$,

(R1) $1 \rightarrow x = x$,

(R2) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$;

(A3) $x \rightarrow y = 1 \Longleftrightarrow x \leq y$, *for all* $x, y \in A$;

(A4) $x \leq y \Longrightarrow z \rightarrow x \leq z \rightarrow y$, *for all* $x, y, z \in A$.

Then, $\mathcal{A}$ is a reversed left-BCK algebra.

ii) Conversely, every reversed left-BCK algebra satisfies (A1) - (A4).

By this theorem we've got the following equivalent definition of reversed left-BCK algebras:

Definition 2.5 [38]

A *reversed left-BCK algebra* is an algebra $\mathcal{A} = (A, \geq, \rightarrow, 1)$ such that the above (A1) - (A4) hold.

* * *

Definition 2.6 [38]

A *reversed left-BCK algebra with condition (P)* (i.e. *with product*) or a *reversed left-BCK(P) algebra* for short, is a reversed left-BCK algebra $\mathcal{A} = (A, \geq, \rightarrow, 1)$ satisfying the condition (P):

(P) there exists, for all $x, y \in A$, $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$.

Proposition 2.7 *(see [38], Theorem 2.13) Let $\mathcal{A}$ be a reversed left-BCK(P) algebra, where*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}.$$

Then, the condition (RP) holds:

(RP) $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$, *for all* x, y, z.

Proposition 2.8 *[43] Let us consider the reversed left-BCK(P) algebra $\mathcal{A} = (A, \geq, \rightarrow, 1)$, where*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}.$$

Then, for all $x, y, z \in A$:

$$x \odot y \leq x,\ y \tag{8}$$

$$x \odot (x \rightarrow y) \leq x,\ y \tag{9}$$

$$y \leq x \rightarrow (x \odot y) \tag{10}$$

$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z), \tag{11}$$

$$(y \rightarrow z) \odot x \leq y \rightarrow (z \odot x). \tag{12}$$

$$(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z, \tag{13}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z, \tag{14}$$

$$z \rightarrow x \leq (y \rightarrow z) \rightarrow (y \rightarrow x), \tag{15}$$

$$(x \odot z) \rightarrow (y \odot z) \leq x \rightarrow (z \rightarrow y). \tag{16}$$

$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z) \leq z \rightarrow (x \rightarrow y), \tag{17}$$

$$x \rightarrow (x \wedge y) = x \rightarrow y, \tag{18}$$

$$x \leq y \quad \Rightarrow \quad x \odot z \leq y \odot z. \tag{19}$$

Proposition 2.9 *[38] Let $\mathcal{A} = (A, \geq, \to, 1)$ be a left-BCK(P) algebra, where for all $x, y \in A$:*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}.$$

Then the algebra $(A, \geq, \odot, 1)$ is a partially ordered, abelian (i.e. commutative), integral (left-) monoid, or, equivalently, the operation $\odot$ is a t-norm on the poset $(A, \geq, 1)$ with greatest element 1.

Reversed left-BCK(P) algebras are categorically equivalent with pocrims (partially ordered, commutative, residuated, integral monoids) [38].

* * *

Definition 2.10 [49]
If there is an element, 0, of a reversed left-BCK algebra $\mathcal{A} = (A, \geq, \to, 1)$, satisfying $0 \leq x$ (i.e. $0 \to x = 1$), for all $x \in A$, then 0 is called the *zero* of $\mathcal{A}$.
A reversed left-BCK algebra with zero is called to be *bounded* and it is denoted by: $(A, \geq, \to, 0, 1)$.

Proposition 2.11 *[43] Let us consider the bounded reversed left-BCK(P) algebra $\mathcal{A} = (A, \geq, \to, 0, 1)$. Then, for all $x, y, z \in A$:*

$$0 \odot x (= x \odot 0) = 0. \tag{20}$$

Let $\mathcal{A} = (A, \geq, \to, 0, 1)$ be a bounded reversed left-BCK algebra. Define, for all $x \in A$, a negation $^-$, by [49]: for all $x \in A$,

$$x^- \overset{def}{=} x \to 0. \tag{21}$$

Proposition 2.12 *In a bounded reversed left-BCK algebra $\mathcal{A}$ the following properties hold, for all $x, y \in A$ [49]:*

$$1^- = 0,\ 0^- = 1, \tag{22}$$
$$x \leq (x^-)^-, \tag{23}$$
$$x \to y \leq y^- \to x^-, \tag{24}$$
$$x \leq y \Rightarrow y^- \leq x^-, \tag{25}$$
$$y \to x^- = x \to y^-, \tag{26}$$
$$((x^-)^-)^- = x^-, \tag{27}$$

Remarks 2.13
(1) The negation $^-$ defined by 21 depends on $\to$ and 0.
(2) The negation $^-$ is a **weak** negation, by (23).

Definition 2.14 [38]
If a bounded reversed left-BCK algebra $\mathcal{A} = (A, \geq, \to, 0, 1)$ verifies, for every $x \in A$:

$$(x^-)^- = x,$$

then we shall say that $\mathcal{A}$ is *with condition (DN) (double negation).*

Remark 2.15 If $\mathcal{A}$ is with condition (DN), then the negation $^-$ is a **strong** one.

Lemma 2.16 *Let $\mathcal{A}$ be a bounded reversed left-BCK algebra with condition (DN). Then, for all $x, y \in A$ (see [49]):*

$$x \leq y \Leftrightarrow y^- \leq x^-, \tag{28}$$
$$x \to y = y^- \to x^-, \tag{29}$$
$$y^- \to x = x^- \to y. \tag{30}$$

Remark 2.17 The property (29) of $\rightarrow$ is called "the contrapositive symmetry with respect to the strong negation" in [21].

Theorem 2.18 *[38] Let $\mathcal{A} = (A, \geq, \rightarrow, 0, 1)$ be a bounded reversed left-BCK algebra with condition (DN). Then $\mathcal{A}$ is with condition (P) and, for all $x, y \in A$, we have:*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = (x \rightarrow y^-)^-, \tag{31}$$

$$x \rightarrow y = (x \odot y^-)^-. \tag{32}$$

Theorem 2.19 *Let $\mathcal{A} = (A, \geq, \rightarrow, 0, 1)$ be a bounded reversed left-BCK algebra with condition (DN). Then, for all $x, y, z \in A$, the condition (P2) from Definition 1.6 is satisfied, where:*

(P2) $(z^-)^- \odot [(x \odot z) \rightarrow (y \odot z)] \leq x \rightarrow y$.

Proof.
By preceeding Theorem, $\mathcal{A}$ is with condition (P) and $x \odot y = (x \rightarrow y^-)^-$.
Then, (P2) becomes:
$(z^-)^- \odot [(x \odot z) \rightarrow (y \odot z)] \leq x \rightarrow y \overset{comm. of \odot}{\Longleftrightarrow}$
$[(x \odot z) \rightarrow (y \odot z)] \odot z \leq x \rightarrow y \overset{(RP)}{\Longleftrightarrow}$
$[(x \odot z) \rightarrow (y \odot z)] \leq z \rightarrow (x \rightarrow y)$.
But,
$(x \odot z) \rightarrow (y \odot z) = (x \rightarrow z^-)^- \rightarrow (y \rightarrow z^-)^- \overset{(29)}{=} (y \rightarrow z^-) \rightarrow (x \rightarrow z^-) \overset{(DN)}{=}$
$((y^-)^- \rightarrow z^-) \rightarrow ((x^-)^- \rightarrow z^-) \overset{(29)}{=} (z \rightarrow y^-) \rightarrow (z \rightarrow x^-) \overset{(3)}{=} z \rightarrow [(z \rightarrow y^-) \rightarrow x^-]$.
Thus, we must prove that

$$z \rightarrow [(z \rightarrow y^-) \rightarrow x^-] \leq z \rightarrow (x \rightarrow y). \tag{33}$$

But, by (5), $y^- \leq z \rightarrow y^-$. Hence, by (1),

$$(z \rightarrow y^-) \rightarrow x^- \leq y^- \rightarrow x^- \overset{(29)}{=} x \rightarrow y. \tag{34}$$

From (34), by (7), we get (33). □

Proposition 2.20 *Let $\mathcal{A}$ be a bounded reversed left-BCK(P) algebra. Then,*

$$x \odot x^- = 0.$$

Proof.
$x \odot x^- = 0 \Leftrightarrow x \odot x^- \leq 0 \overset{(RP)}{\Leftrightarrow} x \leq x^- \rightarrow 0 = (x^-)^-$, which is true. □

* * *

In a reversed left-BCK algebra $\mathcal{A}$ we define, for all $x, y \in A$ (see [49]):

$$x \vee y \overset{def}{=} (x \rightarrow y) \rightarrow y. \tag{35}$$

Proposition 2.21 *Let $\mathcal{A}$ be a bounded reversed left-BCK algebra, Then, for all $x \in A$ (see [49]):*

$$0 \vee x = x, \tag{36}$$

$$x \vee 0 = (x^-)^-. \tag{37}$$

Definition 2.22 If $x \vee y = y \vee x$, for all $x, y \in A$, then the reversed left-BCK algebra $\mathcal{A}$ is called to be *commutative* (see [49]) or, better, $\vee$-*commutative* [38].

Lemma 2.23 *(see [49]) A reversed left-BCK algebra is ($\vee$-) commutative iff it is a semilattice with respect to $\vee$ (under $\geq$).*

Corollary 2.24 *(see [49]) Let $\mathcal{A}$ be a bounded, ($\vee$-) commutative reversed left-BCK algebra. Then, $\mathcal{A}$ is with condition (DN).*

In a bounded, ($\vee$-) commutative reversed left-BCK algebra $\mathcal{A}$, define, for all $x, y \in A$ (see [49]):

$$x \wedge y \overset{def}{=} (x^- \vee y^-)^-. \tag{38}$$

Proposition 2.25 *(see [49]) If a reversed left-BCK algebra is bounded and ($\vee$-) commutative, then it is a lattice with respect to $\vee$, $\wedge$ (under $\geq$).*

The bounded, ($\vee$-)commutative reversed left-BCK (left-BCK(P)) algebra is an equivalent definition of Wajsberg algebra [58] (see [38]).

From now on we shall simply say "BCK algebra (BCK(P) algebra)", instead of "reversed left-BCK algebra (reversed left-BCK(P) algebra)" .

2.2 BCK(P) lattices (residuated lattices)

Note that we consider the case when the lattices are bounded (i.e. with greatest element 1, but with smallest element 0 also, under $\geq$), in order to be able to define a negation. The more general case, when there is no 0, is already considered in the literature when speaking about reversed left-BCK(P) lattices (they are called "BCK-lattices with condition (S)" in [34]; note that it should be (P) instead of (S) in that paper).

Definition 2.26 (see [38])

(1) Let $\mathcal{A} = (A, \geq, \rightarrow, 0, 1)$ be a bounded BCK algebra. If the poset $(A, \geq)$ is a lattice, then we shall say that $\mathcal{A}$ is a *reversed left-BCK lattice.*

(2) Let $\mathcal{A} = (A, \geq, \rightarrow, 0, 1)$ be a bounded BCK(P) algebra, where for all $x, y \in A$:

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}.$$

If the poset $(A, \geq)$ is a lattice, then we shall say that $\mathcal{A}$ is a *reversed left-BCK(P) lattice.*

From now on we shall simply say "BCK(P) lattice", instead of "reversed left-BCK(P) lattice".
Denote by **BCK(P)-L** the class of BCK(P) lattices.
A BCK lattice (BCK(P) lattice) will be denoted:

$$\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1).$$

BCK(P) lattices are categorically equivalent with residuated lattices [38].
We write: **BCK(P)-L** $\cong$ **R-L**.

Proposition 2.27 *Let $\mathcal{A}$ be a BCK(P) lattice. Then the following properties hold, for all $x, y, z \in A$ [52]:*

$$\text{if } \vee Z \text{ exists, then } x \odot \vee Z = \vee\{x \odot z \mid z \in Z\}, \tag{39}$$

$$\text{if } \vee Z \text{ exists, then } \vee Z \rightarrow x = \wedge\{z \rightarrow x \mid z \in Z\}, \tag{40}$$

$$\text{if } \wedge Z \text{ exists, then } x \rightarrow \wedge Z = \wedge\{x \rightarrow z \mid z \in Z\}, \tag{41}$$

$$y \rightarrow z = \max\{x \mid x \odot y \leq z\}. \tag{42}$$

Proposition 2.28 *Let $\mathcal{A}$ be a BCK(P) lattice. Then we have [38]:*

$$x \odot (x \rightarrow y) \leq x \wedge y. \tag{43}$$

Proposition 2.29 *In a BCK(P) lattice we have the properties [38]:*

$$(x \vee y)^- = x^- \wedge y^-, \tag{44}$$

$$x^- \vee y^- \leq (x \wedge y)^-, \tag{45}$$

$$x \to y^- = (y \odot x)^- \tag{46}$$

* * *

We say that a BCK(P) lattice $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is *with condition* **(C)** if, for all $x, y \in A$ [38],

$$x \vee y = (x \to y) \to y.$$

In what follows we shall concentrate on some important subclasses of BCK(P) lattices: those satisfying the condition (DN), those satisfying the condition (WNM) and those satisfying the condition (G).

We say that a BCK(P) lattice $\mathcal{A}$ is *with condition (DN)* or a BCK(P)$_{(DN)}$ lattice, for short, if the associated bounded reversed left-BCK(P) algebra is with condition (DN) [38].

Note that BCK(P) lattices with condition (DN) are categorically equivalent with residuated lattices with condition (DN), also named "Girard monoids" [33].

Corollary 2.30 *[38] Let $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ be a BCK(P) lattice with condition* **(C)**. *Then $\mathcal{A}$ is a with condition (DN).*

Theorem 2.31 *[38] The BCK(P) lattice with condition* **(C)** *is an equivalent definition of Wajsberg algebra.*

Proposition 2.32 *Let $\mathcal{A}$ be a BCK(P)$_{(DN)}$ lattice. Then we have [38]:*

$$(x \wedge y)^- = x^- \vee y^-, \tag{47}$$

$$x \wedge y = (x^- \vee y^-)^-. \tag{48}$$

Theorem 2.33 *Let $\mathcal{A} = (A, \geq, \to, 0, 1)$ be a BCK(P)$_{(DN)}$ lattice. Then, for all $x, y \in A$, we have:*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-, \tag{49}$$

$$x \to y = (x \odot y^-)^-. \tag{50}$$

Proof. By Theorem 2.18. □

Theorem 2.34 *Let $\mathcal{A}$ be a BCK(P)$_{(DN)}$ lattice. Then $\mathcal{A}$ satisfies the condition (P2): for all $x, y, z \in A$,*

(P2) $(z^-)^- \odot [(x \odot z) \to (y \odot z)] \leq x \to y$.

Proof By Theorem 2.19. □

Proposition 2.35 *Let $\mathcal{A}$ be a BCK(P)$_{(DN)}$ lattice which satisfies the condition (P1): for all $x \in A$,*

(P1) $x \wedge x^- = 0$.

Then $\mathcal{A}$ is a Boolean algebra.

Proof. If $x \wedge x^- = 0$, it follows that we also have:
$x \vee x^- = (x^-)^- \vee x^- = (x^- \wedge x)^- = 0^- = 1$, by Proposition 2.32, hence $\mathcal{A}$ is a Boolean algebra. □

Definition 2.36 We say that a BCK(P) lattice $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a $_{(WNM)}$ *BCK(P) lattice* if it satisfies the condition (WNM) (weak nilpotent minimum): for all $x, y \in A$,

(WNM) $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 1$.

Definition 2.37 We shall say that a BCK(P) lattice is a $_{(WNM)}BCK(P)_{(DN)}$ *lattice* if it verifies both conditions (DN) and (WNM).

Definition 2.38 We say that a BCK(P) lattice $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is *of Gödel type* if it satisfies the condition (G): for all $x \in A$,

(G) $x \odot x = x$.

Proposition 2.39 *Let $\mathcal{A}$ be a BCK(P) lattice of Gödel type. Then, for all $x, y \in A$,*

$$x \odot y = x \wedge y.$$

Proof.
By Proposition 2.9, $\odot$ is commutative and associative. By condition (G), it is also idempotent.
We also have the two absorbtions:
- $x \vee (x \odot y) = x$, since $x \odot y \leq x$, by (8).
- $x \odot (x \vee y) = x$; indeed, $x \odot (x \vee y) \leq x$, by (8) and since $x \leq x \vee y$ it follows that $x = x \odot x \leq x \odot (x \vee y)$, by (19); thus, $x \odot (x \vee y) = x$.

Finally, $x \leq y \Leftrightarrow x \odot y = x$. Indeed,
- $x \leq y$ implies by (19) that $x = x \odot x \leq x \odot y$; then, by (8), it follows that $x \odot y = x$.
- $x \rightarrow y = (x \odot y) \rightarrow y = x \rightarrow (y \rightarrow y) = x \rightarrow 1 = 1$, by (14), hence $x \leq y$, by (VI).

Thus, $x \odot y = x \wedge y$. □

Proposition 2.40 *Let $\mathcal{A}$ be a BCK(P) lattice of Gödel type. Then $\mathcal{A}$ verifies the condition (P1).*

Proof. By Proposition 2.39, we have

$$x \odot y = x \wedge y$$

and by Proposition 2.20, we have

$$x \odot x^- = 0.$$

Then, $x \wedge x^- = x \odot x^- = 0$. □

Theorem 2.41 *Let $\mathcal{A}$ be a BCK(P) lattice of Gödel type. Then $\mathcal{A}$ verifies the condition (WNM) (i.e. it is a $_{(WNM)}BCK(P)$ lattice) .*

Proof.
By Proposition 2.39, $x \odot y = x \wedge y$, for all $x, y \in A$, hence:

$$(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = (x \odot y)^- \vee [(x \odot y) \rightarrow (x \odot y)] = (x \odot y)^- \vee 1 = 1,$$

by (III), (VI). □

Theorem 2.42 *Let $\mathcal{A}$ be a BCK(P) lattice verifying the following:*

for each $x, y \in A$, such that $x \odot y \neq 0$, we have $x \odot y = x \wedge y$.
Then, $\mathcal{A}$ is a $_{(WNM)}BCK(P)$ lattice.

Proof. For all $x, y \in A$, there are two cases:

(1) $x \odot y = 0$; then the condition (WNM) is satisfied:

$$(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 0^- \vee [(x \wedge y) \rightarrow 0] = 1 \vee [(x \wedge y) \rightarrow 0] = 1.$$

(2) $x \odot y \neq 0$; then, by hypothesis, $x \odot y = x \wedge y$; then the condition (WNM) is satisfied:

$$(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = (x \odot y)^- \vee [(x \odot y) \rightarrow (x \odot y)] = (x \odot y)^- \vee 1 = 1.$$

□

2.3 Hájek(P) algebras (BL algebras) and related structures

Definition 2.43 [38]

A *reversed left-Hájek(P) algebra* is an algebra

$$\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$$

such that:

(B1) $\mathcal{A}$ is a BCK(P) lattice,

(B2) for all $x, y \in A$, $\quad x \wedge y = x \odot (x \rightarrow y)$ (divisibility);

(B3) for all $x, y \in A$, $\quad (x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

From now on we shall simply say "Hájek(P) algebra", instead of "reversed left-Hájek(P) algebra".

Let **Ha(P)** denote the class of reversed left-Hájek(P) algebras.

The Hájek(P) algebras are categorically equivalent with BL algebras [38].

We write: $\quad$ **Ha(P)** $\cong$ **BL**.

Consequently, the class of Hájek(P) algebras contain the Wajsberg, the Product and the Gödel algebras.

Proposition 2.44 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a Hájek(P) algebra. Then, $\mathcal{L}(\mathcal{A}) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.*

Proposition 2.45 *Let $\mathcal{A}$ be a Hájek(P) algebra. Then for all $x, y \in A$:*
$x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$.

Proposition 2.46 *Let $\mathcal{A}$ be a Hájek(P) algebra satisfying the condition (B3). Then, for all $x, y \in A$,*

$$(x \wedge y)^{-} = x^{-} \vee y^{-}.$$

Proof. By (45), we have the inequality $x^{-} \vee y^{-} \leq (x \wedge y)^{-}$. It remains to prove the converse inequality:

$$(x \wedge y)^{-} \leq x^{-} \vee y^{-}. \tag{51}$$

Indeed, we have: $x \rightarrow y \overset{(18)}{=} x \rightarrow (x \wedge y) \overset{(24)}{\leq} (x \wedge y)^{-} \rightarrow x^{-}$. Hence, by (RP), $(x \rightarrow y) \odot (x \wedge y)^{-} \leq x^{-}$. Similarly, $(y \rightarrow x) \odot (x \wedge y)^{-} \leq y^{-}$. It follows that:

$$(x \wedge y)^{-} = 1 \odot (x \wedge y)^{-} \overset{(B3)}{=} [(x \rightarrow y) \vee (y \rightarrow x)] \odot (x \wedge y)^{-} \overset{(39)}{=} [(x \wedge y)^{-} \odot (x \rightarrow y)] \vee [(x \wedge y)^{-} \odot (y \rightarrow x)] \leq x^{-} \vee y^{-},$$

i.e. (51) holds. □

It follows immediately the well known result:

Corollary 2.47 *In a Hájek(P) algebra (BL algebra) we have*

$$(x \wedge y)^{-} = x^{-} \vee y^{-}.$$

We say that a Hájek(P) (BL) algebra is *with condition (C)* if the associated BCK(P) (residuated) lattice is with condition (C) [38].

We say that a Hájek(P) (BL) algebra is *with condition (DN) (double negation)* or a *Hájek(P)$_{(DN)}$ (BL$_{(DN)}$) algebra* for short, if the associated BCK(P) (residuated) lattice is *with condition (DN)*, i.e. is BCK(P)$_{(DN)}$ lattice.

We say that a Hájek(P) (BL) algebra is *with condition (WNM) (weak nilpotent minimum)* or a *$_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebra* for short, if the associated BCK(P) lattice is with condition (WNM), i.e. is $_{(WNM)}$BCK(P) lattice.

When the Hájek(P) (BL) algebra satisfies both (DN) and (WNM) conditions, then we shall say that is a $_{(WNM)}$Hájek(P)$_{(DN)}$ ($_{(WNM)}$BL$_{(DN)}$) algebra.

In this paper, part III, we shall put in evidence the importance $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras.

Let **Ha(P)**$_{(DN)}$ (**BL**$_{(DN)}$) denote the subclass of H'ajek(P)$_{(DN)}$ (BL$_{(DN)}$) algebras. We write:

$$\mathbf{Ha(P)}_{(DN)} = \mathbf{Ha(P)} + (\mathrm{DN}) \equiv \mathbf{W}, \quad \mathbf{BL}_{(DN)} = \mathbf{BL} + (\mathrm{DN}) \equiv \mathbf{MV}.$$

Let $_{(WNM)}\mathbf{Ha(P)}$ ($_{(WNM)}\mathbf{BL}$) denote the subclass of $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras and $_{(WNM)}\mathbf{Ha(P)}_{(DN)}$ ($_{(WNM)}\mathbf{BL}_{(DN)}$) denote the subclass of $_{(WNM)}$Hájek(P)$_{(DN)}$ ($_{(WNM)}$BL$_{(DN)}$) algebras. We write:

$$_{(WNM)}\mathbf{Ha(P)} = \mathbf{Ha(P)} + (\mathrm{WNM}), \quad _{(WNM)}\mathbf{BL} = \mathbf{BL} + (\mathrm{WNM}).$$

Theorem 2.48 *[38] A Hájek(P) algebra is with condition* **(C)** *iff it is with condition (DN).*

Recall [30] that a Hájek(P) algebra (BL algebra) is a Wajsberg algebra (MV algebra) iff it is with condition (DN). We write:

$$\mathbf{Ha(P)} + (\mathrm{DN}) \cong \mathbf{W}, \quad \mathbf{BL} + (\mathrm{DN}) \cong \mathbf{MV}.$$

Proposition 2.49 *Every Wajsberg (MV) algebra satisfies the condition (P2) from the definition of Product algebras.*

Proof. Every Wajsberg (MV) algebra satisfies the condition (DN) (double negation); then apply Theorem 2.19. □

Proposition 2.50 *A Wajsberg (MV) algebra* $\mathcal{A} = (A, \rightarrow, ^{-}, 1)$ *which satisfies the condition (P1) from Definition 1.6 is a Boolean algebra.*

Proof. By Proposition 2.35. □

In this paper, part III, we shall put in evidence the important subclass of those Wajsberg (MV) algebras verifying the condition (WNM), named as $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras.

Let $_{(WNM)}\mathbf{W}$ ($_{(WNM)}\mathbf{MV}$) denote the class of $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras.

We have:

$$_{(WNM)}\mathbf{Ha(P)}_{(DN)} = \mathbf{Ha(P)} + (\mathrm{WNM}) + (\mathrm{DN}) \equiv \mathbf{W} + (\mathrm{WNM}) = {}_{(WNM)}\mathbf{W},$$

$$_{(WNM)}\mathbf{BL}_{(DN)} = \mathbf{BL} + (\mathrm{WNM}) + (\mathrm{DN}) \equiv \mathbf{MV} + (\mathrm{WNM}) = {}_{(WNM)}\mathbf{MV}.$$

Recall the followings:
- Any Boolean algebra is a Wajsberg (MV) algebra, a Product algebra and a Gödel algebra.
- The only finite Product algebras are the finite Boolean algebras.
- Any Gödel algebra satisfies the condition (P1), by Proposition 2.40, hence is a SBL algebra.
- Any Gödel algebra satisfies the condition (WNM), by Theorem 2.41.

Open problem 2.51 It will be interesting to study the class of those Hájek(P) (BL) algebras which verify the condition (P2) (i.e. the SSBL algebras). The exemples in this paper, Part III, will be useful.

* * *

By Proposition 2.46, we get immediately that:

Corollary 2.52 *In a weak-BL algebra (MTL algebra) we have*

$$(x \wedge y)^{-} = x^{-} \vee y^{-}.$$

Let us introduce the following definitions:

Definition 2.53

(1) A divisible BCK(P) lattice is said to be of *Gödel type* if the BCK(P) lattice is of of Gödel type. We shall give examples of such algebras in Part III.

(2) A divisible BCK(P) lattice is said to be of *Product type* if it satisfies the conditions (P1) and (P2). It is an open problem if there exist any proper such algebras (i.e. which are not Product algebras).

(3) A divisible BCK(P) lattice is said to be *proper* if it is not a Hájek (BL) algebra, if it is not of Product or of Gödel type and if it does not satisfy the conditions (DN) and (WNM).

3 The decomposition of the conditions (B2) and (B3). Consequences

Let us consider the two conditions (B2) ($x \wedge y = x \odot (x \to y)$) and (B3) ($(x \to y) \vee (y \to x) = 1$) from the definition of a Hájek (BL) algebra.

Theorem 3.1 *Let $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ be a BCK(P) lattice. Then, the condition (B2) is equivalent with the following cancellative condition:*

(C_c) for any $x, y, z \in A$, if $z \to x = z \to y$ and $z \geq x, y$, then $x = y$.

Proof.

(B2) $\Longrightarrow$ (C_c) [68]: Since $x \leq z$ and $y \leq z$, then $z \wedge x = x$ and $z \wedge y = y$; hence $x = z \wedge x = z \odot (z \to x) = z \odot (z \to y) = z \wedge y = y$.

(C_c) $\Longrightarrow$ (B2): By (8), $x \odot (x \to y) \leq x$. We also have $x \wedge y \leq x$. We shall prove that

$$x \to [x \odot (x \to y)] = x \to (x \wedge y)(= x \to y). \tag{52}$$

By (43), $x \odot (x \to y) \leq x \wedge y$, hence , by (7),

$$x \to [x \odot (x \to y)] \leq x \to (x \wedge y). \tag{53}$$

On the other hand, by Proposition 2.9, $x \odot 1 = 1 \odot x = x$, hence $x \to [x \odot (x \to y)] = [1 \odot x] \to [(x \to y) \odot x] \overset{(11)}{\geq} 1 \to (x \to y) \overset{(6)}{=} x \to y$. Hence,

$$x \to y \leq x \to [x \odot (x \to y)]. \tag{54}$$

But, by (18),

$$x \to (x \wedge y) \leq x \to y. \tag{55}$$

By (54) and (55), we get:

$$x \to (x \wedge y) \leq x \to [x \odot (x \to y)]. \tag{56}$$

Consequently, by (53) and (56), (52) holds. It follows, by (C_c), that $x \wedge y = x \odot (x \to y)$. □

Note that it is much easier to check on the table of $\to$ the equivalent condition (C_c) than the condition (B2), in finite examples.

Note also that condition (B3) is easilly checked on the table of $\to$.

Remarks 3.2

(i) Recall that the divisibility condition (B2) is also equivalent with the following condition (C_d), which gave the name "divisibility" ([33], Lemma 2.5):

(C_d) for all $x, y \in A$, if $y \leq x$, then there exists $z \in A$ such that $y = x \odot z$.

(ii) Note that condition (C_c) is expressed in terms of "$\to$", while the equivalent condition (C_d) is expressed in terms of "$\odot$".

(iii) Note that "divisible BCK(P) lattices" are categorically equivalent to divisible residuated lattices (or "divisible integral, residuated, commutative l-monoids", in [33], Lemma 2.5).

Note also that, while the condition (B2) is expressed either in terms of "$\rightarrow$" or in terms of "$\odot$" (besides "$\geq$"), the condition (B3) is expressed only in terms of "$\rightarrow$" (besides "$\vee$" and "1"). This is an important reason to work with "$\rightarrow$" as primitive operation, and not with "$\odot$", when dealing with the involved algebras.

Open problem 3.3 Find a simple equivalent condition of condition (B3) in terms of $\odot$.

Proposition 3.4 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a BCK(P) lattice. Then, the condition (B2) is equivalent with the following two conditions:*
$(C_\wedge)$ $\quad x \wedge y = [x \odot (x \rightarrow y)] \vee [y \odot (y \rightarrow x)]$,
$(C_\odot)$ $\quad x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$.

Proof. Obvious. □

Proposition 3.5 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a BCK(P) lattice. Then, the condition $(C_\odot)$ is equivalent with the following two conditions:*
$(C_\rightarrow)$ $\quad (x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x$,
(C_X) $\quad x \odot [(y \rightarrow x) \rightarrow (x \rightarrow y)] = y \odot [(x \rightarrow y) \rightarrow (y \rightarrow x)]$.

Proof.
• $(C_\odot) \Longrightarrow (C_\rightarrow)$: By (5), we have $x \leq (x \rightarrow y) \rightarrow x$; hence, by (1), we get:

$$[(x \rightarrow y) \rightarrow x] \rightarrow y \leq x \rightarrow y. \tag{57}$$

On the other hand, by (III), $(x \rightarrow y) \rightarrow x \leq (x \rightarrow y) \rightarrow x$, hence, by (4), we get:

$$x \rightarrow y \leq [(x \rightarrow y) \rightarrow x] \rightarrow x. \tag{58}$$

By (57),(58) and (2) and by Proposition 2.7, we get
$[(x \rightarrow y) \rightarrow x] \rightarrow y \leq [(x \rightarrow y) \rightarrow x] \rightarrow x \overset{(RP)}{\Leftrightarrow} ([(x \rightarrow y) \rightarrow x] \rightarrow y) \odot [(x \rightarrow y) \rightarrow x] \leq x \Leftrightarrow$
$\Leftrightarrow [(x \rightarrow y) \rightarrow x] \odot ([(x \rightarrow y) \rightarrow x] \rightarrow y) \leq x \overset{(C_\odot)}{\Leftrightarrow} y \odot (y \rightarrow [(x \rightarrow y) \rightarrow x]) \leq x \overset{(VI)}{\Leftrightarrow}$
$\overset{(VI)}{\Leftrightarrow} (y \odot (y \rightarrow [(x \rightarrow y) \rightarrow x])) \rightarrow x = 1 \overset{comm. of \odot}{\Leftrightarrow} ((y \rightarrow [(x \rightarrow y) \rightarrow x]) \odot y) \rightarrow x = 1$
$\overset{(14)}{\Leftrightarrow} (y \rightarrow [(x \rightarrow y) \rightarrow x]) \rightarrow (y \rightarrow x) = 1 \overset{(VI)}{\Leftrightarrow} y \rightarrow [(x \rightarrow y) \rightarrow x] \leq y \rightarrow x \overset{(3)}{\Leftrightarrow} (x \rightarrow y) \rightarrow (y \rightarrow x) \leq y \rightarrow x$.
But, we also have, by (5), $y \rightarrow x \leq (x \rightarrow y) \rightarrow (y \rightarrow x)$. Thus, by (V), $(C_\rightarrow)$) holds.
• $(C_\odot) \Longrightarrow (C_X)$: since $(C_\odot)$ implies $(C_\rightarrow)$, it follows immediately (C_X).
• (C_X) and $(C_\rightarrow)$ implies obviously $(C_\odot)$. □

Remark that in a BCK(P) lattice, if (C_X) holds, then $(C_\odot) \Leftrightarrow (C_\rightarrow)$ and if $(C_\rightarrow)$ holds, then $(C_\odot) \Leftrightarrow (C_X)$.

By Propositions 3.4 and 3.5 we get the following

Theorem 3.6 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a BCK(P) lattice. Then, the condition (B2) is equivalent with the three conditions $(C_\wedge)$, $(C_\rightarrow)$ and (C_X).*

□

Theorem 3.7 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a BCK(P) lattice. Then, the condition (B3) is equivalent with the conditions $(C_\rightarrow)$ and $(C_\vee)$, where:*
$(C_\vee)$ $\quad x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$.

Proof.
• (B3) $\Longrightarrow (C_\vee)$: Denote by "α" the right side of $(C_\vee)$.
We have $x \leq (x \rightarrow y) \rightarrow y$, by (II) and $y \leq (x \rightarrow y) \rightarrow y$, by (5); hence $x \vee y \leq (x \rightarrow y) \rightarrow y$. Similarly, $x \vee y \leq (y \rightarrow x) \rightarrow x$. It follows that $x \vee y \leq \alpha$.

By Proposition 2.9,

$\alpha = \alpha \odot 1 \overset{(B3)}{=} \alpha \odot [(x \to y) \vee (y \to x)] \overset{(39)}{=} [\alpha \odot (x \to y)] \vee [\alpha \odot (y \to x)]$.

But, $\alpha \odot (x \to y) = ([(x \to y) \to y] \wedge [(y \to x) \to x]) \odot (x \to y)$

$\overset{(19)}{\leq} [(x \to y) \to y] \odot (x \to y) = (x \to y) \odot [(x \to y) \to y] \overset{(43)}{\leq} (x \to y) \wedge y \leq y$.

Similarly, $\alpha \odot (y \to x) \leq [(y \to x) \to x] \odot (y \to x) \leq (y \to x) \wedge x \leq x$.

Thus, $\alpha = [\alpha \odot (x \to y)] \vee [\alpha \odot (y \to x)] \leq y \vee x$. It follows that $x \vee y = \alpha$.

• (B3) $\Longrightarrow (C_{\to})$: Since (B3) implies $(C_{\vee})$, it follows:

$1 = (x \to y) \vee (y \to x) = [((x \to y) \to (y \to x)) \to (y \to x)] \wedge [((y \to x) \to (x \to y)) \to (x \to y)] \leq [((x \to y) \to (y \to x)) \to (y \to x)]$.

Hence, $((x \to y) \to (y \to x)) \to (y \to x) = 1$, i.e., by (VI), $(x \to y) \to (y \to x) \leq y \to x$; since we also have, by (5), that $y \to x \leq (x \to y) \to (y \to x)$, it follows that $(x \to y) \to (y \to x) = y \to x$, i.e. $(C_{\to})$ holds.

• $(C_{\to})$ and $(C_{\vee})$ imply $(B3)$: $(x \to y) \vee (y \to x) \overset{(C_{\vee})}{=}$

$[((x \to y) \to (y \to x)) \to (y \to x)] \wedge [((y \to x) \to (x \to y)) \to (x \to y)] \overset{(C_{\to})}{=}$

$[(y \to x) \to (y \to x)] \wedge [(x \to y) \to (x \to y)] = 1 \wedge 1 = 1$. □

Theorem 3.8 *Let $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ be a BCK(P) lattice. If $\mathcal{A}$ is a chain (i.e it is linearly ordered), then it satisfies the conditions $(C_{\to})$, $(C_{\vee})$, $(C_{\wedge})$.*

Proof.

Let $x, y \in A$; then either $x \leq y$ or $y \leq x$, i.e. either $x \to y = 1$ or $y \to x = 1$, respectively.

• $(C_{\to})$: We prove that $(x \to y) \to (y \to x) = y \to x$. Indeed,

- if $x \leq y$, then $(x \to y) \to (y \to x) = 1 \to (y \to x) = y \to x$ and
- if $y \leq x$, then $(x \to y) \to (y \to x) = (x \to y) \to 1 = 1 = y \to x$.

• $(C_{\vee})$: We prove that $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$. Indeed, if $x \leq y$, for instance, then $[(x \to y) \to y] \wedge [(y \to x) \to x] = [1 \to y] \wedge [(y \to x) \to x] = y \wedge [(y \to x) \to x] = y = x \vee y$, since, by (II), $y \leq (y \to x) \to x$.

• $(C_{\wedge})$: We prove that $x \wedge y = [x \odot (x \to y)] \vee [y \odot (y \to x)]$. Indeed, if $x \leq y$, for instance, then $[x \odot (x \to y)] \vee [y \odot (y \to x)] = [x \odot 1] \vee [y \odot (y \to x)] = x \vee [y \odot (y \to x)] = x = x \wedge y$, since, by (43), $y \odot (y \to x) \leq x \wedge y = x$. □

By Theorems 3.6 and 3.7 we immediately get the following:

Theorem 3.9 *A BCK(P) lattice is a Hájek(P) (BL) algebra if and only if it satisfies the four conditions $(C_{\to})$, $(C_{\vee})$, $(C_{\wedge})$, (C_X)*

By this Theorem, we immediately get the following

Corollary 3.10 *Let $\mathcal{A}$ be a BCK(P) lattice. Then we have:*

$(B2) + (B3) \Longleftrightarrow (C_{\to}) + (C_{\vee}) + (C_{\wedge}) + (C_X) \Longleftrightarrow (B2) + (C_{\vee}) \Longleftrightarrow (B3) + (C_{\wedge}) + (C_X)$.

□

Proposition 3.11 *Let $\mathcal{A}$ be a BCK(P) lattice. Then, we have the equivalence:*

$$(B2) + (DN) \Longleftrightarrow (C).$$

Proof.

$\Longrightarrow$: By Proposition 2.32 and Theorem 2.18, $x \vee y = (x^-)^- \vee (y^-)^- = (x^- \wedge y^-)^- = [x^- \odot (x^- \to y^-)]^- \overset{(29)}{=} [x^- \odot (y \to x)]^- = (y \to x) \to x$.

$\Longleftarrow$: By Corollary 2.30, (C) $\Longrightarrow$ **(DN)**. It remains to prove that (C) $\Longrightarrow$ (B2). Indeed, by Theorem 2.18, $x \wedge y = (x^- \vee y^-)^- = [(y^- \to x^-) \to x^-]^- \overset{(29)}{=} [(x \to y) \to x^-]^- = (x \to y) \odot x = x \odot (x \to y)$.

□

Corollary 3.12 *Any divisible BCK(P) lattice satisfying the condition (DN) is a Wajsberg (MV) algebra.*

Proof. Since the BCK(P) lattice satisfying condition (C) is an equivalent definition of Wajsberg algebra. □

Proposition 3.13 *A linearly ordered divisible BCK(P) lattice is a Hájek(P) (BL) algebra (chain).*

Proof. By Theorem 3.8, any linearly ordered BCK(P) lattice satisfies the condition $(C_\vee)$; then apply Corollary 3.10. □

Proposition 3.14
Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a BCK(P) lattice. Then,
$(C_\wedge)$ + *(DN)* $\Longleftrightarrow$ $(C_\vee)$ + *(DN)* $\Longleftrightarrow$ $(C_\wedge)$ + $(C_\vee)$ + *(DN).*

Proof. It is sufficient to prove the following two implications:

• $(C_\wedge)$ + (DN) $\Longrightarrow$ $(C_\vee)$: Indeed,
$x \vee y \overset{(DN)}{=} ((x \vee y)^-)^- \overset{(44)}{=} (x^- \wedge y^-)^- \overset{(C_\wedge)}{=} [[x^- \odot (x^- \rightarrow y^-)] \vee [y^- \odot (y^- \rightarrow x^-)]]^- \overset{(29)}{=} [[x^- \odot (y \rightarrow x)] \vee [y^- \odot (x \rightarrow y)]]^- = [[(y \rightarrow x) \odot x^-] \vee [(x \rightarrow y) \odot y^-]]^- \overset{(44)}{=} [(y \rightarrow x) \odot x^-]^- \wedge [(x \rightarrow y) \odot y^-]^- \overset{(32)}{=} [(y \rightarrow x) \rightarrow x] \wedge [(x \rightarrow y) \rightarrow y]$.

• $(C_\vee)$ + (DN) $\Longrightarrow$ $(C_\wedge)$: Indeed,
$x \wedge y \overset{(48)}{=} (x^- \vee y^-)^- \overset{(C_\vee)}{=} [[(x^- \rightarrow y^-) \rightarrow y^-] \wedge [(y^- \rightarrow x^-) \rightarrow x^-]]^- \overset{(29)}{=} [[(y \rightarrow x) \rightarrow y^-] \wedge [(x \rightarrow y) \rightarrow x^-]]^- \overset{(47)}{=} [(y \rightarrow x) \rightarrow y^-]^- \vee [(x \rightarrow y) \rightarrow x^-]^- \overset{(31)}{=} [(y \rightarrow x) \odot y] \vee [(x \rightarrow y) \odot x] = [x \odot (x \rightarrow y)] \vee [y \odot (y \rightarrow x)]$. □

Remarks 3.15
1) Among the four conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$, (C_X), the first three are very important, since any linearly ordered BCK(P) lattice satisfies them, by Theorem 3.8.
2) Among the important three conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$, two are very important, $(C_\vee)$ and $(C_\wedge)$, since they are dual, i.e. in a BCK(P) lattice with condition (DN), $(C_\vee) \Longleftrightarrow (C_\wedge)$, by Proposition 3.14.

We then immediately get the following consequences.

Corollary 3.16 *Let $\mathcal{A}$ be a residuated lattice (BCK(P) lattice). Then,*

(B3) + (DN) $\Longleftrightarrow$ $(C_\rightarrow)$ + $(C_\vee)$ + $(C_\wedge)$+ *(DN).*

Corollary 3.17 *Let $\mathcal{A}$ be a residuated lattice (BCK(P) lattice). Then,*

(C) $\Longleftrightarrow$ $(C_\rightarrow)$ + $(C_\vee)$ + $(C_\wedge)$+ (C_X) + *(DN)* $\Longleftrightarrow$ *(B2) + (B3) + (DN).*

Proof.
By Proposition 3.11, Theorem 3.6, Proposition 3.14 and Corollary 3.10, we get:

(C) $\Longleftrightarrow$ (B2) + (DN) $\Longleftrightarrow$ $(C_\wedge)$ + $(C_\rightarrow)$ + (C_X) + (DN)
$\Longleftrightarrow$ $(C_\wedge)$ + $(C_\rightarrow)$ + (C_X) + $(C_\vee)$ + (DN) $\Longleftrightarrow$ (B2) + (B3) + (DN). □

Corollary 3.18
IMTL = **MTL** + *(DN)*= **R-L** + *(B3)* + *(DN)* = **R-L** + $(C_\rightarrow)$ + $(C_\vee)$ + $(C_\wedge)$+ *(DN).*

Corollary 3.19
(1) **MV** $\cong$ **IMTL** + (C_X).
(1') **W** $\equiv$ **weak-R$_0$** + (C_X).

Proof.
$\mathbf{MV} \cong \mathbf{BL}$ + (DN) = **R-L** + (B2) +(B3) + (DN)=
= **R-L** + $(C_{\rightarrow})$ + $(C_{\vee})$ + $(C_{\wedge})$ + (C_X) + (DN) = **IMTL** + (C_X).
Thus, (1) holds.
(1') follows immediately, by (1). □

We also have:

NM = **IMTL** + (WNM) or, equivalently $\mathbf{R_0}$= **weak** R_0 + (R6).

We shall prove in Part III that the class of MV algebras (Wajsberg algebras) and the class of NM algebras (R_0 algebras) are incomparable (not included one in the other), but have "something" in common, namely the subclass ${}_{(WNM)}\mathbf{MV}$ (${}_{(WNM)}\mathbf{W}$):

${}_{(WNM)}\mathbf{MV}$=**MV** + (WNM) $\cong$ **NM** + (C_X).

Indeed, ${}_{(WNM)}\mathbf{MV}$=**MV** + (WNM) $\cong$ [**IMTL** + (C_X)] +(WNM)=[**IMTL** + (WNM)] +(C_X)= **NM** + (C_X).

Remarks 3.20

(1) ${}_{(WNM)}\mathbf{W}$ = **W** + (WNM) $\equiv$ [**Ha(P)** + (DN)] + (WNM)= [**Ha(P)** + (WNM)] + (DN)=${}_{(WNM)}\mathbf{Ha(P)}$ + (DN).

(1') ${}_{(WNM)}\mathbf{MV}$ = **MV** + (WNM) $\cong$ [**BL** + (DN)] + (WNM)= [**BL** + (WNM)] + (DN)=${}_{(WNM)}\mathbf{BL}$ + (DN).

We shall give examples of ${}_{(WNM)}$Wajsberg algebras (${}_{(WNM)}$MV algebras) and of ${}_{(WNM)}$Hájek algebras (${}_{(WNM)}$BL algebras) in the third part of this paper.

Very recently, Y.L. Liu and S.Y. Liu, have introduced the notions of *normal (weak) R_0-algebra* (cf. [54]):

Definition 3.21 A *normal (weak) R_0-algebra* or, (NWR_0) NWR_0 for short is an (weak) R_0-algebra verifying:

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

Since $WR_0 \cong$ IMTL, it follows that condition $(C_{\vee})$ is verified in (weak) R_0-algebras; consequently, in a normal (weak) R_0 algebra, we have: $x \vee y = (x \rightarrow y) \rightarrow y$, i.e. condition (C) holds.

We then obtain, by Corollary 3.17 and Theorem 2.31, that:

Theorem 3.22

(1) The normal weak R_0-algebra is an equivalent definition of Wajsberg algebra (MV algebra), i.e. $\mathbf{NWR_0} \equiv \mathbf{W}$*;*

(2) The class of normal R_0-algebras is a subclass of the class of Wajsberg algebras (MV algebras), namely we have:
$\mathbf{NR_0} \equiv \mathbf{W}$ *+ (R6) =* **W** *+ (WNM)=* ${}_{(WNM)}\mathbf{W}$.

Let (PIM) (positive implicative) be the following condition (see [49]):

(PIM) $x \rightarrow (x \rightarrow y) = x \rightarrow y$, for all x, y.

By [54], we have:

$\mathbf{NR_0}$ + (PIM) $\Longleftrightarrow$ **Boolean** or, equivalently,

${}_{(WNM)}\mathbf{W}$ + (PIM) $\Longleftrightarrow$ **Boolean**.

Definition 3.23

(1) We shall say that a Hájek(P) algebra (BL algebra) is *proper* if it is not a Wajsberg (MV), a Product or a Gödel algebra and if it does not verify the condition (WNM).

(2) We shall say that a $_{(WNM)}$Hájek(P) algebra ($_{(WNM)}$BL algebra) is *proper* if it is not a $_{(WNM)}$Wajsberg ($_{(WNM)}$MV), a Product or a Gödel algebra.

Resuming, we have for the moment the generalizations and the particular cases of Hájek(P) (BL) algebras from Figure 2.

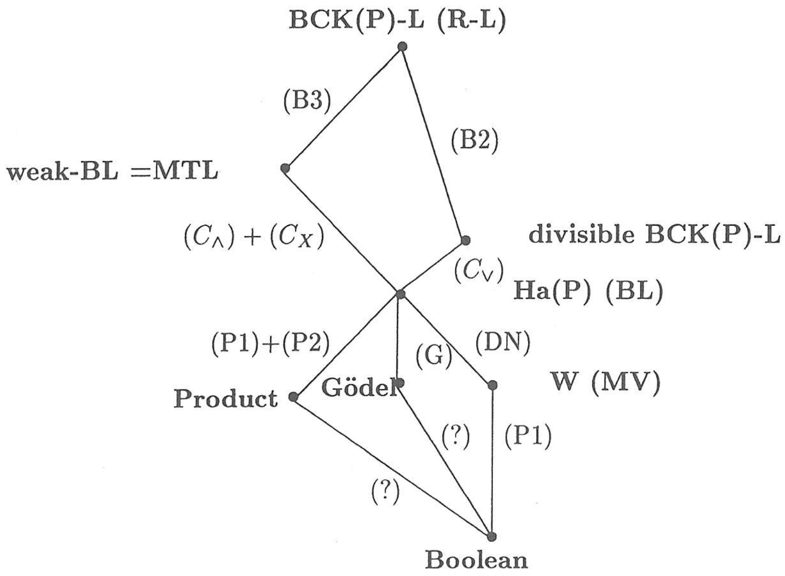

Figure 2: Generalizations and particular cases of Hájek(P) (BL) algebras

When adding the condition (DN), we get the hierarchy from Figure 3.

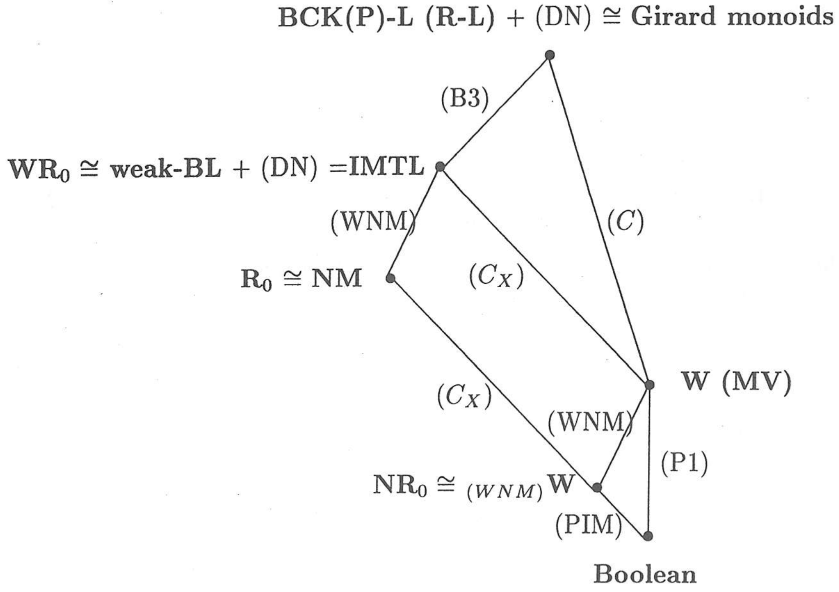

Figure 3: Generalizations and particular cases of Wajsberg (MV) algebras

4 The ordinal sum of BCK(P) lattices

We generalize here the notion of "ordinal sum" of two BL chains in the following way.

Definition 4.1 Let $\mathcal{M}_i = (M_i, \geq_i, \rightarrow_i, 0_i, 1_i)$, $0_i \neq 1_i$, $i \in \{1,2\}$ be two BCK(P) lattices such that $1_1 = 0_2$ and $(M_1 \setminus \{1_1\}) \bigcap (M_2 \setminus \{0_2\}) = \emptyset$.

The *ordinal sum* of $\mathcal{M}_1$ and $\mathcal{M}_2$ (in this order) is the structure:

$$\mathcal{M}_1 \bigoplus \mathcal{M}_2 = (M_1 \bigcup M_2, \geq, \rightarrow, 0, 1),$$

where:
$x \geq y$ if ($x, y \in M_1$ and $x \geq_1 y$) or ($x, y \in M_2$ and $x \geq_2 y$) or ($x \in M_1$ and $y \in M_2$), i.e. we have the lattice reprezented by diagram Hasse from Figure 4;
$0 = 0_1$, $1 = 1_2$;

$$x \rightarrow y = \begin{cases} 1, & \text{if} \quad x \leq y \\ x \rightarrow_i y, & \text{if} \quad x > y, \quad x, y \in M_i, \quad , i \in \{1,2\} \\ y, & \text{if} \quad x > y, \quad x \in M_2,\ y \in M_1 \setminus \{1_1\}, \end{cases}$$

or, equivalently

$$x \rightarrow y = \begin{cases} 1, & \text{if} \quad x \leq y \\ x \rightarrow_i y, & \text{if} \quad x > y, \quad x, y \in M_i, \quad , i \in \{1,2\} \\ y, & \text{if} \quad x > y, \quad x \in M_2 \setminus \{0_2\},\ y \in M_1 \setminus \{1_1\}, \end{cases}$$

since in M_1, $1_1 \rightarrow y_1 = y_1$.

Hence, $\rightarrow$ has the following table:

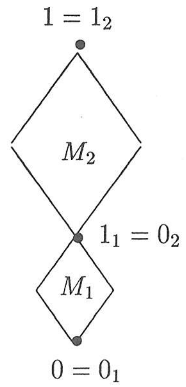

Figure 4: The ordinal sum $M_1 \bigoplus M_2$

$M_1 \bigoplus M_2$

$x \to y$	$0 = 0_1$	$\dots$	y_1	$\dots$	$1_1 = 0_2$	$\dots$	y_2	$\dots$	$1 = 1_2$
$0 = 0_1$	1	$\dots$	1	$\dots$	1	$\dots$	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_1	$x_1 \to_1 0_1$	$\dots$	$x_1 \to_1 y_1$	$\dots$	1	$\dots$	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1_1 = 0_2$	0	$\dots$	y_1	$\dots$	1	$\dots$	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_2	0	$\dots$	y_1	$\dots$	$x_2 \to_2 0_2$	$\dots$	$x_2 \to_2 y_2$	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1 = 1_2$	0	$\dots$	y_1	$\dots$	$1_1 = 0_2$	$\dots$	y_2	$\dots$	1

where $x_1, y_1 \in M_1 \setminus \{0 = 0_1, 1_1\}$, $x_2, y_2 \in M_2 \setminus \{0_2, 1 = 1_2\}$.

Remarks 4.2 Remark that the table of $\to$ in $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ contains:
1) the initial table of $\to$ in M_2,
2) the initial table of $\to$ in M_1, modified in the sense that 1_1 is replaced by 1,
3) if $x \in M_1 \setminus \{1_1\}$ and $y \in M_2$, then $x \to y = 1$, since $x < y$,
4) if $x \in M_2 \setminus \{0_2\}$ and $y \in M_1 \setminus \{1_1\}$, then $x \to y = y$.

Note that if M_1 and M_2 are BL chains, we get the well-known definition of ordinal sum of two BL chains (written sometimes in an ambigous way).

Then we have the following

Theorem 4.3 *Let $\mathcal{M}_i = (M_i, \geq_i, \to_i, 0_i, 1_i)$, $i \in \{1, 2\}$ be two BCK(P) lattices. Then $\mathcal{M}_1 \bigoplus \mathcal{M}_2$, is a BCK(P) lattice.*

Proof. Obviously, $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ is a lattice with first element $0 = 0_1$ and last element $1 = 1$.

We prove now that it is a BCK algebra:

• First we prove that

$$1 \to x = x. \tag{59}$$

Indeed, since $1 \in M_2$, it follows that $1 \to x = 1 \to_2 x = x$, if $x \in M_2$ and $1 \to x = x$, if $x \in M_1 \setminus \{1_1\}$. Thus, (59) holds.

• Then we prove that

$$x \leq y \to x. \tag{60}$$

Indeed, if $y \leq x$, then $y \to x = 1 \geq x$; if $y, x \in M_i$, $i \in \{1,2\}$, then (60) holds; if $y \in M_2$ and $x \in M_1 \setminus \{1_1\}$, then $y = rax = x \geq x$. Thus (60) holds.

• Then we need that

$$x \to x = 1, \tag{61}$$

which is true since $x \leq x$.

• Now we can prove (II), i.e. $(y \to x) \to x \geq y$.

Indeed,

- if $y \leq x$, then $y \to x = 1$; hence $(y \to x) \to x = 1 \to x \overset{(59)}{=} x \geq y$.
- if $y, x \in M_i$, $i \in \{1,2\}$, then (II) holds.
- if $y \in M_2$, $x \in M_1 \setminus \{1_1\}$, then $y \to x = x$ and hence $(y \to x) \to x = x \to x \overset{(61)}{=} 1 \geq y$. Thus, (II) holds.

• Now we prove (I), i.e. $(z \to x) \to (y \to x) \geq Y \to z$:

Denote

$$T1 \overset{notation}{=} (z \to x) \to (y \to x), \qquad T2 \overset{notation}{=} y \to z.$$

We must prove that $T1 \geq T2$. Indeed,

if $y, z \in M_1$, then:
- if $x \in M_1$, then (I) holds, since it holds in M_1;
- if $x \in M_2 \setminus \{0_2\}$, then $x \geq y, z$, hence $z\ rax = 1 = y \to x$ and thus $T1 = 1$; it follows that $T2 \leq T1$.

if $y, z \in M_2$, then:
- if $x \in M_2$, then (I) holds, since it holds in M_2;
- if $x \in M_1 \setminus \{1_1\}$, then $z \to x = x$ and $y \to x = x$, hence $T1 = x \to x \overset{(61)}{=} 1 \geq T2$.

if $y \in M_2$, $z \in M_1 \setminus \{1_1\}$, then $T2 = y \to z = z$;
- if $x \in M_2$, then $z \leq x$ and hence $z \to x = 1$; then $T1 = (z \to x) \to (y \to x) = 1 \to (y \to x) \overset{(59)}{=} y \to x \in M_2$; hence $T1 > T2 = z$;
- if $x \in M_1 \setminus \{1_1\}$, then $y \to x = x$ and then $T1 = (z \to x) \to (y \to x) = (z \to x) \to x \overset{(II)}{\geq} z = T2$.

if $z \in M_2$, $y \in M_1 \setminus \{1_1\}$, then $y \leq z$, hence $T2 = y \to x = 1$;
- if $x \in M_2$, then $x \geq y$, hence $y \to x = 1$ and consequently $T1 = (z \to x) \to (y \to x) = (z \to x) \to 1 = 1 = T2$, since $z \to x \leq 1$;
- if $x \in M_1 \setminus \{1_1\}$, then $z \to x = x$ and since by (60) $x \leq y \to x$, it follows that $T1 = (z \to x) \to (y \to x) = x \to (y \to x) = 1 = T2$.

Thus, (I) holds and consequently $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ is a BCK lattice.

• Finally, we must prove that the condition (P) is satisfied, i.e. for all $x, y \in M_1 \bigcup M_2$, there exists

$$x \odot y = \min\{z \mid x \leq y \to z\}.$$

Indeed, we get that

$$x \odot y = \begin{cases} x \odot_i y, & \text{if} \quad x, y \in M_i \\ x, & \text{if} \quad x \in M_1 \setminus \{1_1\},\ y \in M_2, \end{cases}$$

or, equivalently, since for any $x_1 \in M_1 \setminus \{1_1\}$, we have $x_1 \odot 1_1 = x_1$,

$$x \odot y = \begin{cases} x \odot_i y, & \text{if} \quad x, y \in M_i,\ i \in \{1,2\} \\ x, & \text{if} \quad x \in M_1 \setminus \{1_1\},\ y \in M_2 \setminus \{0_2\}, \end{cases}$$

or, equivalently, since $\odot$ is commutative,

$$x \odot y = \begin{cases} x \odot_i y, & \text{if} \quad x, y \in M_i,\ i \in \{1,2\} \\ x, & \text{if} \quad x \in M_1 \setminus \{1_1\},\ y \in M_2 \setminus \{0_2\} \\ y, & \text{if} \quad x \in M_2 \setminus \{0_2\},\ y \in M_1 \setminus \{1_1\}, \end{cases}$$

i.e. we have the following table:

$M_1 \bigoplus M_2$

$x \odot y$	$0 = 0_1$	$\ldots$	y_1	$\ldots$	$1_1 = 0_2$	$\ldots$	y_2	$\ldots$	$1 = 1_2$
$0 = 0_1$	0	$\ldots$	0	$\ldots$	0	$\ldots$	0	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_1	0	$\ldots$	$x_1 \odot_1 y_1$	$\ldots$	x_1	$\ldots$	x_1	$\ldots$	x_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1_1 = 0_2$	0	$\ldots$	y_1	$\ldots$	0_2	$\ldots$	0_2	$\ldots$	$1_1 = 0_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_2	0	$\ldots$	y_1	$\ldots$	0_2	$\ldots$	$x_2 \odot_2 y_2$	$\ldots$	x_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$1 = 1_2$	0	$\ldots$	y_1	$\ldots$	0_2	$\ldots$	y_2	$\ldots$	1

□

Remark 4.4 The ordinal sum is not commutative, but it is associative, by definition of $\to$.

Theorem 4.5 *If the BCK(P) lattices $\mathcal{M}_1$ and $\mathcal{M}_2$ both satisfy the condition (B2), then $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ also satisfies (B2).*

Proof.
Obvious, if we consider instead of (B2), the equivalent cancellative condition (C_c). □

Theorem 4.6 *If the BCK(P) lattices $\mathcal{M}_1$ and $\mathcal{M}_2$ both satisfy the condition (B3), then:*
(i) if $\mathcal{M}_1$ is linearly ordered (a chain), then $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ also satisfies (B3);
(ii) if $\mathcal{M}_1$ is not linearly ordered, then $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ does not satisfy (B3).

Proof.
Obvious.
(i) is obvious, since in this case, above the principal diagonal of the table of $\to$ we have always 1 and thus (B3) holds.
(ii) Since $\mathcal{M}_1$ is not linearly ordered, there are $a, b \in M_1 \setminus \{0 = 0_1, 1_1 = 0_2\}$ such that they are incomparable, but since $\mathcal{M}_1$ satisfies (B3), we have $(a \to b) \vee (b \to a) = 1_1 \neq 1 = 1_2$. Thus, (B3) is not satisfied. □

Corollary 4.7 *If the BCK(P) lattices $\mathcal{M}_1$ and $\mathcal{M}_2$ both satisfy both conditions (B2) and (B3), then:*
(i) if $\mathcal{M}_1$ is linearly ordered (a chain), then $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ also satisfies (B2) and (B3);
(ii) if $\mathcal{M}_1$ is not linearly ordered, then $\mathcal{M}_1 \bigoplus \mathcal{M}_2$ satisfies (B2), but does not satisfy (B3), i.e. does not satisfy $(C_\vee)$.

Proof.
Obvious, by the previous two Theorems. □

Remarks 4.8
(i) Any MV, Gödel or Product algebra is a BL algebra.
(ii) The ordinal sum "liniar BL algebra $\bigoplus$ BL algebra " is a BL algebra, by Corollary 4.7.
(iii) The ordinal sum "non-liniar BL algebra $\bigoplus$ BL algebra " is a divisible BCK(P) lattice, by Corollary 4.7.
(iv) The ordinal sum "linearly ordered Gödel algebra $\bigoplus$ Gödel algebra " is a Gödel algebra, by (ii).
(v) The ordinal sum "non-linearly ordered Gödel algebra $\bigoplus$ Gödel algebra " is a divisible BCK(P) lattice of Gödel type, by (iii).
(vi) The ordinal sum of two BCK(P) lattices with condition (DN) is no more a BCK(P) lattice with condition (DN), by the definition of the ordinal sum. Consequently, the ordinal sum "MV algebra $\bigoplus$ MV algebra " is never an MV algebra (there is $1_1 = 0_2$ such that $(1_1^-)^- = 0_1^- = 1 \neq 1_1$).
(vii) Note that the ordinal sum $M_1 \bigoplus M_2$ preserves $(C_\wedge)$ and if M_1 is non-linearly ordered, then it does not preserves $(C_\vee)$.

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