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CLASSES OF BCK ALGEBRAS – PART II

by

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BUCUREŞTI

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Classes of BCK algebras - Part II

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Dedicated to Grigore C. Moisil (1905-1973)

(10 January 2004)

Abstract

In this paper we study the BCK algebras and their particular classes: the BCK(P) (residuated) lattices, the Hájek(P) (BL) algebras and the Wajsberg (MV) algebras, we introduce new classes of BCK(P) lattices, we establish hierarchies and we give many examples. The paper has five parts.

In the first part, the most important part, we decompose the divisibility and the pre-linearity conditions from the definition of a BL algebra into four new conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). We study the additional conditions (WNM) (weak nilpotent minimum) and (DN) (double negation) on a BCK(P) lattice. We introduce the ordinal sum of two BCK(P) lattices and prove in what conditions we get BL algebras or other structures, more general, or more particular than BL algebras.

In part II, we give examples of some finite bounded BCK algebras. We introduce new generalizations of BL algebras, named α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). By adding the conditions (WNM) and (DN) to these classes, we get more classes; among them, we get many generalizations of Wajsberg (MV) algebras and of R_0 (NM) algebras. The subclasses of $_{(WNM)}$ Wajsberg algebras ($_{(WNM)}$MV algebras) and of $_{(WNM)}$Hájek algebras ($_{(WNM)}$BL algebras) are introduced. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I.

In part III, we give examples of finite MV and $_{(WNM)}$MV algebras, of Hájek(P) (i.e. BL) algebras and $_{(WNM)}$BL algebras and of $\alpha\gamma\delta$ (i.e. divisible BCK(P) lattices (divisible residuated lattices or divisible integral, residuated, commutative l-monoids)) and of divisible $_{(WNM)}$BCK(P) lattices.

In part IV, we stress the importance of $\alpha\beta\gamma$ algebras versus $\alpha\beta$ (i.e. MTL) algebras algebras and of R_0 (i.e. NM) algebras versus Wajsberg (i.e. MV) algebras and of $_{(WNM)}\alpha\beta\gamma$ algebras versus BL algebras and of $\alpha\gamma$ versus $\alpha\gamma\delta$ algebras. We give examples of finite IMTL algebras and of $_{(WNM)}$IMTL (i.e. NM) algebras), of $\alpha\beta\gamma$ algebras and of $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras and finally of $\alpha\gamma$ algebras.

In part V, we give other examples of finite BCK(P) lattices, finding examples for the others remaining an open problem. We make final remarks and formulate final open problems.

Keywords MV algebra, Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, divisible BCK(P) lattice, α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, $_{(WNM)}$MV, $_{(WNM)}$BL, $_{(WNM)}$ $\alpha\beta\gamma$, Roman algebra

Part II has two sections.

In Section 5, we give examples of some finite bounded BCK algebras.

In Section 6, we introduce new generalizations of Hájek(P) (BL) algebras, named α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$, (C_X) found in Part I. We make the connections with MTL algebras [18] and with divisible integral, residuated, commutative l-monoids [33]. By adding the conditions (WNM) and (DN) to these classes, we get more classes: of $_{(WNM)}\alpha$ algebras, $\alpha_{(DN)}$ algebras, $_{(WNM)}\alpha_{(DN)}$ algebras etc.

Thus, we get generalizations of BL and $_{(WNM)}$BL algebras, and of Wajsberg (MV) algebras and of NR$_0$ algebras. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I and Part II. We make the connections with MTL, WNM, IMTL and NM algebras [18], [21] and with R$_0$ [73], [63] and NR_0 algebras [54]. See the Remarks 6.15, where we introduce the name "Roman algebra".

5 Examples of finite bounded BCK algebras

In this section we shall give many examples, but we shall present the proof only in one case, as an example of proof.

5.1 Examples of bounded BCK algebras which are not lattices

We give here three examples.

1. Example of bounded BCK algebra which is not a lattice and does not satisfy the condition (P)

Let us consider the set $A = \{0, a, b, c, d, n, 1\}$ organized as a poset which is not a lattice as in Hasse diagramme from Figure 1 and as a BCK algebra with the operation $\to$ as in the following table:

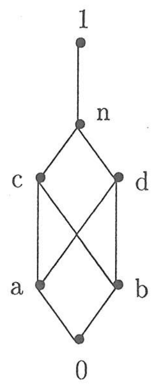

Figure 1: Bounded BCK algebra without or with condition (P), which is not a lattice

$\to$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	0	1	n	1	1	1	1
b	0	n	1	1	1	1	1
c	0	n	n	1	n	1	1
d	0	n	n	n	1	1	1
n	0	n	n	n	n	1	1
1	0	a	b	c	d	n	1

This bounded BCK algebra does not satisfy the condition (P) since does not exist

$$a \odot b \overset{notation}{=} \min\{z \mid a \le b \to z\} = \min\{a, b, c, d, n, 1\}.$$

2. Example of bounded BCK algebra which is not a lattice and which satisfies the condition (P)

Let us consider the set $A = \{0, a, b, c, d, n, 1\}$ organized as a poset which is not a lattice as in Figure 1 and as a BCK algebra with condition (P) with the operation $\rightarrow$ and

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$$

as in the following tables:

$\rightarrow$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	n	1	n	1	1	1	1
b	n	n	1	1	1	1	1
c	n	n	n	1	n	1	1
d	n	n	n	n	1	1	1
n	n	n	n	n	n	1	1
1	0	a	b	c	d	n	1

$\odot$	0	a	b	c	d	n	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	0	0	0	0	b
c	0	0	0	0	0	0	c
d	0	0	0	0	0	0	d
n	0	0	0	0	0	0	n
1	0	a	b	c	d	n	1

Thus, $\mathcal{A}$ is a BCK(P) algebra which is not a lattice.

3. Example of bounded BCK algebra which is not a lattice and which satisfies the conditions (P) and (DN)

This example answers to the open problem raised in [42], Remarks 2.40(2). Let us consider the set $A = \{0, m, a, b, c, d, n, 1\}$ organized as a poset which is not a lattice as in Figure 2 and as a BCK algebra with condition (P) with the operation $\rightarrow$ and

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$$

as in the following tables:

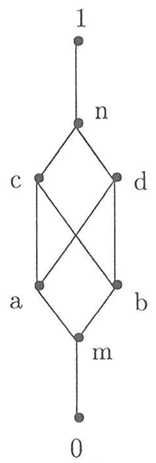

Figure 2: Bounded BCK algebra with conditions (P) and (DN), which is not a lattice

$\to$	0	m	a	b	c	d	n	1
0	1	1	1	1	1	1	1	1
m	n	1	1	1	1	1	1	1
a	c	n	1	n	1	1	1	1
b	d	n	n	1	1	1	1	1
c	a	n	n	n	1	n	1	1
d	b	n	n	n	n	1	1	1
n	m	n	n	n	n	n	1	1
1	0	m	a	b	c	d	n	1

$\odot$	0	m	a	b	c	d	n	1
0	0	0	0	0	0	0	0	0
m	0	0	0	0	0	0	0	m
a	0	0	0	0	0	m	m	a
b	0	0	0	0	m	0	m	b
c	0	0	0	m	m	m	m	c
d	0	0	m	0	m	m	m	d
n	0	0	m	m	m	m	m	n
1	0	m	a	b	c	d	n	1

Hence, for all $u, v \in A$,

$$u \to v = \begin{cases} 1, & u \leq v \\ n, & 0 < v < u < 1 \\ n, & u = m, v = 0 \\ c, & u = a, v = 0 \\ d, & u = b, v = 0 \\ a, & u = c, v = 0 \\ b, & u = d, v = 0 \\ m, & u = n, v = 0 \\ v, & u = 1 \\ n, & (u,v) \in \{(a,b),(b,a),(c,d),(d,c)\}. \end{cases}$$

To prove that $(A, \geq, \to, 1)$ is a BCK algebra it is sufficient to prove (I), i.e. that for all $x, y, z \in A$, we have $(z \to x) \to (y \to x) \geq y \to z$.

Let us denote

$$T1 \overset{notation}{=} (z \to x) \to (y \to x) \quad \text{and} \quad T2 \overset{notation}{=} y \to z.$$

Hence, we have to prove that for all $x, y, z \in A$, $T1 \geq T2$. Indeed, we have the following cases:
1) $y \leq z$,
2) $0 < z < y < 1$,
3) $z = 0$ and $y \in \{m, a, b, c, d, n\}$,
4) $y = 1$,
5) $(y, z) \in \{(a,b),(b,a),(c,d),(d,c)\}$.

• 1). If $y \leq z$, then $z \to x \leq y \to x$, hence $T1 = 1$, $T2 = 1$, thus (I) holds.
• 2). If $0 < z < y < 1$, then we have the following subcases:
2.1) $y = a$, $z = m$,
2.2) $y = b$, $z = m$,
2.3) $y = c, z \in \{m, a, b\}$,
2.4) $y = d, z \in \{m, a, b\}$,
2.5) $y = n, z \in \{m, a, b, c, d\}$.

2.1): $y = a$, $z = m$. Then $T2 = y \to z = a \to m = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (m \to 0) \to (a \to 0) = n \to c = n = T2$.
- If $x = m$, then $T1 = (z \to x) \to (y \to x) = (m \to m) \to (a \to m) = 1 \to n = n = T2$.
- If $x \in \{a, c, d, n, 1\}$, then $x \geq y = a$; hence $y \to x = 1$ and consequently, $T1 = (z \to x) \to (y \to x) = (z \to x) \to 1 = 1 > T2$.
- If $x = b$, then $T1 = (z \to x) \to (y \to x) = (m \to b) \to (a \to b) = 1 \to n = n = T2$.
Thus, (I) holds.

2.2) $y = b$, $z = m$. Similar to 2.1).

2.3) $y = c, z \in \{m, a, b\}$:

2.3.1) $y = c,\ z = m$. Then $T2 = y \to z = c \to m = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (m \to 0) \to (c \to 0) = n \to a = n = T2$.
- If $x \in \{m, a, b\}$, then we have $0 < z \leq x < y < 1$ and hence $T1 = 1 \to n = n = T2$.
- If $x = d$, then $T1 = (m \to d) \to (c \to d) = 1 \to n = n = T2$.
- If $x \in \{c, n, 1\}$, then we have $x \geq y = c$, hence $y \to x = 1$ and consequently, $T1 = 1 > T2$.

2.3.2) $y = c,\ z = a$. Then $T2 = y \to z = c \to a = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (a \to 0) \to (c \to 0) = c \to a = n = T2$.
- If $x = m$, then $T1 = (a \to m) \to (c \to m) = n \to n = 1 > T2$.
- If $x = a$, then $T1 = (a \to a) \to (c \to a) = 1 \to n = n = T2$.
- If $x = b$, then $T1 = (a \to b) \to (c \to m) = n \to n = 1 > T2$.
- If $x = d$, then $T1 = (a \to d) \to (c \to d) = 1 \to n = n = T2$.
- If $x \in \{c, n, 1\}$, then $x \geq y = c$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

2.3.3) $y = c,\ z = b$. Similar to 2.3.2).

2.4) $y = d, z \in \{m, a, b\}$: Similar to 2.3).

2.5) $y = n, z \in \{m, a, b, c, d\}$:

2.5.1) $y = n,\ z = m$. Then $T2 = y \to z = n \to m = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (m \to 0) \to (n \to 0) = n \to m = n = T2$.
- If $x = m$, then $T1 = (m \to m) \to (n \to m) = 1 \to n = n = T2$.
- If $x \in \{a, b, c, d\}$, then $0 < z < x < y < 1$, hence $T1 = 1 \to n = n = T2$.
- If $x \in \{n, 1\}$, then $x \geq y = n$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

2.5.2) $y = n,\ z = a$. Then $T2 = y \to z = n \to a = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (a \to 0) \to (n \to 0) = c \to m = n = T2$.
- If $x = m$, then $T1 = (a \to m) \to (n \to m) = n \to n = 1 > T2$.
- If $x \in \{a, c, d\}$, then $0 < z \leq x < y < 1$, hence $T1 = 1 \to n = n = T2$.
- If $x = b$, then $T1 = (a \to b) \to (n \to b) = n \to n = 1 > T2$.
- If $x \in \{n, 1\}$, then $x \geq y = n$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

2.5.3) $y = n,\ z = b$. Similar to 2.5.2).

2.5.4) $y = n,\ z = c$. Then $T2 = y \to z = n \to c = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (c \to 0) \to (n \to 0) = a \to m = n = T2$.
- If $x \in \{m, a, b\}$, then $0 < x < z < y < 1$, hence $T1 = n \to n = 1 > T2$.
- If $x = d$, then $T1 = (c \to d) \to (n \to d) = n \to n = 1 > T2$.
- If $x = c$, then $T1 = (c \to c) \to (n \to c) = 1 \to n = n = T2$.
- If $x \in \{n, 1\}$, then $x \geq y = n$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

2.5.5) $y = n,\ z = d$. Similar to 2.5.4).

- 3). $z = 0$ and $y \in \{m, a, b, c, d, n\}$:

3.1) $z = 0,\ y = m$. Then $T2 = y \to z = m \to 0 = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (0 \to 0) \to (m \to 0) = 1 \to n = n = T2$.
- If $x = m$, then $T1 = (0 \to m) \to (m \to m) = 1 \to 1 = 1 > T2$.
- If $x \in \{a, b, c, d, n, 1\}$, then $x \geq y$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

3.2) $z = 0,\ y = a$. Then $T2 = y \to z = a \to 0 = c$. - If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (0 \to 0) \to (a \to 0) = 1 \to c = c = T2$.
- If $x = m$, then $T1 = (0 \to m) \to (a \to m) = 1 \to n = n > T2$.
- If $x = b$, then $T1 = (0 \to b) \to (a \to b) = 1 \to n = n > T2$.
- If $x \in \{a, c, d, n, 1\}$, then $x \geq y$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

3.3) $z = 0,\ y = b$. Similar to 3.2).

3.4) $z = 0,\ y = c$. Then $T2 = y \to z = c \to 0 = a$. - If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (0 \to 0) \to (c \to 0) = 1 \to a = a = T2$.
- If $x \in \{m, a, b\}$, then $0 = z < x < y = c < 1$, hence $T1 = 1 \to n = n > T2$.
- If $x = d$, then $T1 = (0 \to d) \to (c \to d) = 1 \to n = n > T2$.
- If $x \in \{c, n, 1\}$, then $x \geq y$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

3.5) $z = 0,\ y = d$. Similar to 3.4).

3.6) $z = 0,\ y = n$. Then $T2 = y \to z = n \to 0 = m$. - If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (0 \to 0) \to (n \to 0) = 1 \to m = m = T2$.
- If $x \in \{m, a, b, c, d\}$, then $0 = z < x < y = n$, hence $T1 = 1 \to n = n > T2$.
- If $x \in \{n, 1\}$, then $x \geq y$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

- 4) $y = 1$. Then $T2 = y \to z = 1 \to z = z$, for all $z \in A$:

4.1) $y = 1,\ z = T2 = 0$.
- $T1 = (z \to x) \to (y \to x) = (0 \to x) \to (1 \to x) = 1 \to x = x > T2$, for all $x \in A$.

4.2) $y = 1,\ z = T2 = m$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (m \to 0) \to (1 \to 0) = n \to 0 = m = T2$.
- If $x \in \{m, a, b, c, d, n, 1\}$, then $z \leq x \leq y = 1$, hence $T1 = 1 \to (1 \to x) = x \geq T2$.

4.3) $y = 1,\ z = T2 = a$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (a \to 0) \to (1 \to 0) = c \to 0 = a = T2$.
- If $x = m$, then $T1 = (a \to m) \to (1 \to m) = n \to m = n > T2$.
- If $x \in \{a, c, d, n, 1\}$, then $z \leq x \leq y = 1$, hence $T1 = 1 \to (1 \to x) = x \geq T2$.
- If $x = b$, then $T1 = (a \to b) \to (1 \to b) = n \to b = n > T2$.

4.4) $y = 1,\ z = T2 = b$. Similar to 4.3).

4.5) $y = 1,\ z = T2 = c$. - If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (c \to 0) \to (1 \to 0) = a \to 0 = c = T2$.
- If $x \in \{m, a, b\}$, then $0 < x < z = c < y = 1$, hence $T1 = n \to (1 \to x) = n \to x = n > T2$.
- If $x = d$, then $T1 = (c \to d) \to (1 \to d) = n \to d = n > T2$.
- If $x \in \{c, n, 1\}$, then $z \leq x \leq y = 1$, hence $T1 = 1 \to (1 \to x) = 1 \to x = x \geq T2$.

4.6) $y = 1,\ z = T2 = d$. Similar to 4.5).

4.7) $y = 1,\ z = T2 = n$. - If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (n \to 0) \to (1 \to 0) = m \to 0 = n = T2$.
- If $x \in \{m, a, b, c, d\}$, then $0 < x < z = n < y = 1$, hence $T1 = n \to (1 \to x) = n \to x = n = T2$.
- If $x \in \{n, 1\}$, then $z \leq x \leq y = 1$, hence $T1 = 1 \to (1 \to x) = 1 \to x = x \geq T2$.

- 5) $(y, z) \in \{(a, b), (b, a), (c, d), (d, c)\}$:

5.1) $(y, z) = (a, b)$. Then $T2 = y \to z = a \to b = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (b \to 0) \to (a \to 0) = d \to c = n = T2$.
- If $x = m$, then $T1 = (b \to m) \to (a \to m) = n \to n = 1 > T2$.
- If $x = a$, then $T1 = (b \to a) \to (a \to a) = n \to 1 = 1 > T2$.
- If $x = b$, then $T1 = (b \to b) \to (a \to b) = 1 \to n = n = T2$.
- If $x \in \{c, d, n, 1\}$, then $x \geq y = a$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

5.2) $(y, z) = (b, a)$. Similar to 5.1).

5.3) $(y, z) = (c, d)$. Then $T2 = y \to z = c \to d = n$.
- If $x = 0$, then $T1 = (z \to x) \to (y \to x) = (d \to 0) \to (c \to 0) = b \to a = n = T2$.
- If $x \in \{m, a, b\}$, then $0 < x \leq y, z < 1$, hence $T1 = n \to n = 1 > T2$.

- If $x = c$, then $T1 = (d \to c) \to (c \to c) = n \to 1 = 1 > T2$.
- If $x = d$, then $T1 = (d \to d) \to (c \to d) = 1 \to n = n = T2$.
- If $x \in \{n, 1\}$, then $x \geq y = c$, hence $y \to x = 1$ and consequently $T1 = 1 > T2$.

5.4) $(y, z) = (d, c)$. Similar to 5.3). Here the proof of (I) comes to the end.

Note that another proof of (I) consists in taking all the triples $(y, z, x) \in A \times A \times A$, in the direct lexicographical order.

Thus, $\mathcal{A}$ is a bounded BCK(P) algebra with condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0), which is not $\vee$-commutative.

5.2 Examples of bounded BCK algebras which are lattices and which do not satisfy the condition (P)

We shall give four examples, very known as lattices.

Example 1

Let us consider the set $A = \{-1, 0, a, b, 1\}$ organized as a lattice as in Figure 3 and as a BCK algebra with the operation $\to$ as in the following table:

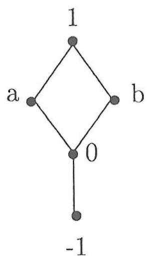

Figure 3: Example 1 of BCK lattice without (P)

Example 1

$\to$	-1	0	a	b	1
-1	1	1	1	1	1
0	-1	1	1	1	1
a	-1	0	1	b	1
b	-1	0	a	1	1
1	-1	0	a	b	1

Remark $(A, \wedge, \vee, \to, 0, 1)$ is a BCK lattice which does not satisfy the condition (P), since there are $a, b \in A$ such that

$$a \odot b \overset{notation}{=} \min\{z \mid a \leq b \to z\} = \min\{a, b, 1\}$$

does not exist.

Example 2 (see [34])

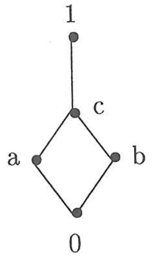

Figure 4: Example 2 of BCK lattice without (P)

Let us consider the set $A = \{0, a, b, c, 1\}$ organized as a lattice as in Figure 4 and as a BCK algebra with the operation $\rightarrow$ as in the following table:

Example 2

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	c	1	1
b	0	c	1	1	1
c	0	c	c	1	1
1	0	a	b	c	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK lattice which does not satisfy the condition (P), since there are $b, a \in A$ such that

$$b \odot a \overset{notation}{=} \min\{z \mid b \leq a \rightarrow z\} = \min\{a, b, c, 1\}$$

does not exist.

Example 3 (see [35])

Let us consider the set $A = \{0, a, b, c, 1\}$ organized as a lattice as in Figure 5 and as a BCK algebra with the operation $\rightarrow$ as in the following table:

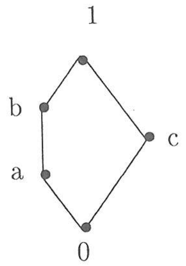

Figure 5: Example 3 of BCK lattice without (P)

Example 3

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	c	1
b	0	a	1	c	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK lattice which does not satisfy the condition (P), since there are $b, c \in A$ such that

$$b \odot c \overset{notation}{=} \min\{z \mid b \leq c \rightarrow z\} = \min\{b, c, 1\}$$

does not exist.

Example 4

Let us consider the set $A = \{0, a, b, c, 1\}$ organized as a lattice as in Figure 6 and as a BCK algebra with the operation $\rightarrow$ as in the following table:

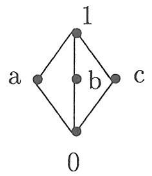

Figure 6: Example 4 of BCK lattice without (P)

Example 4

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK lattice which does not satisfy the condition (P), since there are $c, b \in A$ such that

$$c \odot b \overset{notation}{=} \min\{z \mid c \leq b \rightarrow z\} = \min\{b, c, 1\}$$

does not exist.

5.3 Concluding remark

Hence, we have the situation from Figure 7.

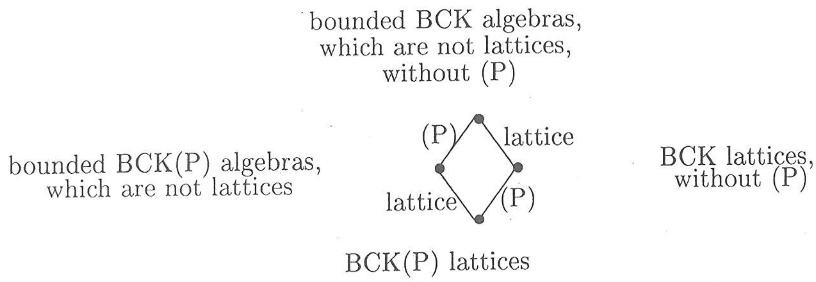

Figure 7: Classes of bounded BCK algebras

If we add the condition **(DN)**, then, by Theorem ??, we get the situation from Figure 8.

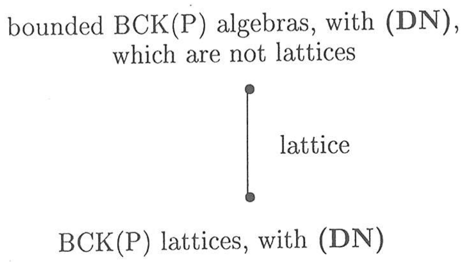

Figure 8: Classes of bounded BCK(P) algebras, with **(DN)**

6 New classes of BCK(P) lattices (residuated lattices)

Recall [42], [43] that a Hájek(P) (BL) algebra is an algebra

$$\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$$

such that:
(B1) $\mathcal{A}$ is a BCK(P) lattice (residuated lattice),
(B2) for all $x, y \in A$, $x \wedge y = x \odot (x \rightarrow y)$ (divisibility);
(B3) for all $x, y \in A$, $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (pre-linearity).

Recall also that in [43], it was proved that:
(B2) is equivalent with $(C_{\rightarrow}) + (C_{\wedge}) + (C_X)$, while
(B3) is equivalent with $(C_{\rightarrow}) + (C_{\vee})$, hence
(B2)+(B3) is equivalent with $(C_{\vee}) + (C_{\wedge}) + (C_{\rightarrow}) + (C_X)$, where:

$(C_{\rightarrow})$ $\quad (x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x$,
$(C_{\vee})$ $\quad x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$,
$(C_{\wedge})$ $\quad x \wedge y = [x \odot (x \rightarrow y)] \vee [y \odot (y \rightarrow x)]$,
(C_X) $\quad x \odot [(y \rightarrow x) \rightarrow (x \rightarrow y)] = y \odot [(x \rightarrow y) \rightarrow (y \rightarrow x)]$.

We shall introduce now new algebras, which are particular cases of BCK(P) lattices (residuated lattices) and generalizations of Hájek(P) algebras (BL algebras).

- 1) We define first the algebras satisfying one of the above conditions; we get four algebras:

Definition 6.1
An α *algebra* is a BCK(P) lattice satisfying the condition $(C_{\rightarrow})$.
A β *algebra* is a BCK(P) lattice satisfying the condition $(C_{\vee})$.
A γ *algebra* is a BCK(P) lattice satisfying the condition $(C_{\wedge})$.
A δ *algebra* is a BCK(P) lattice satisfying the condition (C_X).

- 2) Now we define the algebras satisfying two of the above conditions; we have thus six algebras:

Definition 6.2
An $\alpha\beta$ *algebra* is a BCK(P) lattice satisfying both conditions $(C_{\rightarrow})$ and $(C_{\vee})$, i.e. satisfying the condition (B3).
An $\alpha\gamma$ *algebra* is a BCK(P) lattice satisfying both conditions $(C_{\rightarrow})$ and $(C_{\wedge})$.
An $\alpha\delta$ *algebra* is a BCK(P) lattice satisfying both conditions $(C_{\rightarrow})$ and (C_X).
A $\beta\gamma$ *algebra* is a BCK(P) lattice satisfying both conditions $(C_{\vee})$ and $(C_{\wedge})$.
A $\beta\delta$ *algebra* is a BCK(P) lattice satisfying both conditions $(C_{\vee})$ and (C_X).
A $\gamma\delta$ *algebra* is a BCK(P) lattice satisfying both conditions $(C_{\wedge})$ and (C_X).

Remark 6.3 The $\alpha\beta$ algebra is just the weak-BL algebra [20] (the MTL algebra [18]). Thus, "weak-BL algebra", "MTL algebra" and "$\alpha\beta$ algebra" are duplicate names for the same algebra.

- 3) Now we define the algebras satisfying three of the above conditions; we have four algebras:

Definition 6.4
An $\alpha\beta\gamma$ *algebra* is a BCK(P) lattice satisfying the conditions $(C_{\rightarrow})$, $(C_{\vee})$ and $(C_{\wedge})$.
An $\alpha\beta\delta$ *algebra* is a BCK(P) lattice satisfying the conditions $(C_{\rightarrow})$, $(C_{\vee})$ and (C_X).
An $\alpha\gamma\delta$ *algebra* is a BCK(P) lattice satisfying the conditions $(C_{\rightarrow})$, $(C_{\wedge})$ and (C_X), i.e. satisfying the condition (B2).
A $\beta\gamma\delta$ *algebra* is a BCK(P) lattice satisfying the conditions $(C_{\vee})$, $(C_{\wedge})$ and (C_X).

Remarks 6.5
(i) Recall ([43], Theorem 3.7) that a linearly ordered BCK(P) lattice (BCK(P) chain) satisfies the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$, i.e. it is an $\alpha\beta\gamma$ algebra. Consequently, all the examples of weak-BL algebras [20] (duplicate name: MTL algebras [18]), i.e. of $\alpha\beta$ algebras, given in the literature are in fact examples of linearly ordered $\alpha\beta\gamma$ algebras (A is the real unit interval $[0,1]$ and $\odot$ is a left-continuous t-norm on $[0,1]$).
(ii) It remains an open problem to find examples of proper $\alpha\beta$ (duplicate names: MTL, weak-BL) algebras, i.e which are not $\alpha\beta\gamma$ algebras.
(iii) There exist $\alpha\beta\gamma$ algebras which are not linearly ordered (see in the sequel).

Remark 6.6 The $\alpha\gamma\delta$ algebra is just (a duplicate name for) the "divisible BCK(P) lattice" ("divisible residuated lattice", "divisible integral, residuated, commutative l-monoid" [33]).

- 4) Finally, we have:

Definition 6.7 An $\alpha\beta\gamma\delta$ *algebra* is a BCK(P) lattice satysfying all the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$, (C_X).

Remark 6.8 An $\alpha\beta\gamma\delta$ algebra is just (a duplicate name for) a Hajek(P) algebra (BL algebra).

Remark 6.9
We have used the short names "α", ..., "$\alpha\beta\gamma\delta$" = Hájek(P) algebras instead of the long names "reversed left-α(P)",..., "reversed left-$\alpha\beta\gamma\delta$(P)" algebras. We remind you that we have decided to work with algebras from the world of "$\rightarrow, 1$" (see the first column from the table from [43], Figure 1).
When working with algebras from the world of "$\odot, \rightarrow, 1$" (i.e. in the 3^{rd} column from the table from [43], Figure 1), the corresponding names will be: "X-α(RP)", ..., "X-$\alpha\beta\gamma\delta$(RP)"=BL algebras, respectively.

Remark 6.10 Recall the important result ([43], Theorem 3.8) that a linearly ordered BCK(P) lattice (BCK(P) chain) satisfies the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$, i.e. it is an $\alpha\beta\gamma$ algebra. Consequently, only $\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$ = Hájek (BL) algebras can be linearly ordered, all the others are not linearly ordered. Consequently, note that:
- the four conditions are divided into two groups: the three conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ on one side and the condition (C_X) on the other side;
- since most of the above defined algebras are not linearly ordered, it is important that we know to make the ordinal sum between two BCK(P) lattices, not necessarily linearly ordered.

We add now the conditions (WNM) and (DN) to α, β, ..., $\alpha\beta\gamma\delta$ algebras. Thus, we give the following definitions.

Definition 6.11
(1) We shall name ${}_{(WNM)}\alpha$, ${}_{(WNM)}\beta$, ..., ${}_{(WNM)}\alpha\beta\gamma\delta$ *algebras* those α, β, ..., $\alpha\beta\gamma\delta$ algebras, respectively, which satisfy the condition (WNM).
(2) We shall name $\alpha_{(DN)}$, $\beta_{(DN)}$, ..., $\alpha\beta\gamma\delta_{(DN)}$ *algebras* those α, β, ..., $\alpha\beta\gamma\delta$ algebras, respectively, which satisfy the condition (DN).
(3) We shall name ${}_{(WNM)}\alpha_{(DN)}$, ${}_{(WNM)}\beta_{(DN)}$, ..., ${}_{(WNM)}\alpha\beta\gamma\delta_{(DN)}$ *algebras* those α, β, ..., $\alpha\beta\gamma\delta$ algebras, respectively, which satisfy both conditions (WNM) and (DN).

We do not know exactly how the condition (WNM) reacts with the four conditions $(C_\to)$, $(C_\vee)$, $(C_\wedge)$, (C_X), but we know how condition (DN) reacts. Recall for this the following result from [43]:

Proposition 6.12
Let $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ be a BCK(P) lattice. Then,
$(C_\wedge)$ + *(DN)* $\Longleftrightarrow$ $(C_\vee)$ + *(DN)* $\Longleftrightarrow$ $(C_\wedge)$ + $(C_\vee)$ + *(DN)*.

Remark 6.13 In the group of the three conditions $(C_\to)$, $(C_\vee)$, $(C_\wedge)$ the last two, $(C_\vee)$ and $(C_\wedge)$, are very special, since in a BCK(P) lattice with condition (DN) they are equivalent (dual). Consequently, by Proposition 6.12,
- the $\beta_{(DN)}$ algebras, the $\gamma_{(DN)}$ algebras and the $\beta\gamma_{(DN)}$ algebras coincide;
- the $\alpha\beta_{(DN)}$ algebras, the $\alpha\gamma_{(DN)}$ algebras and the $\alpha\beta\gamma_{(DN)}$ algebras coincide;
- the $\beta\delta_{(DN)}$ algebras, the $\gamma\delta_{(DN)}$ algebras and the $\beta\gamma\delta_{(DN)}$ coincide;
- the $\alpha\beta\delta_{(DN)}$ algebras, the $\alpha\gamma\delta_{(DN)}$ algebras and the $\alpha\beta\gamma\delta_{(DN)}$ algebras coincide; they are equivalent definitions of Wajsberg algebra (MV algebra).

We write:
$\beta_{(DN)} = \gamma_{(DN)} = \beta\gamma_{(DN)}$,
$\alpha\beta_{(DN)} = \alpha\gamma_{(DN)} = \alpha\beta\gamma_{(DN)}$,
$\beta\delta_{(DN)} = \gamma\delta_{(DN)} = \beta\gamma\delta_{(DN)}$,
$\alpha\beta\delta_{(DN)} = \alpha\gamma\delta_{(DN)} = \alpha\beta\gamma\delta_{(DN)} = \mathbf{W}$ **(MV)**.

Note that $\alpha\beta_{(DN)}$ algebras, i.e. $\alpha\beta\gamma_{(DN)}$ algebras, are already studied in the literature under the names "IMTL algebras" (Involutive Monoidal t-norm based Logic), introduced in 2001 by Esteva and Godo [18] or "weak-R_0" algebras, introduced in 1997 by G.J. Wang [73]; note also that Pei [63] proved that IMTL and weak-R$_0$ algebras coincide (are categorically equivalent).

Recall [18] that a particular case of IMTL algebras are the "NM algebras", i.e those IMTL algebras satisfying the condition (WNM) (or those WNM algebras satisfying the condition (DN)).

Recall also [63] that NM and R_0 algebras are categorically equivalent and also IMTL and weak R_0 are categorically equivalent.

Finally, recall that:
${}_{(WNM)}$**MV**= **MV** + (WNM), ${}_{(WNM)}$**W** = **W** + (WNM).

Hence, we have:
(1) **NM** = **IMTL** + (WNM)= **WNM** + (DN)=**MTL** + (DN) + (WNM),

(1') $\mathbf{R_0}$ = **weak-R$_0$** + (R6) $\equiv \alpha\beta$ +(DN) + (WNM) = $\alpha\beta\gamma$ + (DN) + (WNM) = $\alpha\beta\gamma_{(DN)}$ + (WNM)= ${}_{(WNM)}\alpha\beta\gamma$ + (DN).

(2) **W** $\equiv$ **Ha(P)** + (DN)=[$\alpha\beta\gamma$ + (C_X)] + (DN)= [$\alpha\beta\gamma$ + (DN)] + (C_X)=[$\alpha\beta$ + (DN)] + (C_X) $\equiv$ **weak-R$_0$** + (C_X).

(2') **MV** $\cong$ **BL** + (DN)=[$X-\alpha\beta\gamma$(RP) + (C_X)] + (DN)= [X-$\alpha\beta\gamma$(RP) + (DN)] + (C_X)=[X-$\alpha\beta$(RP) + (DN)] + (C_X) = [**MTL** + (DN)] + (C_X)= **IMTL** + (C_X).

(3) ${}_{(WNM)}\mathbf{W}$ = $\mathbf{W}$ + (WNM) $\overset{(2)}{\equiv}$ [[$\alpha\beta\gamma$ + (DN)] + (C_X)] + (WNM)= [$\alpha\beta\gamma$ + (DN) + (WNM)] + (C_X) $={}_{(WNM)}\alpha\beta\gamma_{(DN)}$ + $(C_X) \equiv \mathbf{R}_0$ + (C_X).

(3') ${}_{(WNM)}$**MV**=**MV** + (WNM) $\overset{(2')}{\cong}$ [**IMTL** + (C_X)] + (WNM)= [**IMTL** + (WNM)] + (C_X)= **NM** + (C_X).

Definition 6.14

We shall say that a BCK(P) lattice (residuated lattice) is *proper* if it does not verify the four conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$, (C_X), the condition (WNM) and the condition (DN). Their class is denoted by **BCK(P)-L** (**R-L**, respectively).

We shall say that a ${}_{(WNM)}$BCK(P) lattice is *proper* if it does not verify the four conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$, (C_X) and the condition (DN). Their class is denoted by ${}_{(WNM)}$**BCK(P)-L** (${}_{(WNM)}$**R-L**, respectively).

We shall say that a BCK(P)$_{(DN)}$ lattice is *proper* if it does not verify the four conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$, (C_X) and the condition (WNM). Their class is denoted by **BCK(P)-L**$_{(DN)}$ (**R-L**$_{(DN)}$, respectively).

We shall say that a ${}_{(WNM)}$BCK(P)$_{(DN)}$ lattice is *proper* if it does not verify the four conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$, (C_X). Their class is denoted by ${}_{(WNM)}$**BCK(P)-L**$_{(DN)}$ (${}_{(WNM)}$**R-L**$_{(DN)}$, respectively).

We shall say that the algebras α, β, ... , $\alpha\beta\gamma\delta$ are *proper* if they satisfy only the condition (conditions) from their definition. We define similarly a proper ${}_{(WNM)}\alpha$ algebra, a proper $\alpha_{(DN)}$ algebra and a proper ${}_{(WNM)}\alpha_{(DN)}$ algebra, etc.

Note that **BCK(P)-L**$_{(DN)}$ (i.e. BCK(P) lattices with condition (DN)) are categorically equivalent with residuated lattices with condition (DN), also named "Girard monoids" [33].

Consequently, we have:

- in Figure 9, the plane ("map") P, of the hierarchies of the BCK(P) lattices which are generalizations of Hajek(P) algebras (BL algebras);
- in Figure 10, the plane ${}_{(WNM)}$P, of the hierarchies of the the ${}_{(WNM)}$BCK(P) lattices which are generalizations of ${}_{(WNM)}$Hajek(P) algebras (${}_{(WNM)}$BL algebras);
- in Figure 11, the plane P$_{(DN)}$, of the hierarchies of the BCK(P)$_{(DN)}$ lattices which are generalisations of Ha(P)$_{(DN)}$ algebras (i.e. of Wajsberg algebras);
- in Figure 12, the plane ${}_{(WNM)}$P$_{(DN)}$, of the hierarchies of the ${}_{(WNM)}$BCK(P)$_{(DN)}$ lattices which are generalisations of ${}_{(WNM)}$Hájek(P)$_{(DN)}$ algebras (i.e. of ${}_{(WNM)}$Wajsberg algebras).

In Figures 9, 10, 11, 12, the sign "=" means duplicate names, the sign " $\equiv$" means equivalent definitions, while the sign "$\cong$" means that the corresponding categories are equivalent.

In those four Figures also, those classes for which we didn't found any examples, without or with condition (WNM), are marked by the sign "??" and those for which we didn't found examples are marked by the sign "?"; thus, it remains an open problem to find examples for that classes.

We give in Figure 13 the spacial vue of the four planes P (see Figure 9), ${}_{(WNM)}P$ (see Figure 10), $P_{(DN)}$ (see Figure 11) and ${}_{(WNM)}P_{(DN)}$ (see Figure 12).

By cutting with vertical planes, we get the following hierarchies, for examples:

In Part III and Part IV we shall give examples of algebras from Figure **??** and in Part V we shall give examples of algebras from the other previous Figures.

Remarks 6.15

(1) The following pairs of BCK(P) lattices satisfying the condition (DN) seem to be very important; the algebras of each pair seem to be incomparable (under inclusion):

(1.1) ${}_{(WNM)}\alpha_{(DN)}$ algebras and $\alpha\delta_{(DN)}$ (??) algebras from Figure 19; we have only examples of ${}_{(WNM)}\alpha_{(DN)}$ algebras (see Part V);

(1.2) ${}_{(WNM)}\beta\gamma_{(DN)}$ (?) algebras and $\beta\gamma\delta_{(DN)}$ (??) algebras from Figures 23 and 24; it remains an open problem to find examples;

(1.3) $\mathrm{R}_0 \equiv {}_{(WNM)}\alpha\beta\gamma_{(DN)}$ (NM) algebras and $\alpha\beta\gamma\delta_{(DN)} \equiv$ Wajsberg (MV) algebras from Figures 25 and 26 (another Figure appears in Part IV). We have examples (see Part III and Part IV) which prove that

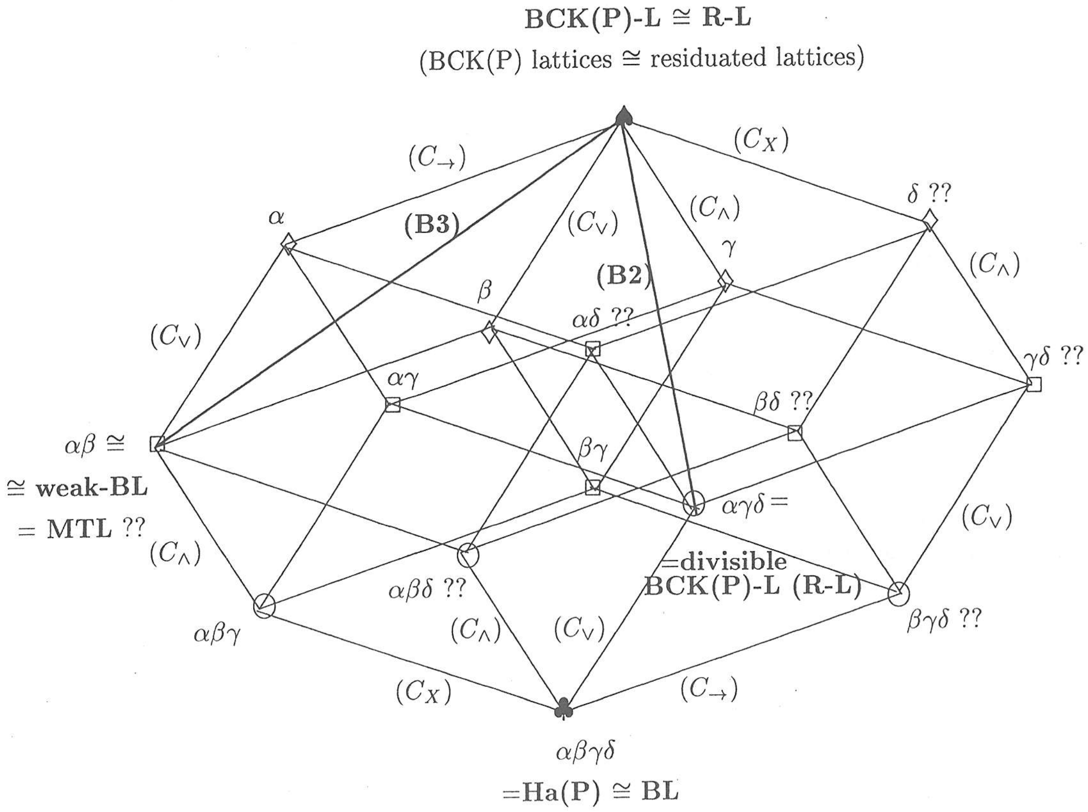

Figure 9: Plane P (Classes of BCK(P) lattices (residuated lattices), generalizations of Hájek(P) algebras (BL algebras))

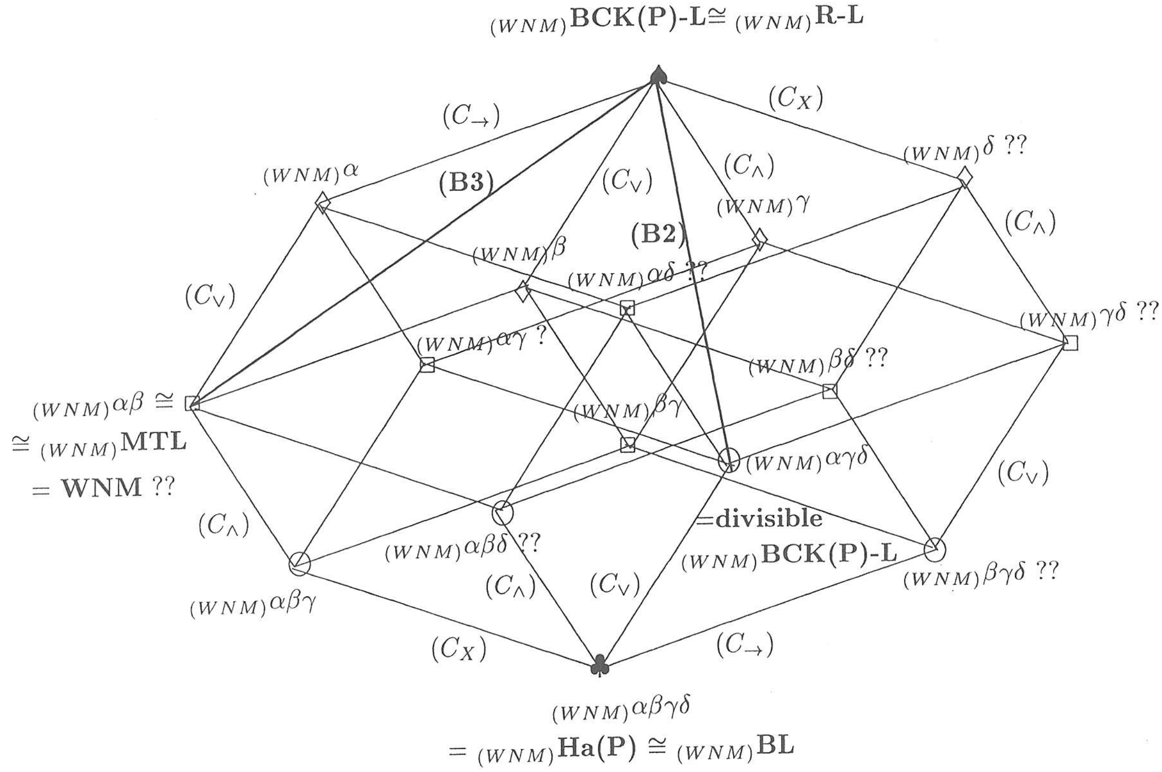

Figure 10: Plane $_{(WNM)}P$ (Classes of BCK(P)-L (R-L), generalizations of Ha(P) (BL), all with condition (WNM))

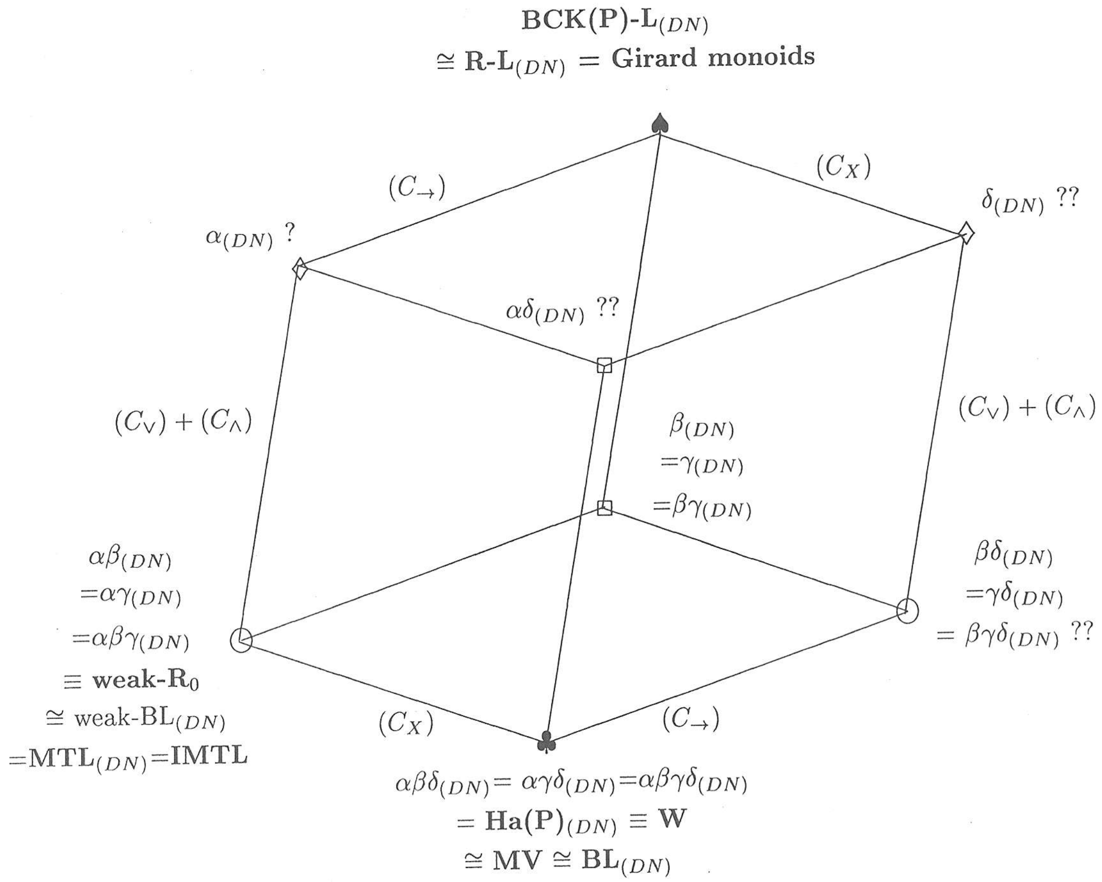

Figure 11: Plane P$_{(DN)}$ (Classes of BCK(P)$_{(DN)}$ lattices (residuated lattices with condition (DN)), generalizations of Wajsberg algebras (MV algebras))

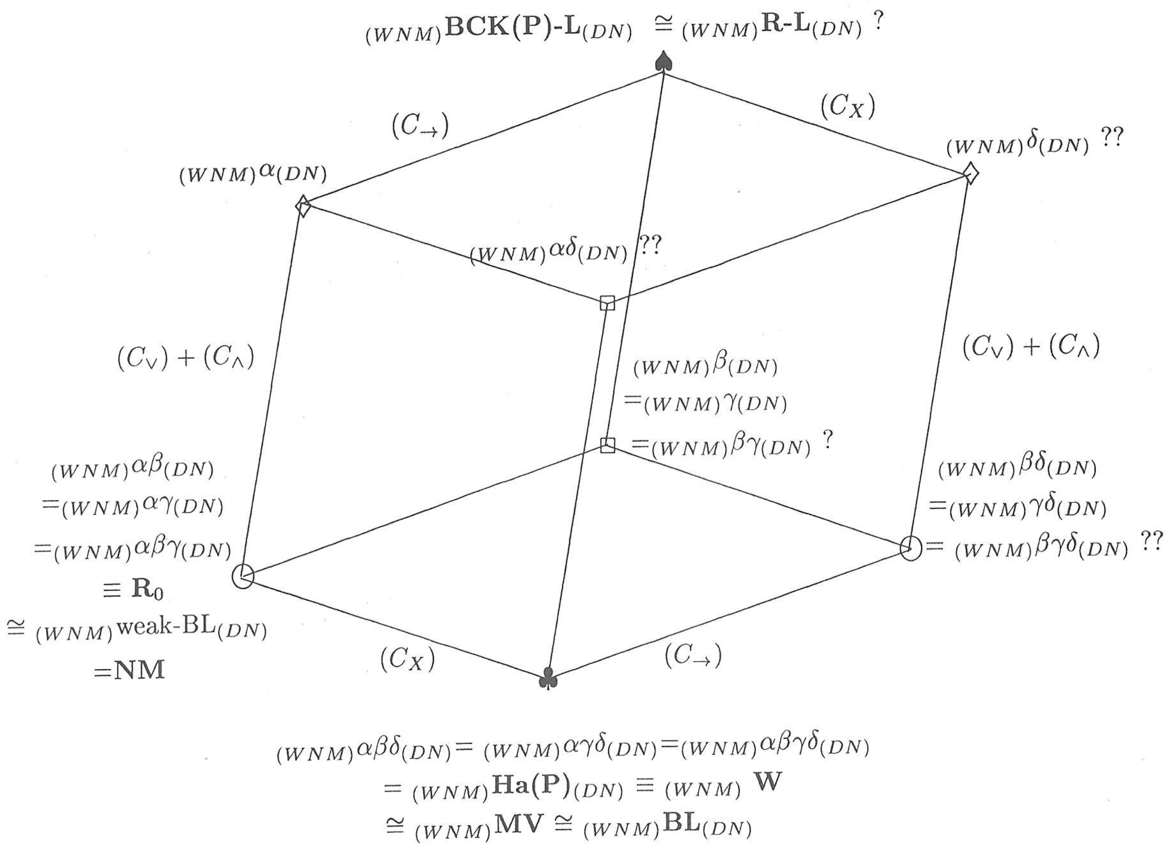

Figure 12: Plane $_{(WNM)}P_{(DN)}$ (Classes of $_{(WNM)}$BCK(P)-L$_{(DN)}$ (residuated lattices with conditions (WNM) and (DN)), generalizations of NR_0 algebras (Wajsberg algebras (MV algebras) with condition (WNM)))

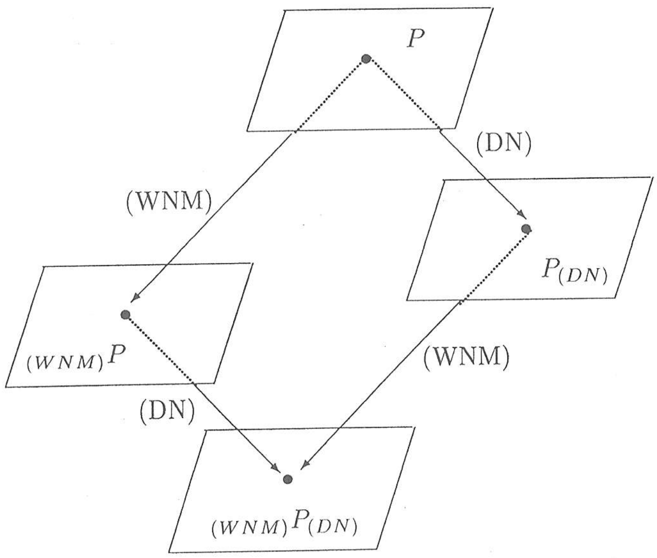

Figure 13: Spatial vue of the four planes P, $_{(WNM)}P$, $P_{(DN)}$, $_{(WNM)}P_{(DN)}$

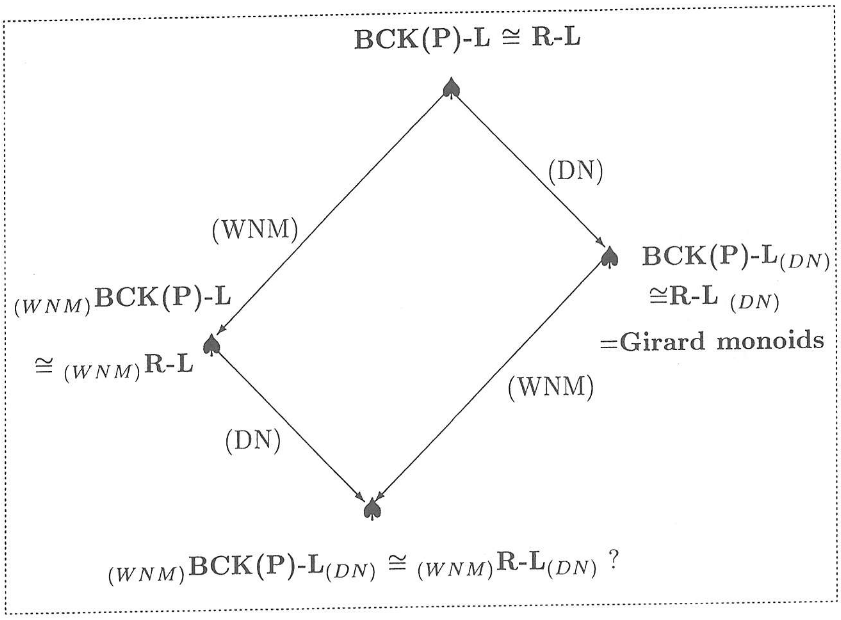

Figure 14: Vertical section through BCK(P) lattices (residuated lattices)

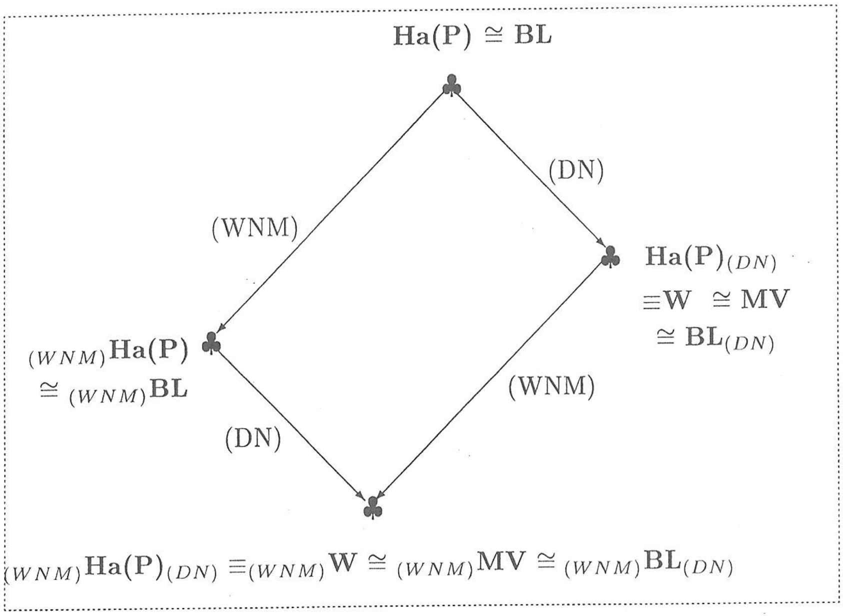

Figure 15: Vertical section through Ha(P) (BL) algebras

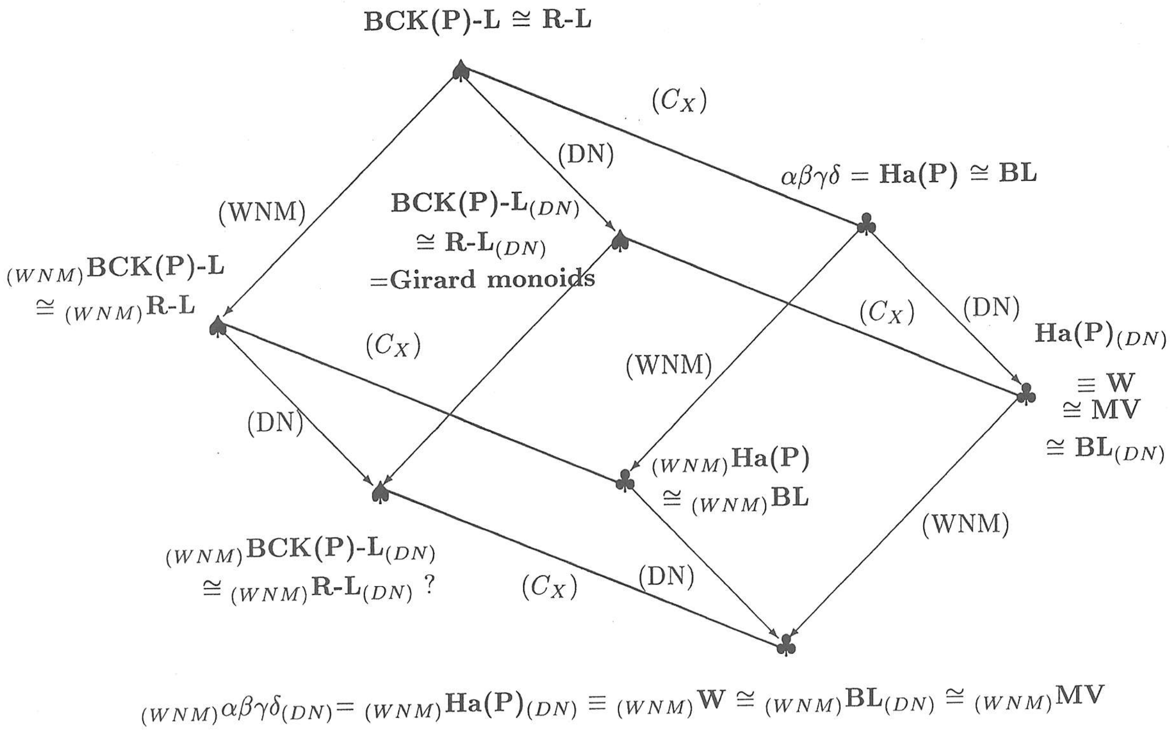

Figure 16: Vertical sections through residuated lattices and BL algebras

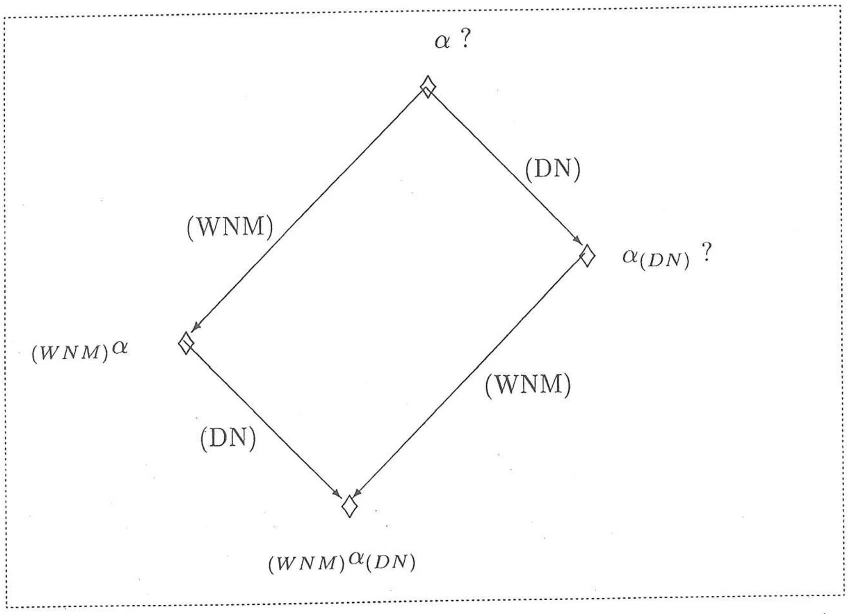

Figure 17: Vertical section through α algebras

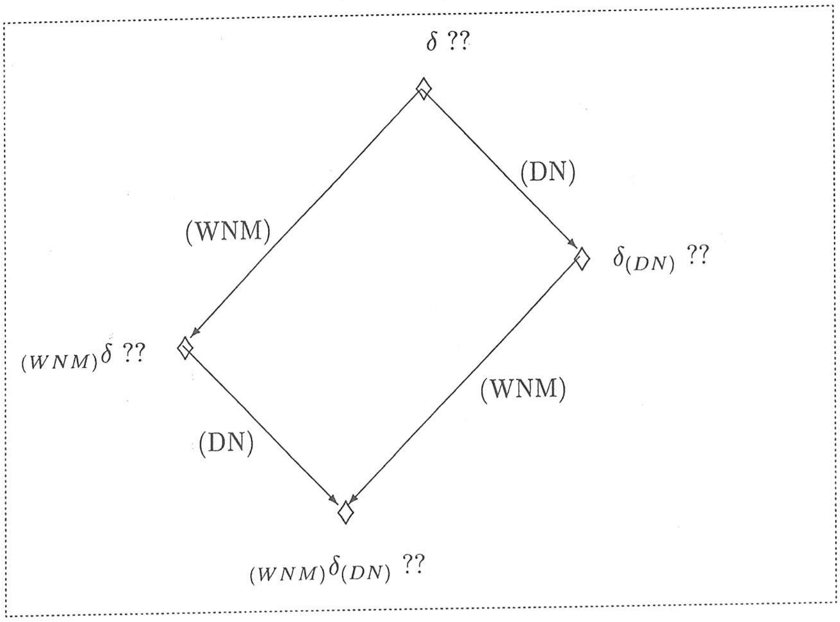

Figure 18: Vertical section through δ algebras

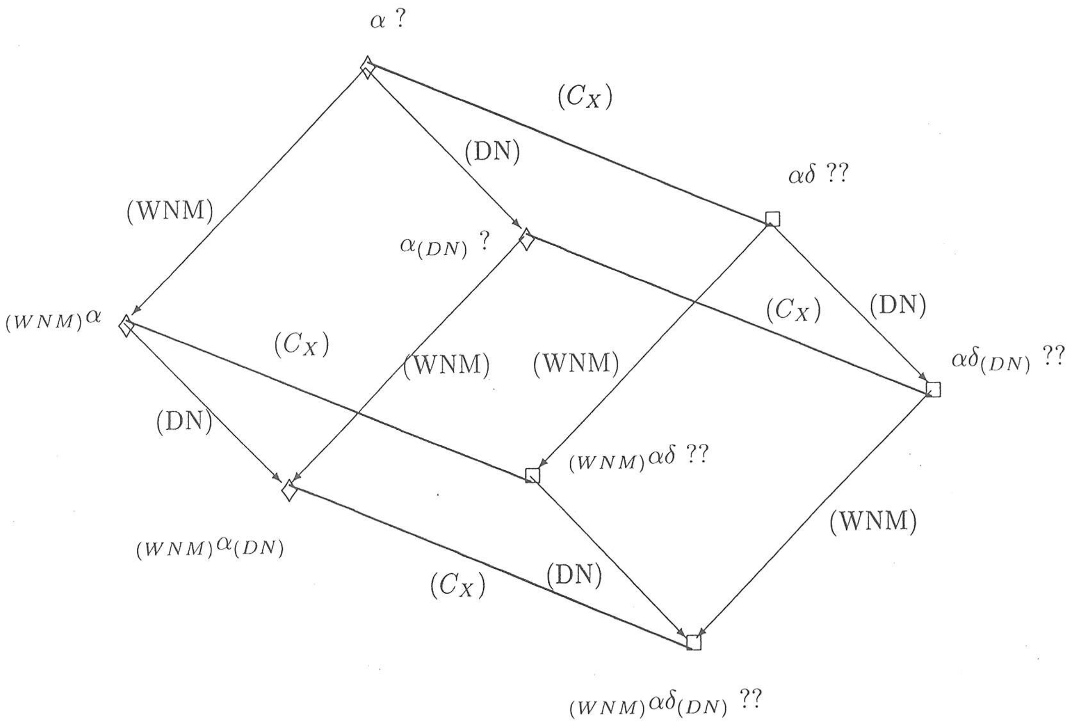

Figure 19: Vertical sections through α and $\alpha\delta$ algebras

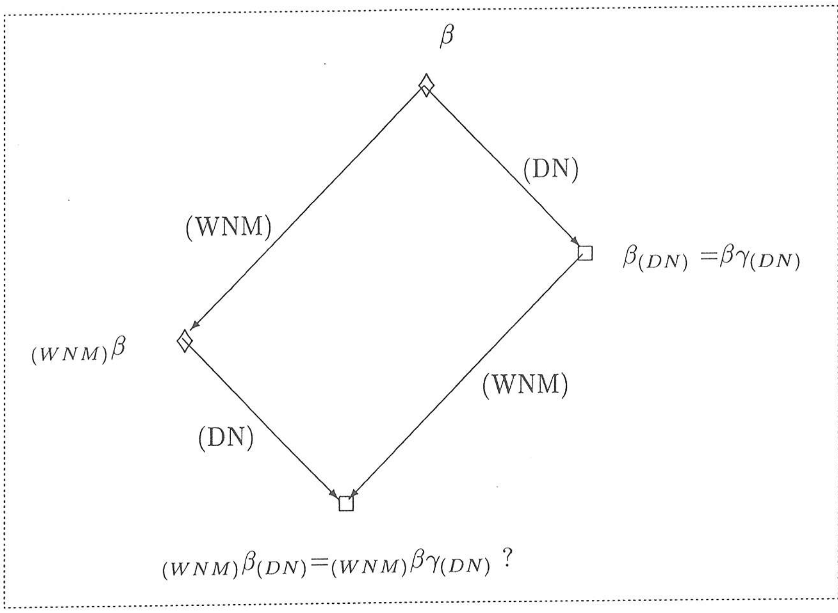

Figure 20: Vertical section through β algebras

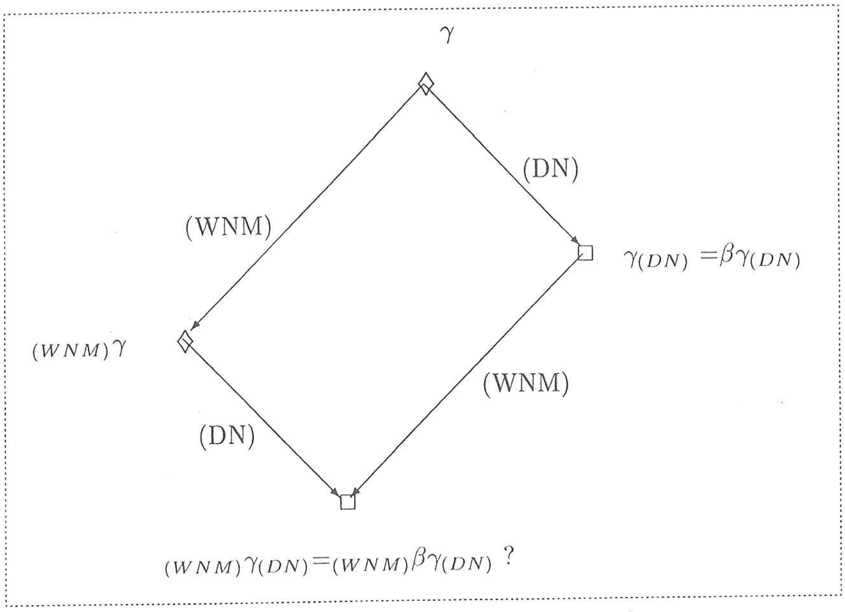

Figure 21: Vertical section through γ algebras

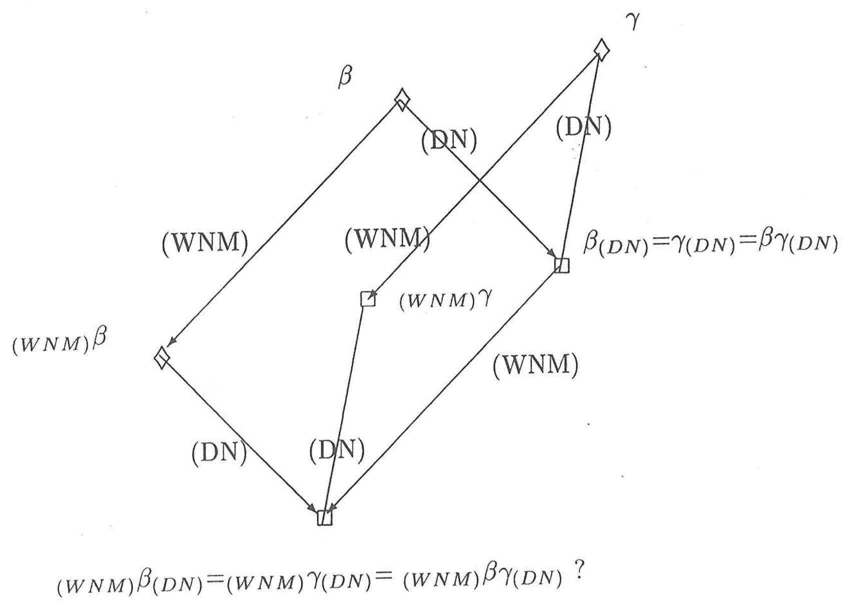

Figure 22: "Vertical" sections through β and γ algebras

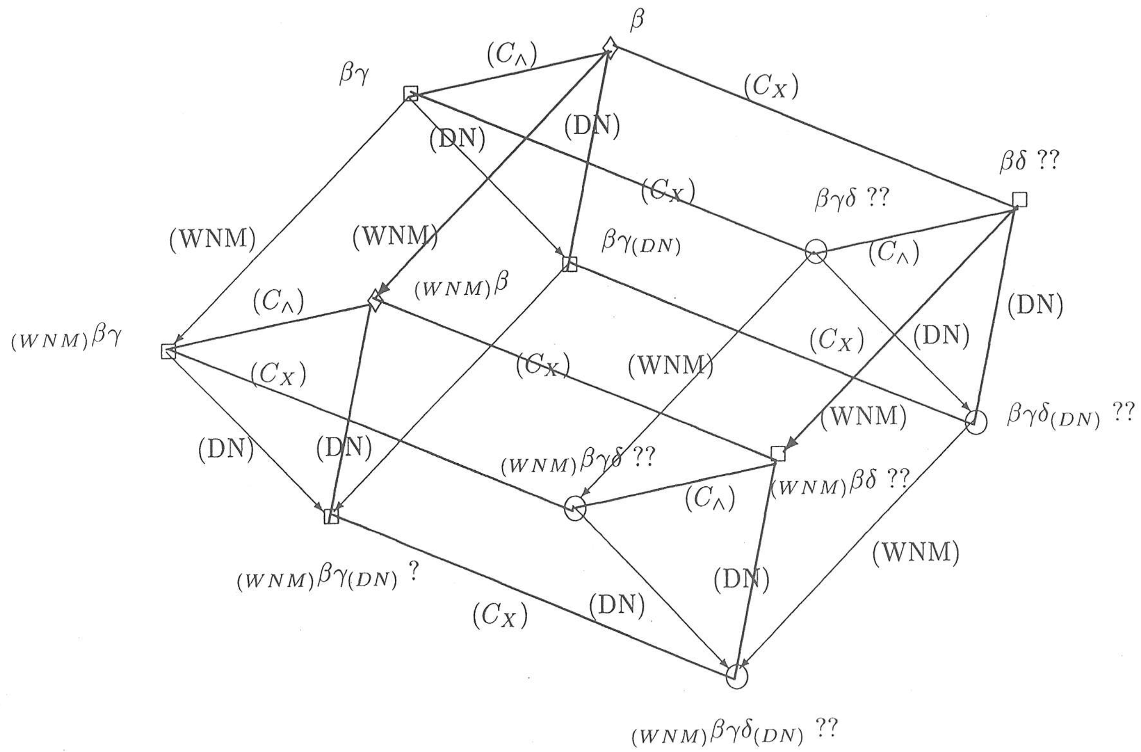

Figure 23: "Vertical" sections through β, $\beta\gamma$, $\beta\delta$ and $\beta\gamma\delta$ algebras

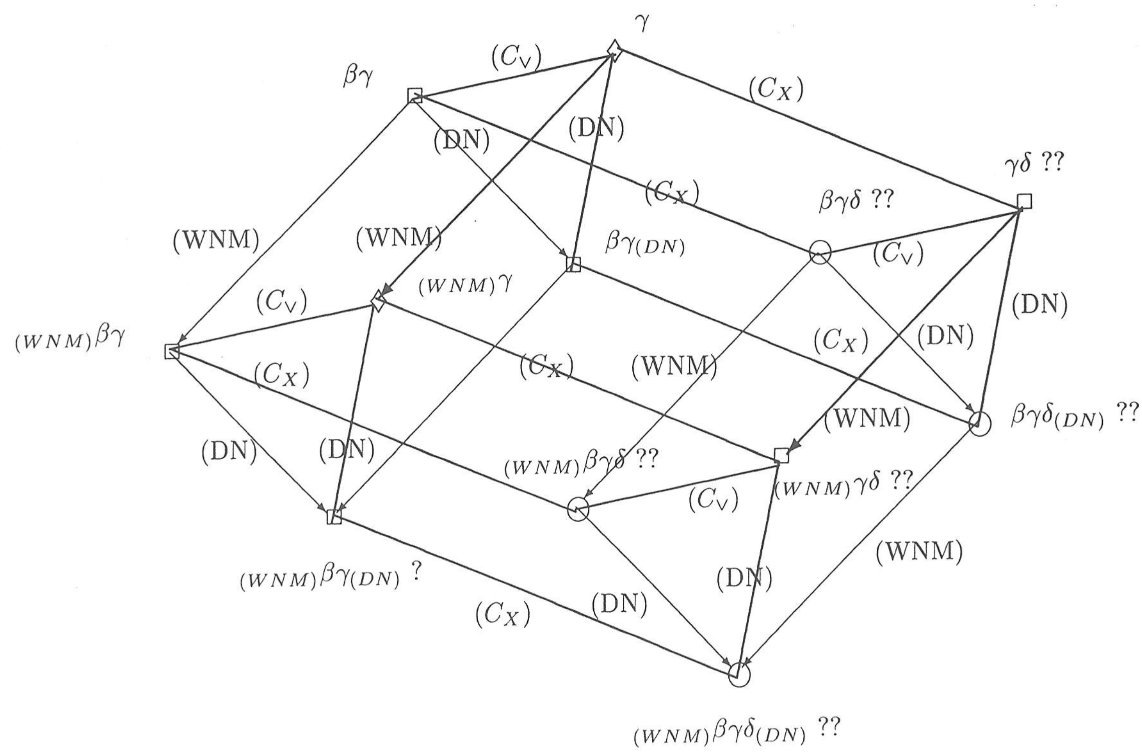

Figure 24: "Vertical" sections through γ, $\beta\gamma$, $\gamma\delta$ and $\beta\gamma\delta$ algebras

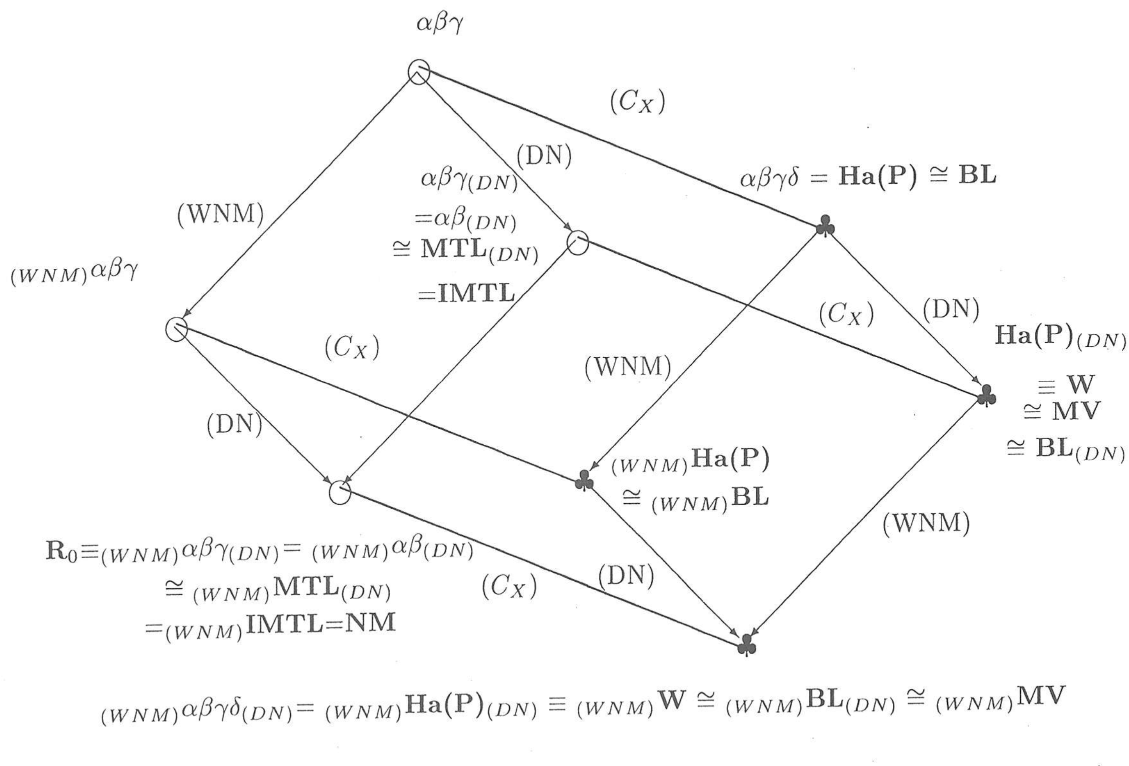

Figure 25: Vertical sections through $\alpha\beta\gamma$ and Hájek(P) (BL) algebras

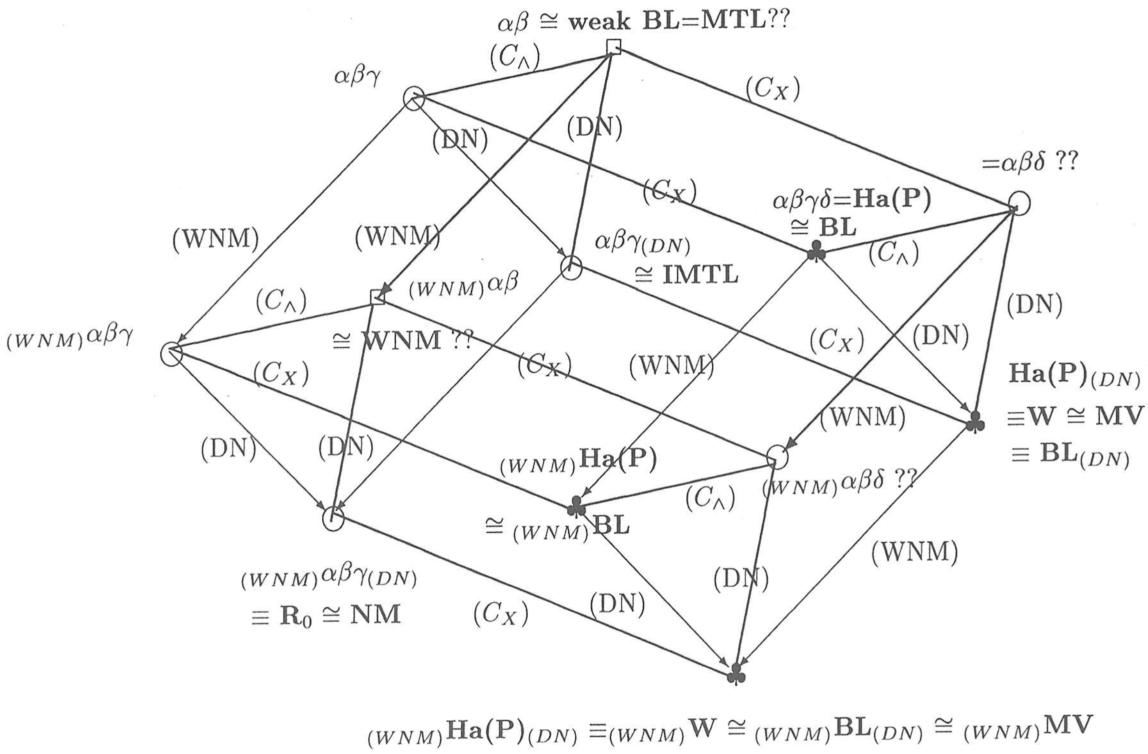

Figure 26: "Vertical" sections through $\alpha\beta$, $\alpha\beta\gamma$, $\alpha\beta\delta$ and $\alpha\beta\gamma\delta$ (BL) algebras

they are incomparable.

(2) Consequently, the following pairs of BCK(P) lattices not satisfying the condition (DN) seem to be very important; the algebras of each pair seem to be incomparable (under inclusion):
(2.1) $_{(WNM)}\alpha$ algebras and $\alpha\delta$ (??) algebras from Figure 19; we have only examples of $_{(WNM)}\alpha$ algebras (see Part V);
(2.2) $_{(WNM)}\beta\gamma$ algebras and $\beta\gamma\delta$ (??) algebras from Figures 23 and 24; it remains an open problem to find examples;
(2.3) $_{(WNM)}\alpha\beta\gamma$ algebras and $\alpha\beta\gamma\delta$ = Hájek (BL) algebras from Figures 25 and 26 (another Figure appears in Part IV). We have examples (see Part III and Part IV) which prove that they are incomparable. We shall call as "Roman algebras" the X-$_{(WNM)}\alpha\beta\gamma$(RP) algebras, i.e. we have:

Roman $\cong$ $_{(WNM)}\alpha\beta\gamma$ just as:

BL $\cong$ $\alpha\beta\gamma\delta$=**Ha**.

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