INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

CLASSES OF BCK ALGEBRAS – PART III

by

AFRODITA IORGULESCU

Preprint nr. 3/2004

BUCUREȘTI

CLASSES OF BCK ALGEBRAS – PART III

by

AFRODITA IORGULESCU[1]

Ianuarie, 2004

1 Academy of Economic Studies, Department of Computer Science, Piata Romana, No. 6 – R 70167, Oficiul Postal 22, Bucharest, Romania, e–mail: afrodita@inforec.ase.ro

Classes of BCK algebras - Part III

Afrodita Iorgulescu

Department of Computer Science, Academy of Economic Studies,
Piaţa Romană Nr.6 - R 70167, Oficiul Poştal 22, Bucharest, Romania
E-mail: afrodita@inforec.ase.ro

Dedicated to Grigore C. Moisil (1905-1973)
(10 January 2004)

Abstract

In this paper we study the BCK algebras and their particular classes: the BCK(P) (residuated) lattices, the Hájek(P) (BL) algebras and the Wajsberg (MV) algebras, we introduce new classes of BCK(P) lattices, we establish hierarchies and we give many examples. The paper has five parts.

In the first part, the most important part, we decompose the divisibility and the pre-linearity conditions from the definition of a BL algebra into four new conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). We study the additional conditions (WNM) (weak nilpotent minimum) and (DN) (double negation) on a BCK(P) lattice. We introduce the ordinal sum of two BCK(P) lattices and prove in what conditions we get BL algebras or other structures, more general, or more particular than BL algebras.

In part II, we give examples of some finite bounded BCK algebras. We introduce new generalizations of BL algebras, named α, β, γ, δ, $\alpha\beta$, $\ldots$, $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). By adding the conditions (WNM) and (DN) to these classes, we get more classes; among them, we get many generalizations of Wajsberg (MV) algebras and of R_0 (NM) algebras. The subclasses of $_{(WNM)}$ Wajsberg algebras ($_{(WNM)}$MV algebras) and of $_{(WNM)}$Hájek(P) algebras ($_{(WNM)}$BL algebras) are introduced. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I.

In part III, we give examples of finite MV and $_{(WNM)}$MV algebras, of Hájek(P) (i.e. BL) algebras and $_{(WNM)}$BL algebras and of $\alpha\gamma\delta$ (i.e. divisible BCK(P) lattices (divisible residuated lattices or divisible integral, residuated, commutative l-monoids)) and of divisible $_{(WNM)}$BCK(P) lattices.

In part IV, we stress the importance of $\alpha\beta\gamma$ algebras versus $\alpha\beta$ (i.e. MTL) algebras algebras and of R_0 (i.e. NM) algebras versus Wajsberg (i.e. MV) algebras and of $_{(WNM)}\alpha\beta\gamma$ algebras versus BL algebras and of $\alpha\gamma$ versus $\alpha\gamma\delta$ algebras. We give examples of finite IMTL algebras and of $_{(WNM)}$IMTL (i.e. NM) algebras), of $\alpha\beta\gamma$ algebras and of $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras and finally of $\alpha\gamma$ algebras.

In part V, we give other examples of finite BCK(P) lattices, finding examples for the others remaining an open problem. We make final remarks and formulate final open problems.

Keywords MV algebra, Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, divisible BCK(P) lattice, α, β, γ, δ, $\alpha\beta$, $\ldots$, $\alpha\beta\gamma\delta$ algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, $_{(WNM)}$MV, $_{(WNM)}$BL, $_{(WNM)}$ $\alpha\beta\gamma$, Roman algebra

Part III has seven sections.

In Section 7 we give examples of finite Wajsberg (MV) algebras, useful in the next sections.

In Section 8 we give examples of finite linearly ordered reversed left-Hájek(P) algebras (BL algebras) which are not Wajsberg (MV) algebras.

In Section 9 we give examples of finite non-linearly ordered reversed left-Hájek(P) algebras (BL) algebras which are not Wajsberg (MV) algebras.

In Section 10 we give examples of infinite proper BL algebras, obtained as ordinal sums of two product algebras.

In Section 11 we give examples of finite divisible reversed left-BCK(P) lattices (divisible residuated lattices).

In Section 12 we give an example of infinite proper divisible reversed left-BCK(P) lattice, as an ordinal sum of two product algebras.

In Section 13 we present two open problems.

In this section we shall give examples of algebras from the hierarchy from Figure 1, which is a part of the hierarchy from Figure 2.

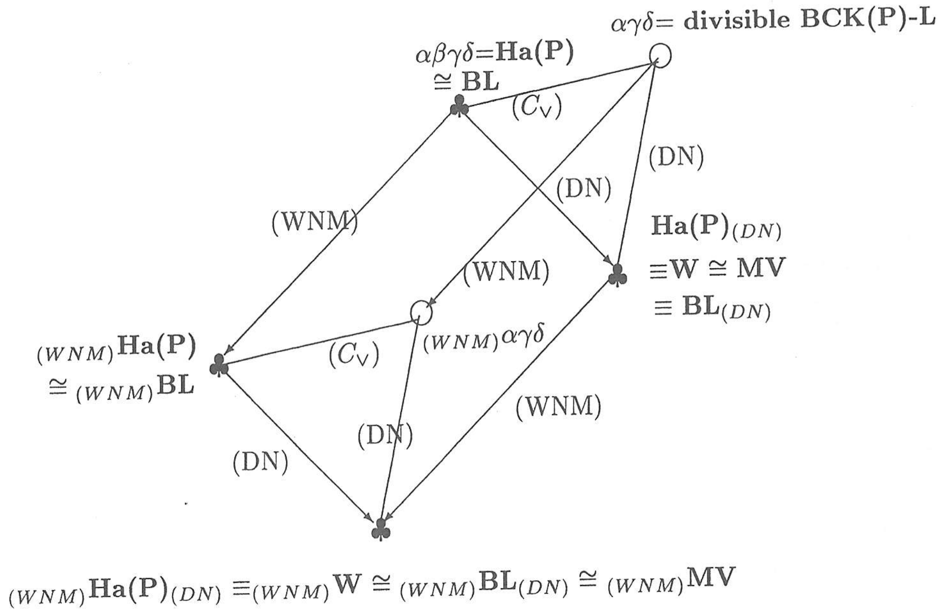

Figure 1: "Vertical" sections through $\alpha\gamma\delta$ and $\alpha\beta\gamma\delta$ (BL) algebras

7 Examples of finite Wajsberg (MV) and $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras

In the examples we shall indicate not only the primitive operation, $\to$, but the derived one, $\odot$, too. Recall that by [23], Proposition 2.32, any Wajsberg algebra satisfies the condition (P2).
We shall sometimes use only the shorter name, MV algebra, in the sequel.

7.1 Examples of linearly ordered Wajsberg (MV) and $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras

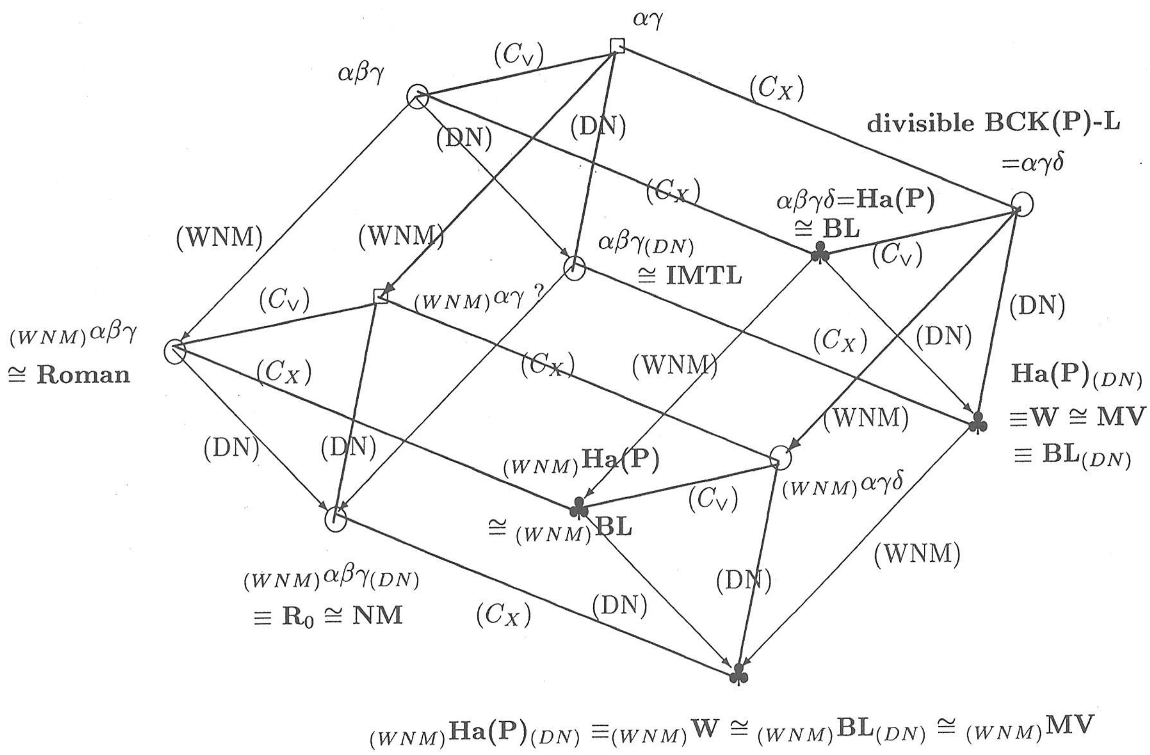

Figure 2: "Vertical" sections through $\alpha\gamma$, $\alpha\beta\gamma$, $\alpha\gamma\delta$ (divisible residuated lattices) and $\alpha\beta\gamma\delta$ (BL) algebras

Recall that the linearly ordered set $L_{n+1} = \{0, 1, 2, \ldots, n\}$, $(n \geq 1)$, organized as a lattice with $\wedge = \min$ and $\vee = \max$, can be organized:

- as (right-)MV algebra: $(L_{n+1}, \oplus, ^{-R}, 0)$,

with
$x \oplus y = \min(n, x + y)$,
$x^{-R} = n - x$,
$x \rightarrow_R y = (x \oplus y^{-R})^{-R}$;

- as left-MV algebra: $\mathcal{L}_{n+1} = (L_{n+1}, \odot, ^-, n)$,

with
$x^- = n - x$,
$x \odot y = (x^- \oplus y^-)^- = n - (x^- \oplus y^-) = n - \min(n, x^- \oplus y^-) = n - \min(n, (n - x) + (n - y)) = n - \min(n, 2n - x - y) = 0 - \min(0, n - x - y) = \max(0, x + y - n)$,
$x \rightarrow y = x \rightarrow_L y = (x \odot y^-)^- = \min(n, y - x + n)$;

- as Wajsberg algebra (or, equivalently, as bounded, ($\vee$-)commutative BCK algebra): $\mathcal{L}_{n+1} = (L_{n+1}, \rightarrow, ^-, n)$,

with
$x \rightarrow y = \min\{n, y - x + n\}$,
$x^- = x \rightarrow 0 = \min\{n, n - x\} = n - x, \quad (0 = n^-)$ and
$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = (x \rightarrow y^-)^- = \max\{0, x + y - n\}$.

Note also that the algebra $(L_{n+1}, \vee = \max, \wedge = \min, \odot, \rightarrow, 0, n)$ is a Hajek(P) (BL) algebra with condition (DN).

Hence, for $n = 1, 2, 3, 4, 5$, we have the linearly ordered Wajsberg (left-MV) algebras $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$, $\mathcal{L}_6$, whose tables are the following:

$\mathcal{L}_2$

$\rightarrow$	0	1
0	1	1
1	0	1

$\odot$	0	1
0	0	0
1	0	1

$\mathcal{L}_3$

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

$\odot$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\mathcal{L}_4$

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	2	3	3
3	0	1	2	3

$\odot$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	1	2
3	0	1	2	3

$\mathcal{L}_5$

$\rightarrow$	0	1	2	3	4
0	4	4	4	4	4
1	3	4	4	4	4
2	2	3	4	4	4
3	1	2	3	4	4
4	0	1	2	3	4

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	2
3	0	0	1	2	3
4	0	1	2	3	4

$\mathcal{L}_6$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	4	5	5	5	5
3	2	3	4	5	5	5
4	1	2	3	4	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	2
3	0	0	0	1	2	3
4	0	0	1	2	3	4
5	0	1	2	3	4	5

Remarks 7.1

(1) For $n = 1, 2$, the Wajsberg (MV) algebras $\mathcal{L}_2$ and $\mathcal{L}_3$ verify the condition (WNM), hence they are examples of $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras.

(2) For $n = 3$, the Wajsberg (MV) algebra $\mathcal{L}_4$ does not verify the condition (WNM) for 2, for example:

$$(2 \odot 2)^- \vee [(2 \wedge 2) \rightarrow (2 \odot 2)] = 1^- \vee [2 \rightarrow 1] = 2 \vee 2 = 2 \neq 3.$$

Hence, $\mathcal{L}_4$ is a proper Wajsberg (MV) algebra.

(3) For $n \geq 4$, the Wajsberg (MV) algebra $\mathcal{L}_{n+1}$ does not verify the condition (WNM). Indeed, the condition (WNM) fails for the elements $n-2$, $n-1$:

$$[(n-2)\odot(n-1)]^- \vee [(n-2)\wedge(n-1) \rightarrow (n-2)\odot(n-1)] = (n-3)^- \vee [(n-2) \rightarrow (n-3)] = 3 \vee (n-1) = n-1 \neq n,$$

since:
$(n-2) \odot (n-1) = \max(0, (n-2) + (n-1) - n) = \max(0, n-3) = n-3$, since $(n-3) \geq (4-3) = 1$,
$(n-3)^- = n - (n-3) = 3$,
$(n-2) \rightarrow (n-3) = \min(n, (n-3) - (n-2) + n) = \min(n, n-1) = n-1$ and
$(n-1) \geq (4-1) = 3$.

Remarks 7.2

(i) Recall that the algebra $\mathcal{B}_2 = (L_2 = \{0,1\}, \vee = \max, \wedge = \min, ^-, 0, 1)$ is the standard (canonical) Boolean algebra, where $x^- = x \to 0 = 1 - x$, for all $x \in L_2$. Remark that corresponding BL algebra, $(L_2, \vee, \wedge, \odot, \to, 0, 1)$, is not only a Wajsberg (MV) algebra, but it is also a Gödel and a Product algebra (since every Boolean algebra is a Product algebra) in the same time.

(ii) Remark also that for $n > 1$, the corresponding BL algebra $(L_{n+1}, \vee, \wedge, \odot, \to, 0, 1)$ is not a Gödel algebra, since there is $1 \in L_{n+1}$ such that $1 \odot 1 = \max(0, 1 + 1 - n) = 0 \neq 1$ and it is not a Product algebra, since condition (P1) is not satisfied (there is $x = 1 \in L_{n+1}$ such that $1 \wedge 1^- = 1 \wedge (n-1) = \min(1, n-1) = 1 \neq 0$).

(iii) For $n = 3$, there is another one (and only one) structure of BCK(P) lattice with condition (DN) on the chain L_4, which satisfies the conditions $(C_\to)$, $(C_\vee)$, $(C_\wedge)$. It will be presented in the third part of this paper (as example of linearly ordered $\alpha\beta\gamma$+(DN) algebra (IMTL algebra)).

(iv) For $n = 4$ there are another two (and only two) structures of BCK(P) lattices with condition (DN) on the chain L_5, which satisfy the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$. They will be presented in the third part of this paper (as examples of linearly ordered $\alpha\beta\gamma$+(DN) algebras (IMTL algebras)).

(iv) For $n = 5$ there are another six (and only six) structures of BCK(P) lattices with condition (DN) on the chain L_6, which satisfy the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$. They will be presented in the third part of this paper (as examples of linearly ordered $\alpha\beta\gamma$+(DN) algebras (IMTL algebras)).

Remark 7.3

Remark that $\mathcal{L}_2$ satisfies the condition (P1), while $\mathcal{L}_{n+1}$, for $n > 1$, do not satisfy (P1).

7.2 Examples of non-linearly ordered Wajsberg (MV) and $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras

We give five examples.

Example 1

The set

$$L_{2\times 2} = \{0, a, b, 1\} \cong L_2 \times L_2 = \{0,1\} \times \{0,1\} = \{(0,0), (0,1), (1,0), (1,1)\},$$

organized as a lattice as in Figure 3 and as a BCK(P) algebra with the operation $\to$ and

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$$

as in the following tables, is a non-linearly ordered MV algebra, denoted $\mathcal{L}_{2\times 2}$.

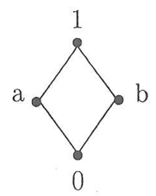

Figure 3: The non-linearly ordered MV (Boolean) algebra $\mathcal{L}_{2\times 2}$

$\mathcal{L}_{2\times 2}$

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Remark that $\mathcal{L}_{2\times 2}$ is a Boolean algebra. It satisfies the condition (WNM), hence is a proper $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebra.

Example 2

The set

$$L_{3\times 2} = \{0, a, b, c, d, 1\} \cong L_3 \times L_2 = \{0, 1, 2\} \times \{0, 1\} = \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\},$$

organized as a lattice as in Figure 4 and as a BCK(P) algebra with the operation $\rightarrow$ and

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = (x \rightarrow y^{-})^{-}$$

as in the following tables, is a non-linearly ordered MV algebra, denoted by $\mathcal{L}_{3\times 2}$.

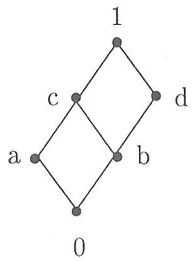

Figure 4: The non-linearly ordered MV algebra $\mathcal{L}_{3\times 2}$ and BL algebra $\mathcal{H}_{(2,2)\times 2}$

$\mathcal{L}_{3\times 2}$

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

It verifies the condition (WNM), hence is a proper $_{(WNM)}$Wajsberg algebra.

Example 3

The set

$$L_{2\times 3} = \{0, a, b, c, d, 1\} \cong L_2 \times L_3 = \{0, 1\} \times \{0, 1, 2\} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\},$$

organized as a lattice as in Figure 5 and as a BCK(P) algebra with the operation $\rightarrow$ and

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = (x \rightarrow y^{-})^{-}$$

as in the following tables, is a non-linearly ordered MV algebra, denoted by $\mathcal{L}_{2\times 3}$.

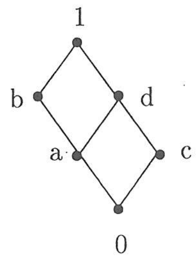

Figure 5: The non-linearly ordered MV algebra $\mathcal{L}_{2\times 3}$ and BL algebra $\mathcal{H}_{2\times(2,2)}$

$\mathcal{L}_{2\times 3}$

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

Note that $\mathcal{L}_{3\times 2}$ and $\mathcal{L}_{2\times 3}$ are isomorphic.

Example 4

The set

$$L_{3\times 3} = \{0, a, b, c, d, e, f, g, 1\} \cong L_3 \times L_3 = \{0, 1, 2\} \times \{0, 1, 2\} =$$
$$= \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\},$$

organized as a lattice as in Figure 6 and as a BCK(P) algebra with the implication $\rightarrow$ and

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = (x \rightarrow y^-)^-$$

as in the following tables, is a non-linearly ordered MV algebra, denoted by $\mathcal{L}_{3\times 3}$.

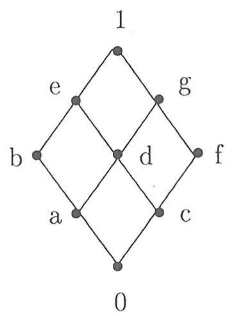

Figure 6: The non-linearly ordered MV algebra $\mathcal{L}_{3\times 3}$ and BL algebra $\mathcal{H}_{(2,2)\times(2,2)}$

$\mathcal{L}_{3\times 3}$

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

It verifies the condition (WNM), hence is a proper $_{(WNM)}$Wajsberg algebra.

Example 5
The algebra $\mathcal{L}_{4\times 2} \cong \mathcal{L}_4 \times \mathcal{L}_2$, with $A = \{0, a, b, c, d, e, f, 1\}$, does not verify the condition (WNM):

$$(e \odot e)^- \vee [(e \wedge e) \to (e \odot e)] = d \vee e = e \neq 1.$$

Hence, it is a non-linearly ordered proper Wajsberg (MV) algebra.

Remarks 7.4
(i) Recall that the algebra $\mathcal{B}_{2\times 2} = (L_{2\times 2} = \{0, a, b, 1\}, \wedge = \min, \vee = \max, ^-, 0, 1)$ is a Boolean algebra, where $x^- = x \to 0$, for all $x \in L_{2\times 2}$. Remark that corresponding BL algebra, $(L_{2\times 2}, \wedge, \vee, \odot, \to, 0, 1)$, is not only a Wajsberg (MV) algebra, but it is also a Gödel and a Product algebra (since every Boolean algebra is a Product algebra) in the same time.
(ii) Remark also that for $n > 1$ or $m > 1$, the corresponding BL algebra

$$(L_{(n+1)\times(m+1)}, \wedge, \vee, \odot, \to, 0, 1) \cong (L_{n+1} \times L_{m+1}, \wedge, \vee, \odot, \to, (0,0), (n,m)),$$

where $L_{n+1} \times L_{m+1} = \{0, 1, \ldots, n\} \times \{0, 1, \ldots, m\} =$
$= \{(0,0), \ldots, (0,m), \quad (1,0), \ldots, (1,m), \quad \ldots, \ldots, \ldots \quad (n,0), \ldots, (n,m)\}$,
is only a Wajsberg (MV) algebra; it is not a Gödel algebra, since, if for instance $n > 1$, then there is $(1,0) \in L_{n+1} \times L_{m+1}$, such that

$$(1,0) \odot (1,0) = (\max(0, 1+1-n), \max(0, 0+0-m) = (0,0) \neq (1,0);$$

it is not a Product algebra, since it does not satisfy the condition (P1): if for instance $n > 1$, there is $(1,0) \in L_{n+1} \times L_{m+1}$ such that

$$(1,0) \wedge (1,0)^- = (1,0) \wedge (n-1, m-0) = (\min(1, n-1), \min(0,m)) = (1,0) \neq (0,0).$$

Final remark for non-linearly ordered MV algebras
Remark that $\mathcal{L}_{2\times 2}$ satisfies the condition (P1), while the other three examples and in general, $\mathcal{L}_{(n+1)\times(m+1)}$, with $n > 1$ or $m > 1$, do not satisfy condition (P1).

8 Examples of finite, linearly ordered, proper Hájek(P) (BL) algebras and $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras

We shall sometimes use the shorter names, BL and MV algebras.
The examples are of one of the following forms:
8.1 linearly ordered MV $\bigoplus$ linearly ordered MV,
8.2 linearly ordered MV $\bigoplus$ linearly ordered BL or linearly ordered BL $\bigoplus$ linearly ordered MV,
8.3 linearly ordered BL $\bigoplus$ linearly ordered BL.

8.1 Examples of the form: linearly ordered MV $\oplus$ linearly ordered MV

Denote $\mathcal{H}_{m+1,n+1} = \mathcal{L}_{m+1} \bigoplus \mathcal{L}_{n+1}$, for $m, n \geq 1$.

8.1.1 Examples of the form: $\mathcal{H}_{2,n+1} = \mathcal{L}_2 \bigoplus \mathcal{L}_{n+1}$, for $n \geq 1$

Denote $H_{2,n+1} = L_2 \bigcup L_{n+1} = \{-1, 0\} \bigcup \{0, 1, 2, \ldots, n\} = \{-1, 0, 1, 2, \ldots, n\}$.

For $n = 1, 2, 3, 4, 5$, we have the linearly ordered Hájek(P) (BL) algebras $\mathcal{H}_{2,2} = \mathcal{L}_2 \bigoplus \mathcal{L}_2$, $\mathcal{H}_{2,3} = \mathcal{L}_2 \bigoplus \mathcal{L}_3$, $\mathcal{H}_{2,4} = \mathcal{L}_2 \bigoplus \mathcal{L}_4$, $\mathcal{H}_{2,5} = \mathcal{L}_2 \bigoplus \mathcal{L}_5$, $\mathcal{H}_{2,6} = \mathcal{L}_2 \bigoplus \mathcal{L}_6$, whose tables are the following:

$\mathcal{H}_{2,2}$

$\rightarrow$	-1	0	1
-1	1	1	1
0	-1	1	1
1	-1	0	1

$\odot$	-1	0	1
-1	-1	-1	-1
0	-1	0	0
1	-1	0	1

$\mathcal{H}_{2,3}$

$\rightarrow$	-1	0	1	2
-1	2	2	2	2
0	-1	2	2	2
1	-1	1	2	2
2	-1	0	1	2

$\odot$	-1	0	1	2
-1	-1	-1	-1	-1
0	-1	0	0	0
1	-1	0	0	1
2	-1	0	1	2

$\mathcal{H}_{2,4}$

$\rightarrow$	-1	0	1	2	3
-1	3	3	3	3	3
0	-1	3	3	3	3
1	-1	2	3	3	3
2	-1	1	2	3	3
3	-1	0	1	2	3

$\odot$	-1	0	1	2	3
-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0
1	-1	0	0	0	1
2	-1	0	0	1	2
3	-1	0	1	2	3

$\mathcal{H}_{2,5}$

$\rightarrow$	-1	0	1	2	3	4
-1	4	4	4	4	4	4
0	-1	4	4	4	4	4
1	-1	3	4	4	4	4
2	-1	2	3	4	4	4
3	-1	1	2	3	4	4
4	-1	0	1	2	3	4

$\odot$	-1	0	1	2	3	4
-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0
1	-1	0	0	0	0	1
2	-1	0	0	0	1	2
3	-1	0	0	1	2	3
4	-1	0	1	2	3	4

$\mathcal{H}_{2,6}$

$\rightarrow$	-1	0	1	2	3	4	5
-1	5	5	5	5	5	5	5
0	-1	5	5	5	5	5	5
1	-1	4	5	5	5	5	5
2	-1	3	4	5	5	5	5
3	-1	2	3	4	5	5	5
4	-1	1	2	3	4	5	5
5	-1	0	1	2	3	4	5

$\odot$	-1	0	1	2	3	4	5
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
1	-1	0	0	0	0	0	1
2	-1	0	0	0	0	1	2
3	-1	0	0	0	1	2	3
4	-1	0	0	1	2	3	4
5	-1	0	1	2	3	4	5

Note that $\mathcal{H}_{2,2}$ and $\mathcal{H}_{2,3}$ satisfy the condition (WNM), hence they are proper $_{(WNM)}$BL algebras, while the other do not satisfy the condition (WNM) ($\mathcal{H}_{2,4}$ for 2, $\mathcal{H}_{2,5}$ for 2, 3, $\mathcal{H}_{2,6}$ for 3, 4), hence they are proper BL algebras.

Note also that the subalgebra of $\mathcal{H}_{2,n+1}$ ($n \geq 2$) obtained by subtracting "0" is $\mathcal{H}_{2,n}$.

Remark 8.1 $\mathcal{H}_{2,n+1}$, for $n \geq 1$ is not a Wajsberg (MV) algebra, since there is $0 \in H_{2,n+1}$ such that $(0^-)^- = (-1)^- = n \neq 0$, where $x^- = x \to -1$.

Remarks 8.2
(i) $\mathcal{H}_{2,2}$ is a Gödel algebra.
(ii) $\mathcal{H}_{2,n+1}$, for $n > 1$ is not a Gödel algebra, since there is $1 \in H_{2,n+1}$ such that $1 \odot 1 = \max\{0, 1 + 1 - n\} = 0 \neq 1$.

Remarks 8.3
(i) Since the first algebra of the ordinal sum is $\mathcal{L}_2$, it follows that $\mathcal{H}_{2,n+1}$, $n \geq 1$, verify the condition (P1), since if $x = -1$, then $x^- = x \to -1 = n$ and if $x \neq -1$, then $x^- = -1$, by Definition ??.
(ii) But, remark that $\mathcal{H}_{2,n+1}$, $n \geq 1$, is not a Product algebra, since the condition (P2) is not satisfied: there are $x = n$, $y = 0$, $z = 0 \in H_{2,n+1}$ such that

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = n \odot [0 \to 0] = n \odot n = n \not\leq x \to y = 0.$$

8.1.2 Examples of the form: $\mathcal{H}_{3,n+1} = \mathcal{L}_3 \bigoplus \mathcal{L}_{n+1}$, for $n \geq 1$

Denote $H_{3,n+1} = L_3 \bigcup L_{n+1} = \{-2, -1, 0\} \bigcup \{0, 1, 2, \ldots, n\} = \{-2, -1, 0, 1, 2, \ldots, n\}$.

Example 1 The linearly ordered Hájek(P) (BL) algebra $\mathcal{H}_{3,2} = \mathcal{L}_3 \bigoplus \mathcal{L}_2$, whose tables are the following:

$\mathcal{H}_{3,2}$

$\to$	-2	-1	0	1
-2	1	1	1	1
-1	-1	1	1	1
0	-2	-1	1	1
1	-2	-1	0	1

$\odot$	-2	-1	0	1
-2	-2	-2	-2	-2
-1	-2	-2	-1	-1
0	-2	-1	0	0
1	-2	-1	0	1

It satisfies the condition (WNM).

Remark that $\mathcal{H}_{3,2} = \mathcal{L}_3 \bigoplus \mathcal{L}_2$ is not a Wajsberg (MV) algebra; it is not a Gödel algebra; it does not satisfies evidently condition (P1) and it does not satisfy also condition (P2): there are $x = 1$, $y = z = 0$ such that

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = 1 \odot [0 \to 0] = 1 \odot 1 = 1 \not\leq 0 = x \to y;$$

consequently, it is not a Product algebra. Hence, it is the proper, liniarly ordered, Hájek(P) (BL) with the smallest number of elements.

Example 2 The linearly ordered Hájek(P) (BL) algebra $\mathcal{H}_{3,3} = \mathcal{L}_3 \bigoplus \mathcal{L}_3$, whose tables are the following:

$\mathcal{H}_{3,3}$

$\to$	-2	-1	0	1	2
-2	2	2	2	2	2
-1	-1	2	2	2	2
0	-2	-1	2	2	2
1	-2	-1	1	2	2
2	-2	-1	0	1	2

$\odot$	-2	-1	0	1	2
-2	-2	-2	-2	-2	-2
-1	-2	-2	-1	-1	-1
0	-2	-1	0	0	0
1	-2	-1	0	0	1
2	-2	-1	0	1	2

It does not satisfy the condition (WNM) for 1.

Remark that $\mathcal{H}_{3,3} = \mathcal{L}_3 \bigoplus \mathcal{L}_3$ is not a Wajsberg (MV) algebra; it is not a Gödel algebra; it does not satisfies evidently condition (P1) and it does not satisfy also condition (P2): there are $x = 1$, $y = z = 0$ such that

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = 2 \odot [0 \to 0] = 2 \odot 2 = 2 \not\leq 1 = x \to y;$$

consequently, it is not a Product algebra. Hence, it is a proper, liniarly ordered, Hájek(P) (BL).

Remark 8.4 $\mathcal{H}_{3,n+1}$, for $n \geq 1$ is not a Wajsberg (MV) algebra, since there is $0 \in H_{3,n+1}$ such that $(0^-)^- = (-2)^- = n \neq 0$, where $x^- = x \to -2$.

Remark 8.5 $\mathcal{H}_{3,n+1}$, for $n \geq 1$ is not a Gödel algebra, since there is $-1 \in H_{3,n+1}$ such that $-1 \odot -1 = -2 \neq -1$.

Remarks 8.6
(i) Since the first algebra of the ordinal sum is $\mathcal{L}_3$, it follows that $\mathcal{H}_{3,n+1}$, $n \geq 1$, does not verify the condition (P1), since there is $x = -1$, such that $x \wedge x^- = -1 \wedge (-1)^- = -1 \wedge -1 = -1 \neq -2$. It does not satisfy the condition (P2) also, since there are $x = n$, $y = z = 0$ such that

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = n \odot [0 \to 0] = n \not\leq 0 = x \to y.$$

(ii) By (i), $\mathcal{H}_{3,n+1}$, $n \geq 1$, is not a Product algebra.

8.2 Examples of the form: linearly ordered MV $\bigoplus$ linearly ordered BL or linearly ordered BL $\bigoplus$ linearly ordered MV

Denote $\mathcal{H}_{m+1,n+1,p+1} = \mathcal{L}_{m+1} \bigoplus \mathcal{H}_{n+1,p+1} = \mathcal{L}_{m+1} \bigoplus (\mathcal{L}_{n+1} \bigoplus \mathcal{L}_{p+1}) =$
$= (\mathcal{L}_{m+1} \bigoplus \mathcal{L}_{n+1}) \bigoplus \mathcal{L}_{p+1} = \mathcal{H}_{m+1,n+1} \bigoplus \mathcal{L}_{p+1}$, by associativity of $\bigoplus$.
Example For $m = n = p = 1$, the set

$$H_{2,2,2} = L_2 \bigcup H_{2,2} = \{-1,0\} \bigcup \{0,1,2\} = H_{2,2} \bigcup L_2 = \{-1,0,1\} \bigcup \{1,2\} = \{-1,0,1,2\},$$

organized as a lattice in an obvious way and as the ordinal sum $\mathcal{H}_{2,2,2} = \mathcal{H}_{2,2} \bigoplus \mathcal{L}_2$, with the tables:

$\mathcal{H}_{2,2,2}$

$\to$	-1	0	1	2
-1	2	2	2	2
0	-1	2	2	2
1	-1	0	2	2
2	-1	0	1	2

$\odot$	-1	0	1	2
-1	-1	-1	-1	-1
0	-1	0	0	0
1	-1	0	1	1
2	-1	0	1	2

is a BL algebra; it is a Gödel algebra, hence it verifies the condition (WNM); it satisfies condition (P1), but not (P2): there are $x = 2$, $y = z = 0$ such that

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = 2 \odot [0 \to 0] = 2 \odot 2 = 2 \not\leq 0 = x \to y.$$

Hence, $\mathcal{H}_{2,2,2}$ is not a Product algebra.

8.3 Examples of the form: linearly ordered BL $\bigoplus$ linearly ordered BL or equivalent forms

Denote $\mathcal{H}_{m+1,n+1,p+1,q+1} = \mathcal{H}_{m+1,n+1} \bigoplus \mathcal{H}_{p+1,q+1} = (\mathcal{L}_{m+1} \bigoplus \mathcal{L}_{n+1}) \bigoplus (\mathcal{L}_{p+1} \bigoplus \mathcal{L}_{q+1}) =$
$= \mathcal{H}_{m+1,n+1,p+1} \bigoplus \mathcal{L}_{q+1} = \mathcal{L}_{m+1} \bigoplus \mathcal{H}_{n+1,p+1,q+1}$, by associativity of $\bigoplus$.
Example For $m = n = p = q = 1$, the set

$$H_{2,2,2,2} = H_{2,2} \bigcup H_{2,2} = H_{2,2,2} \bigcup L_2 = \{-1,0,1,2\} \bigcup \{2,3\} = \{-1,0,1,2,3\} =$$

$$= L_2 \bigcup H_{2,2,2} = \{-1, 0\} \bigcup \{0, 1, 2, 3\},$$

organized as a lattice in an obvious way and as the ordinal sum $\mathcal{H}_{2,2,2,2}$, with the tables:

$\mathcal{H}_{2,2,2,2}$

$\to$	-1	0	1	2	3
-1	3	3	3	3	3
0	-1	3	3	3	3
1	-1	0	3	3	3
2	-1	0	1	3	3
3	-1	0	1	2	3

$\odot$	-1	0	1	2	3
-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0
1	-1	0	1	1	1
2	-1	0	1	2	2
3	-1	0	1	2	3

is a BL algebra; it is a Gödel algebra, hence satisfies the condition (WNM); it satisfies (P1), but not (P2): there are $x = 3$, $y = z = 0$ such that

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = 3 \odot [0 \to 0] = 3 \odot 3 = 3 \not\leq 0 = x \to y.$$

Hence, it is not a Product algebra.

9 Examples of finite, non-linearly ordered, proper Hájek(P) (BL) algebras and $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras

We shall sometimes use the shorter names, BL and MV algebras.

The examples are of one of the following forms:

9.1 linearly ordered MV $\bigoplus$ non-linearly ordered MV,

9.2 isomorphic copies of direct products of two linearly ordered MV/BL algebras,

9.3 linearly ordered MV $\bigoplus$ non-linearly ordered BL or linearly ordered BL $\bigoplus$ non-linearly ordered MV,

9.4 linearly ordered BL $\bigoplus$ non-linearly ordered BL.

9.1 Examples of the form: linearly ordered MV $\bigoplus$ non-linearly ordered MV

Denote $\mathcal{H}_{p+1,(n+1)\times(m+1)} = \mathcal{L}_{p+1} \bigoplus \mathcal{L}_{(n+1)\times(m+1)}$, for $p, n, m \geq 1$.

We give two families of examples.

9.1.1 Examples of the form: $\mathcal{H}_{2,(n+1)\times(m+1)} = \mathcal{L}_2 \bigoplus \mathcal{L}_{(n+1)\times(m+1)}$, for $n, m \geq 1$

Denote $H_{2,(n+1)\times(m+1)} = L_2 \bigcup L_{(n+1)\times(m+1)}$, with $n, m \geq 1$.

We give four examples.

Example 1

The set

$$H_{2,2\times 2} = L_2 \bigcup L_{2\times 2} = \{-1, 0\} \bigcup \{0, a, b, 1\} = \{-1, 0, a, b, 1\},$$

organized as a lattice as in Figure 7 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables, is a non-linearly ordered Hájek(P) (BL) algebra, denoted $\mathcal{H}_{2,2\times 2} = \mathcal{L}_2 \bigoplus \mathcal{L}_{2\times 2}$.

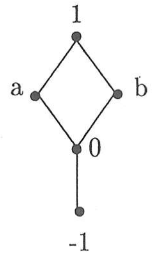

Figure 7: The non-linearly ordered Hájek(P) (BL) algebra $\mathcal{H}_{2,2\times 2}$

$\mathcal{H}_{2,2\times 2}$

$\rightarrow$	-1	0	a	b	1
-1	1	1	1	1	1
0	-1	1	1	1	1
a	-1	b	1	b	1
b	-1	a	a	1	1
1	-1	0	a	b	1

$\odot$	-1	0	a	b	1
-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0
a	-1	0	a	0	a
b	-1	0	0	b	b
1	-1	0	a	b	1

Remark that $\mathcal{H}_{2,2\times 2}$ is a Gödel algebra, hence verifies the condition (WNM) and thus is a $_{(WNM)}$ BL algebra.

It is not an MV algebra, since there is $0 \in \{-1, 0, a, b, 1\}$ such that $(0^-)^- = (-1)^- = 1 \neq 0$.

It is not a Product algebra since it satisfies (P1), but it does not satisfy (P2): there are $x = a$, $y = b$, $z = a$ such that:

$$(z^-)^- \odot [(x \odot z) \rightarrow (y \odot z)] = 1 \odot [0 \rightarrow 0] = 1 \not\leq x \rightarrow y = b.$$

Example 2

The set
$$H_{2,3\times 2} = L_2 \bigcup L_{3\times 2} = \{-1, 0\} \bigcup \{0, a, b, c, d, 1\} = \{-1, 0, a, b, c, d, 1\},$$

organized as a lattice as in Figure 8 and as a BCK(P) algebra with the operations $\rightarrow$ and $\odot$ as in the following tables, is a non-linearly ordered BL algebra, denoted by $\mathcal{H}_{2,3\times 2} = \mathcal{L}_2 \bigoplus \mathcal{L}_{3\times 2}$.

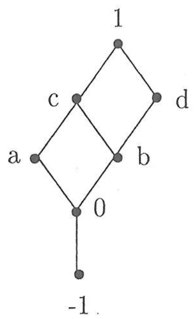

Figure 8: The non-linearly ordered BL algebras $\mathcal{H}_{2,3\times 2}$ and $\mathcal{H}_{2,(2,2)\times 2}$

$\mathcal{H}_{2,3\times 2}$

$\rightarrow$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	d	1	d	1	d	1
b	-1	c	c	1	1	1	1
c	-1	b	c	d	1	d	1
d	-1	a	a	c	c	1	1
1	-1	0	a	b	c	d	1

$\odot$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	0	0	b	b
c	-1	0	a	0	a	b	c
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

Remark that $\mathcal{H}_{2,3\times 2}$ is not a Gödel algebra, since there is b such that $b \odot b = 0 \neq b$. It does not verify the condition (WNM) for b.

It is not an MV algebra, since there is 0 such that $(0^-)^- = (-1)^- = 1 \neq 0$.

It is not a Product algebra since it satisfies (P1), but not (P2): there are $x = a$, $y = z = d$ such that:

$$(z^-)^- \odot [(x \odot z) \rightarrow (y \odot z)] = 1 \odot [0 \rightarrow d] = 1 \odot 1 = 1 \not\leq d = x \rightarrow y.$$

Example 3

The set

$$H_{2,3\times 3} = L_2 \bigcup L_{3\times 3} = \{-1, 0\} \bigcup \{0, a, b, c, d, e, f, g, 1\} = \{-1, 0, a, b, c, d, e, f, g, 1\},$$

organized as a lattice as in Figure 9 and as a BCK(P) algebra with the implication $\rightarrow$ and $\odot$ as in the following tables, is a non-linearly ordered BL algebra, denoted by $\mathcal{H}_{2,3\times 3} = \mathcal{L}_2 \bigoplus \mathcal{L}_{3\times 3}$.

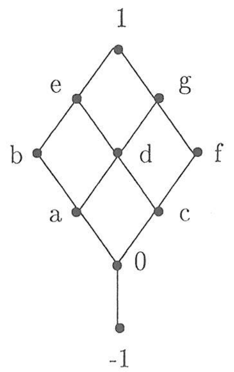

Figure 9: The non-linearly ordered BL algebra $\mathcal{H}_{2,3\times 3}$

$\mathcal{H}_{2,3\times 3}$

$\rightarrow$	-1	0	a	b	c	d	e	f	g	1
-1	1	1	1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1	1	1	1
a	-1	g	1	1	g	1	1	g	1	1
b	-1	f	g	1	f	g	1	f	g	1
c	-1	e	e	e	1	1	1	1	1	1
d	-1	d	e	e	g	1	1	g	1	1
e	-1	c	d	e	f	g	1	f	g	1
f	-1	b	b	b	e	e	e	1	1	1
g	-1	a	b	b	d	e	e	g	1	1
1	-1	0	a	b	c	d	e	f	g	1

$\odot$	-1	0	a	b	c	d	e	f	g	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0	0	0	0
a	-1	0	0	a	0	0	a	0	0	a
b	-1	0	a	b	0	a	b	0	a	b
c	-1	0	0	0	0	0	0	c	c	c
d	-1	0	0	a	0	0	a	c	c	d
e	-1	0	a	b	0	a	b	c	d	e
f	-1	0	0	0	c	c	c	f	f	f
g	-1	0	0	a	c	c	d	f	f	g
1	-1	0	a	b	c	d	e	f	g	1

Remark that $\mathcal{H}_{2,3\times 3}$ is not a Gödel algebra, since there is c such that $c\odot c = 0 \neq c$. It does not verify the condition (WNM) for a, d.

It is not an MV algebra, since there is 0 such that $(0^-)^- = (-1)^- = 1 \neq 0$.

It is not a Product algebra since there are $x = b$, $y = z = f$ such that:

$$(z^-)^- \odot [(x\odot z) \to (y \odot z)] = 1 \odot [0 \to f] = 1 \odot 1 = 1 \not\leq f = x \to y.$$

Example 4

The set

$$H_{2,4\times 2} = L_2 \bigcup L_{4\times 2} = \{-1, 0\} \bigcup \{0, a, b, c, d, e, f, 1\} = \{-1, 0, a, b, c, d, e, f, 1\}$$

is a BL algebra. It does not verify the condition (WNM) for e.

Remark 9.1 $\mathcal{H}_{2,(n+1)\times(m+1)}$, for $n, m \geq 1$, is not a Wajsberg (MV) algebra, since there is 0 such that $(0^-)^- = (-1)^- = 1 \neq 0$, where $x^- = x \to -1$.

Remark 9.2 $\mathcal{H}_{2,(n+1)\times(m+1)}$, for $n > 1$ or $m > 1$, is not a Gödel algebra, since if $n > 1$ for instance, then

$$H_{2,(n+1)\times(m+1)} \cong \{-1, 0\} \bigcup \{0, 1, \ldots, n\} \times \{0, \ldots, m\} =$$

$$= \{-1, 0\} \bigcup \{(0,0), \ldots, (0, m), \quad (1,0), \ldots, (1, m), \quad \ldots, \ldots, \ldots, \quad (n, 0), \ldots, (n, m)\}$$

and there is $(1,0)$ such that $(1,0) \odot (1,0) = (1 \odot 1, 0 \odot 0) = (\max(0, 1 + 1 - n), \max(0, 0 + 0 - m)) = (0,0) \neq (1,0)$.

Remarks 9.3

(i) Since the first algebra of the ordinal sum is $\mathcal{L}_2$, it follows that $\mathcal{H}_{2,(n+1)\times(n+1)}$, $n, m \geq 1$, verify the condition (P1), since if $x = -1$, then $x^- = x \to -1 = 1$ and if $x \neq -1$, then $x^- = -1$, by Definition ??.

(ii) But, remark that $\mathcal{H}_{2,(n+1)\times(m+1)}$, $n, m \geq 1$, is not a Product algebra, since the condition (P2) is not satisfied: there are $x = (0, m)$, $y = z = (n, 0)$ such that:
$(z^-)^- = (-1)^- = (n, m)$,
$x \odot z = (0, m) \odot (n, 0) = (0 \odot n, m \odot 0) = (\max(0, 0 + n - n), \max(0, m + 0 - m)) = (0, 0)$,
$y \odot z = (n, 0) \odot (n, 0) = (n \odot n, 0 \odot 0) = (\max(0, n + n - n), \max(0, 0 + 0 - m)) = (n, 0)$, hence

$$(z^-)^- \odot [(x \odot z) \to (y \odot z)] = (n, m) \odot [(0,0) \to (n, 0)] = (n, m) \odot (0 \to n, 0 \to 0) =$$

$$= (n, m) \odot (\min(n, n - 0 + n), \min(m, 0 - 0 + m)) = (n, m) \odot (n, m) = (n, m) \not\leq x \to y =$$

$$= (0, m) \to (n, 0) = (\min(n, n - 0 + n), \min(m, 0 - m + m)) = (n, 0).$$

9.1.2 Examples of the form: $\mathcal{H}_{3,(n+1)\times(m+1)} = \mathcal{L}_3 \bigoplus \mathcal{L}_{(n+1)\times(m+1)}$, for $n, m \geq 1$

We give here only one example, when $n = m = 1$.

The set

$$H_{3,2\times 2} = L_3 \bigcup L_{2\times 2} = \{-2, -1, 0\} \bigcup \{0, a, b, 1\} = \{-2, -1, 0, a, b, 1\},$$

organized as a lattice as in Figure 10 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables, is a non-linearly ordered Hájek(P) (BL) algebra, denoted $\mathcal{H}_{3,2\times 2} = \mathcal{L}_3 \bigoplus \mathcal{L}_{2\times 2}$.

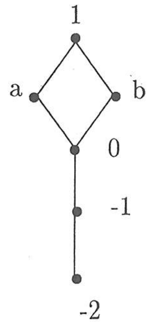

Figure 10: The non-linearly ordered Hájek(P) (BL) algebras $\mathcal{H}_{3,2\times2}$ and $\mathcal{H}_{2,2,2\times2}$

$\mathcal{H}_{3,2\times2}$

$\rightarrow$	-2	-1	0	a	b	1
-2	1	1	1	1	1	1
-1	-1	1	1	1	1	1
0	-2	-1	1	1	1	1
a	-2	-1	b	1	b	1
b	-2	-1	a	a	1	1
1	-2	-1	0	a	b	1

$\odot$	-2	-1	0	a	b	1
-2	-2	-2	-2	-2	-2	-2
-1	-2	-2	-1	-1	-1	-1
0	-2	-1	0	0	0	0
a	-2	-1	0	a	0	a
b	-2	-1	0	0	b	b
1	-2	-1	0	a	b	1

Remark that $\mathcal{H}_{3,2\times2}$ is not a Gödel algebra, since there is -1 such that $-1 \odot -1 = -2 \neq -1$. But, since for $x \odot y \neq -2$ we have $x \wedge y = x \odot y$, it follows that it verifies the condition (WNM), hence it is a $_{(WNM)}$BL algebra.

It is not an MV algebra, since there is $0 \in \{-2, -1, 0, a, b, 1\}$ such that $(0^-)^- = (-2)^- = 1 \neq 0$.

It does not satisfy the condition (P1), since there is $x = -1$ such that $x \wedge x^- = -1 \wedge -1 = -1 \neq -2$.

It does not satisfy the condition (P2) too: there are $x = a$, $y = b$, $z = 0$ such that:

$$(z^-)^- \odot [(z \odot x) \rightarrow (z \odot y)] = 1 \not\leq a \rightarrow b = b.$$

Hence, it is not a Product algebra.

Remark 9.4 $\mathcal{H}_{3,2\times2} = \mathcal{L}_3 \bigoplus \mathcal{L}_{2\times2}$ is the proper non-liniar $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebra with the smallest number of elements.

9.2 Examples of the form: isomorphic copies of direct products of two linearly ordered MV/BL algebras

Note that at least one of the two BL algebras must not be an MV algebra to get a direct product which is not an MV algebra. We give two classes of examples.

9.2.1 Examples of the form: isomorphic coppy of linearly ordered MV $\times$ linearly ordered BL or linearly ordered BL $\times$ linearly ordered MV

Denote, for any $n, m, p \geq 1$, the Hájek(P) (BL) algebras:

$$\mathcal{H}_{(p+1)\times(n+1,m+1)} \cong \mathcal{L}_{p+1} \times \mathcal{H}_{n+1,m+1}$$

and

$$\mathcal{H}_{(n+1,m+1)\times(p+1)} \cong \mathcal{H}_{n+1,m+1} \times \mathcal{L}_{p+1}.$$

Example For $n = m = p = 1$, $\mathcal{H}_{(2,2)\times 2} \cong \mathcal{H}_{2,2} \times \mathcal{L}_2$.
The set

$$H_{(2,2)\times 2} = \{0, a, \quad b, c, \quad d, 1\} \cong H_{2,2} \times L_2 = \{-1, 0, 1\} \times \{0, 1\} = \{(-1, 0), (-1, 1), \quad (0, 0), (0, 1), \quad (1, 0), (1, 1)\}$$

organized as a lattice as in Figure 4 and as a BCK(P) lattice with the operations $\to$ and $\odot$ as in the following tables, is a BL algebra, denoted by $\mathcal{H}_{(2,2)\times 2}$.

$\mathcal{H}_{(2,2)\times 2}$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Remark that $\{0, 1\}$ and $S = \{0, c, 1\}$ are its BL subalgebras and that $\mathcal{S}$ is just $\mathcal{H}_{2,2}$.

9.2.2 Examples of the form: isomorphic copy of linearly ordered BL $\times$ linearly ordered BL

Denote, for any $n, m, p, q \geq 1$, the Hájek(P) (BL) algebra:

$$\mathcal{H}_{(n+1,m+1)\times(p+1,q+1)} \cong \mathcal{H}_{n+1,m+1} \times \mathcal{H}_{p+1,q+1}.$$

Example For $n = m = p = q = 1$, $\mathcal{H}_{(2,2)\times(2,2)} \cong \mathcal{H}_{2,2} \times \mathcal{H}_{2,2}$.
The set

$$H_{(2,2)\times(2,2)} = \{0, a, b, \quad c, d, e, \quad f, g, 1\} \cong H_{2,2} \times H_{2,2} = \{-1, 0, 1\} \times \{-1, 0, 1\} =$$

$$= \{(-1, -1), (-1, 0), (-1, 1), \quad (0, -1), (0, 0), (0, 1), \quad (1, -1), (1, 0), (1, 1)\}$$

organized as a lattice as in Figure 6 and as a BCK(P) lattice with the operations $\to$ and $\odot$ as in the following tables, is a BL algebra, denoted by $\mathcal{H}_{(2,2)\times(2,2)}$.

$\mathcal{H}_{(2,2)\times(2,2)}$

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

Corollary 9.5 *The BL algebra $\mathcal{H}_{2,2\times 2}$ is isomorphic with a subdirect product of $\mathcal{H}_{2,2} \times \mathcal{H}_{2,2}$.*

Proof. Remark first that the BL subalgebra $S = \{0, d, e, g, 1\}$ of $\mathcal{H}_{(2,2)\times(2,2)}$ is isomorphic with $\mathcal{H}_{2,2\times 2}$. Remark more that S is a subdirect product of $\mathcal{H}_{2,2} \times \mathcal{H}_{2,2}$, since

$$S \cong \{(-1,-1),(0,0),(0,1),(1,0),(1,1)\}$$

and $pr_1(S) = H_{2,2} = \{-1,0,1\}$, $pr_2(S) = H_{2,2} = \{-1,0,1\}$, where pr_1 and pr_2 are the projection functions of the direct product $\mathcal{H}_{2,2} \times \mathcal{H}_{2,2}$. □

9.3 Examples of the form: linearly ordered MV $\bigoplus$ non-linearly ordered BL or linearly ordered BL $\bigoplus$ non-linearly ordered MV

Denote, for $u, v, n, m \geq 1$, the BL algebras:

$$\mathcal{H}_{u+1,v+1,(n+1)\times(m+1)} = \mathcal{L}_{u+1} \bigoplus \mathcal{L}_{v+1} \bigoplus \mathcal{L}_{(n+1)\times(m+1)} =$$

$$= \mathcal{L}_{u+1} \bigoplus \mathcal{H}_{v+1,(n+1)\times(m+1)} = \mathcal{H}_{u+1,v+1} \bigoplus \mathcal{L}_{(n+1)\times(m+1)},$$

by the associativity of $\bigoplus$.

We give two examples.

Example 1 For $u = v = n = m = 1$, consider the BL algebra

$$\mathcal{H}_{2,2,2\times 2} = \mathcal{L}_2 \bigoplus \mathcal{H}_{2,2\times 2} = \mathcal{H}_{2,2} \bigoplus \mathcal{L}_{2\times 2}.$$

The support set, $\{-2,-1,0,a,b,1\}$, of the lattice from Figure 10 can be considered either as the union of sets:

$$H_{(2,2),2\times 2} = [\{-2,-1\} \bigcup \{-1,0\}] \bigcup \{0,a,b,1\} = [L_2 \bigcup L_2] \bigcup L_{2\times 2} = H_{2,2} \bigcup L_{2\times 2}$$

or as the union

$$H_{2,(2,2\times 2)} = \{-2,-1\} \bigcup [\{-1,0\} \bigcup \{0,a,b,1\}] = L_2 \bigcup [L_2 \bigcup L_{2\times 2}] = L_2 \bigcup H_{2,2\times 2}.$$

It has the following tables:

$\mathcal{H}_{2,2,2\times 2}$

$\rightarrow$	-2	-1	0	a	b	1
-2	1	1	1	1	1	1
-1	-2	1	1	1	1	1
0	-2	-1	1	1	1	1
a	-2	-1	b	1	b	1
b	-2	-1	a	a	1	1
1	-2	-1	0	a	b	1

$\odot$	-2	-1	0	a	b	1
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0
a	-2	-1	0	a	0	a
b	-2	-1	0	0	b	b
1	-2	-1	0	a	b	1

Remark that $\mathcal{H}_{2,2,2\times 2} = H_{(2,2),2\times 2} = H_{2,(2,2\times 2)}$ is a Gödel algebra.

It is not an MV algebra, since there is -1 such that $((-1)^-)^- = (-2)^- = 1 \neq -1$.

It satisfies the condition (P1), but it does not satisfy the condition (P2), since there are $x = a$, $y = b$, $z = -1$ such that

$$(z^-)^- \odot [(x \odot z) \rightarrow (y \odot z)] = 1 \odot [-1 \rightarrow -1] = 1 \not\leq x \rightarrow y = b.$$

Hence, it is not a Product algebra.

Example 2 For $u = v = n = m = 1$ also, consider the BL algebra $\mathcal{H}_{2,(2,2)\times 2} = \mathcal{L}_2 \bigoplus \mathcal{H}_{(2,2)\times 2}$.

The set

$$H_{2,(2,2)\times 2} = L_2 \bigcup H_{(2,2)\times 2} = \{-1,0\} \bigcup \{0,a,b,c,d,1\} = \{-1,0,a,b,c,d,1\},$$

organized as a lattice as in Hasse diagramme from Figure 8 and as a BCK(P) lattice with the operations $\rightarrow$ and $\odot$ from the following tables is a BL algebra, denoted $\mathcal{H}_{2,(2,2)\times 2}$.

$\mathcal{H}_{2,(2,2)\times 2}$

$\rightarrow$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	d	1	d	1	d	1
b	-1	a	a	1	1	1	1
c	-1	0	a	d	1	d	1
d	-1	a	a	c	c	1	1
1	-1	0	a	b	c	d	1

$\odot$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	0	a
b	-1	0	0	b	b	b	b
c	-1	0	a	b	c	b	c
d	-1	0	0	b	b	d	d
1	-1	0	a	b	c	d	1

9.4 Examples of the form: linearly ordered BL $\oplus$ non-linearly ordered BL or equivalent forms

Denote, for $u, v, n, m, p \geq 1$, the BL algebras:

$$\mathcal{H}_{u+1,v+1,(n+1,m+1)\times(p+1)} = \mathcal{H}_{u+1,v+1} \bigoplus \mathcal{H}_{(n+1,m+1)\times(p+1)}.$$

Example For $u = v = n = m = p = 1$, consider the BL algebra

$$\mathcal{H}_{2,2,(2,2)\times 2} = \mathcal{H}_{2,2} \bigoplus \mathcal{H}_{(2,2)\times 2} = (\mathcal{L}_2 \bigoplus \mathcal{L}_2) \bigoplus \mathcal{H}_{(2,2)\times 2} = \mathcal{L}_2 \bigoplus \mathcal{H}_{2,(2,2)\times 2},$$

with the support set

$$H_{2,2,(2,2)\times 2} = H_{2,2} \bigcup H_{(2,2)\times 2} = \{-2,-1,0\} \bigcup \{0,a,b,c,d,1\} = \{-2,-1,0,a,b,c,d,1\},$$

organized as a lattice as in Figure 11 and as a BCK(P) algebra with the operations $\rightarrow$ and $\odot$ as in the following tables:

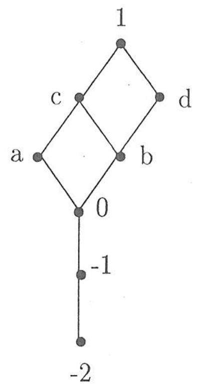

Figure 11: The non-linearly ordered BL algebras $\mathcal{H}_{2,2,(2,2)\times 2}$

$\mathcal{H}_{2,2,(2,2)\times 2}$

$\rightarrow$	-2	-1	0	a	b	c	d	1
-2	1	1	1	1	1	1	1	1
-1	-1	1	1	1	1	1	1	1
0	-2	-1	1	1	1	1	1	1
a	-2	-1	d	1	d	1	d	1
b	-2	-1	a	a	1	1	1	1
c	-2	-1	0	a	d	1	d	1
d	-2	-1	a	a	c	c	1	1
1	-2	-1	0	a	b	c	d	1

$\odot$	-2	-1	0	a	b	c	d	1
-2	-2	-2	-2	-2	-2	-2	-2	-2
-1	-2	-2	-1	-1	-1	-1	-1	-1
0	-2	-1	0	0	0	0	0	0
a	-2	-1	0	a	0	a	0	a
b	-2	-1	0	0	b	b	b	b
c	-2	-1	0	a	b	c	b	c
d	-2	-1	0	0	b	b	d	d
1	-2	-1	0	a	b	c	d	1

* * *

Note that an important class of BL algebras is that of those BL algebras satisfying the condition (P1), where 0 is the first element of the lattice. Recall that BL algebras satisfying condition (P1) are called *SBL* algebras.

Remark that any Gódel and any Product algebra is a SBL algebra, but not any MV algebra is, only the Boolean algebras.

10 Examples of infinite proper BL algebras

Recall first the following examples of Product algebras [2].

Let $\mathcal{G} = (G, \wedge, \vee, +, -, 0)$ be an abelian l-group (i.e. lattice ordered group) and let $G^- = \{x \in G, x \leq 0\}$ be the negative cone of $\mathcal{G}$. Let $\perp$ be an element not belonging to G. On the set $P(G) = G^- \bigcup\{\perp\}$ define the implication $\rightarrow$ by:

$$x \rightarrow y = \begin{cases} 0 \wedge (y - x), & \text{if } \ x, y \in G^- \\ 0, & \text{if } \ x = \perp \\ \perp, & \text{if } \ x \in G^-,\ y = \perp . \end{cases}$$

Then

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = \begin{cases} x + y, & \text{if } x, y \in G^- \\ \perp, & \text{if otherwise.} \end{cases}$$

It follows that $(P(G) = G^- \bigcup\{\perp\}, \wedge, \vee, \rightarrow, \perp, 0)$ is a Product algebra, denoted $\mathcal{P}(G)$.

• For the linearly ordered standard Product algebra $\mathcal{P}(Z) = (P(Z) = Z^- \bigcup\{-\infty\}, \max, \min, \rightarrow, -\infty, 0)$, we get the following tables:

$\mathcal{P}(Z)$

$\rightarrow$	$-\infty$	...	-3	-2	-1	0
$-\infty$	0	...	0	0	0	0
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	0	0	0	0
-2	$-\infty$	...	-1	0	0	0
-1	$-\infty$	...	-2	-1	0	0
0	$-\infty$	...	-3	-2	-1	0

$\odot$	$-\infty$	...	-3	-2	-1	0
$-\infty$	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	-6	-5	-4	-3
-2	$-\infty$	...	-5	-4	-3	-2
-1	$-\infty$	...	-4	-3	-2	-1
0	$-\infty$	...	-3	-2	-1	0

Remark 10.1 The Product algebra $\mathcal{P}(Z)$ does not satisfy the condition (WNM) for $-1, -2$:

$$(-2) \odot (-1)^- \vee [(-2) \wedge (-1) \rightarrow (-2) \odot (-1)] = (-3)^- \vee [(-2) \rightarrow (-3)] = (-\infty) \vee (-1) = (-1) \neq 0.$$

• We shall analyse first the ordinal sum of two Product algebras, the infinite $\mathcal{P}(Z)$ and the finite $\mathcal{L}_2$. We build the two ordinal sums.

Example 1 The linearly ordered set (chain) $H_{P(Z),2} = P(Z) \bigcup L_2 = (Z^- \bigcup \{-\infty\}) \bigcup L_2 = \{-\infty, \dots, -3, -2, -1, 0\} \bigcup \{0, 1\} = \{-\infty, \dots, -3, -2, -1, 0, 1\}$,
with the operations $\rightarrow$ and $\odot$ defined by the following tables, is a linearly ordered Hájek(P) (BL) algebra, denoted by $\mathcal{H}_{P(Z),2} = \mathcal{P}(Z) \bigoplus \mathcal{L}_2$.

$\rightarrow$	$-\infty$	...	-3	-2	-1	0	1
$-\infty$	1	...	1	1	1	1	1
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	1	1	1	1	1
-2	$-\infty$	...	-1	1	1	1	1
-1	$-\infty$	...	-2	-1	1	1	1
0	$-\infty$	...	-3	-2	-1	1	1
1	$-\infty$	...	-3	-2	-1	0	1

$\odot$	$-\infty$	...	-3	-2	-1	0	1
$-\infty$	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	-6	-5	-4	-3	-3
-2	$-\infty$	...	-5	-4	-3	-2	-2
-1	$-\infty$	...	-4	-3	-2	-1	-1
0	$-\infty$	...	-3	-2	-1	0	0
1	$-\infty$	...	-3	-2	-1	0	1

The BL algebra $\mathcal{H}_{P(Z),2}$ is not Gödel, since $-3 \odot -3 = -6 \neq -3$.
It satisfies (P1); it does not satisfy (P2), since there is $x = 1,\ y = 0,\ z = 0$ such that

$$(z^-)^- \odot [(z \odot x) \rightarrow (z \odot y)] = 1 \odot (0 \rightarrow 0) = 1 \not\leq x \rightarrow y = 0.$$

Hence, it is no more a Product algebra.
Thus, it is an infinite, proper, BL chain.

Example 2 The linearly ordered set (chain) $H_{2,P(Z)} = L_2 \bigcup P(Z) = L_2 \bigcup (Z^- \bigcup \{-\infty\}) = \{-\infty_0, -\infty\} \bigcup \{-\infty, \dots, -3, -2, -1, 0\} = \{-\infty_0, -\infty, \dots, -3, -2, -1, 0\}$,
with the operations $\rightarrow$ and $\odot$ defined by the following tables, is a linearly ordered Hájek(P) (BL) algebra, denoted by $\mathcal{H}_{2,P(Z)} = \mathcal{L}_2 \bigoplus \mathcal{P}(Z)$.

$\rightarrow$	$-\infty_0$	$-\infty$	...	-3	-2	-1	0
$-\infty_0$	0	0	...	0	0	0	0
$-\infty$	$-\infty_0$	0	...	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty_0$	$-\infty$	...	0	0	0	0
-2	$-\infty_0$	$-\infty$	...	-1	0	0	0
-1	$-\infty_0$	$-\infty$	...	-2	-1	0	0
0	$-\infty_0$	$-\infty$	...	-3	-2	-1	0

$\odot$	$-\infty_0$	$-\infty$	...	-3	-2	-1	0
$-\infty_0$	$-\infty_0$	$-\infty_0$	...	$-\infty_0$	$-\infty_0$	$-\infty_0$	$-\infty_0$
$-\infty$	$-\infty_0$	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty_0$	$-\infty$	...	-6	-5	-4	-3
-2	$-\infty_0$	$-\infty$	...	-5	-4	-3	-2
-1	$-\infty_0$	$-\infty$	...	-4	-3	-2	-1
0	$-\infty_0$	$-\infty$	...	-3	-2	-1	0

The BL algebra $\mathcal{H}_{2,P(Z)}$ is not Gödel, since $-3 \odot -3 = -6 \neq -3$.
It satisfies (P1); it does not satisfy (P2), since there is $x = 3,\ y = z = -\infty 0$, such that

$$(z^-)^- \odot [(z \odot x) \rightarrow (z \odot y)] = 0 \odot (-\infty \rightarrow -\infty) = 0 \not\leq x \rightarrow y = -\infty.$$

Hence, it is no more a Product algebra.
Thus, it is an infinite, proper, BL chain.

- Now we give an example of infinite proper non-linearly ordered BL algebra.

Example Consider the non-linear Wajsberg (MV) algebra $\mathcal{L}_{2\times 2}$ from Figure 3. Then, the set $H_{P(Z),2\times 2} = P(Z) \bigcup L_{2\times 2} = (Z^- \bigcup \{-\infty\}) \bigcup L_{2\times 2} =$

$\{-\infty, \ldots, -3, -2, -1, 0\} \bigcup \{0, a, b, 1\} = \{-\infty, \ldots, -3, -2, -1, 0, a, b, 1\}$,
organized as a lattice as in Figure 12, with the operations $\rightarrow$ and $\odot$ defined by the following tables, is a non-linearly ordered Hájek(P) (BL) algebra, denoted by $\mathcal{H}_{P(Z),2\times 2} = \mathcal{P}(Z) \bigoplus \mathcal{L}_{2\times 2}$.

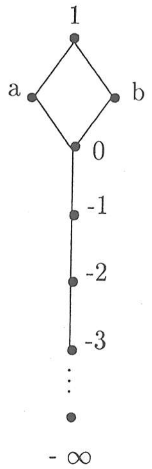

Figure 12: The infinite, non-linearly ordered, proper BL algebra $\mathcal{H}_{\mathcal{P}(Z),2\times 2}$

$\mathcal{H}_{\mathcal{P}(Z),2\times 2}$

$\rightarrow$	$-\infty$	...	-3	-2	-1	0	a	b	1
$-\infty$	1	...	1	1	1	1	1	1	1
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	1	1	1	1	1	1	1
-2	$-\infty$	...	-1	1	1	1	1	1	1
-1	$-\infty$	...	-2	-1	1	1	1	1	1
0	$-\infty$	...	-3	-2	-1	1	1	1	1
a	$-\infty$	...	-3	-2	-1	b	1	b	1
b	$-\infty$	...	-3	-2	-1	a	a	1	1
1	$-\infty$	...	-3	-2	-1	0	a	b	1

$\mathcal{H}_{\mathcal{P}(Z),2\times 2}$

$\odot$	$-\infty$	...	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	...	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	...	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	...	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	...	-3	-2	-1	0	0	0	0
a	$-\infty$	...	-3	-2	-1	0	a	0	a
b	$-\infty$	...	-3	-2	-1	0	0	b	b
1	$-\infty$	...	-3	-2	-1	0	a	b	1

The BL algebra $\mathcal{H}_{P(Z),2\times 2}$ is not Gödel, since $-3 \odot -3 = -6 \neq -3$.
It satisfies (P1); it does not satisfy (P2), since there is $x = a$, $y = z = b$, such that

$$(z^-)^- \odot [(z \odot x) \to (z \odot y)] = 1 \odot (0 \to b) = 1 \not\leq x \to y = b.$$

Hence, it is not a Product algebra.
Thus, $\mathcal{H}_{\mathcal{P}(Z),2\times 2}$ is an infinite proper non-linearly ordered BL algebra.

11 Examples of finite, proper divisible BCK(P) lattices and $_{(WNM)}$BCK(P) lattices

The examples will be of the form: non-linearly ordered MV/BL algebra $\bigoplus$ MV/BL algebra, more precisely of one of the following forms:
11.1 non-linearly ordered MV $\bigoplus$ linearly ordered MV,
11.2 non-linearly ordered MV $\bigoplus$ non-linearly ordered MV,
11.3 non-linearly ordered MV $\bigoplus$ linearly ordered BL,
11.4 non-linearly ordered MV $\bigoplus$ non-linearly ordered BL;
11.5 non-linearly ordered BL $\bigoplus$ linearly ordered MV,
11.6 non-linearly ordered BL $\bigoplus$ non-linearly ordered MV,
11.7 non-linearly ordered BL $\bigoplus$ linearly ordered BL,
11.8 non-linearly ordered BL $\bigoplus$ non-linearly ordered BL.

It follows that they are not MV algebras and are not linearly ordered.
In the sequel we shall simply say "BL algebra" instead of "BL algebra which is not an MV algebra".

11.1 Examples of the form: non-linearly ordered MV $\bigoplus$ linearly ordered MV

Denote, for $p, q, n \geq 1$,

$$\mathcal{D}_{(p+1)\times(q+1),n+1} = \mathcal{L}_{(p+1)\times(q+1)} \bigoplus \mathcal{L}_{n+1}.$$

We give three examples.
Example 1 For $p = q = n = 1$, the divisible BCK(P) lattice

$$\mathcal{D}_{2\times 2,2} = \mathcal{L}_{2\times 2} \bigoplus \mathcal{L}_2,$$

with the support set

$$D_{2\times 2,2} = L_{2\times 2} \bigcup L_2 = \{0, a, b, c\} \bigcup \{c, 1\} = \{0, a, b, c, 1\},$$

is organized as a lattice as in Figure 13 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables.

$\mathcal{D}_{2\times 2,2}$

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

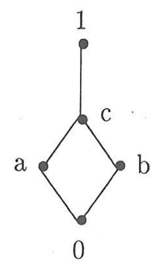

Figure 13: The divisible BCK(P) lattice $\mathcal{D}_{2\times 2,2}$

Note that the condition $(C_\vee)$ is not verified, since there are a, b such that:

$$c = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (b \to b) \wedge (a \to a) = 1.$$

The divisible BCK(P) lattice $\mathcal{D}_{2\times 2,2}$ is of Gödel type, namely is the divisible BCK(P) lattice of Gödel type with the smallest number of elements. Hence, it verifies the condition (WNM), i.e. it is a divisible $_{(WNM)}$BCK(P) lattice.

It verifies the condition (P1).

It does not verify the condition (P2): there are $x = 1$, $y = z = c$ such that

$$(z^-)^- \odot [(z \odot x) \to (z \odot y)] = 1 \odot [c \to c] = 1 \not\leq x \to y = c.$$

Hence, it is not of Product type.

It follows that it is not proper.

Example 2 For $p = q = 1$, $n = 2$, the divisible BCK(P) lattice

$$\mathcal{D}_{2\times 2,3} = \mathcal{L}_{2\times 2} \bigoplus \mathcal{L}_3,$$

with the support set

$$D_{2\times 2,3} = L_{2\times 2} \bigcup L_3 = \{0, a, b, c\} \bigcup \{c, d, 1\} = \{0, a, b, c, d, 1\},$$

is organized as a lattice as in Figure 14 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables.

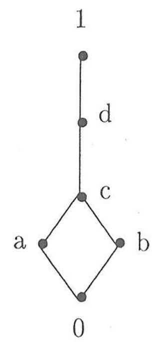

Figure 14: The divisible BCK(P) lattices $\mathcal{D}_{2\times 2,3}$ and $\mathcal{D}_{2\times 2,2,2}$

$\mathcal{D}_{2\times 2,3}$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

The condition $(C_\vee)$ is not verified, since there are a, b such that:

$$c = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (b \to b) \wedge (a \to a) = 1.$$

The divisible BCK(P) lattice $\mathcal{D}_{2\times 2,3}$ is not of Gödel type: there is d such that $d \odot d = c \neq d$.
It verifies the condition (P1).
It does not verify the condition (P2): there are $x = d$ $y = z = c$ such that

$$(z^-)^- \odot [(z \odot x) \to (z \odot y)] = 1 \odot [c \to c] = 1 \not\leq x \to y = d.$$

Hence, it is not of Product type.
It follows that it is proper, namely is the proper divisible BCK(P) lattice with the smallest number of elements.
Example 3 For $p = 1,\ q = 2,\ n = 1$, the divisible BCK(P) lattice

$$\mathcal{D}_{2\times 3,2} = \mathcal{L}_{2\times 3} \bigoplus \mathcal{L}_2,$$

with the support set

$$D_{2\times 3,2} = L_{2\times 3} \bigcup L_2 = \{0, a, b, c, d, n\} \bigcup \{n, 1\} = \{0, a, b, c, d, n, 1\},$$

is organized as a lattice as in Figure 15 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables.

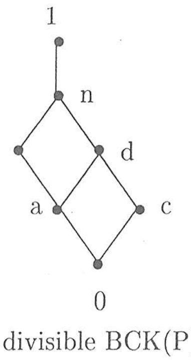

Figure 15: The divisible BCK(P) lattice $\mathcal{D}_{2\times 3,2}$

$\mathcal{D}_{2\times 3,2}$

$\rightarrow$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	d	1	1	d	1	1	1
b	c	d	1	c	d	1	1
c	b	b	b	1	1	1	1
d	a	b	b	d	1	1	1
n	0	a	b	c	d	1	1
1	0	a	b	c	d	n	1

$\odot$	0	a	b	c	d	n	1
0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	a
b	0	a	b	0	a	b	b
c	0	0	0	c	c	c	c
d	0	0	a	c	c	d	d
n	0	a	b	c	d	n	n
1	0	a	b	c	d	n	1

The condition $(C_\vee)$ is not verified, since there are b, d such that:

$$n = b \vee d \neq [(b \rightarrow d) \rightarrow d] \wedge [(d \rightarrow b) \rightarrow b] = (d \rightarrow d) \wedge (b \rightarrow b) = 1.$$

The divisible BCK(P) lattice $\mathcal{D}_{2\times 3,2}$ is not of Gödel type, since there is $a \odot a = 0 \neq a$.
It does not satisfies (P1), since there is $a \wedge a^- = a \wedge d = a \neq 0$.
It does not satisfies (P2), since there are $x = 1$, $y = z = n$ such that

$$(z^-)^- \odot [(z \odot x) \rightarrow (z \odot y)] = 1 \odot [n \rightarrow n] = 1 \not\leq x \rightarrow y = n.$$

Hence, it is not of Product type.
Hence, it is a proper divisible BCK(P) lattice.

11.2 Examples of the form: non-linearly ordered MV $\bigoplus$ non-linearly ordered MV

For $n, m, u, v \geq 1$, denote

$$\mathcal{D}_{(n+1)\times(m+1),(u+1)\times(v+1)} = \mathcal{L}_{(n+1)\times(m+1)} \bigoplus \mathcal{L}_{(u+1)\times(v+1)}.$$

We shall present only the case $n = m = u = v = 1$.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2\times 2,2\times 2} = \mathcal{L}_{2\times 2} \bigoplus \mathcal{L}_{2\times 2},$$

with the support set

$$D_{2\times 2,2\times 2} = L_{2\times 2} \bigcup L_{2\times 2} = \{0, a, b, n\} \bigcup \{n, c, d, 1\} = \{0, a, b, n, c, d, 1\},$$

is organized as a lattice as in Figure 16 and as a BCK(P) algebra with the operations $\rightarrow$ and $\odot$ as in the following tables.

$\mathcal{D}_{2\times 2,2\times 2}$

$\rightarrow$	0	a	b	n	c	d	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	a	a	1	1	1	1	1
n	0	a	b	1	1	1	1
c	0	a	b	d	1	d	1
d	0	a	b	c	c	1	1
1	0	a	b	n	c	d	1

$\odot$	0	a	b	n	c	d	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n
c	0	a	b	n	c	n	c
d	0	a	b	n	n	d	d
1	0	a	b	n	c	d	1

The divisible BCK(P) lattice $\mathcal{D}_{2\times 2,2\times 2}$ is of Gödel type.

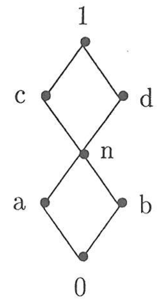

Figure 16: The divisible BCK(P) lattice $\mathcal{D}_{2\times 2,2\times 2}$

It satisfies the condition (P1).
It does not satisfy the condition (P2), since there are: $x = c$, $y = z = n$ such that

$$(Z^-)^- \odot [(z \odot x) \to (z \odot y)] = 1 \odot [n \to n] = 1 \not\leq x \to y = d.$$

Hence, it is not of Product type.
Hence, it is not proper.

11.3 Examples of the form: non-linearly ordered MV $\bigoplus$ linearly ordered BL or equivalent forms

Denote, for any $p, q, n, m \geq 1$,

$$\mathcal{D}_{(p+1)\times(q+1),n+1,m+1} = \mathcal{L}_{(p+1)\times(q+1)} \bigoplus \mathcal{H}_{n+1,m+1}.$$

We shall give here only one example, the case $p = q = n = m = 1$.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2\times 2,2,2} = \mathcal{L}_{2\times 2} \bigoplus \mathcal{H}_{2,2} = \mathcal{L}_{2\times 2} \bigoplus (\mathcal{L}_2 \bigoplus \mathcal{L}_2) = \mathcal{D}_{2\times 2,2} \bigoplus \mathcal{L}_2,$$

with the support set

$$D_{2\times 2,2,2} = L_{2\times 2} \bigcup H_{2,2} = \{0, a, b, c\} \bigcup \{c, d, 1\} = \{0, a, b, c, d, 1\},$$

is organized as a lattice as in Figure 14 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables.

$\mathcal{D}_{2\times 2,2,2}$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	a	a	1	1	1	1
c	0	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

The divisible BCK(P) lattice $\mathcal{D}_{2\times 2,2,2}$ is of Gödel type.
It verifies the condition (P1).

It does not verify the condition (P2): there are $x = d$, $y = z = c$ such that

$$(z^-)^- \odot [(z \odot x) \to (z \odot y)] = 1 \odot [c \to c] = 1 \not\leq x \to y = c.$$

Hence, it is not of Product type.
It follows that it is not proper.

11.4 Examples of the form: non-linearly ordered MV $\oplus$ non-linearly ordered BL or equivalent forms

Denote, for $m, n, p, u, v \geq 1$, the divisible BCK(P) lattice

$$\mathcal{D}_{(m+1)\times(n+1),p+1,(u+1)\times(v+1)} = \mathcal{L}_{(m+1)\times(n+1)} \bigoplus \mathcal{H}_{(p+1),(u+1)\times(v+1)}.$$

We shall consider only one example, for $m = n = p = u = v = 1$.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2\times2,2,2\times2} = \mathcal{L}_{2\times2} \bigoplus \mathcal{H}_{2,2\times2} = \mathcal{L}_{2\times2} \bigoplus(\mathcal{L}_2 \bigoplus \mathcal{L}_{2\times2}) = (\mathcal{L}_{2\times2} \bigoplus \mathcal{L}_2) \bigoplus \mathcal{L}_{2\times2} = (\mathcal{D}_{2\times2,2} \bigoplus(\mathcal{L}_{2\times2},$$

with the support set

$$D_{2\times2,2,2\times2} = L_{2\times2} \bigcup H_{2,2\times2} = \{0, a, b, p\} \bigcup \{p, n\} \bigcup \{n, c, d, 1\} = \{0, a, b, p, n, c, d, 1\},$$

is organized as a lattice as in Figure 17 and as a BCK(P) algebra with the operation $\to$ and $\odot$ as in the following tables.

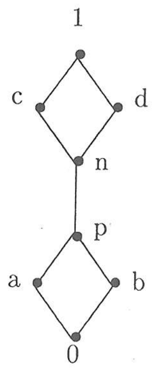

Figure 17: The divisible BCK(P) lattice $\mathcal{D}_{2\times2,2,2\times2}$

$\mathcal{D}_{2\times2,2,2\times2}$

$\to$	0	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
p	0	a	b	1	1	1	1	1
n	0	a	b	p	1	1	1	1
c	0	a	b	p	d	1	d	1
d	0	a	b	p	c	c	1	1
1	0	a	b	p	n	c	d	1

$\odot$	0	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
p	0	a	b	p	p	p	p	p
n	0	a	b	p	n	n	n	n
c	0	a	b	p	n	c	n	c
d	0	a	b	p	n	n	d	d
1	0	a	b	p	n	c	d	1

11.5 Examples of the form: non-linearly ordered BL $\oplus$ linearly ordered MV or equivalent forms

We consider here only one example among the very many possible ones.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2,2\times2,2} = \mathcal{H}_{2,2\times2} \bigoplus \mathcal{L}_2 = (\mathcal{L}_2 \bigoplus \mathcal{L}_{2\times2}) \bigoplus \mathcal{L}_2 = \mathcal{L}_2 \bigoplus (\mathcal{L}_{2\times2} \bigoplus \mathcal{L}_2) = \mathcal{L}_2 \bigoplus \mathcal{D}_{2\times2,2},$$

with the support set

$$D_{2,2\times2,2} = H_{2,2\times2} \bigcup L_2 = \{0, n, a, b, m\} \bigcup \{m, 1\} = \{0, n, a, b, m, 1\},$$

is organized as a lattice as in Figure 18 and as a BCK(P) algebra with the operations $\rightarrow$ and $\odot$ as in the following tables.

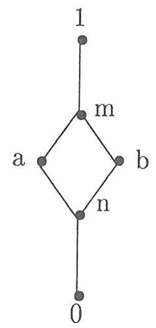

Figure 18: The divisible BCK(P) lattice $\mathcal{D}_{2,2\times2,2}$

$\mathcal{D}_{2,2\times2,2}$

$\rightarrow$	0	n	a	b	m	1
0	1	1	1	1	1	1
n	0	1	1	1	1	1
a	0	b	1	b	1	1
b	0	a	a	1	1	1
m	0	n	a	b	1	1
1	0	n	a	b	m	1

$\odot$	0	n	a	b	m	1
0	0	0	0	0	0	0
n	0	n	n	n	n	n
a	0	n	a	n	a	a
b	0	n	n	b	b	b
m	0	n	a	b	m	m
1	0	n	a	b	m	1

It does not verify the condition $(C_\vee)$, since there a, b such that:

$$m = a \vee b \neq [(a \rightarrow b) \rightarrow b] \wedge [(b \rightarrow a) \rightarrow a] = 1.$$

The divisible BCK(P) lattice $\mathcal{D}_{2,2\times2,2}$ is of Gödel type.
It verifies the condition (P1).
It does not verify the condition (P2): there are $x = 1$, $y = z = m$ such that

$$(z^-)^- \odot [(z \odot x) \rightarrow (z \odot y)] = 1 \odot [m \rightarrow m] = 1 \not\leq x \rightarrow y = m.$$

Hence, it is not of Product type.
It follows that it is not proper.

11.6 Examples of the form: non-linearly ordered BL $\oplus$ non-linearly ordered MV or equivalent forms

We shall present only one example.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2,2\times2,2\times2} = \mathcal{H}_{2,2\times2} \bigoplus \mathcal{L}_{2\times2} = (\mathcal{L}_2 \bigoplus \mathcal{L}_{2\times2}) \bigoplus \mathcal{L}_{2\times2} == \mathcal{L}_2 \bigoplus (\mathcal{L}_{2\times2}) \bigoplus \mathcal{L}_{2\times2}) == \mathcal{L}_2 \bigoplus \mathcal{D}_{2\times2,2\times2},$$

with the support set

$$D_{2,2\times2,2\times2} = H_{2,2\times2} \bigcup L_{2\times2} = \{-1, 0, a, b, n\} \bigcup \{n, c, d, 1\} = \{-1, 0, a, b, n, c, d, 1\},$$

is organized as a lattice as in Figure 19 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables.

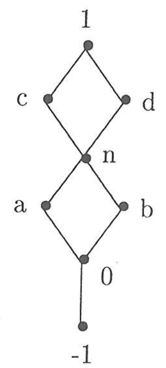

Figure 19: The divisible BCK(P) lattice $\mathcal{D}_{2,2\times2,2\times2}$

$\mathcal{D}_{2,2\times2,2\times2}$

$\to$	-1	0	a	b	n	c	d	1
-1	1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1	1
a	-1	b	1	b	1	1	1	1
b	-1	a	a	1	1	1	1	1
n	-1	0	a	b	1	1	1	1
c	-1	0	a	b	d	1	d	1
d	-1	0	a	b	c	c	1	1
1	-1	0	a	b	n	c	d	1

$\odot$	-1	0	a	b	n	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0	0
a	-1	0	a	0	a	a	a	a
b	-1	0	0	b	b	b	b	b
n	-1	0	a	b	n	n	n	n
c	-1	0	a	b	n	c	n	c
d	-1	0	a	b	n	n	d	d
1	-1	0	a	b	n	c	d	1

11.7 Examples of the form: non-linearly ordered BL $\oplus$ linearly ordered BL or equivalent forms

We shall present only one example.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2,2\times2,2,2} = \mathcal{H}_{2,2\times2} \bigoplus \mathcal{H}_{2,2} == (\mathcal{L}_2 \bigoplus \mathcal{L}_{2\times2}) \bigoplus (\mathcal{L}_2 \bigoplus \mathcal{L}_2) == \mathcal{L}_2 \bigoplus \mathcal{D}_{2\times2,2} \bigoplus \mathcal{L}_2,$$

with the support set

$$D_{2,2\times 2,2,2} = H_{2,2\times 2} \bigcup H_{2,2} = \{-1, 0, a, b, c\} \bigcup \{c, d, 1\} = \{-1, 0, a, b, c, d, 1\},$$

is organized as a lattice as in Figure 20 and as a BCK(P) algebra with the operations $\to$ and $\odot$ as in the following tables.

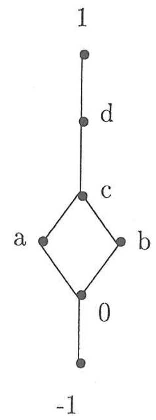

Figure 20: The divisible BCK(P) lattice $\mathcal{D}_{2,2\times 2,2,2}$

$\mathcal{D}_{2,2\times 2,2,2}$

$\to$	-1	0	a	b	c	d	1
-1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1
a	-1	b	1	b	1	1	1
b	-1	a	a	1	1	1	1
c	-1	0	a	b	1	1	1
d	-1	0	a	b	d	1	1
1	-1	0	a	b	c	d	1

$\odot$	-1	0	a	b	c	d	1
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0
a	-1	0	a	0	a	a	a
b	-1	0	0	b	b	b	b
c	-1	0	a	b	c	c	c
d	-1	0	a	b	c	c	d
1	-1	0	a	b	c	d	1

11.8 Examples of the form: non-linearly ordered BL $\bigoplus$ non-linearly ordered BL or equivalent forms

We shall give here only one example, among the very many possible ones.
Example The divisible BCK(P) lattice

$$\mathcal{D}_{2,2\times 2,2,2\times 2} = \mathcal{H}_{2,2\times 2} \bigoplus \mathcal{H}_{2,2\times 2} = (\mathcal{L}_2 \bigoplus \mathcal{L}_{2\times 2}) \bigoplus (\mathcal{L}_2 \bigoplus \mathcal{L}_{2\times 2}) =$$

$$= \mathcal{L}_2 \bigoplus \mathcal{D}_{2\times 2,2} \bigoplus \mathcal{L}_{2\times 2} == \mathcal{D}_{2,2\times 2,2} \bigoplus \mathcal{L}_{2\times 2},$$

with the support set

$$D_{2,2\times 2,2} = H_{2,2\times 2} \bigcup H_{2,2\times 2} = \{0, m, a, b, p\} \bigcup \{p, n, c, d, 1\} = \{0, m, a, b, p, n, c, d, 1\},$$

is organized as a lattice as in Figure 21 and as a BCK(P) algebra with the operation $\to$ and $\odot$ as in the following tables.

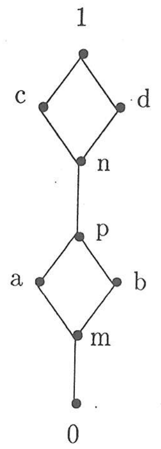

Figure 21: The divisible BCK(P) lattice $\mathcal{D}_{2,2\times 2,2,2\times 2}$

$\mathcal{D}_{2,2\times 2,2,2\times 2}$

$\rightarrow$	0	m	a	b	p	n	c	d	1
0	1	1	1	1	1	1	1	1	1
m	0	1	1	1	1	1	1	1	1
a	0	b	1	b	1	1	1	1	1
b	0	a	a	1	1	1	1	1	1
p	0	m	a	b	1	1	1	1	1
n	0	m	a	b	p	1	1	1	1
c	0	m	a	b	p	d	1	d	1
d	0	m	a	b	p	c	c	1	1
1	0	m	a	b	p	n	c	d	1

$\odot$	0	m	a	b	p	n	c	d	1
0	0	0	0	0	0	0	0	0	0
m	0	m	m	m	m	m	m	m	m
a	0	m	a	m	a	a	a	a	a
b	0	m	m	b	b	b	b	b	b
p	0	m	a	b	p	p	p	p	p
n	0	m	a	b	p	n	n	n	n
c	0	m	a	b	p	n	c	n	c
d	0	m	a	b	p	n	n	d	d
1	0	m	a	b	p	n	c	d	1

The divisible BCK(P) lattice $\mathcal{D}_{2,2\times 2,2,2\times 2}$ is of Gödel type.
It verifies the condition (P1).
It does not verify the condition (P2): there are $x = 1$, $y = z = m$ such that

$$(z^-)^- \odot [(z \odot x) \rightarrow (z \odot y)] = 1 \odot [m \rightarrow m] = 1 \not\leq x \rightarrow y = m.$$

Hence, it is not of Product type.
It follows that it is not proper.

12 Example of infinite proper divisible BCK(P) latice

Consider the non-linear Wajsberg (MV) algebra $\mathcal{L}_{2\times 2}$ from Figure 3. Then, the set $D_{2\times 2,P(Z)} = L_{2\times 2} \bigcup P(Z) = L_{2\times 2} \bigcup (Z^- \bigcup \{-\infty\}) =$
$\{0_{-\infty}, a, b, -\infty\} \bigcup \{-\infty, \ldots, -3, -2, -1, 0\} = \{0_{-\infty}, a, b, -\infty, \ldots, -3, -2, -1, 0\}$,
organized as a lattice as in Figure 22, with the operations $\rightarrow$ and $\odot$ defined by the following tables, is a non-linearly ordered divisible BCK(P) lattice, denoted by $\mathcal{D}_{2\times 2,P(Z)} = \mathcal{L}_{2\times 2} \bigoplus \mathcal{P}(Z)$.

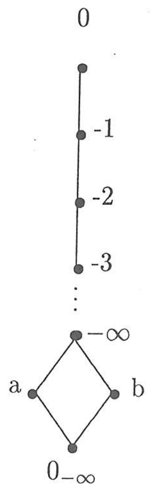

Figure 22: The infinite, proper, divisible BCK(P) lattice $\mathcal{D}_{2\times 2,\mathcal{P}(Z)}$

$\mathcal{D}_{2\times 2,P(Z)}$

$\to$	$0_{-\infty}$	a	b	$-\infty$	...	-3	-2	-1	0
$0_{-\infty}$	0	0	0	0	...	0	0	0	0
a	b	0	b	0	...	0	0	0	0
b	a	a	0	0	...	0	0	0	0
$-\infty$	$0_{-\infty}$	a	b	0	...	0	0	0	0
$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
-3	$0_{-\infty}$	a	b	$-\infty$	...	0	0	0	0
-2	$0_{-\infty}$	a	b	$-\infty$	...	-1	0	0	0
-1	$0_{-\infty}$	a	b	$-\infty$	...	-2	-1	0	0
0	$0_{-\infty}$	a	b	$-\infty$	...	-3	-2	-1	0

$\mathcal{D}_{2\times 2,P(Z)}$

$\odot$	$0_{-\infty}$	a	b	$-\infty$	...	-3	-2	-1	0
$0_{-\infty}$	$0_{-\infty}$	$0_{-\infty}$	$0_{-\infty}$	$0_{-\infty}$	...	$0_{-\infty}$	$0_{-\infty}$	$0_{-\infty}$	$0_{-\infty}$
a	$0_{-\infty}$	a	$0_{-\infty}$	a	...	a	a	a	a
b	$0_{-\infty}$	$0_{-\infty}$	b	b	...	b	b	b	b
$-\infty$	$0_{-\infty}$	a	b	$-\infty$	...	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$0_{-\infty}$	a	b	$-\infty$	...	-6	-5	-4	-3
-2	$0_{-\infty}$	a	b	$-\infty$	...	-5	-4	-3	-2
-1	$0_{-\infty}$	a	b	$-\infty$	...	-4	-3	-2	-1
0	$0_{-\infty}$	a	b	$-\infty$	...	-3	-2	-1	0

Note that it is not of Gödel type. It satisfies (P1), but not (P2), since there are $x = -3$, $y = z = -\infty$, such that:

$$(z^-)^- \odot [(z \odot x) \to (z \odot y)] = 0 \odot (-\infty \to -\infty) = 0 \not\leq x \to y = -\infty.$$

Hence, it is not of Product type eather. Hence, $\mathcal{D}_{2\times 2,P(Z)}$ is an infinite, proper, divisible BCK(P) lattice.

13 Conclusions and open problems

We have the following hierarchies:

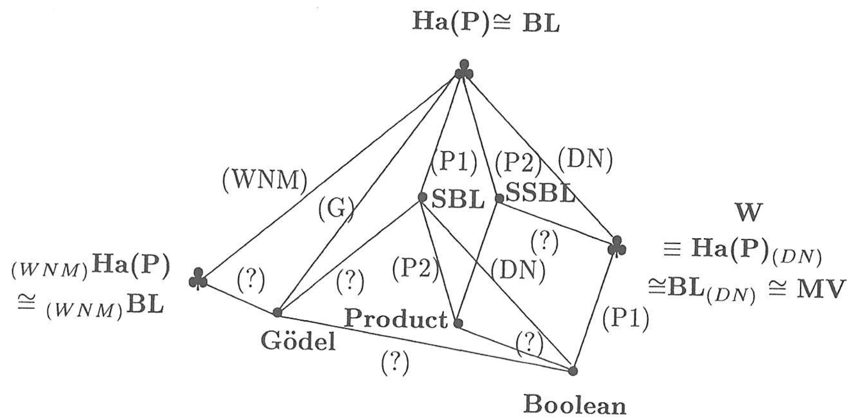

Figure 23: Particular cases of Hájek(P) (BL) algebras

By examining the given examples of Wajsberg (MV) and Hájek(P) (BL) algebras, we conclude the followings:

1) The Product algebras and $_{(WNM)}$ Wajsberg (MV) algebras are incomparable. Indeed, $\mathcal{L}_3$ is a $_{(WNM)}$MV algebra not satisfying (P1) and the Product algebra $\mathcal{P}(Z)$ does not verify both the conditions (DN) and (WNM).

2) In the chain $\mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_{n+1}$ $(n \geq 1)$ of liniarly ordered Wajsberg (MV) algebras, The first two, $\mathcal{L}_2$ and $\mathcal{L}_3$, are $_{(WNM)}$Wajsberg (MV) algebras, $\mathcal{L}_2$ being even a Boolean algebra.

3) By [16], we have:

$\mathbf{NR}_0$ + (PIMP) $\equiv$ **Boolean,**

where the condition (PIMP) is:

$$(PIMP) \qquad x \to (x \to y) = x \to y.$$

It follows that we have:

$_{(WNM)}$**MV** + (PIMP) $\equiv$ **Boolean.**

Hence, we have the hierachies from Figure 24.

Following the examples from Sections 7-11, two groups of open problems raised.

Open problems 13.1

(0) Is any BL subalgebra of a BL algebra either an MV algebra or a Product algebra or an ordinal sum: "linearly ordered BL algebra $\bigoplus$ BL algebra"?

(1) By (0), it is possible to define reccurently BL algebras ? An ideea is the following:

Reccurent definition of BL algebras ?

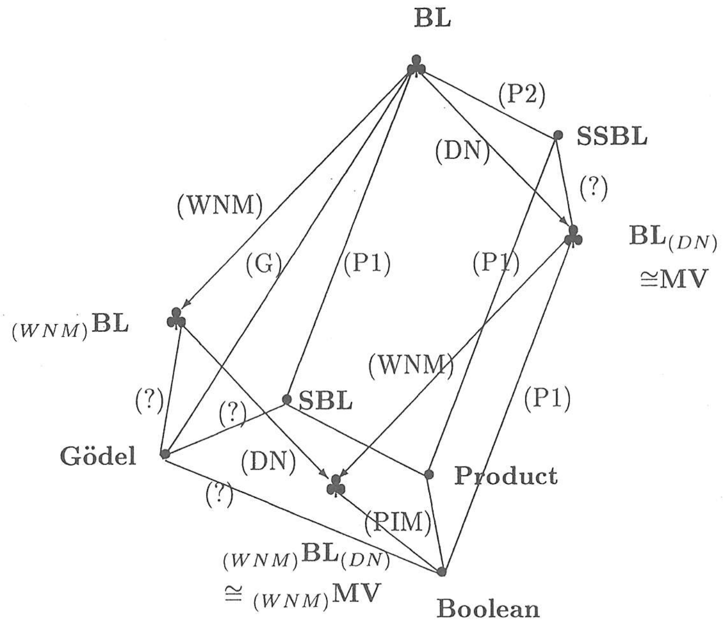

Figure 24: Some descendents (particular cases) of BL algebras

(i) MV algebras and Product algebras are BL algebras;

(ii) The direct product of two linearly ordered BL algebras is a BL algebra;

(iii) The isomorphic immage (coppy) of a BL algebra is a BL algebra;

(iv) The ordinal sum "linearly ordered BL algebra $\bigoplus$ BL algebra" is a BL algebra;

(v) Every BL algebra is obtained by applying the rules (i)-(iv) in a finite number of times.

Open problem 13.2
(1) Is it possible to define reccurently the divisible BCK(P) lattices? An ideea is the following:
Reccurent definition of divisible BCK(P) lattices ?

(j) Every BL algebra is a divisible BCK(P) lattice;

(jj) The direct product of two divisible BCK(P) lattices is a divisible BCK(P) lattice;

(jjj) The isomorphic immage (coppy) of a divisible BCK(P) lattice is a divisible BCK(P) lattice;

(jv) The ordinal sum: "non-linearly ordered divisible BCK(P) lattice $\bigoplus$ linearly ordered divisible BCK(P) lattice (i.e. BL algebra)" is a divisible BCK(P) lattice;

(jjj) Every divisible BCK(P) lattice is obtained by applying the rules (j) - (jv) in a finite number of times.

(2) Find a representation theorem for divisible BCK(P) lattices.

References

[1] C. C. CHANG, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* 88, 1958, 467-490.

[2] R. CIGNOLI, A. TORRENS An algebraic analysis of Product logic, Mult. Val. Logic 5, 2000, 45-65.

[3] R. CIGNOLI, F. ESTEVA, L. GODO, A. TORRENS, Basic Fuzzy Logic is the logic of continuous t-norms and their residua, *Soft Computing*, to appear.

[4] W.H. CORNISH, Lattice-ordered groups and BCK-algebras, *Math. Japonica*, 25, No. 4, 1980, 471-476.

[5] F. ESTEVA, L. GODO, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems*, Vol. 124, No. 3, 2001, 271-288.

[6] P. FLONDOR, G. GEORGESCU, A. IORGULESCU, Pseudo-t-norms and pseudo-BL algebras, *Soft Computing*, 5, No 5, 2001, 355-371.

[7] J.C. FODOR, Contrapositive symmetry of fuzzy implications, Fuzzy Sets and Systems,69, 1995, 141-156.

[8] J.M. FONT, A. J. RODRIGUEZ, A. TORRENS, Wajsberg algebras, *Stochastica* Vol. VIII, No. 1, 1984, 5-31.

[9] S. Gottwald, *A Treatise on Many-Valued Logics.* Studies in Logic and Computation, vol. 9, Research Studies Press: Baldock, Hertfordshire, England, 2001.

[10] R. GRIGOLIA, Algebraic analysis of Lukasiewicz-Tarski's n-valued logical systems, in: Selected Papers on Lukasiewicz Sentential Calculi (R. Wójcicki and G. Malinowski, Eds.), 81-92, *Polish Acad. of Sciences*, Ossolineum, Wroclaw, 1977.

[11] A. GRZAŚLEWICZ, On some problem on BCK-algebras, *Math. Japonica* 25, No. 4, 1980, 497-500.

[12] P. HÁJEK, Metamathematics of fuzzy logic, *Inst. of Comp. Science, Academy of Science of Czech Rep., Technical report* 682, 1996.

[13] P. HÁJEK, Metamathematics of fuzzy logic, Kluwer Acad. Publ., Dordrecht, 1998.

[14] P. HÁJEK, Basic fuzzy logic and BL-algebras, *Soft computing*, 2, 1998, 124-128.

[15] P. M. IDZIAK, Lattice operations in BCK-algebras, *Mathematica Japonica* 29, 1984, 839-846.

[16] Y.L. LIU, S.Y. LIU, X.H. ZHANG, Some Classes of R_0-algebras, to appear.

[17] A. IORGULESCU, Connections between MV_n algebras and n-valued Lukasiewicz-Moisil algebras - I, *Discrete Mathematics*, 181 (1-3),155-177, 1998.

[18] A. IORGULESCU, Connections between MV_n algebras and n-valued Lukasiewicz-Moisil algebras - II, *Discrete Mathematics* , 202, 113-134, 1999.

[19] A. IORGULESCU, Connections between MV_n algebras and n-valued Lukasiewicz-Moisil algebras - III, *Discrete Mathematics*, submitted.

[20] A. IORGULESCU, Connections between MV_n algebras and n-valued Lukasiewicz-Moisil algebras - IV, *Journal of Universal Computer Science*, vol. 6, no I(2000), 139-154.

[21] A. IORGULESCU, Iséki algebras. Connection with BL algebras, *Soft Computing*, to appear.

[22] A. IORGULESCU, Some direct ascendents of Wajsberg and MV algebras, *Scientiae Mathematicae Japonicae*, Vol. 57, No. 3, 2003, 583-647.

[23] A. IORGULESCU, Classes of BCK algebras-Part I, manuscript

[24] A. IORGULESCU, Classes of BCK algebras-Part II, manuscript

[25] K. ISÉKI, S. TANAKA, An introduction to the theory of BCK-algebras, *Math. Japonica* 23, No.1, 1978, 1-26.

[26] J. KALMAN, Lattices with involution, *Trans. Amer. Math. Soc.* 87, 1958, 485-491.

[27] E.P. KLEMENT, R. MESIAR, E. PAP, Triangular norms, Kluwer Academic Publishers, 2000.

[28] T. KOWALSKI, H. ONO, Residuated lattices: An algebraic glimpse at logics without contraction, monograph, 2001

[29] E. TURUNEN, Mathematics Behind Fuzzy Logic, Physica-Verlag, 1999

[30] M. WAJSBERG, Beiträge zum Mataaussagenkalkül, *Monat. Math. Phys.* 42, 1935, p. 240.