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CLASSES OF BCK ALGEBRAS – PART IV

by

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Classes of BCK algebras - Part IV

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Dedicated to Grigore C. Moisil (1905-1973)

(10 January 2004)

Abstract

In this paper we study the BCK algebras and their particular classes: the BCK(P) (residuated) lattices, the Hájek(P) (BL) algebras and the Wajsberg (MV) algebras, we introduce new classes of BCK(P) lattices, we establish hierarchies and we give many examples. The paper has five parts.

In the first part, the most important part, we decompose the divisibility and the pre-linearity conditions from the definition of a BL algebra into four new conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). We study the additional conditions (WNM) (weak nilpotent minimum) and (DN) (double negation) on a BCK(P) lattice. We introduce the ordinal sum of two BCK(P) lattices and prove in what conditions we get BL algebras or other structures, more general, or more particular than BL algebras.

In part II, we give examples of some finite bounded BCK algebras. We introduce new generalizations of BL algebras, named α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$ and (C_X). By adding the conditions (WNM) and (DN) to these classes, we get more classes; among them, we get many generalizations of Wajsberg (MV) algebras and of R_0 (NM) algebras. The subclasses of $_{(WNM)}$ Wajsberg algebras ($_{(WNM)}$MV algebras) and of $_{(WNM)}$Hájek algebras ($_{(WNM)}$BL algebras) are introduced. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I.

In part III, we give examples of finite MV and $_{(WNM)}$MV algebras, of Hájek(P) (i.e. BL) algebras and $_{(WNM)}$BL algebras and of $\alpha\gamma\delta$ (i.e. divisible BCK(P) lattices (divisible residuated lattices or divisible integral, residuated, commutative l-monoids)) and of divisible $_{(WNM)}$BCK(P) lattices.

In part IV, we stress the importance of $\alpha\beta\gamma$ algebras versus $\alpha\beta$ (i.e. MTL) algebras algebras and of R_0 (i.e. NM) algebras versus Wajsberg (i.e. MV) algebras and of $_{(WNM)}\alpha\beta\gamma$ algebras versus BL algebras and of $\alpha\gamma$ versus $\alpha\gamma\delta$ algebras. We give examples of finite IMTL algebras and of $_{(WNM)}$IMTL (i.e. NM) algebras), of $\alpha\beta\gamma$ algebras and of $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras and finally of $\alpha\gamma$ algebras.

In part V, we give other examples of finite BCK(P) lattices, finding examples for the others remaining an open problem. We make final remarks and formulate final open problems.

Keywords MV algebra, Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, divisible BCK(P) lattice, α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, $_{(WNM)}$MV, $_{(WNM)}$BL, $_{(WNM)}$ $\alpha\beta\gamma$, Roman algebra

Part IV has four sections.
In Section 14, we give examples of proper IMTL algebras and of NM algebras.
In Section 15, we give examples of proper $\alpha\beta\gamma$ and of $_{(WNM)}\alpha\beta\gamma$ algebras.
In Section 16, we give examples of proper $\alpha\gamma$ algebras.
In Section 17, we formulate some remarks and open problems.
By cutting with vertical planes, we get the hierarchies from Figures 10 and 2, for examples.

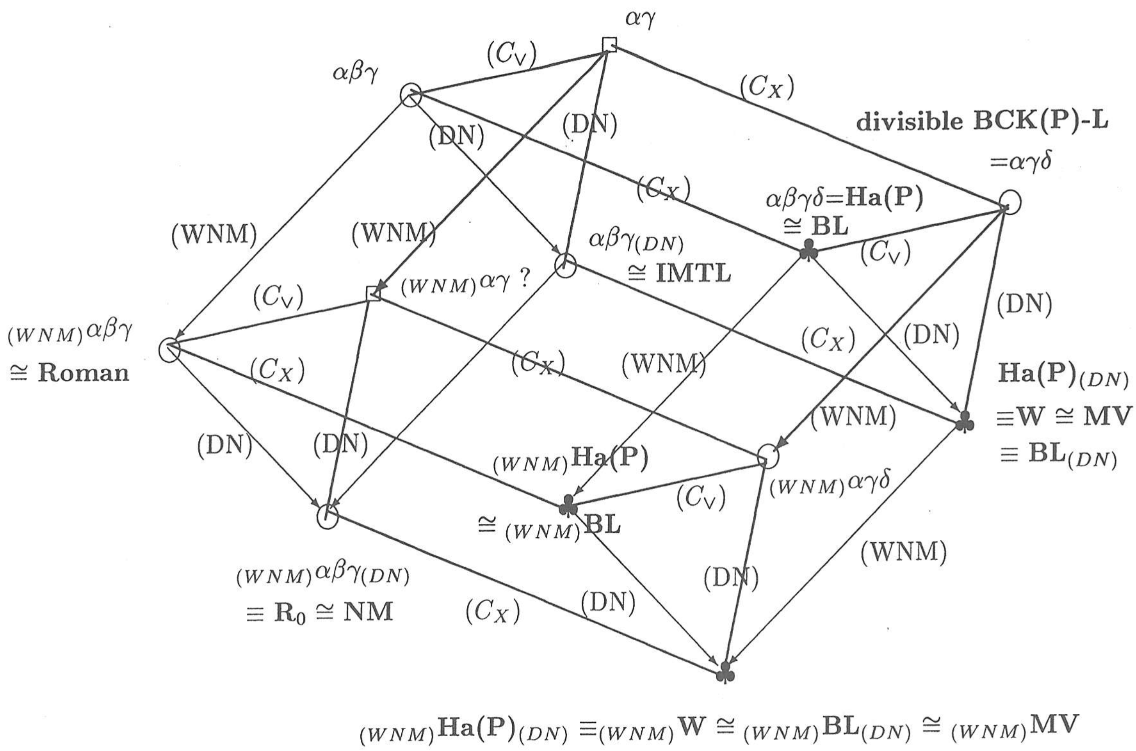

Figure 1: "Vertical" sections through $\alpha\gamma$, $\alpha\beta\gamma$, $\alpha\gamma\delta$ (divisible residuated lattices) and $\alpha\beta\gamma\delta$ (BL) algebras

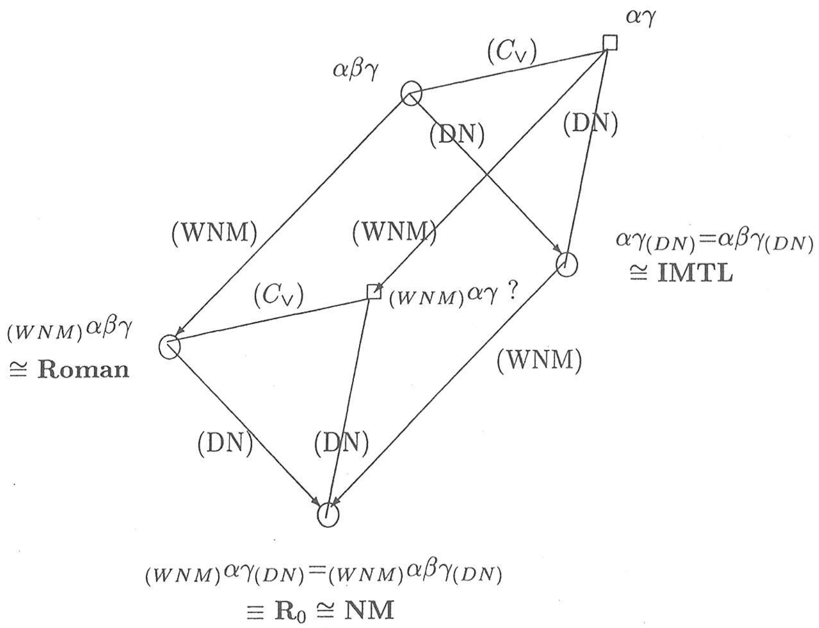

Figure 2: "Vertical" sections through $\alpha\gamma$, $\alpha\beta\gamma$

14 Examples of proper $_{(WNM)}\alpha\beta\gamma_{(DN)}$ (NM) algebras and proper $\alpha\beta\gamma_{(DN)}$ algebras (IMTL algebras)

First recall that, by Proposition 3.14 (obtained in [23]), we have:

$\alpha\beta\gamma_{(DN)}=\alpha\gamma_{(DN)}=\alpha\beta_{(DN)}$ $\equiv$ **weak-R$_0$** $\cong$ **IMTL** =**MTL** +(DN) and

$_{(WNM)}\alpha\beta\gamma_{(DN)}={}_{(WNM)}\alpha\gamma_{(DN)}={}_{(WNM)}\alpha\beta_{(DN)}$ $\equiv$ **R$_0$** =**weak-R$_0$** + (R6)$\cong$ **IMTL** + (WNM)=**NM.**

We give here examples of finite IMTL and NM algebras. You can find examples of infinite IMTL and NM algebras in [4], where the set A is the real interval $[0,1]$.

14.1 Examples of finite linearly ordered, proper NM and IMTL algebras

Recall (see [25]) that for each $n \geq 1$, the chain $L_{n+1} = \{0,1,2,\ldots,n\}$ can be organized as Wajsberg (left-MV) algebra $\mathcal{L}_{n+1}$ by using Lukasiewicz's implication $\to_L$ and t-norm $\odot_L$:

$$x \to_L y = \begin{cases} n, & \text{if } x \leq y \\ (n-x)+y, & \text{if } x > y \end{cases} = \min(n,(n-x)+y), \quad x \odot_L y = (x \to_L y^-)^- = \max(0, x+y-n).$$

14.1.1 Examples of linearly ordered, finite, proper NM algebras

For each $n \geq 1$, let us consider the chain $L_{n+1} = \{0,1,2,\ldots,n\}$, organized as a lattice by $\wedge = \min$ and $\vee = \max$, and as a BCK(P) algebra in the following way: we take the strong negation $^-$, defined on L_{n+1} by $x^- = n - x$, and Fodor's implication and t-norm $\to_F$ and $\odot_F$ [7], [4]:

$$x \to_F y = \begin{cases} n, & \text{if } x \leq y \\ \max(n-x,y), & \text{if } x > y \end{cases} \qquad x \odot_F y = (x \to_F y^-)^- = \begin{cases} 0, & \text{if } x \leq n-y \\ \min(x,y), & \text{if } x > n-y. \end{cases}$$

Hence, for $n = 1,2,3,4,5$, we have the BCK(P) lattices: $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, whose tables are the following:

$\mathcal{F}_2$

$\to_F$	0	1
0	1	1
1	0	1

$\odot_F$	0	1
0	0	0
1	0	1

$\mathcal{F}_3$

$\to_F$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

$\odot_F$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

$\mathcal{F}_4$

$\to_F$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	1	3	3
3	0	1	2	3

$\odot_F$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	**2**	2
3	0	1	2	3

$\mathcal{F}_5$

$\to_F$	0	1	2	3	4
0	4	4	4	4	4
1	3	4	4	4	4
2	2	**2**	4	4	4
3	1	1	**2**	4	4
4	0	1	2	3	4

$\odot_F$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	**2**	2
3	0	0	**2**	**3**	3
4	0	1	2	3	4

$\mathcal{F}_6$

$\to_F$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	**3**	5	5	5	5
3	2	2	**2**	5	5	5
4	1	**1**	**2**	**3**	5	5
5	0	1	2	3	4	5

$\odot_F$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	**2**	2
3	0	0	0	**3**	**3**	3
4	0	0	**2**	**3**	**4**	4
5	0	1	2	3	4	5

Remark 14.1 Note that $\mathcal{F}_2 = \mathcal{L}_2$ and $\mathcal{F}_3 = \mathcal{L}_3$,
i.e. $\mathcal{F}_2$ and $\mathcal{F}_3$ are examples of linearly ordered ${}_{(WNM)}$Wajsberg (${}_{(WNM)}$MV) algebras.
Note that $\mathcal{NM}_2 = \mathcal{L}_2$ is even a Boolean algebra.
Note also that $\mathcal{NM}_3 = \mathcal{L}_3$ verifies the condition (P2), but does not verifies the condition (P1).
Note that $\mathcal{F}_4$ and $\mathcal{F}_5$ appear in [30] as examples of R_0 algebras.

For each $n \geq 1$, $\mathcal{F}_{n+1}$ is a linearly ordered BCK(P) lattice (with condition (DN)), hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

Denote $\to = \to_F$ and $\odot = \odot_F$.

Note that $\mathcal{F}_4$ does not satisfy the condition (C_X), since there exist $1, 2 \in L_4$, such that

$$1 = 1 \odot 3 = 1 \odot (1 \to 3) = 1 \odot [(2 \to 1) \to (1 \to 2)] \neq 2 \odot [(1 \to 2) \to (2 \to 1)] = 2 \odot (3 \to 1) = 2 \odot 1 = 0.$$

Note that $\mathcal{F}_5$ does not satisfy the condition (C_X), since there exist $1, 2 \in L_5$, such that
$1 = 1 \odot 4 = 1 \odot (2 \to 4) = 1 \odot [(2 \to 1) \to (1 \to 2)] \neq$
$2 \odot [(1 \to 2) \to (2 \to 1)] = 2 \odot (4 \to 2) = 2 \odot 2 = 0$.

Note that, for each $n \geq 3$, $\mathcal{F}_{n+1}$ does not satisfy the condition (C_X), since there exist $1, 2 \in L_{n+1}$, such that, since $2 \to 1 = \max(n-2, 10 = n-2$, $1 \to 2 = \max(n-1, 2) = n-1$, then
$1 = \min(1, n) = 1 \odot n = 1 \odot ((n-2) \to (n-1)) = 1 \odot [(2 \to 1) \to (1 \to 2)] \neq$
$2 \odot [(1 \to 2) \to (2 \to 1)] = 2 \odot ((n-1) \to (n-2)) = 2 \odot \max(n-(n-1), n-2) = 2 \odot (n-2) = 0$.

Note that, for each $n \geq 3$, $\mathcal{F}_{n+1}$ satisfy the condition (WNM).

Hence, $\mathcal{F}_{n+1} = (L_{n+1}, \max, \min, \to_F, 0, 1)$ $(n \geq 3)$ is a linearly ordered $R_0 \equiv {}_{(WNM)}\alpha\beta\gamma_{(DN)}$ algebra (NM algebra) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Remark 14.2

The Wajsberg (MV) algebras and the R$_0$ (NM) algebras are incomparable: there are R_0 (NM) algebras (for example $\mathcal{F}_4$) which are not Wajsberg (MV) algebras and there are Wajsberg (MV) algebras which are not R_0 algebras) (for example $\mathcal{L}_4$).

The intersection of the two classes is the subclass of ${}_{(WNM)}$Wajsberg (${}_{(WNM)}$MV) algebras, which contains for example the algebras $\mathcal{F}_2 = \mathcal{L}_2$ and $\mathcal{F}_3 = \mathcal{L}_3$.

Open problem 14.3

We recall that the liniarly ordered Wajsberg (MV) algebras $\mathcal{L}_{n+1}$, $n \geq 1$ (see this paper, Part II), can be organized as MV_{n+1} algebras [10] and that every MV_{n+1} algebra is an $n+1$-valued Lukasiewicz-Moisil algebra [17], [18], [19], [20]. Define similarly NM_{n+1} algebras and define the analogous generalizations of $n+1$-valued Lukasiewicz-Moisil algebras.

14.1.2 Examples of linearly ordered, finite, proper IMTL algebras

The following examples (as well $\mathcal{F}_6$) are taken from [11] (in [11] are given as examples of bounded BCK algebras with condition (DN), which are not ($\vee$-) commutative).

Example 1: n=4

Let us consider the set $L_{n+1} = L_5 = \{0,1,2,3,4\}$ organized as a lattice by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$ as in the following tables:

$IMTL_5$

$\to$	0	1	2	3	4
0	4	4	4	4	4
1	3	4	4	4	4
2	2	**3**	4	4	4
3	1	**3**	**3**	4	4
4	0	1	2	3	4

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	**1**	2
3	0	0	**1**	**1**	3
4	0	1	2	3	4

Then $IMTL_5 = (L_5, \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, with condition (DN), hence $IMTL_5$ satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

Note $IMTL_5$ does not satisfy the condition (C_X), since there exist $2, 3 \in L_5$, such that

$$2 = 2 \odot 4 = 2 \odot (3 \to 4) = 2 \odot [(3 \to 2) \to (2 \to 3)] \neq 3 \odot [(2 \to 3) \to (3 \to 2)] = 3 \odot (4 \to 3) = 3 \odot 3 = 1.$$

Note also that $IMTL_5$ does not satify the condition (WNM), since there exist $2, 3 \in L_5$, such that:

$$(2 \odot 3)^- \vee [(2 \wedge 3) \to (2 \odot 3)] = 1^- \vee [2 \to 1] = 3 \vee 3 = 3 \neq 4.$$

Consequently, $IMTL_5$ is a linearly ordered, proper $\alpha\beta\gamma_{(DN)}$ (IMTL) algebra (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Example 2: n=5

Let us consider the set $L_{n+1} = L_6 = \{0,1,2,3,4,5\}$ organized as a lattice by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ and as a BCK(P) algebra in six different ways, with the operations $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$ as in the following tables:

$IMTL_6^1$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	**4**	5	5	5	5
3	2	**4**	**4**	5	5	5
4	1	**4**	**4**	**4**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	**1**	2
3	0	0	0	**1**	**1**	3
4	0	0	**1**	**1**	**1**	4
5	0	1	2	3	4	5

$IMTL_6^2$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	**4**	5	5	5	5
3	2	**4**	**4**	5	5	5
4	1	**3**	**4**	**4**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	**1**	2
3	0	0	0	**1**	**1**	3
4	0	0	**1**	**1**	**2**	4
5	0	1	2	3	4	5

$IMTL_6^3$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	**4**	5	5	5	5
3	2	**2**	**2**	5	5	5
4	1	**2**	**2**	**4**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	**1**	2
3	0	0	0	**3**	**3**	3
4	0	0	**1**	**3**	**3**	4
5	0	1	2	3	4	5

$IMTL_6^4$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	**3**	5	5	5	5
3	2	**3**	**4**	5	5	5
4	1	**1**	**3**	**3**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	**2**	2
3	0	0	0	**1**	**2**	3
4	0	0	**2**	**2**	**4**	4
5	0	1	2	3	4	5

$IMTL_6^5$

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	4	5	5	5	5	5
2	3	**3**	5	5	5	5
3	2	**2**	**3**	5	5	5
4	1	**1**	**2**	**3**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	**2**	2
3	0	0	0	**2**	**3**	3
4	0	0	**2**	**3**	**4**	4
5	0	1	2	3	4	5

Note that $IMTL_6^1$ - $IMTL_6^5$ do not satisfy the condition (C_X), hence they are not Wajsberg algebras..

Note that $IMTL_6^1$ - $IMTL_6^3$ do not satisfy the condition (WNM), since there exist $2, 4$ such that:

$$(2 \odot 4)^- \vee [(2 \wedge 4) \to (2 \odot 4)] = 1^- \vee [2 \to 1] = 4 \vee 4 = 4 \neq 5.$$

Note that $IMTL_6^4$ does not satisfy the condition (WNM), since there exist $3, 3$ such that:

$$(3 \odot 3)^- \vee [(3 \wedge 3) \to (3 \odot 3)] = 1^- \vee [3 \to 1] = 4 \vee 3 = 4 \neq 5.$$

Note also that $IMTL_6^5$ does not satisfy the condition (WNM), since there exist $3, 3$ such that:

$$(3 \odot 3)^- \vee [(3 \wedge 3) \to (3 \odot 3)] = 2^- \vee [3 \to 2] = 3 \vee 3 = 3 \neq 5.$$

Consequently, $IMTL_6^1$ - $IMTL_6^5$ are linearly ordered, proper IMTL algebras.

Remark 14.4 Let us consider the chain $L_{n+1} = \{0, 1, 2, \ldots, n\}$, for all $n \geq 1$. Note that:
- for $n = 1$, L_2 can be organized in a single way as BCK(P) lattice, namely as Boolean algebra and consequently as $_{(WNM)}$ Wajsberg ($_{(WNM)}$MV) algebra, $\mathcal{B}_2 = \mathcal{F}_2 = \mathcal{L}_2$;
- for $n = 2$, L_3 can be organized in a single way as BCK(P) lattice, namely as $_{(WNM)}$ Wajsberg ($_{(WNM)}$MV) algebra, $\mathcal{F}_3 = \mathcal{L}_3$;
- for $n = 3$, L_4 can be organized in two ways as BCK(P) lattice, namely as Wajsberg (MV) algebra $\mathcal{L}_4$ and as $R_0 \equiv {}_{(WNM)}\alpha\beta\gamma_{(DN)}$ (NM) algebra $\mathcal{F}_4$;
- for $n = 4$, L_5 can be organized in three ways as BCK(P) lattice, namely as Wajsberg (MV) algebra $\mathcal{L}_5$, as $R_0 \equiv {}_{(WNM)}\alpha\beta\gamma_{(DN)}$ (NM) algebra $\mathcal{F}_5$ and as IMTL algebra $IMTL_5$;
- for $n = 5$, L_6 can be organized in seven ways as BCK(P) lattice, namely as Wajsberg (MV) algebra $\mathcal{L}_6$, as $R_0 \equiv {}_{(WNM)}\alpha\beta\gamma_{(DN)}$ (NM) algebra $\mathcal{F}_6$ and as IMTL algebras $IMTL_6^1$ - $IMTL_6^5$.

14.2 Examples of non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma_{(DN)}$ (NM) algebras and $\alpha\beta\gamma_{(DN)}$ (IMTL) algebras

Isomorphic copies of direct products of above mentioned linearly ordered NM (IMTL) algebras will be examples of nonlinearly ordered NM (IMTL respectively) algebras. For instance:

- the following are examples of not proper NM algebras; they are proper $_{(WNM)}$Wajsberg algebras:

$\mathcal{F}_{2\times 2} \cong \mathcal{F}_2 \times \mathcal{F}_2 = \mathcal{L}_2 \times \mathcal{L}_2 \cong \mathcal{L}_{2\times 2}$ (see [25]),

$\mathcal{F}_{3\times 2} \cong \mathcal{F}_3 \times \mathcal{F}_2 = \mathcal{L}_3 \times \mathcal{L}_2 \cong \mathcal{L}_{3\times 2}$ (see [25]),

$\mathcal{F}_{3\times 3} \cong \mathcal{F}_3 \times \mathcal{F}_3 = \mathcal{L}_3 \times \mathcal{L}_3 \cong \mathcal{L}_{3\times 3}$ (see [25]) etc.;

- the following are examples of proper NM algebras:

$\mathcal{F}_{4\times 2} \cong \mathcal{F}_4 \times \mathcal{F}_2 = \mathcal{F}_4 \times \mathcal{L}_2$,

$\mathcal{F}_{4\times 3} \cong \mathcal{F}_4 \times \mathcal{F}_3 = \mathcal{F}_4 \times \mathcal{L}_3$,

$\mathcal{F}_{4\times 4} \cong \mathcal{F}_4 \times \mathcal{F}_4$ etc.;

- the following are examples of proper IMTL algebras:

$IMTL_{5\times 2} \cong IMTL_5 \times \mathcal{F}_2 = IMTL_5 \times \mathcal{L}_2$,

$IMTL_{5\times 3} \cong IMTL_5 \times \mathcal{F}_3$,

$IMTL^F_{5\times 4} \cong IMTL_5 \times \mathcal{F}_4$ and $IMTL^L_{5\times 4} \cong IMTL_5 \times \mathcal{L}_4$,

$IMTL_{5\times 5} \cong IMTL_5 \times IMTL_5$,

$IMTL^F_{5\times 5} \cong IMTL_5 \times \mathcal{F}_5$ and $IMTL^L_{5\times 5} \cong IMTL_5 \times \mathcal{L}_5$ etc.

Other examples:

Example 1 This is an example of non-linearly ordered, proper IMTL algebra. The example is based on the linearly ordered IMTL algebra $IMTL_5$.

Let us consider the set $A_5 = \{0, a, b, i, f, g, h, j, c, d, 1\}$, organized as a lattice as in Figure 3, and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$ as in the following tables: $\mathcal{A}^1_5$

$\to$	0	a	b	**i**	**f**	**g**	**h**	**j**	c	d	1
0	1	1	1	1	1	1	1	1	1	1	1
a	d	1	d	1	1	1	1	1	1	1	1
b	c	c	1	1	1	1	1	1	1	1	1
i	j	c	d	1	1	1	1	1	1	1	1
f	h	h	h	h	1	1	1	1	1	1	1
g	g	g	g	g	**h**	1	1	1	1	1	1
h	f	f	f	f	**h**	**h**	1	1	1	1	1
j	i	i	i	i	f	g	h	1	1	1	1
c	b	i	b	i	f	g	h	d	1	d	1
d	a	a	i	i	f	g	h	c	c	1	1
1	0	a	b	i	f	g	h	j	c	d	1

$\odot$	0	a	b	**i**	**f**	**g**	**h**	**j**	c	d	1
0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	a	0	a
b	0	0	0	0	0	0	0	0	0	b	b
i	0	0	0	0	0	0	0	0	a	b	i
f	0	0	0	0	0	0	0	f	f	f	f
g	0	0	0	0	0	0	**f**	g	g	g	g
h	0	0	0	0	0	**f**	**f**	h	h	h	h
j	0	0	0	0	f	g	h	j	j	j	j
c	0	a	0	a	f	g	h	j	c	j	c
d	0	0	b	b	f	g	h	j	j	d	d
1	0	a	b	i	f	g	h	j	c	d	1

Then $\mathcal{A}^1_5 = (A_5, \wedge, \vee, \to, 0, 1)$ is a non-linearly ordered BCK(P) lattice with condition (DN), where $\mathcal{A}^1_5$ satisfies the conditions $(C_\to)$, $(C_\wedge)$, $(C_\vee)$.

Note that $\mathcal{A}^1_5$ does not satisfy the condition (C_X), since there exist $i, g \in A_5$, such that

$$i = i \odot 1 = i \odot (g \to 1) = i \odot [(g \to i) \to (i \to g)] \neq g \odot [(i \to g) \to (g \to i)] = g \odot (1 \to g) = g \odot g = 0.$$

Note also that $\mathcal{A}^1_5$ does not satisfy the condition (WNM), since there are $g, h \in A_5$ such that:

$$(g \odot h)^- \vee [(g \wedge h) \to (g \odot h)] = f^- \vee [g \to f] = h \vee h = h \neq 1.$$

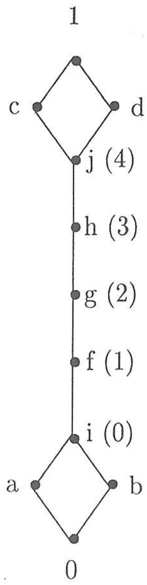

Figure 3: Examples of non-linearly ordered, proper IMTL and NM algebras

Consequently, $\mathcal{A}_5^1$ is a non-linearly ordered, proper $\alpha\beta\gamma_{(DN)}$ (IMTL) algebra (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Example 2 This is an example of non-linearly ordered, proper NM algebra. The example is based on the linearly ordered NM algebra $\mathcal{F}_5$.

Let us consider the set $A_5 = \{0, a, b, i, f, g, h, j, c, d, 1\}$, organized as a lattice as in Figure 3, and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$ as in the following tables: $\mathcal{A}_5^2$

$\to$	0	a	b	**i**	**f**	**g**	**h**	**j**	c	d	1
0	1	1	1	1	1	1	1	1	1	1	1
a	d	1	d	1	1	1	1	1	1	1	1
b	c	c	1	1	1	1	1	1	1	1	1
i	j	c	d	1	1	1	1	1	1	1	1
f	h	h	h	h	1	1	1	1	1	1	1
g	g	g	g	g	**g**	1	1	1	1	1	1
h	f	f	f	f	**f**	**g**	1	1	1	1	1
j	i	i	i	i	f	g	h	1	1	1	1
c	b	i	b	i	f	g	h	d	1	d	1
d	a	a	i	i	f	g	h	c	c	1	1
1	0	a	b	i	f	g	h	j	c	d	1

$\odot$	0	a	b	**i**	**f**	**g**	**h**	**j**	c	d	1
0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	a	0	a
b	0	0	0	0	0	0	0	0	0	b	b
i	0	0	0	0	0	0	0	0	a	b	i
f	0	0	0	0	0	0	0	f	f	f	f
g	0	0	0	0	0	0	**g**	g	g	g	g
h	0	0	0	0	0	**g**	**h**	h	h	h	h
j	0	0	0	0	f	g	h	j	j	j	j
c	0	a	0	a	f	g	h	j	c	j	c
d	0	0	b	b	f	g	h	j	j	d	d
1	0	a	b	i	f	g	h	j	c	d	1

Then $\mathcal{A}_5^2 = (A_5, \wedge, \vee, \to, 0, 1)$ is a non-linearly ordered BCK(P) lattice with condition (DN), which satisfies the conditions $(C_\to)$, $(C_\wedge)$, $(C_\vee)$.

Note that $\mathcal{A}_5^2$ satisfies the condition (WNM).

Note also that $\mathcal{A}_5^2$ does not satisfy the condition (C_X), since there exist $i, g \in A_5$, such that

$$i = i \odot 1 = i \odot (g \to 1) = i \odot [(g \to i) \to (i \to g)] \neq g \odot [(i \to g) \to (g \to i)] = g \odot (1 \to g) = g \odot g = 0.$$

Consequently, $\mathcal{A}_5^2$ is a non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma_{(DN)}$ algebra (NM) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Remark 14.5 We hope that it is obvious the way we can build examples of finite, non-linearly ordered, proper IMTL or NM algebras from the corresponding finite, linearly ordered, proper IMTL or NM algebras. It is routine to build, for example, the non-linearly ordered, proper NM algebra $\mathcal{A}_4$, based on the linearly ordered, proper NM algebra $\mathcal{F}_4$, the non-linearly ordered, proper IMTL algebras $\mathcal{A}_6^1$ - $\mathcal{A}_6^5$, based on the linearly-ordered IMTL algebras $IMTL_6^1$ - $IMTL_6^5$ respectively, and non-linearly ordered, proper NM algebra $\mathcal{A}_6^6$, based on the linearly ordered, proper NM algebra $\mathcal{F}_6$.

15 Examples of proper $\alpha\beta\gamma$ and $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras

15.1 Examples of linearly ordered, proper $\alpha\beta\gamma$ and $_{(WNM)}\alpha\beta\gamma$ algebras

15.1.1 Examples of linearly ordered, proper $\alpha\beta\gamma$ algebras

The examples are of the form: ordinal sums and (isomorphic copies of) subalgebras of ordinal sums. This part can be developed as it was for linearly ordered BL algebras (Part III).

• For the moment, we say that examples of linearly ordered, proper $\alpha\beta\gamma$ algebras are, for examples:

$\mathcal{F}_2 \bigoplus \mathcal{F}_4$, $\mathcal{F}_3 \bigoplus \mathcal{F}_4$, $\mathcal{F}_4 \bigoplus \mathcal{F}_4$, $\mathcal{F}_4 \bigoplus \mathcal{F}_3$, etc.

• The following examples of linearly ordered $\alpha\beta\gamma$ algebras are get from the corresponding examples of linearly ordered $\alpha\beta\gamma_{(DN)}$ (IMTL) algebras, by modifying the column of 0 in the tables of $\to$ (i.e. by modifying the values of x^-); namely, we put:

$$0^- = n, \qquad x^0 = 0, \ \forall x \neq n, \ (n \geq 3).$$

Example 1: $n = 4$ **The** $\alpha\beta\gamma$ **algebra named** $\mathcal{A}_5$**, obtained from** $IMTL_5$.

Let us consider the chain $L_5 = \{0, 1, 2, 3, 4\}$ organized as a lattice by $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\mathcal{A}_5$

$\to$	0	1	2	3	4
0	4	4	4	4	4
1	0	4	4	4	4
2	0	**3**	4	4	4
3	0	**3**	**3**	4	4
4	0	1	2	3	4

$\odot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	1	1	1
2	0	1	1	**1**	2
3	0	1	**1**	**1**	3
4	0	1	2	3	4

Then, $\mathcal{A}_5 = (L_5, \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, hence $\mathcal{A}_5$ satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

It does not satisfy the condition (C_X); indeed, there exist $2, 3 \in L_5$, such that

$$2 = 2 \odot 4 = 2 \odot (3 \to 4) = 2 \odot [(3 \to 2) \to (2 \to 3)] \neq 3 \odot [(2 \to 3) \to (3 \to 2)] = 3 \odot (4 \to 3) = 3 \odot 3 = 1.$$

Note also that it does not satisfy the condition (WNM), since there are $2, 3 \in L_5$ such that:

$$(2 \odot 3)^- \vee [(2 \wedge 3) \to (2 \odot 3)] = 1^- \vee [2 \to 1] = 0 \vee 3 = 3 \neq 4.$$

Hence, $\mathcal{A}_5$ is a linearly ordered, proper proper $\alpha\beta\gamma$ algebra.

Remark 15.1 Note that $\mathcal{A}_5$ is just a isomorphic copy of the subalgebra $\{-1,1,2,3,4\}$ of the ordinal sum $\mathcal{F}_2 \bigoplus IMTL_5$ $(\{-1,0\} \bigcup \{0,1,2,3,4\})$.

Example 2: $n=5$ The $\alpha\beta\gamma$ algebras named $\mathcal{A}_6^1$ - $\mathcal{A}_6^4$ and $\mathcal{A}_6$, obtained from $IMTL_6^1$ - $IMTL_6^4$ and $\mathcal{F}_6$, respectively.

Let us consider the set $L_{n+1} = L_6 = \{0,1,2,3,4,5\}$ organized as a lattice by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ and as a BCK(P) algebra in five different ways, with the operations $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

$\mathcal{A}_6^1$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	5	5	5	5
2	0	**4**	5	5	5	5
3	0	**4**	**4**	5	5	5
4	0	**4**	**4**	**4**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	**1**	2
3	0	1	1	**1**	**1**	3
4	0	1	**1**	**1**	**1**	4
5	0	1	2	3	4	5

$\mathcal{A}_6^2$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	5	5	5	5
2	0	**4**	5	5	5	5
3	0	**4**	**4**	5	5	5
4	0	**3**	**4**	**4**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	**1**	2
3	0	1	1	**1**	**1**	3
4	0	1	**1**	**1**	**2**	4
5	0	1	2	3	4	5

$\mathcal{A}_6^3$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	5	5	5	5
2	0	**4**	5	5	5	5
3	0	**2**	**2**	5	5	5
4	0	**2**	**2**	**4**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	**1**	2
3	0	1	1	**3**	**3**	3
4	0	1	**1**	**3**	**3**	4
5	0	1	2	3	4	5

$\mathcal{A}_6^4$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	5	5	5	5
2	0	**3**	5	5	5	5
3	0	**3**	**4**	5	5	5
4	0	**1**	**3**	**3**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	**2**	2
3	0	1	1	**1**	**2**	3
4	0	1	**2**	**2**	**4**	4
5	0	1	2	3	4	5

$\mathcal{A}_6$

$\rightarrow$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	5	5	5	5	5
2	0	**3**	5	5	5	5
3	0	**2**	**2**	5	5	5
4	0	**1**	**2**	**3**	5	5
5	0	1	2	3	4	5

$\odot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	**2**	2
3	0	1	1	**3**	**3**	3
4	0	1	**2**	**3**	**4**	4
5	0	1	2	3	4	5

Note that $\mathcal{A}_6^1$ - $\mathcal{A}_6^3$ do not satisfy the condition (WNM), since there exists 2 such that:

$$(2 \odot 2)^- \vee [(2 \wedge 2) \to (2 \odot 2)] = 1^- \vee [2 \to 1] = 0 \vee 4 = 4 \neq 5.$$

Note that $\mathcal{A}_6^4$ and $\mathcal{A}_6$ do not satisfy the condition (WNM), since there exists 2 such that:

$$(2 \odot 2)^- \vee [(2 \wedge 2) \to (2 \odot 2)] = 1^- \vee [2 \to 1] = 0 \vee 3 = 3 \neq 5.$$

Hence, they are lineraly ordered, proper $\alpha\beta\gamma$ algebras.

Remark 15.2 Note that $\mathcal{A}_6^1$ - $\mathcal{A}_6^4$ and $\mathcal{A}_6$ are just isomorphic copies of the subalgebra $\{-1, 1, 2, 3, 4, 5\}$ of the ordinal sums $\mathcal{F}_2 \bigoplus IMTL_6^1$, $\mathcal{F}_2 \bigoplus IMTL_6^2$, $\mathcal{F}_2 \bigoplus IMTL_6^3$, $\mathcal{F}_2 \bigoplus IMTL_6^4$, $\mathcal{F}_2 \bigoplus \mathcal{F}_6$, respectively.

Remarks 15.3
(i) For $n = 3$, from $\mathcal{F}_4$ we get, by the above preocedure, the BL algebra $\mathcal{H}_{2,2,2} = \mathcal{L}_2 \bigoplus \mathcal{H}_{2,2}$.
(ii) For $n = 4$, from $\mathcal{F}_5$ we get, by the above procedure, the BL algebra $\mathcal{H}_{2,3,2} = \mathcal{H}_{2,3} \bigoplus \mathcal{L}_2$.
(iii) For $n = 5$, from $IMTL_6^5$ we get the BL algebra $\mathcal{H}_{2,4,2} = \mathcal{H}_{2,4} \bigoplus \mathcal{L}_2$.

15.1.2 Examples of linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras

This part can also be developed as it was for linearly ordered BL algebras (Part III).
We found two categories of examples:
• Examples of the forms: ordinal sums and (isomorphic copies of) subalgebras of ordinal sums.
Here are the following examples:

$\mathcal{R}_{4,2} = \mathcal{F}_4 \bigoplus \mathcal{F}_2$, $\mathcal{R}_{4,2,2} = \mathcal{R}_{4,2} \bigoplus \mathcal{F}_2 = [\mathcal{F}_4 \bigoplus \mathcal{F}_2] \bigoplus \mathcal{F}_2$, etc.;
$\mathcal{R}_{5,2} = \mathcal{F}_5 \bigoplus \mathcal{F}_2$, $\mathcal{R}_{5,2,2} = \mathcal{R}_{5,2} \bigoplus \mathcal{F}_2 = [\mathcal{F}_5 \bigoplus \mathcal{F}_2] \bigoplus \mathcal{F}_2$, etc.
• The following examples are get from the left-continuous t-norm T_0, which is not continuous on $[0, 1]$, from [6]: let $0 < p < 1$ and let

$$T_0(x, y) = x \odot y = \begin{cases} 0, & 0 \leq x \leq p,\ 0 \leq y \leq p \\ \min(x, y), & otherwise \end{cases}$$

and

$$x \to y = \begin{cases} p, & x \leq p,\ x > y \\ y, & x > p,\ x > y \\ 1, & x \leq y, \end{cases}$$

Then, it was writing in [6], $([0, 1], \sup, = inf, T_0, \to, 0, 1)$ is a weak-BL algebra (i.e. MTL algebra, i.e. $\alpha\beta$ algebra).
Note that the above algebra is derived from the 3-valued $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebra (i.e. in the same time Wajsberg (MV) and R_0 (NM) algebra):

$\mathcal{F}_3 = \mathcal{L}_3$

$\to$	0	p	1
0	1	1	1
p	p	1	1
1	0	p	1

$\odot$	0	p	1
0	0	0	0
p	0	0	p
1	0	p	1

By taking a finite number of elements (numbers) in the real interval $[0, 1]$, we distinguish three cases.

•• **Case 1**: $A(x,\mathbf{p})=\{0 \leq x \leq \mathbf{p} < 1\}$.

Example 1.1 Let us consider the chain $A(a_1,\mathbf{p})= \{0 < a_1 < \mathbf{p} < 1\}$ organized as a lattice by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ and as a BCK(P) algebra with the operations $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

$A(a_1,\mathbf{p})$

$\rightarrow$	0	a_1	p	1
0	1	1	1	1
a_1	p	1	1	1
p	p	p	1	1
1	0	a_1	p	1

$\odot$	0	a_1	p	1
0	0	0	0	0
a_1	0	0	0	a_1
p	0	0	0	p
1	0	a_1	p	1

Then, $(A(a_1,\mathbf{p}), \wedge, \vee, \rightarrow, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_{\rightarrow})$, $(C_{\vee})$ and $(C_{\wedge})$.

It also satisfies the condition (WNM).

- It does not satisfy the condition (C_X); indeed, there exist a_1, p, such that

$$a_1 = a_1 \odot 1 = a_1 \odot (p \rightarrow 1) = a_1 \odot [(p \rightarrow a_1) \rightarrow (a_1 \rightarrow p)] \neq p \odot [(a_1 \rightarrow p) \rightarrow (p \rightarrow a_1)] = p \odot (1 \rightarrow p) = p \odot p = 0.$$

- It does not satisfy the condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Consequently, $A(a_1,\mathbf{p})$ is a proper linearly ordered ${}_{(WNM)}\alpha\beta\gamma$ algebra.

Example 1.2 Let us consider the chain $A(a_1,a_2,\mathbf{p})= \{0 < a_1 < a_2 < \mathbf{p} < 1\}$ organized as a BCK(P) algebra with the operations $\rightarrow$ and $x \odot y$ as in the following tables:

$A(a_1,a_2,\mathbf{p})$

$\rightarrow$	0	a_1	a_2	p	1
0	1	1	1	1	1
a_1	p	1	1	1	1
a_2	p	p	1	1	1
p	p	p	p	1	1
1	0	a_1	a_2	p	1

$\odot$	0	a_1	a_2	p	1
0	0	0	0	0	0
a_1	0	0	0	0	a_1
a_2	0	0	0	0	a_2
p	0	0	0	0	p
1	0	a_1	a_2	p	1

Then, $(A(a_1,a_2,\mathbf{p}), \wedge, \vee, \rightarrow, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_{\rightarrow})$, $(C_{\vee})$ and $(C_{\wedge})$.

It also satisfies the condition (WNM).

- It does not satisfy the condition (C_X); indeed, there exist a_1, p, such that

$$a_1 = a_1 \odot 1 = a_1 \odot (p \rightarrow 1) = a_1 \odot [(p \rightarrow a_1) \rightarrow (a_1 \rightarrow p)] \neq p \odot [(a_1 \rightarrow p) \rightarrow (p \rightarrow a_1)] = p \odot (1 \rightarrow p) = p \odot p = 0.$$

- It does not satisfy the condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Consequently, $A(a_1,a_2,\mathbf{p})$ is a proper linearly ordered ${}_{(WNM)}\alpha\beta\gamma$ algebra.

•• **Case 2**: $A(\mathbf{p},x)=\{0 < \mathbf{p} \leq x \leq 1\}$.

Example 2.1 Let us consider the chain $A(\mathbf{p}, b_1)= \{0 < \mathbf{p} < b_1 < 1\}$ organized as a lattice by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ and as a BCK(P) algebra with the operations $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

$A(\mathbf{p}, b_1)$

$\to$	0	p	b_1	1
0	1	1	1	1
p	p	1	1	1
b_1	0	p	1	1
1	0	p	b_1	1

$\odot$	0	p	b_1	1
0	0	0	0	0
p	0	0	p	p
b_1	0	p	b_1	b_1
1	0	p	b_1	1

Then, $(A(\mathbf{p},b_1), \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.
It satisfies the conditions (C_X) and (WNM).

Remark 15.4 Note that it is an isomorphic copy of the $_{(WNM)}$BL algebra $\mathcal{H}_{3,2} = \mathcal{L}_3 \bigoplus \mathcal{L}_2 = \mathcal{F}_3 \bigoplus \mathcal{F}_2 = \mathcal{R}_{3,2}$, i.e. it is not a proper linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebra.

Example 2.2 Let us consider the chain $A(\mathbf{p}, b_1, b_2) = \{0 < \mathbf{p} < b_1 < b_2 < 1\}$ organized as a lattice by $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ and as a BCK(P) algebra with the operations $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$A(\mathbf{p}, b_1, b_2)$

$\to$	0	p	b_1	b_2	1
0	1	1	1	1	1
p	p	1	1	1	1
b_1	0	p	1	1	1
b_2	0	p	b_1	1	1
1	0	p	b_1	b_2	1

$\odot$	0	p	b_1	b_2	1
0	0	0	0	0	0
p	0	0	p	p	p
b_1	0	p	b_1	b_1	b_1
b_2	0	p	b_1	b_2	b_2
1	0	p	b_1	b_2	1

Then, $(A(\mathbf{p},b_1,b_2), \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.
It satisfies the conditions (C_X) and (WNM).

Remark 15.5 Note that it is an isomorphic copy of the $_{(WNM)}$BL algebra $\mathcal{H}_{3,2,2} = \mathcal{L}_3 \bigoplus \mathcal{L}_2 \bigoplus \mathcal{L}_2 = \mathcal{F}_3 \bigoplus \mathcal{F}_2 \bigoplus \mathcal{F}_2 = \mathcal{R}_{3,2,2}$, i.e. it is not a proper linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebra.

•• **Case 3**: $A(x,\mathbf{p},y) = \{0 \leq x \leq \mathbf{p} \leq y \leq 1\}$.

Example 3.1 Let us consider the chain $A(a_1, \mathbf{p}, b_1) = \{0 < a_1 < \mathbf{p} < b_1 < 1\}$ organized as a lattice by $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ and as a BCK(P) algebra with the operations $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$A(a_1,\mathbf{p},b_1)$

$\to$	0	a_1	p	b_1	1
0	1	1	1	1	1
a_1	p	1	1	1	1
p	p	p	1	1	1
b_1	0	a_1	p	1	1
1	0	a_1	p	b_1	1

$\odot$	0	a_1	p	b_1	1
0	0	0	0	0	0
a_1	0	0	0	a_1	a_1
p	0	0	0	p	p
b_1	0	a_1	p	b_1	b_1
1	0	a_1	p	b_1	1

Then, $(A(a_1,\mathbf{p}, b_1), \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.
It also satisfies the condition (WNM).
- It does not satisfy the condition (C_X); indeed, there exist a_1, p, such that

$$a_1 = a_1 \odot 1 = a_1 \odot (p \to 1) = a_1 \odot [(p \to a_1) \to (a_1 \to p)] \neq p \odot [(a_1 \to p) \to (p \to a_1)] = p \odot (1 \to p) = p \odot p = 0.$$

- It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $A(a_1,\mathrm{p},b_1)$ is a proper linearly ordered ${}_{(WNM)}\alpha\beta\gamma$ algebra.

Remark 15.6 Note that $A(a_1,\mathrm{p},b_1)$ is a subalgebra of the ordinal sum $A(a_1,\mathrm{p}) \bigoplus A(\mathrm{q}, b_1)$.

Example 3.2 Let us consider the chain $A(a_1, a_2,\mathrm{p}, b_1) = \{0 < a_1 < a_2 < \mathrm{p} < b_1 < 1\}$ organized as a lattice by $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ and as a BCK(P) algebra with the operations $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \le y \to z\}$ as in the following tables:

$A(a_1, a_2,\mathrm{p},b_1)$

$\to$	0	a_1	a_2	p	b_1	1
0	1	1	1	1	1	1
a_1	p	1	1	1	1	1
a_2	p	p	1	1	1	1
p	p	p	p	1	1	1
b_1	0	a_1	a_2	p	1	1
1	0	a_1	a_2	p	b_1	1

$\odot$	0	a_1	a_2	p	b_1	1
0	0	0	0	0	0	0
a_1	0	0	0	0	a_1	a_1
a_2	0	0	0	0	a_2	a_2
p	0	0	0	0	p	p
b_1	0	a_1	a_2	p	b_1	b_1
1	0	a_1	a_2	p	b_1	1

Then, $(A(a_1, a_2,\mathrm{p}, b_1), \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

It also satisfies the condition (WNM).

- It does not satisfy the condition (C_X); indeed, there exist a_1, p, such that

$$a_1 = a_1 \odot 1 = a_1 \odot (p \to 1) = a_1 \odot [(p \to a_1) \to (a_1 \to p)] \neq p \odot [(a_1 \to p) \to (p \to a_1)] = p \odot (1 \to p) = p \odot p = 0.$$

- It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $A(a_1, a_2,\mathrm{p},b_1)$ is a proper linearly ordered ${}_{(WNM)}\alpha\beta\gamma$ algebra.

Remark 15.7 Note that $A(a_1, a_2,\mathrm{p},b_1)$ is a subalgebra of the ordinal sum $A(a_1, a_2,\mathrm{p}) \bigoplus A(\mathrm{q}, b_1)$.

Example 3.3 Let us consider the chain $A(a_1,\mathrm{p}, b_1, b_2) = \{0 < a_1 < \mathrm{p} < b_1 < b_2 < 1\}$ organized as a lattice by $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ and as a BCK(P) algebra with the operations $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \le y \to z\}$ as in the following tables:

$A(a_1,\mathrm{p},b_1, b_2)$

$\to$	0	a_1	p	b_1	b_2	1
0	1	1	1	1	1	1
a_1	p	1	1	1	1	1
p	p	p	1	1	1	1
b_1	0	a_1	p	1	1	1
b_2	0	a_1	p	b_1	1	1
1	0	a_1	p	b_1	b_2	1

$\odot$	0	a_1	p	b_1	b_2	1
0	0	0	0	0	0	0
a_1	0	0	0	a_1	a_1	a_1
p	0	0	0	p	p	p
b_1	0	a_1	p	b_1	b_1	b_1
b_2	0	a_1	p	b_1	b_2	b_2
1	0	a_1	p	b_1	b_2	1

Then, $(A(a_1,\mathrm{p}, b_1, b_2), \wedge, \vee, \to, 0, 1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

It also satisfies the condition (WNM).

- It does not satisfy the condition (C_X); indeed, there exist a_1, p, such that

$$a_1 = a_1 \odot 1 = a_1 \odot (p \to 1) = a_1 \odot [(p \to a_1) \to (a_1 \to p)] \neq p \odot [(a_1 \to p) \to (p \to a_1)] = p \odot (1 \to p) = p \odot p = 0.$$

- It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $A(a_1,\mathbf{p},b_1,b_2)$ is a proper linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebra.

Remark 15.8 Note that $A(a_1,\mathbf{p},b_1,b_2)$ is a subalgebra of the ordinal sum $A(a_1,\mathbf{p}) \bigoplus A(\mathbf{q},b_1,b_2)$.

Example 3.4 Let us consider the chain $A(a_1,a_2,\mathbf{p}, b_1,b_2)= \{0 < a_1 < a_2 < \mathbf{p} < b_1 < b_2 < 1\}$ organized as a lattice by $x \vee y = \max\{x,y\}$ and $x \wedge y = \min\{x,y\}$ and as a BCK(P) algebra with the operations $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$A(a_1,a_2,\mathbf{p},b_1,b_2)$

$\to$	0	a_1	a_2	p	b_1	b_2	1
0	1	1	1	1	1	1	1
a_1	p	1	1	1	1	1	1
a_2	p	p	1	1	1	1	1
p	p	p	p	1	1	1	1
b_1	0	a_1	a_2	p	1	1	1
b_2	0	a_1	a_2	p	b_1	1	1
1	0	a_1	a_2	p	b_1	b_2	1

$\odot$	0	a_1	a_2	p	b_1	b_2	1
0	0	0	0	0	0	0	0
a_1	0	0	0	0	a_1	a_1	a_1
a_2	0	0	0	0	a_2	a_2	a_2
p	0	0	0	0	p	p	p
b_1	0	a_1	a_2	p	b_1	b_1	b_1
b_2	0	a_1	a_2	p	b_1	b_2	b_2
1	0	a_1	a_2	p	b_1	b_2	1

Then, $(A(a_1,a_2,\mathbf{p}, b_1,b_2),\wedge,\vee,\to,0,1)$ is a linearly ordered BCK(P) lattice, hence it satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

It also satisfies the condition (WNM).

- It does not satisfy the condition (C_X); indeed, there exist a_1, p, such that

$$a_1 = a_1 \odot 1 = a_1 \odot (p \to 1) = a_1 \odot [(p \to a_1) \to (a_1 \to p)] \neq p \odot [(a_1 \to p) \to (p \to a_1)] = p \odot (1 \to p) = p \odot p = 0.$$

- It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $A(a_1,a_2,\mathbf{p},b_1,b_2)$ is a proper linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebra.

Remark 15.9 Note that $A(a_1,a_2,\mathbf{p},b_1,b_2)$ is a subalgebra of the ordinal sum $A(a_1,a_2,\mathbf{p}) \bigoplus A(\mathbf{q},b_1,b_2)$.

Open problem 15.10 Analyse the other left-continuous t-norms from [6]: T_1, T_2, T_3, T_{2n}, T_{2n+1}, $(n \geq 0)$.

15.2 Examples of non-linearly ordered $\alpha\beta\gamma$ and $_{(WNM)}\alpha\beta\gamma$ algebras

15.2.1 Examples of non-linearly ordered, proper $\alpha\beta\gamma$ algebras

We present two groups of examples:

- The (isomorphic copies of) direct products of linearly ordered $\alpha\beta\gamma$ algebras, as for example:

$\mathcal{F}_2 \bigoplus \mathcal{F}_4 \times \mathcal{F}_2 \bigoplus \mathcal{F}_4$, etc.

• The ordinal sums of the form: linearly ordered $\alpha\beta\gamma$ algebra $\bigoplus$ non-linearly ordered $\alpha\beta\gamma$ algebra, as for example:

$\mathcal{F}_2 \bigoplus \mathcal{F}_{4\times 2}$, etc.

We develop only one example, obtained as ordinal sum of $\mathcal{F}_2$ and the non-linearly ordered $\alpha\beta\gamma_{(DN)}$ algebra $\mathcal{A}_5^1$.

Let us consider the set $B_5 = \{0, n, a, b, i, f, g, h, j, c, d, 1\}$, organized as a lattice as in Figure 4, and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables: $\mathcal{B}_5^1$

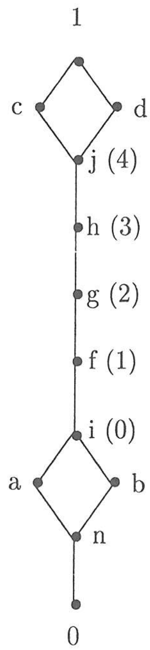

Figure 4: Example of non-linearly ordered, proper $\alpha\beta\gamma$ algebra

$\to$	0	n	a	b	**i**	**f**	**g**	**h**	**j**	c	d	1
0	1	1	1	1	1	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1	1	1	1	1	1
a	0	d	1	d	1	1	1	1	1	1	1	1
b	0	c	c	1	1	1	1	1	1	1	1	1
i	0	j	c	d	1	1	1	1	1	1	1	1
f	0	h	h	h	h	1	1	1	1	1	1	1
g	0	g	g	g	g	**h**	1	1	1	1	1	1
h	0	f	f	f	f	**h**	**h**	1	1	1	1	1
j	0	i	i	i	i	f	g	h	1	1	1	1
c	0	b	i	b	i	f	g	h	d	1	d	1
d	0	a	a	i	i	f	g	h	c	c	1	1
1	0	n	a	b	i	f	g	h	j	c	d	1

$\odot$	0	n	a	b	**i**	**f**	**g**	**h**	**j**	c	d	1
0	0	0	0	0	0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n	n	n	n	n	n
a	0	n	n	n	n	n	n	n	n	a	n	a
b	0	n	n	n	n	n	n	n	n	n	b	b
i	0	n	n	n	n	n	n	n	n	a	b	i
f	0	n	n	n	n	n	n	n	f	f	f	f
g	0	n	n	n	n	n	n	**f**	g	g	g	g
h	0	n	n	n	n	n	**f**	**f**	h	h	h	h
j	0	n	n	n	n	f	g	h	j	j	j	j
c	0	n	a	n	a	f	g	h	j	c	j	c
d	0	n	n	b	b	f	g	h	j	j	d	d
1	0	n	a	b	i	f	g	h	j	c	d	1

Then $\mathcal{B}_5^1 = (B_5, \wedge, \vee, \to, 0, 1)$ is a non-linearly ordered BCK(P) lattice, which satisfies the condition (B3), i.e. $(C_\to)$ and $(C_\vee)$, but it does not satisfy the condition (C_c), i.e. (B2); since it satisfies the condition $(C_\wedge)$, it follows that it does not satisfy the condition (C_X). Consequently, $\mathcal{A}_B^1$ is a proper non-linearly ordered $\alpha\beta\gamma$ algebra (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Note that $\mathcal{B}_5^1$ does not satisfy the condition (WNM), since there are $i, c \in B_5$ such that:

$$(i \odot c)^- \vee [(i \wedge c) \to (i \odot c)] = a^- \vee [i \to a] = 0 \vee c = c \neq 1.$$

15.2.2 Examples of non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras

We present three groups of examples:

• The (isomorphic copies of) direct products of linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebras, as for example:

$\mathcal{A}(a_1,\mathbf{p}) \times \mathcal{A}(a_1,\mathbf{p})$, etc.

• The ordinal sums of the form: linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebra $\bigoplus$ non-linearly ordered $_{(WNM)}\alpha\beta\gamma$ algebra, as for example:

$\mathcal{R}_{4,2\times2} = \mathcal{F}_4 \bigoplus \mathcal{F}_{2\times2}$, $\mathcal{R}_{5,2\times2} = \mathcal{F}_5 \bigoplus \mathcal{F}_{2\times2}$, etc.

• We give three examples, which can be generalized.

Example 1

Let us consider the set $A_1 = \{0, a, b, c, d, 1\}$ organized as a lattice as in Figure 5 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

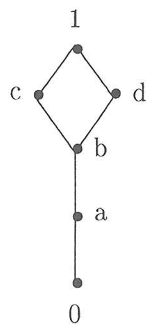

Figure 5: Example 1 of non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ algebra

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Then $\mathcal{A}_1 = (A_1, \wedge, \vee, \to, 0, 1)$ is a non-linearly ordered BCK(P) lattice which satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

Note also that $\mathcal{A}_1$ satisfies the condition (WNM).

$\mathcal{A}_1$ does not satisfy the condition (C_X), since there exist $b, a \in A_1$, such that

$$0 = b \odot a = b \odot (1 \to a) = b \odot [(a \to b) \to (b \to a)] \neq a \odot [(b \to a) \to (a \to b)] = a \odot (a \to 1) = a \odot 1 = a.$$

It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $\mathcal{A}_1$ it is a non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ (Roman) algebra.

Example 2

Let us consider the set $A_2 = \{0, n, a, b, c, d, 1\}$ organized as a lattice as in Figure 6 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

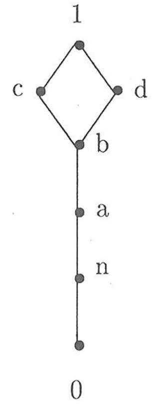

Figure 6: Example 2 of non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ algebra

$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	n	n	a	**1**	**1**	**1**	**1**
c	0	n	a	**d**	**1**	**d**	**1**
d	n	n	a	**c**	**c**	**1**	**1**
1	0	n	a	**b**	**c**	**d**	**1**

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	n	0	n
a	0	0	a	a	a	a	a
b	0	0	a	b	b	b	b
c	0	n	a	b	c	b	c
d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1

Then $\mathcal{A}_2 = (A_2, \wedge, \vee, \to, 0, 1)$ is a non-linearly ordered BCK(P) lattice which satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

Note also that $\mathcal{A}_2$ satisfies the condition (WNM).

Note that $\mathcal{A}_2$ does not satisfy the condition (C_X), since there exist $b, n \in A_2$, such that

$$0 = b \odot n = b \odot (1 \to n) = b \odot [(n \to b) \to (b \to n)] \neq n \odot [(b \to n) \to (n \to b)] = n \odot (n \to 1) = n \odot 1 = n.$$

It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $\mathcal{A}_2$ it is a non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ (Roman) algebra.

Example 3

Let us consider the set $A_3 = \{0, m, n, a, b, c, d, 1\}$ organized as a lattice as in Figure 7 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

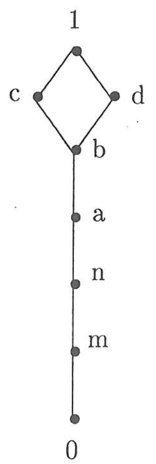

Figure 7: Example 3 of non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ algebra

$\to$	0	m	n	a	b	c	d	1
0	1	1	1	1	1	1	1	1
m	d	1	1	1	1	1	1	1
n	m	n	1	1	1	1	1	1
a	m	m	n	1	1	1	1	1
b	m	m	n	a	**1**	**1**	**1**	**1**
c	0	m	n	a	**d**	**1**	**d**	**1**
d	m	m	n	a	**c**	**c**	**1**	**1**
1	0	m	n	a	**b**	**c**	**d**	**1**

$\odot$	0	m	n	a	b	c	d	1
0	0	0	0	0	0	0	0	0
m	0	0	0	0	0	m	0	m
n	0	0	n	n	n	n	n	n
a	0	0	n	a	a	a	a	a
b	0	0	n	a	b	b	b	b
c	0	m	n	a	b	c	b	c
d	0	0	n	a	b	b	d	d
1	0	m	n	a	b	c	d	1

Then $\mathcal{A}_3 = (A_3, \wedge, \vee, \to, 0, 1)$ is a non-linearly ordered BCK(P) lattice which satisfies the conditions $(C_\to)$, $(C_\vee)$ and $(C_\wedge)$.

Note also that $\mathcal{A}_3$ satisfies the condition (WNM).

Note that $\mathcal{A}_3$ does not satisfy the condition (C_X), since there exist $b, m \in A_3$, such that

$$0 = b\odot m = b\odot(1 \to m) = b\odot[(m \to b) \to (b \to m)] \neq m\odot[(b \to m) \to (m \to b)] = m\odot(m \to 1) = m\odot 1 = m.$$

It does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Consequently, $\mathcal{A}_3$ it is a non-linearly ordered, proper $_{(WNM)}\alpha\beta\gamma$ (Roman) algebra.

16 Examples of proper $\alpha\gamma$ algebras

Recall that these algebras cannot be linearly ordered and proper.

Note that this part can be developped as it was done for divisible BCK(P) lattices (Part III). Note also that we didn't found examples of proper $_{(WNM)}\alpha\gamma$ algebras.

We present two groups of examples of proper $\alpha\gamma$ algebras:

• The ordinal sum of non-linearly ordered NM or IMTL algebra $\bigoplus$ (linearly ordered or non-linearly ordered) NM or IMTL algebra:

$\mathcal{F}_{2\times 2} \bigoplus \mathcal{F}_4$, $\mathcal{F}_{2\times 2} \bigoplus \mathcal{F}_5$ etc., $\mathcal{F}_{4\times 2} \bigoplus \mathcal{F}_2$, $\mathcal{F}_{2\times 2} \bigoplus \mathcal{F}_{4\times 2}$, $\mathcal{A}_5^2 \bigoplus \mathcal{F}_2$, $\mathcal{F}_{2\times 2} \bigoplus$A(p,$b_1$), etc.

• Other examples:

Example 1

Let us consider the set $A = \{0, a, c, d, m, 1\}$ organized as a lattice as in Figure 8 and as a BCK(P) algebra with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

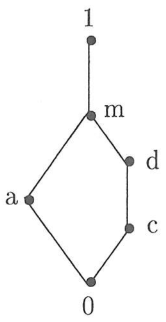

Figure 8: Example 1 of proper $\alpha\gamma$ algebra

$\rightarrow$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

$\odot$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_\rightarrow)$ and $(C_\wedge)$. Consequently, $\mathcal{A}$ is an $\alpha\gamma$ algebra, without condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Note that it is a proper $\alpha\gamma$ algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\vee)$; indeed, there exist $a, d \in A$, such that

$$m = a \vee d \neq [(a \rightarrow d) \rightarrow d] \wedge [(d \rightarrow a) \rightarrow a] = (d \rightarrow d) \wedge (a \rightarrow a) = 1 \wedge 1 = 1.$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $m, d \in A$, such that

$$c = m \odot d = m \odot (1 \rightarrow d) = m \odot [(d \rightarrow m) \rightarrow (m \rightarrow d)] \neq$$

$$\neq d \odot [(m \rightarrow d) \rightarrow (d \rightarrow m)] = d \odot (d \rightarrow 1) = d \odot 1 = d.$$

- it does not satisfy the condition (WNM), since there is d such that:

$$(d \odot d)^- \vee [(d \wedge d) \rightarrow (d \odot d)] = c^- \vee [d \rightarrow c] = a \vee m = m \neq 1.$$

Example 2

Let us consider the set $A = \{0, a, b, c, d, m, 1\}$ organized as a lattice as in Figure 9 and as a BCK(P) algebra with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

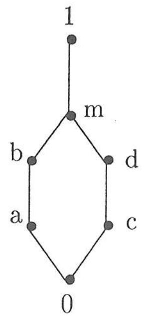

Figure 9: Example 2 of proper $\alpha\gamma$ algebra

$\rightarrow$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

$\odot$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_{\rightarrow})$ and $(C_{\wedge})$. Consequently, $\mathcal{A}$ is an $\alpha\gamma$ algebra, without condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Note that it is a proper $\alpha\gamma$ algebra, since:
- $\mathcal{A}$ does not satisfy the condition $(C_{\vee})$; indeed, there exist $b, d \in A$, such that

$$m = b \vee d \neq [(b \rightarrow d) \rightarrow d] \wedge [(d \rightarrow b) \rightarrow b] = (d \rightarrow d) \wedge (b \rightarrow b) = 1 \wedge 1 = 1.$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $m, d \in A$, such that

$$c = m \odot d = m \odot (1 \rightarrow d) = m \odot [(d \rightarrow m) \rightarrow (m \rightarrow d)] \neq$$

$$\neq d \odot [(m \rightarrow d) \rightarrow (d \rightarrow m)] = d \odot (d \rightarrow 1) = d \odot 1 = d.$$

- it does not satisfy the condition (WNM), since there is b such that:

$$(b \odot b)^- \vee [(b \wedge b) \rightarrow (b \odot b)] = a^- \vee [b \rightarrow a] = d \vee m = m \neq 1.$$

17 Remarks and open problems

1) By combining the hierarchies concerning $\alpha\gamma$ and $\alpha\beta\gamma$ algebras from this paper Part IV with the hierarchies concerning $\alpha\gamma\delta$ = divisible BCK(P) lattices and $\alpha\beta\gamma delta$ = Hájek algebras (BL algebras) from [?], we get the hierarchies from Figure fig:figV-ac-abc-acd-BL.

2) Note that the ordinal sum of two BCK(P) lattices, $M_1 \bigoplus M_2$,preserves $(C_{\wedge})$ and eliminates $(C_{\vee})$, when M_1 is non-linearly ordered.

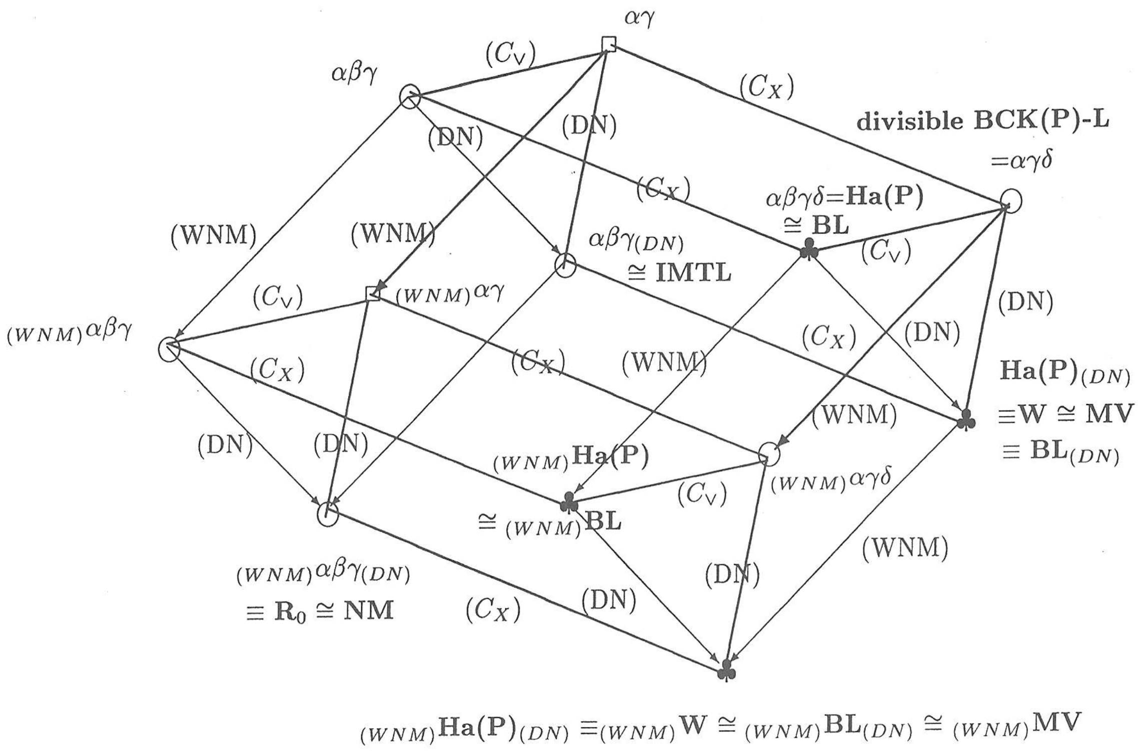

Figure 10: "Vertical" sections through $\alpha\gamma$, $\alpha\beta\gamma$, $\alpha\gamma\delta$ and $\alpha\beta\gamma\delta$ (BL) algebras

3) Find a different type of ordinal sum of two BCK(P) lattices, say $M_1 \bigodot M_2, which$preserves $(C_\vee)$ and eliminates $(C_\wedge)$, when M_1 is non-linearly ordered.

4) The hierarchies concerning $\alpha\beta$ and $\alpha\beta\gamma$ algebras and $\alpha\beta\delta$ and $\alpha\beta\gamma delta$ = Hájek algebras (BL algebras) are the following:

5) Find a representation theorem for $\alpha\gamma$ and $\alpha\beta$ algebras and for $\alpha\gamma\delta$ and $\alpha\beta\delta$ algebras.

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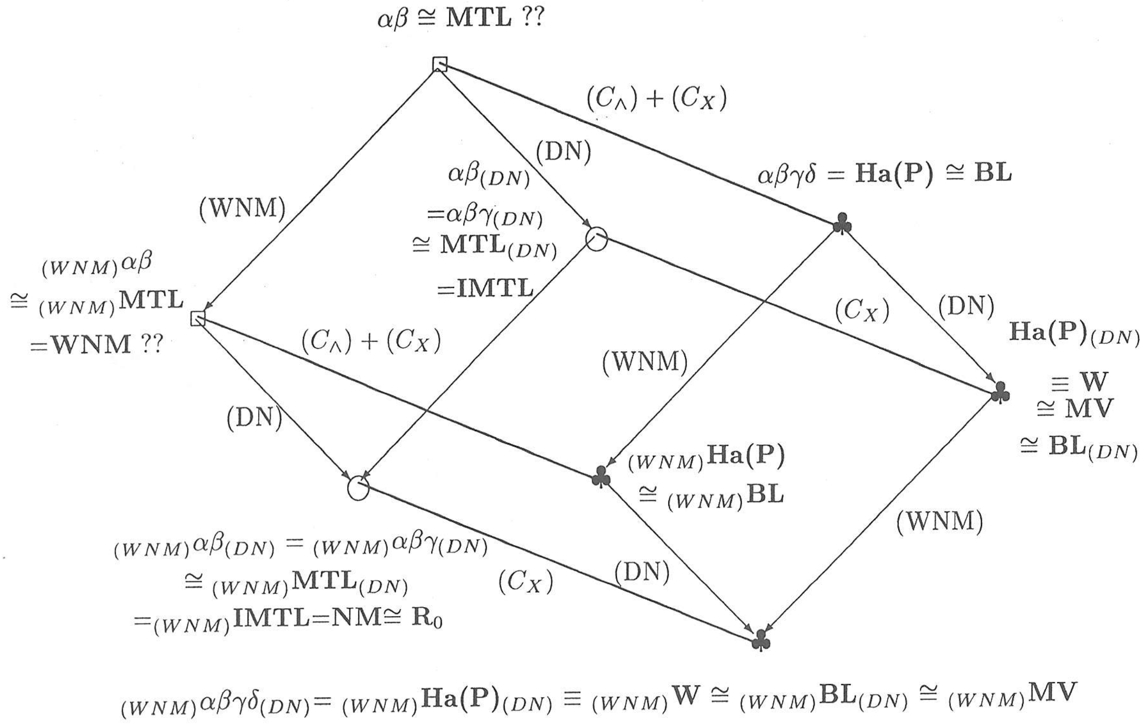

Figure 11: Vertical sections through MTL algebras and BL algebras

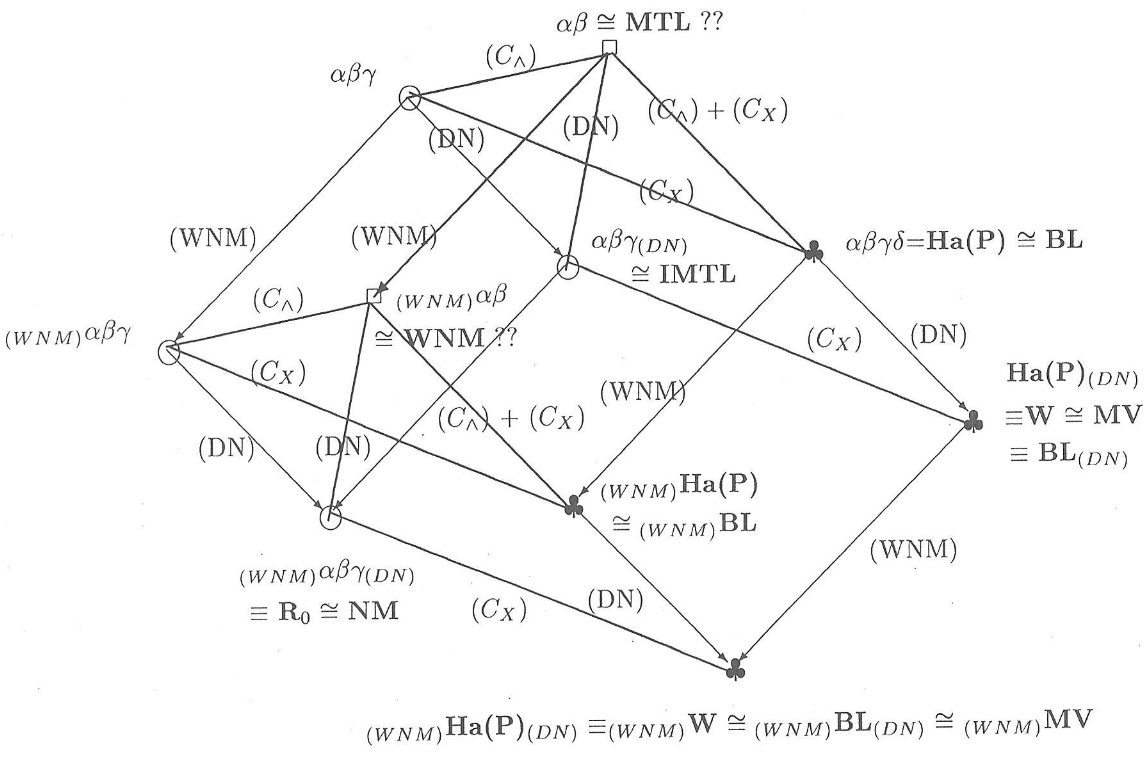

Figure 12: "Vertical" sections through $\alpha\beta$ (MTL), $\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$ (BL) algebras

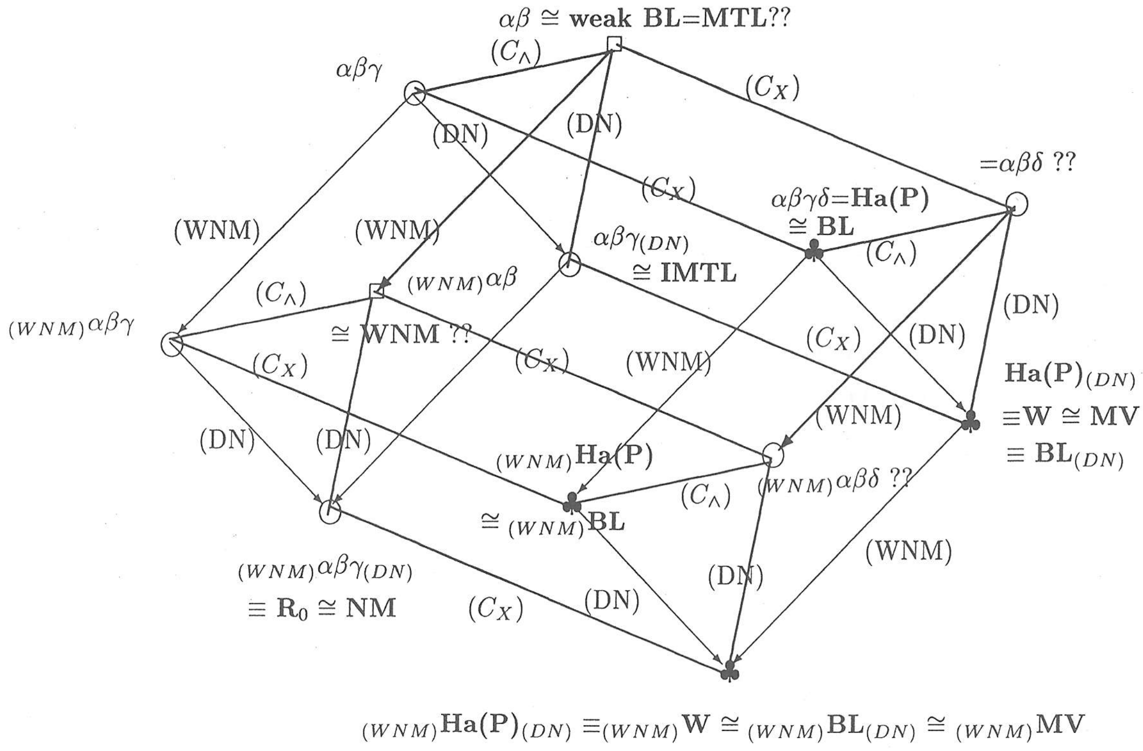

Figure 13: "Vertical" sections through $\alpha\beta$, $\alpha\beta\gamma$, $\alpha\beta\delta$ and $\alpha\beta\gamma\delta$ (BL) algebras

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