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CLASSES OF BCK ALGEBRAS – PART V

by

AFRODITA IORGULESCU

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AFRODITA IORGULESCU[1]

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1 Academy of Economic Studies, Department of Computer Science, Piata Romana, No. 6 – R 70167, Oficiul Postal 22, Bucharest, Romania, e–mail: afrodita@inforec.ase.ro

Classes of BCK algebras - Part V

Afrodita Iorgulescu

Department of Computer Science, Academy of Economic Studies,
Piaţa Română Nr.6 - R 70167, Oficiul Poştal 22, Bucharest, Romania
E-mail: afrodita@inforec.ase.ro

Dedicated to Grigore C. Moisil (1905-1973)
(10 January 2004)

Abstract

In this paper we study the BCK algebras and their particular classes: the BCK(P) (residuated) lattices, the Hájek(P) (BL) algebras and the Wajsberg (MV) algebras, we introduce new classes of BCK(P) lattices, we establish hierarchies and we give many examples. The paper has five parts.

In the first part, the most important part, we decompose the divisibility and the pre-linearity conditions from the definition of a BL algebra into four new conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$ and (C_X). We study the additional conditions (WNM) (weak nilpotent minimum) and (DN) (double negation) on a BCK(P) lattice. We introduce the ordinal sum of two BCK(P) lattices and prove in what conditions we get BL algebras or other structures, more general, or more particular than BL algebras.

In part II, we give examples of some finite bounded BCK algebras. We introduce new generalizations of BL algebras, named α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebras, as BCK(P) lattices (residuated lattices) verifying one, two, three or four of the conditions $(C_\rightarrow)$, $(C_\vee)$, $(C_\wedge)$ and (C_X). By adding the conditions (WNM) and (DN) to these classes, we get more classes; among them, we get many generalizations of Wajsberg (MV) algebras and of R_0 (NM) algebras. The subclasses of $_{(WNM)}$ Wajsberg algebras ($_{(WNM)}$MV algebras) and of $_{(WNM)}$Hájek algebras ($_{(WNM)}$BL algebras) are introduced. We establish connections (hierarchies) between all these new classes and the old classes already pointed out in Part I.

In part III, we give examples of finite MV and $_{(WNM)}$MV algebras, of Hájek(P) (i.e. BL) algebras and $_{(WNM)}$BL algebras and of $\alpha\gamma\delta$ (i.e. divisible BCK(P) lattices (divisible residuated lattices or divisible integral, residuated, commutative l-monoids)) and of divisible $_{(WNM)}$BCK(P) lattices.

In part IV, we stress the importance of $\alpha\beta\gamma$ algebras versus $\alpha\beta$ (i.e. MTL) algebras algebras and of R_0 (i.e. NM) algebras versus Wajsberg (i.e. MV) algebras and of $_{(WNM)}\alpha\beta\gamma$ algebras versus BL algebras and of $\alpha\gamma$ versus $\alpha\gamma\delta$ algebras. We give examples of finite IMTL algebras and of $_{(WNM)}$IMTL (i.e. NM) algebras), of $\alpha\beta\gamma$ algebras and of $_{(WNM)}\alpha\beta\gamma$ (Roman) algebras and finally of $\alpha\gamma$ algebras.

In part V, we give other examples of finite BCK(P) lattices, finding examples for the others remaining an open problem. We make final remarks and formulate final open problems.

Keywords MV algebra, Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, divisible BCK(P) lattice, α, β, γ, δ, $\alpha\beta$, ..., $\alpha\beta\gamma\delta$ algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, $_{(WNM)}$MV, $_{(WNM)}$BL, $_{(WNM)}$ $\alpha\beta\gamma$, Roman algebra

Part V has three parts.

In Section 18, we give other finite examples of generalizations of Wajsberg (MV) algebras and $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras.

In Section 19, we give other finite examples of generalizations of Hájek(P) (BL) algebras and $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras.

In Section 20, we give final remarks and open problems.

18 Examples of other generalisations of Wajsberg (MV) algebras and $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras

18.1 Example of proper BCK(P)$_{(DN)}$ lattice (Girard monoid)

Let us consider the set $A = \{0, a, b, n, c, d, m, 1\}$ organized as a lattice as in Figure 1 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$ as in the following tables:

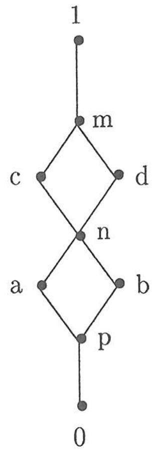

Figure 1: Example of proper BCK(P)$_{(DN)}$ lattice (Girard monoid)

$\to$	0	p	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1
p	m	1	1	1	1	1	1	1	1
a	d	m	1	m	1	1	1	1	1
b	c	m	m	1	1	1	1	1	1
n	n	m	m	m	1	1	1	1	1
c	b	m	m	m	m	1	m	1	1
d	a	m	m	m	m	m	1	1	1
m	p	m	m	m	m	m	m	1	1
1	0	p	a	b	n	c	d	m	1

$\odot$	0	p	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	0	0	p
a	0	0	0	0	0	p	0	p	a
b	0	0	0	0	0	0	p	p	b
n	0	0	0	0	0	p	p	p	n
c	0	0	p	0	p	p	p	p	c
d	0	0	0	p	p	p	p	p	d
m	0	0	p	p	p	p	p	p	m
1	0	p	a	b	n	c	d	m	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P)$_{(DN)}$ lattice (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Note that it is a proper one, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\to)$, since there exist $c, d \in A$, such that

$$(c \to d) \to (d \to c) = m \to m = 1 \neq m = d \to c;$$

- $\mathcal{A}$ does not satisfy the condition $(C_\vee)$, since there exist $a, b \in A$, such that

$$n = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (m \to b) \wedge (m \to a) = m \wedge m = m.$$

- $\mathcal{A}$ does not satisfy the condition $(C_\wedge)$, since there exist $c, d \in A$, such that

$$n = c \wedge d \neq [c \odot (c \to d)] \vee [d \odot (d \to c)] = (c \odot m) \vee (d \odot m) = p \vee p = p;$$

- $\mathcal{A}$ does not satisfy the condition (C_X), since there exist $a, b \in A$, such that

$$a = a\odot 1 = a\odot(m \to m) = a\odot[(b \to a) \to (a \to b)] \neq b\odot[(a \to b) \to (b \to a)] = b\odot(m \to m) = b\odot 1 = b.$$

- $\mathcal{A}$ does not satisfy the condition (WNM), since there are $a, c \in A$, such that:

$$(a \odot c)^- \vee [(a \wedge c) \to (a \odot c)] = p^- \vee [a \to p] = m \vee m = m \neq 1.$$

18.2 Example of proper $_{(WNM)}\alpha_{(DN)}$ algebra

Let us consider the set $A = \{0, a, b, c, d, 1\}$ organized as a lattice as in Figure 2 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-$ as in the following tables:

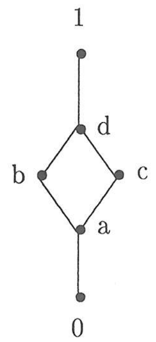

Figure 2: Example of proper $_{(WNM)}\alpha_{(DN)}$ algebra

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	c	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	0	b	b
c	0	0	0	c	c	c
d	0	0	b	c	d	d
1	0	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_\to)$, (DN) and (WNM). Consequently, $\mathcal{A}$ is a α algebra with conditions (DN) and (WNM), i.e. a $_{(WNM)}\alpha_{(DN)}$ algebra (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Note that $\mathcal{A}$ is a proper $_{(WNM)}\alpha_{(DN)}$ algebra, since:

- it does not satisfy the condition $(C_\vee)$:

$$d = b \vee c \neq [(b \to c) \to c] \wedge [(c \to b) \to b] = (c \to c) \wedge (b \to b) = 1;$$

- it does not satisfy the condition $(C_\wedge)$:

$$a = b \wedge c \neq [b \odot (b \to c)] \vee [c \odot (c \to b)] = b \odot c \vee c \odot b = 0;$$

- it does not satisfy the condition (C_X):

$$a = a \odot 1 = a \odot (c \to 1) = a \odot [(b \to a) \to (a \to b)] \neq b \odot [(a \to b) \to (b \to a)] = b \odot [1 \to c] = b \odot c = 0.$$

18.3 Example of proper $\beta\gamma_{(DN)}$ algebra

Let us consider the set $A = \{0, a, b, c, d, 1\}$ organized as a lattice as in Figure 3 and as a BCK(P) algebra with the operation $\rightarrow$ and $x \odot y \stackrel{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = (x \rightarrow y^-)^-$ as in the following tables:

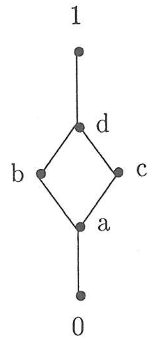

Figure 3: Example of proper $\beta\gamma_{(DN)}$ algebra

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	d	1	d	1	1
c	b	d	d	1	1	1
d	a	d	d	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	0	a	b
c	0	0	0	a	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_\vee)$, $(C_\wedge)$ and (DN). Consequently, $\mathcal{A}$ is a $\beta\gamma$ algebra with condition (DN), i.e. a $\beta\gamma_{(DN)}$ algebra (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Note that $\mathcal{A}$ is a proper $\beta\gamma_{(DN)}$ algebra , since:

- it does not satisfy the condition $(C_\rightarrow)$:

$$(b \rightarrow c) \rightarrow (c \rightarrow b) = d \rightarrow d = 1 \neq d;$$

- it does not satisfy the condition (C_X):

$$b = b \odot 1 = b \odot [d \rightarrow d] = b \odot [(c \rightarrow b) \rightarrow (b \rightarrow c)] \neq c \odot [(b \rightarrow c) \rightarrow (c \rightarrow b)] = c \odot [d \rightarrow d] = c \odot 1 = c;$$

- it does not satisfy the condition (WNM), since there is b such that:

$$(b \odot b)^- \vee [(b \wedge b) \rightarrow (b \odot b)] = a^- \vee [b \rightarrow a] = d \vee d = d \neq 1.$$

19 Examples of other new generalizations of Hájek(P) (BL) algebras and $_{(WNM)}$Hájek(P) ($_{(WNM)}$BL) algebras

19.1 Example of proper BCK(P) and $_{(WNM)}$BCK(P) lattices

19.1.1 Example of proper $_{(WNM)}$BCK(P) lattice

Let us consider the set $A = \{0, a, b, n, c, d, m, 1\}$ organized as a lattice as in Figure 4 and as a BCK(P) algebra with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

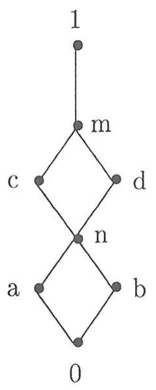

Figure 4: Example of proper $_{(WNM)}$BCK(P) lattice

$\rightarrow$	0	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1
a	m	1	m	1	1	1	1	1
b	m	m	1	1	1	1	1	1
n	m	m	m	1	1	1	1	1
c	m	m	m	m	1	m	1	1
d	m	m	m	m	m	1	1	1
m	m	m	m	m	m	m	1	1
1	0	a	b	n	c	d	m	1

$\odot$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
n	0	0	0	0	0	0	0	n
c	0	0	0	0	0	0	0	c
d	0	0	0	0	0	0	0	d
m	0	0	0	0	0	0	0	m
1	0	a	b	n	c	d	m	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which satisfies the condition (WNM); it is a proper $_{(WNM)}$BCK(P) lattice, since:

- $\mathcal{A}$ does not satisfy the condition $(C_{\rightarrow})$: there exist $c, d \in A$, such that

$$(c \rightarrow d) \rightarrow (d \rightarrow c) = m \rightarrow m = 1 \neq m = d \rightarrow c;$$

- $\mathcal{A}$ does not satisfy the condition $(C_{\wedge})$: there exist $c, d \in A$, such that

$$n = c \wedge d \neq [c \odot (c \rightarrow d)] \vee [d \odot (d \rightarrow c)] = (c \odot m) \vee (d \odot m) = 0 \vee 0 = 0;$$

- $\mathcal{A}$ does not satisfy the condition $(C_{\vee})$: there exist $a, b \in A$, such that

$$n = a \vee b \neq [(a \rightarrow b) \rightarrow b] \wedge [(b \rightarrow a) \rightarrow a] = (m \rightarrow b) \wedge (m \rightarrow a) = m \wedge m = m.$$

- $\mathcal{A}$ does not satisfy the condition (C_X): there exist $a, b \in A$, such that

$$a = a \odot 1 = a \odot (m \rightarrow m) = a \odot [(b \rightarrow a) \rightarrow (a \rightarrow b)] \neq b \odot [(a \rightarrow b) \rightarrow (b \rightarrow a)] = b \odot (m \rightarrow m) = b \odot 1 = b;$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

19.1.2 Examples of proper BCK(P) lattices (residuated lattices)

We give three examples.

• **Example 1** Let $\mathcal{A}_1$ be the ordinal sum of $\mathcal{L}_2$ and the above $_{(WNM)}$BCK(P) lattice $\mathcal{A}$, i.e. $A_1 = \{0, p\} \bigcup \{p, a, b, n, c, d, m, 1\} = \{0, p, a, b, n, c, d, m, 1\}$, organized as a lattice as in Figure 5 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

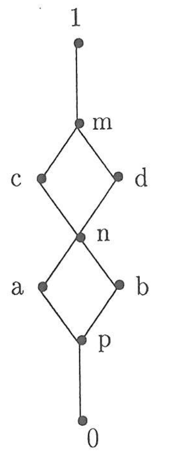

Figure 5: Example 1 of proper BCK(P) lattice

$\to$	0	p	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1
p	0	1	1	1	1	1	1	1	1
a	0	m	1	m	1	1	1	1	1
b	0	m	m	1	1	1	1	1	1
n	0	m	m	m	1	1	1	1	1
c	0	m	m	m	m	1	m	1	1
d	0	m	m	m	m	m	1	1	1
m	0	m	m	m	m	m	m	1	1
1	0	p	a	b	n	c	d	m	1

$\odot$	0	p	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0	0
p	0	p	p	p	p	p	p	p	p
a	0	p	p	p	p	p	p	p	a
b	0	p	p	p	p	p	p	p	b
n	0	p	p	p	p	p	p	p	n
c	0	p	p	p	p	p	p	p	c
d	0	p	p	p	p	p	p	p	d
m	0	p	p	p	p	p	p	p	m
1	0	p	a	b	n	c	d	m	1

Then $\mathcal{A}_1 = (A_1, \wedge, \vee, \to, 0, 1)$ is a proper BCK(P) lattice, since:

- $\mathcal{A}_1$ does not satisfy the condition $(C_\to)$: there exist $c, d \in A_1$, such that

$$(c \to d) \to (d \to c) = m \to m = 1 \neq m = d \to c;$$

- $\mathcal{A}_1$ does not satisfy the condition $(C_\wedge)$: there exist $c, d \in A_1$, such that

$$n = c \wedge d \neq [c \odot (c \to d)] \vee [d \odot (d \to c)] = (c \odot m) \vee (d \odot m) = p \vee p = p;$$

- $\mathcal{A}_1$ does not satisfy the condition $(C_\vee)$: there exist $a, b \in A_1$, such that

$$n = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (m \to b) \wedge (m \to a) = m \wedge m = m.$$

- $\mathcal{A}_1$ does not satisfy the condition (C_X): there exist $a, b \in A_1$, such that

$$a = a \odot 1 = a \odot (m \to m) = a \odot [(b \to a) \to (a \to b)] \neq b \odot [(a \to b) \to (b \to a)] = b \odot (m \to m) = b \odot 1 = b;$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0);
- it does not satisfy the condition (WNM):

$$(a \odot a)^- \vee [a \wedge a \to a \odot a] = p^- \vee [a \to p] = 0 \vee m = m \neq 1.$$

• **Example 2** Let $\mathcal{A}_2$ be the ordinal sum of $\mathcal{L}_2$ and the proper BCK(P)$_{(DN)}$ lattice $\mathcal{A}$ from the previous section, i.e. $A_2 = \{-1, 0\} \bigcup \{0, p, a, b, n, c, d, m, 1\} = \{-1, 0, p, a, b, n, c, d, m, 1\}$, organized as a lattice as in Figure 6 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

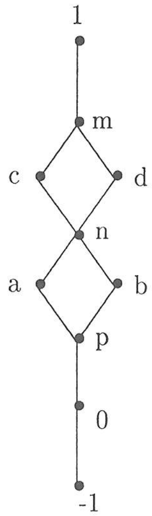

Figure 6: Example 2 of proper BCK(P) lattice

$\to$	-1	0	p	a	b	n	c	d	m	1
-1	1	1	1	1	1	1	1	1	1	1
0	-1	1	1	1	1	1	1	1	1	1
p	-1	m	1	1	1	1	1	1	1	1
a	-1	d	m	1	m	1	1	1	1	1
b	-1	c	m	m	1	1	1	1	1	1
n	-1	n	m	m	m	1	1	1	1	1
c	-1	b	m	m	m	m	1	m	1	1
d	-1	a	m	m	m	m	m	1	1	1
m	-1	p	m	m	m	m	m	m	1	1
1	-1	0	p	a	b	n	c	d	m	1

$\odot$	-1	0	p	a	b	n	c	d	m	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	-1	0	0	0	0	0	0	0	0	0
p	-1	0	0	0	0	0	0	0	0	p
a	-1	0	0	0	0	0	p	0	p	a
b	-1	0	0	0	0	0	0	p	p	b
n	-1	0	0	0	0	0	p	p	p	n
c	-1	0	0	p	0	p	p	p	p	c
d	-1	0	0	0	p	p	p	p	p	d
m	-1	0	0	p	p	p	p	p	p	m
1	-1	0	p	a	b	n	c	d	m	1

Then $\mathcal{A}_2 = (A_2, \wedge, \vee, \to, -1, 1)$ is a proper BCK(P) lattice, since:
- $\mathcal{A}_2$ does not satisfy the condition $(C_\to)$, since there exist $c, d \in A_2$, such that

$$(c \to d) \to (d \to c) = m \to m = 1 \neq m = d \to c;$$

- $\mathcal{A}_2$ does not satisfy the condition $(C_\vee)$, since there exist $a, b \in A_2$ such that

$$n = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (m \to b) \wedge (m \to a) = m \wedge m = m.$$

- $\mathcal{A}_2$ does not satisfy the condition $(C_\wedge)$, since there exist $c, d \in A_2$, such that

$$n = c \wedge d \neq [c \odot (c \to d)] \vee [d \odot (d \to c)] = (c \odot m) \vee (d \odot m) = p \vee p = p;$$

- $\mathcal{A}$ does not satisfy the condition (C_X), since there exist $a, b \in A$, such that

$$a = a \odot 1 = a \odot (m \to m) = a \odot [(b \to a) \to (a \to b)] \neq b \odot [(a \to b) \to (b \to a)] = b \odot (m \to m) = b \odot 1 = b.$$

- $\mathcal{A}_2$ does not satisfy the condition (WNM), since there are $a, c \in A_2$, such that:

$$(a \odot c)^- \vee [(a \wedge c) \to (a \odot c)] = p^- \vee [a \to p] = -1 \vee m = m \neq 1.$$

- it does not satisfy, evidently, the condition (DN).

• **Example 3** Let $\mathcal{A}_3$ be the ordinal sum of the proper BCK(P)$_{(DN)}$ lattice $\mathcal{A}$ from the previous section and $\mathcal{L}_2$, i.e. $A_3 = \{0, p, a, b, n, c, d, m, 1\} \bigcup \{1, 2\} = \{0, p, a, b, n, c, d, m, 1, 2\}$, organized as a lattice as in Figure 7 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

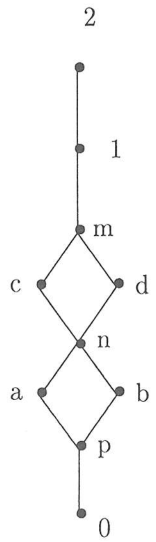

Figure 7: Example 3 of proper BCK(P) lattice

$\to$	0	p	a	b	n	c	d	m	1	2
0	2	2	2	2	2	2	2	2	2	2
p	m	2	2	2	2	2	2	2	2	2
a	d	m	2	m	2	2	2	2	2	2
b	c	m	m	2	2	2	2	2	2	2
n	n	m	m	m	2	2	2	2	2	2
c	b	m	m	m	m	2	m	2	2	2
d	a	m	m	m	m	m	2	2	2	2
m	p	m	m	m	m	m	m	2	2	2
1	0	p	a	b	n	c	d	m	2	2
2	0	p	a	b	n	c	d	m	1	2

$\odot$	0	p	a	b	n	c	d	m	1	2
0	0	0	0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	0	0	p	p
a	0	0	0	0	0	p	0	p	a	a
b	0	0	0	0	0	0	p	p	b	b
n	0	0	0	0	0	p	p	p	n	n
c	0	0	p	0	p	p	p	p	c	c
d	0	0	0	p	p	p	p	p	d	d
m	0	0	p	p	p	p	p	p	m	m
1	0	p	a	b	n	c	d	m	1	1
1	0	p	a	b	n	c	d	m	1	2

Then $\mathcal{A}_3 = (A_3, \wedge, \vee, \rightarrow, 0, 2)$ is a proper BCK(P) lattice, since:

- $\mathcal{A}_3$ does not satisfy the condition $(C_\rightarrow)$, since there exist $c, d \in A_3$, such that

$$(c \rightarrow d) \rightarrow (d \rightarrow c) = m \rightarrow m = 2 \neq m = d \rightarrow c;$$

- $\mathcal{A}_3$ does not satisfy the condition $(C_\vee)$, since there exist $a, b \in A_3$, such that

$$n = a \vee b \neq [(a \rightarrow b) \rightarrow b] \wedge [(b \rightarrow a) \rightarrow a] = (m \rightarrow b) \wedge (m \rightarrow a) = m \wedge m = m.$$

- $\mathcal{A}_3$ does not satisfy the condition $(C_\wedge)$, since there exist $c, d \in A_3$, such that

$$n = c \wedge d \neq [c \odot (c \rightarrow d)] \vee [d \odot (d \rightarrow c)] = (c \odot m) \vee (d \odot m) = p \vee p = p;$$

- $\mathcal{A}_3$ does not satisfy the condition (C_X), since there exist $a, b \in A_3$, such that

$$a = a \odot 2 = a \odot (m \rightarrow m) = a \odot [(b \rightarrow a) \rightarrow (a \rightarrow b)] \neq b \odot [(a \rightarrow b) \rightarrow (b \rightarrow a)] = b \odot (m \rightarrow m) = b \odot 2 = b.$$

- $\mathcal{A}_3$ does not satisfy the condition (WNM), since there are $a, c \in A_3$, such that:

$$(a \odot c)^- \vee [(a \wedge c) \rightarrow (a \odot c)] = p^- \vee [a \rightarrow p] = m \vee m = m \neq 2.$$

- $\mathcal{A}_3$ does not obviously satisfy the condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

19.2 Examples of proper $_{(WNM)}\alpha$ algebras

We shall give two examples.

Example 1 Let us consider the ordinal sum of the proper $_{(WNM)}\alpha_{(DN)}$ from the previous section and of $\mathcal{L}_2$, i.e. let us consider the set $A = \{0, a, b, c, d, n\} \bigcup \{n, 1\} = \{0, a, b, c, d, n, 1\}$, organized as a lattice as in Figure 8 and as a BCK(P) algebra with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

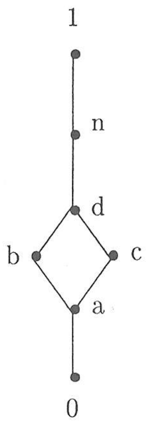

Figure 8: Examples of proper $_{(WNM)}\alpha$, γ and $\beta\gamma$ algebras

$\rightarrow$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	d	1	1	1	1	1	1
b	c	c	1	c	1	1	1
c	b	b	b	1	1	1	1
d	a	a	b	c	1	1	1
n	0	a	b	c	d	1	1
1	0	a	b	c	d	n	1

$\odot$	0	a	b	c	d	n	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	a	a
b	0	0	b	0	b	b	b
c	0	0	0	c	c	c	c
d	0	0	b	c	d	d	d
n	0	a	b	c	d	n	n
1	0	a	b	c	d	n	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which satisfies the condition $(C_{\rightarrow})$ and and the condition (WNM). Consequently, $\mathcal{A}$ is a proper ${}_{(WNM)}\alpha$ algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_{\vee})$; indeed, there exist $b, c \in A$, such that

$$d = b \vee c \neq [(b \rightarrow c) \rightarrow c) \wedge [(c \rightarrow b) \rightarrow b] = (c \rightarrow c) \rightarrow (b \rightarrow b) = 1;$$

- $\mathcal{A}$ does not satisfy the condition $(C_{\wedge})$; indeed, there exist $b, c \in A$, such that

$$a = b \wedge c \neq [b \odot (b \rightarrow c)] \vee [c \odot (c \rightarrow b)] = (b \odot c) \vee (c \odot b) = 0;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $a, b \in A$, such that

$$a = a \odot 1 = a \odot (c \rightarrow 1) = a \odot [(b \rightarrow a) \rightarrow (a \rightarrow b)] \neq b \odot [(a \rightarrow b) \rightarrow (b \rightarrow a)] = b \odot (1 \rightarrow c) = b \odot c = 0.$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Example 2 Let us consider the ordinal sum of $\mathcal{L}_2$ and the proper ${}_{(WNM)}\alpha_{(DN)}$ from the previous section, i.e. let us consider the set $A = \{0, n\} \bigcup \{n, a, b, c, d, 1\} = \{0, n, a, b, c, d, 1\}$, organized as a lattice as in Figure 9 and as a BCK(P) algebra with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

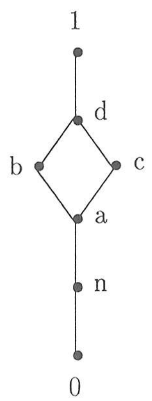

Figure 9: Examples of proper ${}_{(WNM)}\alpha$, β and $\beta\gamma$ algebras

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1
a	0	d	1	1	1	1	1
b	0	c	c	1	c	1	1
c	0	b	b	b	1	1	1
d	0	a	a	b	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n
a	0	n	n	n	n	n	a
b	0	n	n	b	n	b	b
c	0	n	n	n	c	c	c
d	0	n	n	b	c	d	d
1	0	n	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which looses the condition (DN), i.e. it satisfies the conditions $(C_{\rightarrow})$ and (WNM). Consequently, $\mathcal{A}$ is a proper ${}_{(WNM)}\alpha$ algebra, since:
- it does not satisfy the condition $(C_{\vee})$:

$$d = b \vee c \neq [(b \rightarrow c) \rightarrow c] \wedge [(c \rightarrow b) \rightarrow b] = (c \rightarrow c) \wedge (b \rightarrow b) = 1;$$

- it does not satisfy the condition $(C_\wedge)$:

$$a = b \wedge c \neq [b \odot (b \to c)] \vee [c \odot (c \to b)] = b \odot c \vee c \odot b = 0;$$

- it does not satisfy the condition (C_X):

$$a = a \odot 1 = a \odot (c \to 1) = a \odot [(b \to a) \to (a \to b)] \neq b \odot [(a \to b) \to (b \to a)] = b \odot [1 \to c] = b \odot c = 0;$$

- it does not satisfies obviously the condition (DN).

Remark 19.1 Note that in these cases, taking the ordinal sum of $\mathcal{L}_2$ and the above $_{(WNM)}\alpha$ algebras, we do not loose the condition (WNM), i.e. we do not get examples of proper α algebras.

19.3 Examples of proper β and $_{(WNM)}\beta$ algebras

19.3.1 Example of proper $_{(WNM)}\beta$ algebra

Let us consider the set $A = \{0, a, b, c, d, 1\}$ organized as a lattice as in Figure 10 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

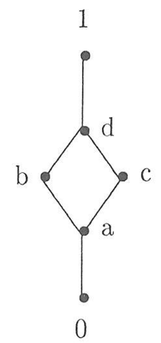

Figure 10: Example of proper $_{(WNM)}\beta$ and $\beta\gamma$ algebras

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	d	d	1	d	1	1
c	d	d	d	1	1	1
d	d	d	d	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the condition $(C_\vee)$ and the condition (WNM). Consequently, $\mathcal{A}$ is a proper $_{(WNM)}\beta$ algebra, since:
- $\mathcal{A}$ does not satisfy the condition $(C_\to)$; indeed, there exist $b, c \in A$, such that

$$(b \to c) \to (c \to b) = d \to d = 1 \neq d;$$

- $\mathcal{A}$ does not satisfy the condition $(C_\wedge)$; indeed, there exist $b, c \in A$, such that

$$a = b \wedge c \neq [b \odot (b \to c)] \vee [c \odot (c \to b)] = (b \odot d) \vee (c \odot d) = 0 \vee 0 = 0;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $a, b \in A$, such that

$$a = a \odot 1 = a \odot (d \to 1) = a \odot [(b \to a) \to (a \to b)] \neq b \odot [(a \to b) \to (b \to a)] = b \odot (1 \to d) = b \odot d = 0.$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

19.3.2 Example of proper β algebra

Let us consider the ordinal sum of $\mathcal{L}_2$ and the above $_{(WNM)}\beta$ algebra, i.e. let us consider the set $A = \{0, n\} \bigcup \{n, a, b, c, d, 1\} = \{0, n, a, b, c, d, 1\}$ organized as a lattice as in Figure 9 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1
a	0	d	1	1	1	1	1
b	0	d	d	1	d	1	1
c	0	d	d	d	1	1	1
d	0	d	d	d	d	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n
a	0	n	n	n	n	n	a
b	0	n	n	n	n	n	b
c	0	n	n	n	n	n	c
d	0	n	n	n	n	n	d
1	0	n	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the condition $(C_\vee)$. Consequently, $\mathcal{A}$ is a proper β algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\to)$; indeed, there exist $b, c \in A$, such that

$$(b \to c) \to (c \to b) = d \to d = 1 \neq d;$$

- $\mathcal{A}$ does not satisfy the condition $(C_\wedge)$; indeed, there exist $b, c \in A$, such that

$$a = b \wedge c \neq [b \odot (b \to c)] \vee [c \odot (c \to b)] = (b \odot d) \vee (c \odot d) = n \vee n = n;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $a, b \in A$, such that

$$a = a \odot 1 = a \odot (d \to 1) = a \odot [(b \to a) \to (a \to b)] \neq b \odot [(a \to b) \to (b \to a)] = b \odot (1 \to d) = b \odot d = n.$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0); - it does not satisfy the condition (WNM): there is $b \in A$, such that

$$(b \odot b)^- \vee [b \wedge b \to b \odot b] = n^- \vee [b \to n] = 0 \vee d = d \neq 1.$$

19.4 Examples of proper γ and $_{(WNM)}\gamma$ algebras

19.4.1 Example of proper $_{(WNM)}\gamma$ algebra

Let us consider the set $A = \{0, b, c, d, n, 1\}$ organized as a lattice as in Figure 11 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

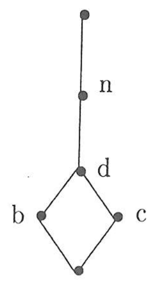

Figure 11: Example of proper $_{(WNM)}\gamma$ algebra

$\to$	0	b	c	d	n	1
0	1	1	1	1	1	1
b	n	1	n	1	1	1
c	n	n	1	1	1	1
d	n	n	n	1	1	1
n	n	n	n	n	1	1
1	0	b	c	d	n	1

$\odot$	0	b	c	d	n	1
0	0	0	0	0	0	0
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
n	0	0	0	0	0	n
1	0	b	c	d	n	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the condition $(C_\wedge)$ and the condition (WNM). Consequently, $\mathcal{A}$ is a proper $_{(WNM)}\gamma$ algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\to)$; indeed, there exist $b, c \in A$, such that

$$(b \to c) \to (c \to b) = n \to n = 1 \neq n;$$

- $\mathcal{A}$ does not satisfy the condition $(C_\vee)$; indeed, there exist $b, c \in A$, such that

$$d = b \vee c \neq [(b \to c) \to c] \wedge [(c \to b) \to b] = (n \to c) \wedge (n \to b) = n \wedge n = n;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $b, c \in A$, such that

$$b = b \odot 1 = b \odot (n \to n) = b \odot [(c \to b) \to (b \to c)] \neq c \odot [(b \to c) \to (c \to b)] = c \odot (n \to n) = c \odot 1 = c.$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

19.4.2 Examples of proper γ algebras

We shall give two examples.

Example 1 Let us consider the ordinal sum of $\mathcal{L}_2$ and of the above $_{(WNM)}\gamma$ algebra, i.e. let us consider the set $A = \{0, a\} \bigcup \{a, b, c, d, n, 1\} = \{0, a, b, c, d, n, 1\}$, organized as a lattice as in Figure 8 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\to$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	n	1	n	1	1	1
c	0	n	n	1	1	1	1
d	0	n	n	n	1	1	1
n	0	n	n	n	n	1	1
1	0	a	b	c	d	n	1

$\odot$	0	a	b	c	d	n	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	a	a	a	c
d	0	a	a	a	a	a	d
n	0	a	a	a	a	a	n
1	0	a	b	c	d	n	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the condition $(C_\wedge)$. Consequently, $\mathcal{A}$ is a proper γ algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\to)$; indeed, there exist $b, c \in A$, such that

$$(b \to c) \to (c \to b) = n \to n = 1 \neq n;$$

- $\mathcal{A}$ does not satisfy the condition $(C_\vee)$; indeed, there exist $b, c \in A$, such that

$$d = b \vee c \neq [(b \to c) \to c] \wedge [(c \to b) \to b] = (n \to c) \wedge (n \to b) = n \wedge n = n;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $b, c \in A$, such that

$$b = b \odot 1 = b \odot (n \to n) = b \odot [(c \to b) \to (b \to c)] \neq c \odot [(b \to c) \to (c \to b)] = c \odot (n \to n) = c \odot 1 = c.$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0);
- it does not satisfy the condition (WNM): there is $b \in A$ such that

$$(b \odot b)^- \vee [b \wedge b \to b \odot b] = a^- \vee [b \to a] = 0 \vee n = n \neq 1.$$

Example 2 Let us consider the ordinal sum of $\mathcal{L}_{2\times 2}$ and of the proper $\beta\gamma_{(DN)}$ algebra from the previous section, i.e. let us consider the set $A = \{0, m, n, p\} \bigcup \{p, a, b, c, d, 1\} = \{0, m, n, p, a, b, c, d, 1\}$, organized as a lattice as in Figure 12

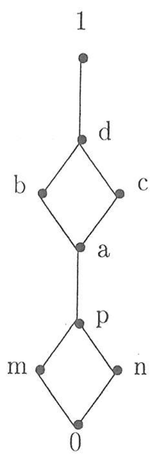

Figure 12: Example 2 of proper γ algebra

$\to$	0	m	n	p	a	b	c	d	1
0	1	1	1	1	1	1	1	1	1
m	n	1	n	1	1	1	1	1	1
n	m	m	1	1	1	1	1	1	1
p	0	m	n	1	1	1	1	1	1
a	0	m	n	d	1	1	1	1	1
b	0	m	n	c	d	1	d	1	1
c	0	m	n	b	d	d	1	1	1
d	0	m	n	a	d	d	d	1	1
1	0	m	n	p	a	b	c	d	1

$\odot$	0	m	n	p	a	b	c	d	1
0	0	0	0	0	0	0	0	0	0
m	0	m	0	m	m	m	m	m	m
n	0	0	n	n	n	n	n	n	n
p	0	m	n	p	p	p	p	p	p
a	0	m	n	p	p	p	p	p	a
b	0	m	n	p	p	a	p	a	b
c	0	m	n	p	p	p	a	a	c
d	0	m	n	p	p	a	a	a	d
1	0	m	n	p	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the condition $(C_\wedge)$. Consequently, $\mathcal{A}$ is a proper γ algebra, since:
- it does not satisfy the condition $(C_\to)$:

$$(b \to c) \to (c \to b) = d \to d = 1 \neq d;$$

- it does not satisfy the condition $(C_\vee)$:

$$p = m \vee n \not= [(mran) \to n] \wedge [(n \to m) \to m] = [n \to n] \wedge [m \to m] = 1 \wedge 1 = 1;$$

- it does not satisfy the condition (C_X):

$$b = b \odot 1 = b \odot [d \to d] = b \odot [(c \to b) \to (b \to c)] \neq c \odot [(b \to c) \to (c \to b)] = c \odot [d \to d] = c \odot 1 = c;$$

- it does not satisfy the condition (WNM), since there is b such that:

$$(b \odot b)^- \vee [(b \wedge b) \to (b \odot b)] = a^- \vee [b \to a] = 0 \vee d = d \neq 1;$$

- it does not satisfy obviously the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Remark 19.2 Note that the ordinal sum $M_1 \bigoplus M_2$ preserves $(C_\wedge)$ and if M_1 is non-linearly ordered, as it was the case in this example, then it does not preserves $(C_\vee)$.

19.5 Examples of proper $\beta\gamma$ and $_{(WNM)}\beta\gamma$ algebras

19.5.1 Example of proper $_{(WNM)}\beta\gamma$ algebra

Let us consider the set $A = \{0, b, c, d, 1\}$ organized as a lattice as in Figure 13 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\to$	0	b	c	d	1
0	1	1	1	1	1
b	d	1	d	1	1
c	d	d	1	1	1
d	d	d	d	1	1
1	0	b	c	d	1

$\odot$	0	b	c	d	1
0	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	0	c
d	0	0	0	0	d
1	0	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_\vee)$ and $(C_\wedge)$ and the condition (WNM). Consequently, $\mathcal{A}$ is a proper $_{(WNM)}\beta\gamma$ algebra, since:

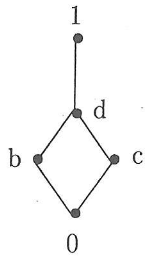

Figure 13: Example of proper $_{(WNM)}\beta\gamma$ algebra

- $\mathcal{A}$ does not satisfy the condition $(C_{\rightarrow})$; indeed, there exist $b, c \in A$, such that

$$(b \to c) \to (c \to b) = d \to d = 1 \neq d = c \to b;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $b, c \in A$, such that

$$b = b \odot 1 = b \odot (d \to d) = b \odot [(c \to b) \to (b \to c)] \neq c \odot [(b \to c) \to (c \to b)] = c \odot (d \to d) = c \odot 1 = c.$$

- it does not satisfy the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

19.5.2 Examples of proper $\beta\gamma$ algebras

We shall give three examples.

Example 1 Let us consider the ordinal sum of $\mathcal{L}_2$ and the above proper $_{(WNM)}\beta\gamma$ algebra, i.e. let us consider the set $A_1 = \{0, a\} \bigcup \{a, b, c, d, 1\} = \{0, a, b, c, d, 1\}$, organized as a lattice as in Figure 10 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	1	1	1	1
b	0	d	1	d	1	1
c	0	d	d	1	1	1
d	0	d	d	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	a	a	a	b
c	0	a	a	a	a	c
d	0	a	a	a	a	d
1	0	a	b	c	d	1

Then $\mathcal{A}_1 = (A_1, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_{\vee})$ and $(C_{\wedge})$. Consequently, $\mathcal{A}_1$ is a proper $\beta\gamma$ algebra, since:

- $\mathcal{A}_1$ does not satisfy the condition $(C_{\rightarrow})$; indeed, there exist $b, c \in A_1$, such that

$$(b \to c) \to (c \to b) = d \to d = 1 \neq d = c \to b;$$

- $\mathcal{A}_1$ does not satisfy the condition (C_X); indeed, there exist $b, c \in A_1$, such that

$$b = b \odot 1 = b \odot (d \to d) = b \odot [(c \to b) \to (b \to c)] \neq c \odot [(b \to c) \to (c \to b)] = c \odot (d \to d) = c \odot 1 = c.$$

- it does not satisfy the condition (WNM), since there is b such that:

$$(b \odot b)^- \vee [(b \wedge b) \to (b \odot b)] = a^- \vee [b \to a] = 0 \vee d = d \neq 1.$$

- it does not obviously the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Example 2

Let us consider the set $A = \{0, a, b, c, d, n, 1\}$ organized as a lattice as in Figure 8 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\to$	0	a	b	c	d	n	1
0	1	1	1	1	1	1	1
a	d	1	1	1	1	1	1
b	c	d	1	d	1	1	1
c	b	d	d	1	1	1	1
d	a	d	d	d	1	1	1
n	0	a	b	c	d	1	1
1	0	a	b	c	d	n	1

$\odot$	0	a	b	c	d	n	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	a	a
b	0	0	a	0	a	b	b
c	0	0	0	a	a	c	c
d	0	0	a	a	a	d	d
n	0	a	b	c	d	n	n
1	0	a	b	c	d	n	1

Then $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_\vee)$ and $(C_\wedge)$. Consequently, $\mathcal{A}$ is a proper $\beta\gamma$ algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\to)$; indeed, there exist $b, c \in A$, such that

$$(b \to c) \to (c \to b) = d \to d = 1 \neq d = c \to b;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $b, c \in A$, such that

$$b = b \odot 1 = b \odot (d \to d) = b \odot [(c \to b) \to (b \to c)] \neq c \odot [(b \to c) \to (c \to b)] = c \odot (d \to d) = c \odot 1 = c.$$

- it does not satisfy the condition (WNM): there is $b \in A$ such that

$$(b \odot b)^- \vee [b \wedge b \to b \odot b] = a^- \vee [b \to a] = d \vee d = d \neq 1.$$

-it does not satisfy obviously the condition (DN) (you have the values of $x^- = x \to 0$ in the table of $\to$, column of 0).

Example 3

Let us consider the set $A = \{0, n, a, b, c, d, 1\}$ organized as a lattice as in Figure 9 and as a BCK(P) algebra with the operation $\to$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\}$ as in the following tables:

$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	d	d	1	1	1	1	1
b	c	c	d	1	d	1	1
c	b	b	d	d	1	1	1
d	a	a	d	d	d	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	n
a	0	0	0	0	0	0	a
b	0	0	0	a	0	a	b
c	0	0	0	0	a	a	c
d	0	0	0	a	a	a	d
1	0	n	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a BCK(P) lattice which satisfies the conditions $(C_\vee)$ and $(C_\wedge)$. Consequently, $\mathcal{A}$ is a proper $\beta\gamma$ algebra, since:

- $\mathcal{A}$ does not satisfy the condition $(C_\rightarrow)$; indeed, there exist $b, c \in A$, such that

$$(b \rightarrow c) \rightarrow (c \rightarrow b) = d \rightarrow d = 1 \neq d = c \rightarrow b;$$

- $\mathcal{A}$ does not satisfy the condition (C_X); indeed, there exist $b, c \in A$, such that

$$b = b \odot 1 = b \odot (d \rightarrow d) = b \odot [(c \rightarrow b) \rightarrow (b \rightarrow c)] \neq c \odot [(b \rightarrow c) \rightarrow (c \rightarrow b)] = c \odot (d \rightarrow d) = c \odot 1 = c.$$

- it does not satisfy the condition (WNM): there is $b \in A$ such that

$$(b \odot b)^- \vee [b \wedge b \rightarrow b \odot b] = a^- \vee [b \rightarrow a] = d \vee d = d \neq 1.$$

-it does not satisfy obviously the condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

20 Final remarks and open problems

We get the hierarchies from Figure 14 of all the descendents (particular cases) of BL algebras mentioned in this paper.

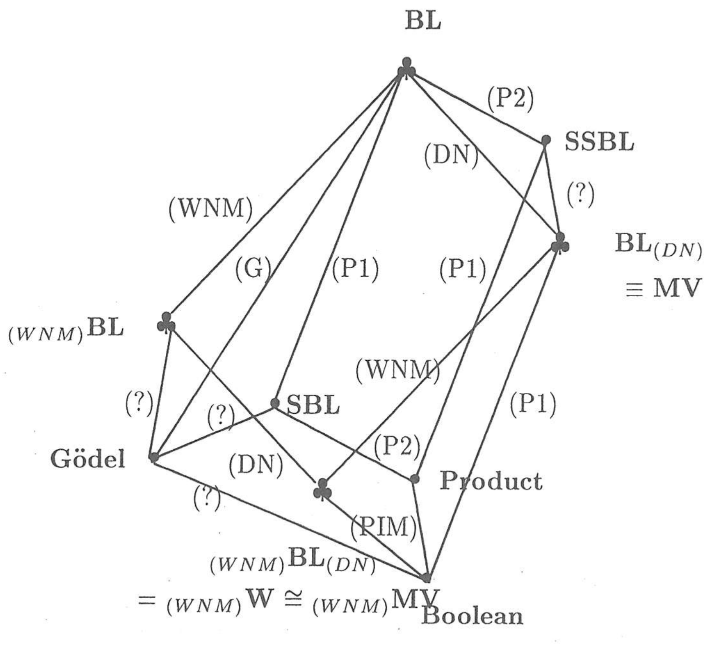

Figure 14: Some descendents (particular cases) of BL algebras

By combining the two hierarchies from Figures ?? and 14, we get the hierarchies from Figure 15.
Final remarks

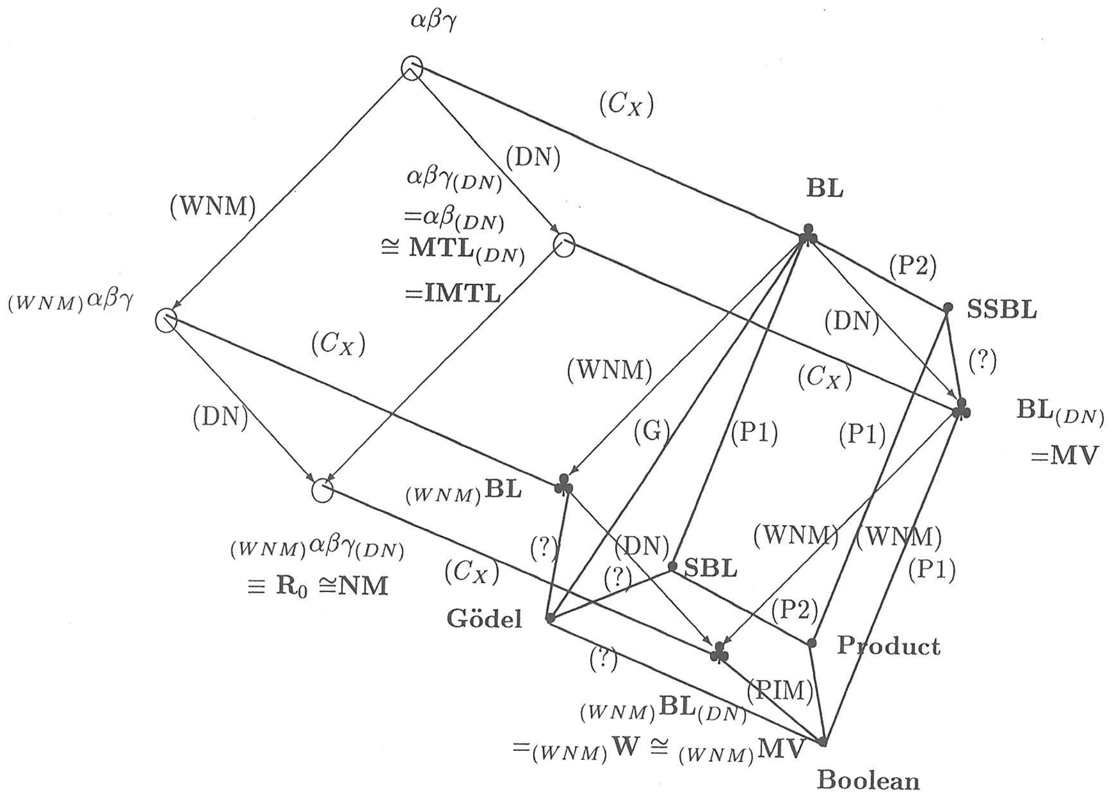

Figure 15: Vertical sections through $\alpha\beta\gamma$ and BL algebras and other algebras

1) The divisible BCK(P) lattices seem to be a significant generalization of BL algebras, very closed connected with BL algebras. Recall that a linearly ordered divisible BCK(P) lattice is a BL algebra and that divisible BCK(P) lattice with condition (DN) and BL algebras with condition (DN) coincide (with Wajsberg (MV) algebras).

2) The $\alpha\beta\gamma$ algebras seem to be a significant generalization of BL algebras also; they deserve to be called "MTL" algebras, not the $\alpha\beta$ algebras. The relation "$\alpha\gamma$ algebras - $\alpha\beta\gamma$ algebras" is similar to the relation "divisible residuated lattices - BL algebras": recall that a linearly ordered $\alpha\gamma$ algebra is an $\alpha\beta\gamma$ algebra and that $\alpha\gamma$ algebras with condition (DN) and $\alpha\beta\gamma$ algebras with condition (DN) coincide (with IMTL algebras).

3) We have not yet examples of proper $\alpha\beta$ (weak-BL =MTL) algebras. The relation "$\alpha\beta$ algebras - $\alpha\beta\gamma$ algebras" is similar to the relation "divisible residuated lattices - BL algebras": recall that a linearly ordered $\alpha\beta$ algebra is an $\alpha\beta\gamma$ algebra and that $\alpha\beta$ algebras with condition (DN) and $\alpha\beta\gamma$ algebras with condition (DN) coincide (with IMTL algebras).

4) The R_0 (NM) algebras seem to be as much important as Wajsberg (MV) algebras are. They are incomparable as classes, by Remark ?? (there are R_0 (NM) algebras which are not Wajsberg (MV) algebras and there are Wajsberg (MV) algebras which are not R_0 (NM) algebras). The intersection of the two classes is the subclass of $_{(WNM)}$Wajsberg ($_{(WNM)}$MV) algebras (see the hierarchies from Figure ??).

We have got the following chain (by set inclusion) of liniarly ordered R_0 (NM) algebras:

$\mathcal{NM}_2 = \mathcal{L}_2,\ \mathcal{NM}_3 = \mathcal{L}_3,\ \mathcal{NM}_4, \ldots, \mathcal{NM}_{n+1}, \ldots\ (n \geq 1)$,

where

$$\mathcal{NM}_{n+1} = (L_{n+2}, \wedge = \min, \vee = \max, \to, ^-, n), \quad (n \geq 1),$$

with $x^- = n - x$ (strong negation) and $\to$ is Fodor's implication [7], [4]:

$$x \to y = \begin{cases} n, & \text{if } x \leq y \\ \max(n-x, y), & \text{if } x > y, \end{cases}$$

while the corresponding (i.e. with the same support sets, the sets L_{n+1}, $n \geq 1$) chain of linearly ordered Wajsberg (left-MV) algebras is:

$\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \ldots, \mathcal{L}_{n+1}, \ldots \ (n \geq 1)$,
where
$$\mathcal{L}_{n+1} = (L_{n+1}, \rightarrow, ^-, n),$$
with $x^- = n - x$ (strong negation) and $\rightarrow$ is Lukasiewicz's implication:
$$x \rightarrow y = \begin{cases} n, & \text{if } x \leq y \\ n - x + y, & \text{if } x > y, \end{cases} = \min(n, n - x + y),$$
$x \vee y = (x \rightarrow y) \rightarrow y = \max(x, y)$ and
$x \wedge y = x \odot (x \rightarrow y) = \min(x, y)$.

The two common algebras (i.e. the two ${}_{(WNM)}$Wajsberg (${}_{(WNM)}$MV) algebras) of the two chains of algebras are $\mathcal{L}_2$ and $\mathcal{L}_3$, where $\mathcal{L}_2$ is even a Boolean algebra.

5) The ${}_{(WNM)}\alpha\beta\gamma$ algebras seem to be as much important as Hájek(P) (BL) algebras are. They are incomparable as classes (there are ${}_{(WNM)}\alpha\beta\gamma$ algebras which are not Hájek(P) (BL) algebras and there are Hájek(P) (BL) algebras which are not ${}_{(WNM)}\alpha\beta\gamma$ algebras algebras). The intersection of the two classes is the subclass of ${}_{(WNM)}$Ha(P) (${}_{(WNM)}$BL) algebras (see the hierarchies from Figure ??).

6) There are cases when the ordinal sum of two BCK(P) lattices with condition (WNM) is no more a BCK(P) lattice with condition (WNM).

Final open problems

1) A first group of open problems concernes the algebras without condition (DN):
- Find example of proper α algebra.
- Find examples of proper $\alpha\beta$ (MTL) algebra and of proper ${}_{(WNM)}\alpha\beta$ (WNM) algebra.
- Find examples for the other proper algebras marked by the signs "??" and "?" in Figures ?? and ??.
- Find a representation theorem for $\alpha\beta$ (MTL) algebras, for $\alpha\gamma$ algebras and for $\alpha\gamma\delta$ algebras (divisible residuated lattices).

2) A second group of open problems concernes the algebras with condition (DN):
- Find examples of proper ${}_{(WNM)}$BCK(P)${}_{(DN)}$ lattice, of proper $\alpha_{(DN)}$ algebra, of proper ${}_{(WNM)}\beta\gamma_{(DN)}$ algebra.
- Find examples for the other proper algebras marked by the signs "??" and "?" in Figures ?? and ??.

3) General open problem:
Study the distributivity of the algebras given as examples (almost all are distributive) and find examples of non-distributive ones.

Acknowledgements

On the begining of December 2003, looking for the best definition of ordinal sum of BL chains, I have found the paper [5] (received in October 2002 from L. Godo); reading it, I have discovered that it contains six examples of residuated lattices and I have noticed that: their Example 1 coincides with my $\mathcal{D}_{2\times2,2}$ from [25], their Example 2 is a good example of proper non-distributive $\beta\gamma_{(DN)}$ algebra, their Example 3 is a good example of proper non-distributive $\beta\gamma$ algebra, their Example 4 coincides with my example of $\alpha_{(DN)}$ algebra, their Example 5 is a proper $\beta\gamma$ algebra and their Example 6 coincides with my example of $\beta\gamma_{(DN)}$ algebra.

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