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H^2 Optimal Control for Linear Stochastic Periodic Systems

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Abstract

This paper presents an optimal solution of the H^2 state-feedback control problem for time varying periodic stochastic linear systems subjected both to jump Markov perturbations and to multiplicative white noise. It is proved that the optimal solution is a static gain which is also optimal in the class of all higher order controllers. This solution is expressed in terms of the stabilizing solution of some suitable system of coupled Riccati type differential equations.

_ _eywords: linear stochastic systems, periodic coefficients, H^2-norms, Riccati differential equations.

1 Introduction

The H^2 and the linear quadratic control problem for linear stochastic systems have been widely studied in the current literature. A particular attention was paid to two classes of stochastic systems, namely to Markov jump linear systems and to systems subjected to multiplicative white noise. When an important and unpredictible variation causes a discrete change in the plant characterization at isolated points in time, a Markov chain with a finite state space is a natural model for the plant parameter processes.

Some illustrative applications of these systems can be found for example in [17, 1, 16, 18, 12] and their references, where stochastic stability properties and useful results concerning controllability, observability and optimal control are presented.

More recently, the H^2 control problem for Markov jump linear stationary systems has been studied in [2] for the state-feedback case using convex analysis and in [4] for the output-feedback case. Here the solutions of the H^2 problem is explicitly expressed in terms of the solutions of some linear matrix inequalities (LMI).

The stochastic systems with multiplicative white noise naturally arise in control problems of linear uncertain systems with stochastic uncertainty (see [21, 11, 15, 20] and the references therein). Results concerning the H^2 control problem for this type of systems are derived for instance in [3, 5].

In the present paper the H^2 optimal state-feedback control problem is addressed for time varying periodic linear stochastic systems subjected both to Markov jumps and to multiplicative white noise. The paper extends to the time varying periodic case the results of [8]. An optimal regulation problem for this class of stochastic systems can be found in [14].

The approach derived in the present paper uses the stabilizing solution of a suitable system of coupled Riccati differential equations (SGRDE). In [6] was proved that if the coefficients of (SGRDE) are periodic functions then its stabilizing solution is a periodic function too. Therefore this solution may be computed applying an iterative procedure on an compact interval equal with a period.

The optimal solution providing the minimum of the H^2 norm of the resulting systems is a zero-order dynamics controller. Moreover, it is proved that this solution is also optimal in the class of all higher order controllers, extending thus the corresponding known result from the deterministic framework ([10]).

2 Problem formulation

A. Consider the system $(\tilde{\mathbf{G}})$ described by the state-space equations:

$$\begin{aligned} dx(t) &= A_0(t,\eta(t))\,x(t) + \sum_{k=1}^{r} A_k(t,\eta(t))\,x(t)\,dw_k(t) + B_v(t,\eta(t))\,dv(t) \quad (2.1) \\ z(t) &= C(t,\eta(t))\,x(t) \end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the state, $\eta(t)$, $t \geq 0$ is a right continuous homogenous Markov chain with the state space the set $\mathcal{D} = \{1, ..., d\}$ and the probability transition matrix $P(t) = [p_{ij}(t)] = e^{Qt}$, $t \geq 0$; here $Q = [q_{ij}]$ with $\sum_{j=1}^{d} q_{ij} = 0$, $i \in \mathcal{D}$ and $q_{ij} \geq 0$ if $i \neq j$ (see [9]); $[w^*(t) \;\; v^*(t)]^*$, $t \geq 0$ is an $(r + m_v)$-dimensional standard Wiener process on a given probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$, $w(t) = (w_1(t), \dots, w_r(t))^*$, $v(t) = (v_1(t), \dots, v_{m_v}(t))^*$ (for more details, see [13, 19]); $A_k(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n\times n}$, $0 \leq k \leq r$, $B_v(\cdot, i) : \mathbf{R} \to \mathbf{R}^{n\times m_v}$, $C(\cdot, i) : \mathbf{R} \to \mathbf{R}^{p\times n}$, $i \in \mathcal{D}$ are continuous and periodic functions with period $\theta > 0$.

Throughout this paper it is assumed that $\{w(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$, $\{\eta(t)\}_{t\geq 0}$ are independent stochastic processes and $P\{\eta(0) = i\} > 0$ for all $i \in \mathcal{D}$.

Let $\mathcal{H}_t$ be the smallest σ-algebra with respect to which all functions $w_j(s)$, $\eta(s)$, $1 \leq j \leq r$, $0 \leq s \leq t$ are measurable and $\hat{\mathcal{H}}_t$ be the smallest σ-algebra with respect to which all functions $w_j(s)$, $1 \leq j \leq r$, $v_l(s)$, $1 \leq l \leq m_v$, $\eta(s)$, $0 \leq s \leq t$ are measurable.

Using a standard argument of successive approximations and the properties of stochastic integral, one can prove that for each $t_0 \geq 0$, $x_0 \in \mathbf{R}^n$ the equation (2.1) has a unique solution $x(t, t_0, x_0)$ with the initial condition $x(t_0, t_0, x_0) = x_0$. Moreover, $t \longmapsto x(t, t_0, x_0)$ is continuous with probability 1 and

$$\sup_{t\in[t_0,T]} E\left[|x(t, t_0, x_0)|^{2a}\right] < \infty,$$

for all integers $a \geq 1$, for all $T \geq t_0$ where E denotes as usually the expectation.

Consider now the homogeneous system:

$$dx(t) = A_0(t,\eta(t))\,x(t)\,dt + \sum_{k=1}^{r} A_k(t,\eta(t))\,x(t)\,dw_k(t) \tag{2.2}$$

associated with (2.1) and denote by $\Phi(t,s)$, $t \geq s \geq 0$ its fundamental matrix solution. This means that its j's column $\Phi_j(t,s) = x(t,s,e_j)$ where $e_j = [0, \ldots, 0, 1, 0, \ldots, 0]^*$ is vector of canonical bases in $\mathbf{R}^n$.

Based on the Itô-type formula, the following representation formula for the solution of (2.1) can be obtained (see [7]):

$$x(t,t_0,x_0) = \Phi(t,t_0)\,x_0 + \Phi(t,t_0)\int_{t_0}^{t} \Phi^{-1}(s,t_0)\,B_v(s,\eta(s))\,dv(s), \tag{2.3}$$

$t \geq t_0 \geq 0$, $x_0 \in \mathbf{R}^n$.

Definition. The zero solution of the equation (2.2) is called *exponentially stable in mean square (ESMS)* or equivalently, the system $(A_0, \ldots, A_r; Q)$ is stable if there exist $\alpha > 0$, $\beta \geq 1$ such that

$$E\left[|\Phi(t,t_0)\,x_0|^2 \,\middle|\, \eta(t_0) = i\right] \leq \beta e^{-\alpha(t-t_0)}\,|x_0|^2$$

for all $t \geq t_0$, $x_0 \in \mathbf{R}^n, i \in \mathcal{D}$, where $E[\cdot|\eta(t_0) = i]$ stands for the conditional expectation with respect to the event $\{\eta(t_0) = i\}$.

B. For the system $(\tilde{\mathbf{G}})$ described by (2.1) we introduce the following norms:

$$\|\tilde{\mathbf{G}}\|_2 = [\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} E|z(t)|^2 dt]^{\frac{1}{2}} \tag{2.4}$$

and

$$|||\tilde{\mathbf{G}}|||_2 = \{\lim_{T\to\infty} \frac{1}{T}\sum_{j=1}^{d}\int_{t_0}^{t_0+T} E[|z(t)|^2|\eta(t_0) = i]dt\}^{\frac{1}{2}} \tag{2.5}$$

respectively, where $E[\cdot|\eta(t_0) = i]$ is the conditional expectation with respect to the event $\eta(t_0) = i$ and E stands for the mathematical expectation.

It can be verified that if the zero solution of (2.2) is (ESMS) then (2.4) and (2.5) are independent of t_0 and x_0 and therefore the norms (2.4) and (2.5) are well defined.

We shall see in the next section that under the assumption that the zero solution of (2.2) is exponentially stable in mean square (ESMS), the above limits exist and do not depend upon t_0 and x_0 and therefore these norms are well defined. The norm (2.5) extends to the periodic case a norm considered in [8] while the norm (2.4) extends to this framework the well known norm widely investigated in the literature (see [2, 10, 3, 5, 8] and references therein).

C. Consider the system $\mathbf{G}$ described by:

$$\begin{aligned} dx(t) &= [A_0(t,\eta(t))\,x(t) + B_0(t,\eta(t))\,u(t)]\,dt \\ &\quad + \sum_{k=1}^{r} [A_k(t,\eta(t))\,x(t) + B_k(t,\eta(t))\,u(t)]\,dw_k(t) \\ &\quad + B_v(t,\eta(t))\,dv(t) \\ z(t) &= C(t,\eta(t))\,x(t) + D(t,\eta(t))\,u(t) \end{aligned} \tag{2.6}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ denotes the vector of control variables, $z \in \mathbf{R}^p$ is the regulated output and $A_k(\cdot,i)$, $B_v(\cdot,i)$, $0 \leq k \leq r$, $C(\cdot,i)$ are as before and $B_k(\cdot,i) : \mathbf{R} \to \mathbf{R}^{n\times m}$, $D(\cdot,i) : \mathbf{R} \to \mathbf{R}^{p\times m}$ are also continuous and θ-periodic functions.

Consider the following family of controllers $\mathbf{G}_c$ described by:

$$\begin{aligned} \dot{x}_c(t) &= A_c(t,\eta(t))\, x_c(t) + B_c(t,\eta(t))\, u_c(t) \\ y_c(t) &= C_c(t,\eta(t))\, x_c(t) + D_c(t,\eta(t))\, u_c(t) \end{aligned} \tag{2.7}$$

where $x_c \in \mathbf{R}^{n_c}, u_c \in \mathbf{R}^n$, $y_c \in \mathbf{R}^m$, $A_c(\cdot,i) : \mathbf{R} \to \mathbf{R}^{n_c\times n_c}, B_c(\cdot,i) : \mathbf{R} \to \mathbf{R}^{n_c\times n}, C_c(\cdot,i) : \mathbf{R} \to \mathbf{R}^{m\times n_c}$ and $D_c(\cdot,i) : \mathbf{R} \to \mathbf{R}^{m\times n}$ are continuous and θ-periodic functionc.

Let us remark that the controller $\mathbf{G}_c$ of form (2.7) is completely determined by the set of parameters $(n_c, A_c(\cdot,i), B_c(\cdot,i), C_c(\cdot,i), D_c(\cdot,i), i \in \mathcal{D})$ where $n_c \geq 0$ denotes the controller order.

In the particular case $n_c = 0$ the controller (2.7) reduces to

$$y_c(t) = D_c(t,\eta(t))\, u_c(t)$$

which shows that the zero order (state-feedback) controllers are included in the set of controllers (2.7).

The resulting system $\mathbf{G}_{cl}$ obtained by coupling a controller of form (2.7) to the system (2.6) by taking $u_c(t) = x(t)$ and $u(t) = y_c(t)$ is:

$$\begin{aligned} dx_{cl}(t) &= A_{0cl}(t,\eta(t))\, x_{cl}(t)\, dt + \sum_{k=1}^{r} A_{kcl}(t,\eta(t))\, x_{cl}(t)\, dw_k(t) \\ &\quad + B_{vcl}(t,\eta(t))\, dv(t) \\ y_{cl}(t) &= C_{cl}(t,\eta(t))\, x_{cl}(t) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} x_{cl} &= \begin{bmatrix} x \\ x_c \end{bmatrix}; \\ A_{0cl}(t,i) &= \begin{bmatrix} A_0(t,i) + B_0(t,i)\, D_c(t,i) & B_0(t,i)\, C_c(t,i) \\ B_c(t,i) & A_c(t,i) \end{bmatrix}; \\ A_{kcl}(t,i) &= \begin{bmatrix} A_k(t,i) + B_k(t,i)\, D_c(t,i) & B_k(t,i)\, C_c(t,i) \\ 0 & 0 \end{bmatrix}; \\ B_{vcl}(t,i) &= \begin{bmatrix} B_v(t,i) \\ 0 \end{bmatrix}; \\ C_{cl}(t,i) &= \left[C(t,i) + D(t,i)\, D_c(t,i) \quad D(t,i)\, C_c(t,i) \right]. \end{aligned}$$

Definition. A controller $\mathbf{G}_c$ of form (2.7) is called *stabilizing* for the system (2.6) if the zero solution of the closed loop system (2.8) (in the absence of the noise v) is (ESMS).

By $\mathcal{K}_s(\mathbf{G})$ it is denoted the set of all stabilizing controllers $\mathbf{G}_c$ of the form (2.7).

Then two optimization problems will be formulated and solved in this paper:

(OP1). Find a stabilizing controller of the form (2.7) minimizing $\|\mathbf{G}_{cl}\|_2$;

(OP2) Find a stabilizing controller of the form (2.7) minimizing $|||\mathbf{G}_{cl}|||_2$.

3 Observability Gramian

Let $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$ be the linear space of $n \times n$ symmetric matrices and $\mathcal{S}_n^d = \mathcal{S}_n \oplus .... \oplus \mathcal{S}_n$ and consider the linear operator

$$\mathcal{L}(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d$$

defined by

$$\mathcal{L}(t)S = [\mathcal{L}_1(t)S \quad \mathcal{L}_2(t)S \ ... \ \mathcal{L}_d(t)S]$$

where

$$\mathcal{L}_i(t)S = A_0(t,i)\,S(i) + S(i)\,A_0^*(t,i) + \sum_{k=1}^{r} A_k(t,i)\,S(i)\,A_k^*(t,i) + \sum_{j=1}^{d} q_{ji}S(j)\,. \quad (3.1)$$

If $\mathcal{L}^*(t) : \mathcal{S}_n^d \to \mathcal{S}_n^d$ is the adjoint operator of $\mathcal{L}(t)$ with respect to the usual inner product on $\mathcal{S}_n^d$:

$$\langle S, H \rangle = \sum_{i=1}^{d} Tr\left[S(i)\,H(i)\right],$$

then by direct computations it follows that

$$\mathcal{L}^*(t)S = [\mathcal{L}_1^*(t)S \quad \mathcal{L}_2^*(t)S \ ... \ \mathcal{L}_d^*(t)S]$$

where

$$\mathcal{L}_i^*(t)S = A_0^*(t,i)\,S(i) + S(i)\,A_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)\,S(i)\,A_k(t,i) + \sum_{j=1}^{d} q_{ij}S(j)$$

for all $S \in \mathcal{S}_n^d$.

Throughout this paper $T(t,t_0)$ is the linear evolution operator over $\mathcal{S}_n^d$ defined by the linear differential equation:

$$\frac{d}{dt}S(t) = \mathcal{L}(t)S(t)\,.$$

Since $\mathcal{L}(t)$ and $\mathcal{L}^*(t)$ are periodic functions with period θ we obtain that $T(t+\theta, s+\theta) = T(t,s)$ and $T^*(t+\theta, s+\theta) = T^*(t,s)$ foe al $t, s \in \mathbf{R}$.

In the developments of the next sections a crucial role is played by the unique periodic solution of the following affine differential equation on $\mathcal{S}_n^d$:

$$\frac{d}{dt}P_o(t) + \mathcal{L}^*(t)P_o(t) + \tilde{C}(t) = 0 \quad (3.2)$$

where $\tilde{C}(t) = (\tilde{C}(t,1) \ \ \tilde{C}(t,d))$, $\ \tilde{C}(t,i) = C^*(t,i)C(t,i)$.

The next result follows from Corollary 4.9 in [7].

Proposition 3.1 *If the zero solution of (2.2) is (ESMS) then (3.2) has unique periodic solution. Moreover this periodic solution is given by*

$$P_o(t) = \int_t^{\infty} T^*(s,t)\tilde{C}(s)ds, \quad t \in \mathbf{R} \quad (3.3)$$

$P_o(t)$ extends to this framework the well known observability Gramian.

4 H^2-norms for linear stochastic systems

In this section we show how the norms (2.4) and (2.5) are expressed in terms of the observability Gramian of the system (2.1).

Firstly we prove:

Lemma 4.1 *If the zero solution of the system (2.2) is (ESMS) then*

$$E[\int_{t_0}^{t_0+T}|z(t)|^2dt|\eta(t_0)=i]=\sum_{j=1}^{d}\int_{t_0}^{t_0+T}Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]p_{ij}(s-t_0)ds+\psi_i(x_0,t_0,T)$$

where

$$\lim_{T\to\infty}\frac{1}{T}\psi_i(x_0,t_0,T)=0$$

for all $t_0\geq 0, x_0\in\mathbf{R}^n, i\in\mathcal{D}, P_o(t)=(P_o(t,1)\ \ \dots\ \ P_o(t,d))$ being the unique periodic solution of equation (3.2).

Proof. Under the considered assumptions we obtain from Proposition 3.1 that the equation (3.2) has a unique periodic solution $P_o:\mathbf{R}\to\mathcal{S}_n^d, P_o(t)=P_o(t,1)\ \ \dots P_o(t,d))$. Applying Theorem 3.1 from [7] to the function $v(t,x,i)=x^*P_o(t,i)x$ and to the system (2.1) one gets

$$E[\int_{t_0}^{t_0+T}|z(t)|^2dt|\eta(t_0)=i]= \tag{4.1}$$

$$E[\int_{t_0}^{t_0+T}Tr\{B_v^*(s,\eta(s))P_o(s,\eta(s))B_v(s,\eta(s))\}ds|\eta(t_0)=i]+\psi(x_0,t_0,T)$$

where

$$\psi_i(x_0,t_0,T)=x_0^*P_o(t_0,i)x_0-E[x^*(t_0+T)P_o(t_0+T,\eta(t_0+T))x(t_0+T)|\eta(t_0)=i]$$

and $x(t)=x(t,t_0,x_0)$. By Theorem 5.1 in [7] $\psi_i(x_0,t_0,\cdot)$ is a bounded function and then we have

$$\lim_{T\to\infty}\frac{1}{T}\psi_i(x_0,t_0,T)=0. \tag{4.2}$$

On the other hand we have:

$$E[\int_{t_0}^{t_0+T}Tr(B_v^*(s,\eta(s))P_o(s,\eta(s))B_v(s,\eta(s))ds|\eta(t_0)=i]$$

$$=\sum_{j=1}^{d}\int_{t_0}^{t_0+T}Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]E[\chi_{\eta(s)=j}|\eta(t_0)=i]ds$$

which leads to

$$E[\int_{t_0}^{t_0+T}Tr[B_v^*(s,\eta(s))P_o(s,\eta(s))B_v(s,\eta(s))ds|\eta(t_0)=i]$$

$$=\sum_{j=1}^{d}\int_{t_0}^{t_0+T}Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]p_{ij}(s-t_0)ds. \tag{4.3}$$

Combining (4.2), (4.3) with (4.1) we obtain the equality in the statement and thus the proof is complete.

In what follows we shall use the notations:

$$\begin{aligned} \pi_i(t) &= \mathcal{P}\{\eta(t) = i\}, \\ \tilde{P} &= \lim_{t\to\infty} P(t) \end{aligned}$$

with elements $\tilde{p}_{ij}$ (the existence of the above limit is proved in [9])

$$\begin{aligned} \pi_i &= \mathcal{P}\{\eta(0) = i\} = \pi_i(0) \\ \pi_{i\infty} &= \sum_{j=1}^{d} \pi_j \tilde{p}_{ji}. \end{aligned}$$

It is obvious that $\pi_i(t) = \sum_{j=1}^{d} \pi_j p_{ji}(t)$ and hence $\lim_{t\to\infty} \pi_i(t) = \pi_{i\infty}$. Set

$$\begin{aligned} \hat{B}_v(s,i) &= \pi_i(s) B_v(s,i) B_v^*(s,i) \\ \tilde{B}_v(s,i) &= \pi_{i\infty} B_v(s,i) B_v^*(s,i). \end{aligned}$$

Theorem 4.2 *If the zero solution of the system (2.2) is (ESMS) then*

$$(\|\tilde{\mathbf{G}}\|_2)^2 = \frac{1}{\theta} \sum_{j=1}^{d} \int_0^\theta \pi_{j\infty} Tr[B_v^*(s,j) P_o(s,j) B_v(s,j)] ds \tag{4.4}$$

Proof. Applying Lemma 4.1 one obtains:

$$\begin{aligned} E\int_{t_0}^{t_0+T} |z(t)|^2 dt = \sum_{i=1}^{d} \pi_i(t_0) E[\int_{t_0}^{t_0+T} |z(t)|^2 dt | \eta(t_0) = i] = \\ \sum_{j=1}^{d}\sum_{i=1}^{d} \pi_i(t_0) \int_{t_0}^{t_0+T} Tr[B_v^*(s,j) P_o(s,j) B_v(s,j)] p_{ij}(s-t_0) ds \\ + \sum_{i=1}^{d} \pi_i(t_0) \psi)_i(x_0, t_0, T). \end{aligned} \tag{4.5}$$

Since $\pi_i(t_0) = \sum_{l=1}^{d} \pi_l p_{li}(t_0)$ one obtains that

$$\sum_{i=1}^{d} \pi_i(t_0) p_{ij}(s-t_0) = \sum_{l=1}^{d} \pi_l p_{lj}(s).$$

Further we have:

$$\begin{aligned} \sum_{i=1}^{d} \pi_i(t_0) \int_{t_0}^{t_0+T} Tr[B_v^*(s,j) P_o(s,j) B_v(s,j)] p_{ij}(s-t_0) ds \\ = \int_{t_0}^{t_0+T} Tr[B_v^*(s,j) P_o(s,j) B_v(s,j)] \pi_{j\infty} ds \\ + \sum_{i=1}^{d} \pi_i \int_{t_0}^{t_0+T} Tr[B_v^*(s,j) P_o(s,j) B_v(s,j)] (p_{ij}(s) - \tilde{p}_{ij}) ds. \end{aligned} \tag{4.6}$$

Using (4.5) and (4.6) together with $\lim_{s\to\infty} p_{ij}(s) = \tilde{p}_{ij}$ one obtains

$$\lim_{T\to\infty} \frac{1}{T} E \int_{t_0}^{t_0+T} |z(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \sum_{j=1}^{d} \pi_{j\infty} \int_{t_0}^{t_0+T} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]ds.$$

Since the integrant is periodic function we may write:

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]\pi_{j\infty} ds =$$

$$\lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]\pi_{j\infty} ds =$$

$$\frac{1}{\theta} \int_{0}^{\theta} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]\pi_{j\infty} ds.$$

Thus we have obtained that

$$\lim_{T\to\infty} \frac{1}{T} E[\int_{t_0}^{t_0+T} |z(t)|^2 ds] = \sum_{j=1}^{d} \frac{1}{\theta} \int_{0}^{\theta} \pi_{j\infty} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]ds$$

and ths the proof is complete.

Theorem 4.3 *If the zero solution of equation (2.2) is (ESMS) then*

$$(|||\tilde{\mathbf{G}}|||_2)^2 = \sum_{j=1}^{d} \frac{1}{\theta} \int_{0}^{\theta} \delta_j Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]ds \tag{4.7}$$

where $\delta_j = \sum_{i=1}^{d} \tilde{p}_{ij}$.

Proof. Using the equality proved in Lemma 4.1 one obtains:

$$\sum_{i=1}^{d} E[\int_{t_0}^{t_0+T} |z(t)|^2 dt|\eta(t_0) = i] =$$

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \tilde{p}_{ij} \int_{t_0}^{t_0+T} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)]ds + \sum_{i=1}^{d}(\psi_i(x_0,t_0,T) + h_i(t_0,T)) \tag{4.8}$$

where $\psi_i(x_0, t_0, T)$ are as in Lemma 4.1 and

$$h_i(t_0,T) = \sum_{j=1}^{d} \int_{t_0}^{t_0+T} Tr[B_v^*(s,j)P_o(s,j)B_v(s,j)](p_{ij}(s-t_0) - \tilde{p}_{ij})ds.$$

The proof goes one as in the previous theorem.

Remark 4.4

a) From (4.4) and (4.7) it follows that the norms (2.4) and (2.5) do not depend upon $(t_0, x_0, i) \in \mathbf{R}_+ \times \mathbf{R}^n \times \mathcal{D}$.

b) Since $\delta_j \geq \pi_{j\infty}$ it follows that $\|\tilde{\mathbf{G}}\|_2 \leq |||\tilde{\mathbf{G}}|||_2$. From (4.7) one can see that $|||\cdot|||_2$ does not depend upon the initial distribution $\pi = (\pi_1,\ \pi_2\ \ldots\pi_d)$ of the Markov chain.

5 Solution of the optimization problems

In this section we solve the optimization problems stated in section 2. For the sake of simplicity we shall unify the notations writing $\|\cdot\|_{2,\ell}$, $\ell = 1, 2$ where $\|\cdot\|_{2,1}$ stands for $\|\cdot\|_2$ defined by (2.4) and $\|\cdot\|_{2,2}$ stands for $|||\cdot|||_2$ defined by (2.5). Thus from Theorems 4.2 and 4.3 we have

$$\|\mathbf{G}_{cl}\|_{2,\ell}^2 = \sum_{i=1}^{d} \varepsilon_i \frac{1}{\theta} \int_0^{\theta} Tr\left(B_{vcl}^*(s,i)\, P_{ocl}(s,i)\, B_{vcl}(s,i)\right) ds \tag{5.1}$$

with

$$\begin{aligned} \varepsilon_i &= \pi_{i\infty} \quad \texttt{for} \quad \ell = 1 \\ \varepsilon_i &= \delta_i \quad \texttt{for} \quad \ell = 2 \end{aligned} \tag{5.2}$$

and $P_{ocl}(s) = (P_{ocl}(s,1), ..., P_{ocl}(s,d))$ is the unique periodic positive semidefinite solution of the Lyapunov type equation on $\mathcal{S}_{n+n_c}^d$:

$$\begin{aligned} &\tfrac{d}{dt}P_{ocl}(t,i) + A_{0cl}^*(t,i)\, P_{ocl}(t,i) + P_{ocl}(t,i)\, A_{0cl}(t,i) + \textstyle\sum_{k=1}^{r} A_{kcl}^*(t,i)\, P_{ocl}(t,i)\, A_{kcl}(t,i) \\ &+ \textstyle\sum_{j=1}^{d} q_{ij} P_{ocl}(t,j) + C_{cl}^*(t,i)\, C_{cl}(t,i) = 0, \ i \in \mathcal{D}. \end{aligned} \tag{5.3}$$

One can associate to the system (2.6) the following *stochastic generalized Riccati differential equations* (SGRDE):

$$\begin{aligned} &\frac{d}{dt}X(t,i) + A_0^*(t,i)\, X(t,i) + X(t,i)\, A_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)\, X(t,i)\, A_k(t,i) \\ &+ \sum_{j=1}^{d} q_{ij} X(t,j) - \left[X(t,i)\, B_0(t,i) + \sum_{k=1}^{r} A_k^*(t,i)\, X(t,i)\, B_k(t,i) + C^*(t,i)\, D(t,i)\right] \\ &\times \left[D^*(t,i)\, D(t,i) + \sum_{k=1}^{r} B_k^*(t,i)\, X(t,i)\, B_k(t,i)\right]^{-1} \\ &\times \left[B_0^*(t,i)\, X(t,i) + \sum_{k=1}^{r} B_k^*(t,i)\, X(t,i)\, A_k(t,i) + D^*(t,i)\, C(t,i)\right] + C^*(t,i)\, C(t,i) = 0, \end{aligned} \tag{5.4}$$

$i \in \mathcal{D}$, which can be written in a compact form as:

$$\frac{d}{dt}X(t) + \mathcal{L}^*(t)X(t) - \mathcal{P}^*(t, X(t))\, \mathcal{R}^{-1}(t, X(t))\, \mathcal{P}(t, X(t)) + \tilde{C}(t) = 0$$

where $\mathcal{L}(t)$ is the Lyapunov operator defined by (3.1) and

$$\mathcal{P}(t,X) = (\mathcal{P}_1(t,X), ..., \mathcal{P}_d(t,X))$$

with

$$\mathcal{P}_i(t,X) = B_0^*(t,i)\, X(i) + \sum_{k=1}^{r} B_k^*(t,i)\, X(i)\, A_k(t,i) + D^*(t,i)\, C(t,i)$$

and $\mathcal{R}(t,X) = (\mathcal{R}_1(t,X), ..., \mathcal{R}_d(t,X))$ with

$$\mathcal{R}_i(t,X) = D^*(t,i)\, D(t,i) + \sum_{k=1}^{r} B_k^*(t,i)\, X(i)\, B_k(t,i).$$

Definition. A solution $\tilde{X}(t) = \left(\tilde{X}(t,1), ..., \tilde{X}(t,d)\right)$ of the equation (5.4) is called *stabilizing solution* if it has the following properties:

a)

$$\mathcal{R}_i\left(t, \tilde{X}(t)\right) > 0, \qquad t \in \mathbf{R}, i \in \mathcal{D}.$$

b) The system $\left(A_0 + B_0\tilde{F}, A_1 + B_1\tilde{F}, ..., A_r + B_r\tilde{F}; Q\right)$ is stable, where $\tilde{F}(t) = \left(\tilde{F}(t,1), \tilde{F}(t,2), ..., \tilde{F}(t,d)\right)$, with

$$\tilde{F}(t,i) = -\mathcal{R}_i^{-1}\left(t, \tilde{X}(t)\right)\mathcal{P}_i\left(t, \tilde{X}(t)\right), \quad t \in \mathbf{R}, i \in \mathcal{D}. \tag{5.5}$$

Remark. The solution $\tilde{X}(t)$ of the system (5.4) is a stabilizing solution if the control $u(t) = \tilde{F}(t, \eta(t))x(t)$ stabilizes the system (2.6) in the absence of the additive noise $v(t)$.

Denote by

$$\mathcal{N}(X) = (\mathcal{N}_1(X), ..., \mathcal{N}_d(X)) \in \mathcal{S}^d_{n+m}, \quad X \in C^1(\mathbf{R}, \mathcal{S}^d_n)$$

the *generalized dissipation matrix*, where

$$\mathcal{N}_i(X) = \begin{bmatrix} \left(\frac{d}{dt}X(t,i) + \mathcal{L}^*(t)X\right)(i) + \tilde{C}(t,i) & \mathcal{P}_i^*(t,X) \\ \mathcal{P}_i(t,X) & \mathcal{R}_i(t,X) \end{bmatrix}.$$

Throughout this section $\mathbf{A} = (A_0, A_1, ..., A_r)$, $\mathbf{B} = (B_0, B_1, ..., B_r)$.

We make the following assumptions:

H1. The system $(\mathbf{A}, \mathbf{B}; Q)$ is stabilizable (the concept of stabilizability for the triple $(\mathbf{A}, \mathbf{B}; Q)$ is defined in the standard way see e.g. [7]).

H2. Assume that there exists a periodic C^1-function $\hat{X}(t) = \left(\hat{X}(t,1),,,\hat{X}(t,d)\right)$ such that $\mathcal{N}\left(\hat{X}(t)\right) > 0$.

Applying Theorem 4.9 and Theorem 5.1 in [6] it follows that the (SGRDE) (5.4) has a unique stabilizing solution $\tilde{X}(t)$ which is θ-periodic function.

If we take $u(t) = \tilde{F}(t, \eta(t))x(t)$ the corresponding closed loop system denoted by $\tilde{\mathbf{G}}_{cl}$ is:

$$\begin{aligned} dx_{cl}(t) &= \left[A_0(t,\eta(t)) + B_0(t,\eta(t))\tilde{F}(t,\eta(t))\right]x(t)\,dt \\ &\quad + \sum_{k=1}^{r}\left[A_k(t,\eta(t)) + B_k(t,\eta(t))\tilde{F}(t,\eta(t))\right]x(t)\,dw_k(t) \\ &\quad + B_v(t,\eta(t))\,dv(t) \\ z(t) &= \left[C(t,\eta(t)) + D(t,\eta(t))\tilde{F}(t,\eta(t))\right]x(t) \end{aligned} \tag{5.6}$$

Then the following result holds:

Proposition 5.1 *Under the assumptions* **H1** *and* **H2** *we have*

$$\left\|\tilde{\mathbf{G}}_{cl}\right\|^2_{2,\ell} = \sum_{j=1}^{d}\varepsilon_j\frac{1}{\theta}\int_0^\theta Tr\left(B_v^*(s,j)\tilde{X}(s,j)B_v(s,j)\right)ds$$

where $\tilde{X}(t)$ is the stabilizing solution of (SGRDE) (5.4).

Proof. By direct algebraic manipulations one obtains that the (SGRDE) (5.4) verified by $\tilde{X}(t)$ can be written in a Lyapunov form as follows:

$$\frac{d}{dt}\tilde{X}(t,i) + \left[A_0(t,i) + B_0(t,i)\tilde{F}(t,i)\right]^* \tilde{X}(t,i) + \tilde{X}(t,i)\left[A_0(t,i) + B_0(t,i)\tilde{F}(t,i)\right]$$

$$+\sum_{k=1}^{r}\left[A_k(t,i) + B_k(t,i)\tilde{F}(t,i)\right]^* \tilde{X}(t,i)\left[A_k(t,i) + B_k(t,i)\tilde{F}(t,i)\right]$$

$$+\sum_{j=1}^{r} q_{ij}\tilde{X}(t,j) + \left[C(t,i) + D(t,i)\tilde{F}(t,i)\right]^*\left[C(t,i) + D(t,i)\tilde{F}(t,i)\right] = 0$$

which shows that the observability Gramian P_{ocl} associated with the closed loop system (5.6) coincides with the stabilizing solution $\tilde{X}(t)$ of the (SGRDE) (5.4). The conclusion in the statement follows from Theorems 4.2 and 4.3.

The main result of this section is:

Theorem 5.2 *Assume that* **H1** *and* **H2** *are fulfilled. Under these conditions,*

$$\min_{\mathbf{G}_c\in\mathcal{K}_s(\mathbf{G})} \|\mathbf{G}_{cl}\|_{2,\ell} = \left[\sum_{j=1}^{d}\varepsilon_j\frac{1}{\theta}\int_0^{\theta} Tr\left(B_v^*(s,j)\tilde{X}(s,j)B_v(s,j)\right)ds\right]^{\frac{1}{2}}$$

and the optimal control is

$$u(t) = \tilde{F}(t,\eta(t))\,x(t)$$

where $\tilde{X}(t)$ is the stabilizing solution of (SGRDE) (5.4), $\tilde{F}(t) = \left(\tilde{F}(t,1),...,\tilde{F}(t,d)\right)$ is the stabilizing feedback gain defined by (5.5) and ε_i are defined in (5.2).

Proof. Let $\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})$ and $\mathbf{G}_{cl}$ the corresponding closed-loop system and $P_{ocl}(t)$ denotes the observability Gramian. Let

$$\begin{bmatrix} U_{11}(t,i) & U_{12}(t,i) \\ U_{12}^*(t,i) & U_{22}(t,i)\end{bmatrix}$$

be a partition of $P_{ocl}(t,i)$ conformably with the partition of the state matrix of the resulting system. Partitioning (5.3) according with the partition of $P_{ocl}(t,i)$ it follows that:

$$\begin{aligned}&\tfrac{d}{dt}U_{11}(t,i) + (A_0(t,i) + B_0(t,i)D_c(t,i))^* U_{11}(t,i) + B_c^*(t,i)U_{12}^*(t,i)\\ &+U_{11}(t,i)(A_0(t,i) + B_0(t,i)D_c(t,i)) + U_{12}(t,i)B_c(t,i)\\ &+\textstyle\sum_{k=1}^{r}(A_k(t,i) + B_k(t,i)D_c(t,i))^* U_{11}(t,i)(A_k(t,i) + B_k(t,i)D_c(t,i))\\ &+\textstyle\sum_{j=1}^{d} q_{ij}U_{11}(t,j) + (C(t,i) + D(t,i)D_c(t,i))^*(C(t,i) + D(t,i)D_c(t,i)) = 0\end{aligned} \tag{5.7}$$

$$\begin{aligned}&\tfrac{d}{dt}U_{12}(t,i)+(A_0(t,i)+B_0(t,i)D_c(t,i))^*U_{12}(t,i)+B_c^*(t,i)U_{22}(t,i)+U_{11}(t,i)B_0(t,i)C_c(t,i)\\ &+U_{12}(t,i)A_c(t,i) + \textstyle\sum_{k=1}^{r}(A_k(t,i) + B_k(t,i)D_c(t,i))^* U_{11}(t,i)B_k(t,i)C_c(t,i)\\ &+\textstyle\sum_{j=1}^{d} q_{ij}U_{12}(t,j) + (C(t,i) + D(t,i)D_c(t,i))^* D(t,i)C_c(t,i) = 0\end{aligned} \tag{5.8}$$

$$\begin{array}{l}\frac{d}{dt}U_{22}(t,i)+C_c^*(t,i)\,B_0^*(t,i)\,U_{12}(t,i)+A_c^*(t,i)\,U_{22}(t,i)+U_{12}^*(t,i)B_0(t,i)\,C_c(t,i)\\ +U_{22}(t,i)\,A_c(t,i)+\sum_{k=1}^{r}C_c^*(t,i)\,B_k^*(t,i)\,U_{11}(t,i)\,B_k(t,i)\,C_c(t,i)\\ +\sum_{j=1}^{d}q_{ij}U_{22}(t,j)+C_c^*(t,i)\,D^*(t,i)\,D(t,i)\,C_c(t,i)=0.\end{array} \tag{5.9}$$

The (SGRDE) (5.4) can be written for the stabilizing solution $\tilde{X}(t)$ in the following Lyapunov form:

$$\begin{array}{l}\frac{d}{dt}\tilde{X}(t,i)+(A_0(t,i)+B_0(t,i)\,D_c(t,i))^*\,\tilde{X}(t,i)+\tilde{X}(t,i)\,(A_0(t,i)+B_0(t,i)\,D_c(t,i))\\ +\sum_{k=1}^{r}(A_k(t,i)+B_k(t,i)\,D_c(t,i))^*\,\tilde{X}(t,i)\,(A_k(t,i)+B_k(t,i)\,D_c(t,i))\\ +\sum_{j=1}^{d}q_{ij}\tilde{X}(t,j)+(C(t,i)+D(t,i)\,D_c(t,i))^*\,(C(t,i)+D(t,i)\,D_c(t,i))\\ -\left(D_c(t,i)-\tilde{F}(t,i)\right)^*\mathcal{R}_i\left(t,\tilde{X}(t)\right)\left(D_c(t,i)-\tilde{F}(t,i)\right)=0.\end{array} \tag{5.10}$$

Denoting by

$$\tilde{U}_{11}(t,i)=U_{11}(t,i)-\tilde{X}(t,i)$$

and subtracting (5.10) from (5.7) one easily obtains that

$$\tilde{U}(t,i)=\begin{bmatrix}\tilde{U}_{11}(t,i) & U_{12}(t,i)\\ U_{12}^*(t,i) & U_{22}(t,i)\end{bmatrix},$$

is the periodic solution of the folowing affine differential equation:

$$\begin{array}{l}\frac{d}{dt}\tilde{U}(t,i)+A_{0cl}^*(t,i)\,\tilde{U}(t,i)+\tilde{U}(t,i)\,A_{0cl}(t,i)+\sum_{k=1}^{r}A_{kcl}^*(t,i)\,\tilde{U}(t,i)\,A_{kcl}(t,i)\\ +\sum_{j=1}^{d}q_{ij}\tilde{U}(t,j)+\Theta^*(t,i)\,\mathcal{R}_i\left(t,\tilde{X}(t)\right)\Theta(t,i)=0,\end{array}\quad i\in\mathcal{D}$$

with

$$\Theta(t,i)=\left[D_c(t,i)-\tilde{F}(t,i)\quad C_c(t,i)\right].$$

Since the system $(A_{0cl},A_{1cl},...,A_{rcl};Q)$ is stable it follows via Proposition 4.8 in [7] that $\tilde{U}(t,i)\geq 0$. Further one obtains

$$\begin{aligned}\|\mathbf{G}_{cl}\|_{2,\ell}^2 &= \sum_{i=1}^{d}\varepsilon_i\frac{1}{\theta}\int_0^\theta Tr\left(B_{vcl}^*(t,i)\,P_{0cl}(t,i)\,B_{vcl}(t,i)\right)dt\\ &= \sum_{i=1}^{d}\varepsilon_i\frac{1}{\theta}\int_0^\theta Tr\left(B_v^*(t,i)\,\tilde{X}(t,i)\,B_v(t,i)\right)dt\\ &+ \sum_{i=1}^{d}\varepsilon_i\frac{1}{\theta}\int_0^\theta Tr\left(B_{vcl}^*(t,i)\,\tilde{U}(t,i)\,B_{vcl}(t,i)\right)dt.\end{aligned}$$

Since $\tilde{U}(t,i)$ is positive semidefinite it follows that

$$\|\mathbf{G}_{cl}\|_{2,\ell}^2\geq\sum_{i=1}^{d}\varepsilon_i\frac{1}{\theta}\int_0^\theta Tr\left(B_v^*(t,i)\,\tilde{X}(t,i)\,B_v(t,i)\right)dt$$

for all stabilizing controllers $\mathbf{G}_c$. Using Proposition 5.1 the conclusion in the statement follows immediately.

Remark. From Theorem 5.2 it follows that both optimization problems (OP1) and (OP2) have the same optimal solution given by the controllers with the set of parameters $n_c=0,\ A_c(t,i)=0,\ B_c(t,i)=0,\ C_c(t,i)=0,\ D_c(t,i)=\tilde{F}(t,i),\ i\in\mathcal{D}$.

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