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Asymptotic behaviour of higher order derivatives of Perron-Frobenius operator

A. Berechet

1. Introduction. Preliminaires.

We consider in this paper the transfer operator $\widetilde{\mathcal{U}}$ of an unidimensional ($\mathcal{B} \subset \mathbb{R}^1$) measurable fibred system $(\mathcal{B}, \Sigma, \tau)$ in the class delimited below. This class generalizes pw m t.s for which are descrbes in [8] associated Markov chains $(s_n)_{n\geq 0}$ and their transition operators. Many properties of such Markov chains can be extended at certain classes of fibred systems. Some of their properties have been used to study conditional expectations and distributions of several associated random variables in [6], [7].

The needed in the description of many fibred systems verification of Gauss' (or Kuzmin's) equation amounts to the proof of fact that the family of functions $\mathcal{U}\chi\langle i\rangle h/h$, $i \in A$ define a transition probability function; here $\langle i\rangle$, $A \in i$ are cylinders of order 1 of the given fibred system, h is the invariant rest the Perron-Frobenius operator $\mathcal{U}$ density. Hence it is essential for the possibility to obtain iterates of a system.

The relation between the Perron-Frobenius operators and the integral Markov operators (in particular sums) is viewed conformly to [5], Ch VI and VIII.

We continue this first section with prerequisites.

Then we delimit the class of fibred systems we deal with and we restate Theorem 1, [3] which sums Kuzmin type theorems proved in [2]. It is valid for fibred systems in the class considered and is essentially used in the proof of Theorem 3 below. We postponed the notations of spaces and norms in an Appendix.

In Section 2 we introduce the condition (E_s) on derivatives of order $s \geq 1$ of τ^l, $l \geq 1$. Then, by Theorem 3, we prove that the s-th derivative on cells,

$h\,|^{(s)}$, of h exists and is Lipschitz continuous on every cell (i.e. $h\,|^{(s)}\in \mathcal{L}$; see Appendix) The proof is reminiscent of methods from last mentioned paper.

The main result, Theorem 5, is a consequence of Lemma 4 from Section 3 . By this theorem one sees that the derivative on cells of order $\bar{s} \leq s$ of $\mathcal{U}^l$, $(\mathcal{U}^l)\,|^{(\bar{s})}$, considered on $\mathcal{C}^s$ can be approximated using the operator we denote $V^{(\bar{s},l)}$, $(l, \bar{s} \in \mathbb{N}, \bar{s} \leq s)$. The analogous in a sense of Kuzmin's equation (concerning as it is well known $\mathcal{U}^l h$, $l \geq 1$) holds asymptotically as $l \to \infty$ for $V^{(\bar{s},l)}h$. On what I know these results are not known even in continued fraction (abbr. RCF) case. In the case of RCF one also shows that $h^{(l)}$ is an eigenvector except a normating function of $V^{(\bar{s},l)}$ and one gives a formula allowing the iterative reduction of order of derivative (of $\bar{s}$).

Let $\mathcal{B} \equiv [c, d]$ be a compact interval of $\mathbb{R}^1$, Σ be the σ algebra of Borel sets of $\mathcal{B}$, λ the Lebesgue measure on Σ. Let A be the index set and $\langle i \rangle$, $i \in A$ the fundamental intervals or cylinders of the measurable fibred system $(\mathcal{B}, \Sigma, \tau)$.

We denote by $\chi\{E; \cdot\} = \chi E$ the indicator function of arbitrary set $E \subset \mathcal{B}$ and given the real map g on $\mathcal{B}$, by $g\,|_E$ the restriction of g at E; we also set $g^{(s)}(y) := \dfrac{d^s g(y)}{dy^s}$, in points $y \in \mathcal{B}$ in which it exists. $s \in \mathbb{N}_*$.

The map $\tau\,|_{<i>} =: \tau_i$ where $i \in A$ is injective by general definition of fibred systems (cf [7]); we assume here that $\langle i \rangle$ are open, $i \in A$ and that τ_i, $i \in A$ are for a $n \geq 1$, C^n-diffeomorphisms $\tau\langle i \rangle \leftrightarrow \langle i \rangle$. We denote $v_i := \tau_i^{-1}$, $i \in A$. Then v_i are strictly monotonic. Given any "block" $i^{(n)} := (i_1, \ldots, i_n) \in A^n \equiv A \times \ldots \times A$ we define iteratively the set $\langle i^{(n)} \rangle := \langle i^{(n-1)} \rangle \cap (\tau^{n-1} \in \langle i_n \rangle)$, $i^{(n)} \in A^{(n)}$. We consider as collection X_n of fundamental intervals or cylinders of order n is the set $X_n := \{\langle i^{(n)} \rangle : i^{(n)} \in A^n \text{ and } \lambda(\langle i^{(n)} \rangle) > 0\}$. We denote $A_{(n)} := \{i^{(n)} \in A^n : \lambda(\langle i^{(n)} \rangle) > 0\}$ and we call its elements admissible blocks. We denote $\mathcal{E} := B \setminus \cup_{t^{(j)} \in A_{(j)}} \langle t^{(j)} \rangle$, $j > 0$, the collection of end points of fundamental intervals of order n.

One defines the label sequence $(a_n)_{n \geq 1}$ of given fibred system by

$$a_1(y) = i \text{ iff } y \in \langle i \rangle; a_n(y) = a_1(\tau^{n-1}(y)), \text{ if } y \notin \mathcal{E}_n$$

Then whatever we have $\langle i^{(n)} \rangle = \{y \in \mathcal{B}, a^{(n)}(y) = i^{(n)}\}$, $i^{(n)} \in A_{(n)}$, $n \geq 1$. Under the assumptions stated below, $\lambda(\mathcal{E}_n) = 0$, $r \geq 1$ and $(a_n)_{n \geq 1}$ is a sequence of A valued random variables on $\mathcal{B}$. Under that assumptions, whatever $n \in \mathbb{N}_*$ the set of cylinders of order n is a partition of $\mathcal{B}$ (mod 0).

Whatever $i^{(n)} \in A^n$ we denote $\widetilde{i^{(n)}} := (i_n, \ldots, i_1)$ and when $\tau^n \langle i^{(n)} \rangle \neq 0$, $v_{i^{(n)}} := v_{i_1} \circ \ldots \circ v_{i_n}$. Hence $\tau^n_{\widetilde{i^{(n)}}} := \tau_{i_n} \circ \ldots \circ \tau_{i_1}$ and $v_{i^{(n)}} = (\tau^n_{\widetilde{i^{(n)}}})^{-1}$.

Under the assumptions below (see namely (C) and (E)), $0 < v^{(1)}_{i^{(n)}}(t) < c$, $t \in \tau^n\langle i^{(n)}\rangle$, $i^{(n)} \in A_{(n)}$. We denote $\omega(i^{(n)}, t) := \left|\frac{d}{dt}v_{i^{(n)}}(t)\right|$, $i^{(n)} \in A_{(n)}$, $t \in \tau^n\langle i^{(n)}\rangle$; $\omega(i^{(n)}, t) = 0$, $y \notin \tau^n\langle i^{(n)}\rangle$ (convention also used in [8]).

In this paper we assume that the next conditions on fibred system enunciated also in [8], [9] and [1] hold

$$(A) \qquad \hat{\sigma}(n) :\equiv \sup \text{ diam } \langle a^{(m)}\rangle \underset{m\to\infty}{\to} 0,$$

where the supremum is considered over $a^{(m)} \in A_{(m)}$.

The finite range condition is:

(B) There is a finite collection of subintervals of $\mathcal{B}$, $\mathcal{S}_t$, $1 \le t \le \hat{l}$ such that, given any $i^{(n)} \in A_{(n)}$ of order n, $\tau^n\langle i^{(n)}\rangle = \mathcal{S}_{\hat{t}}$ for a $\hat{t} = \hat{t}(i^{(n)}) \le \hat{l}$.

The collection of sets $\mathcal{S}_t$, $1 \le t \le \hat{l}$ induces a finite partition of $\mathcal{B}$ denoted $\mathcal{H}$, whose elements are called cells.

(C) (Renyi's condition) There exists a real constant $c \ge 1$ such that

$$\omega(i^{(m)}, y) \le c\omega(i^{(m)}, t), i^{(m)} \in A_{(m)}, t, y \in \tau^m\langle i^{(m)}\rangle.$$

One says that the cylinder $\langle i^{(n)}\rangle$ is proper or full iff $\tau^n\langle i^{(n)}\rangle \equiv \mathcal{B}$

(D) Every set $\mathcal{S}_t$, $1 \le t \le \hat{l}$ contains a proper cylinder.

Let λ_1 be the number defined by $1/\lambda_1 := \min_{\hat{t}\le\hat{l}} \lambda(\mathcal{S}_{\hat{t}})$. By above (B) and (D), $\lambda_1 \ge 1$. We denote $\lambda_1 c := C$.

Using condition (C) we obtain (see e.g. [9])

$$\lambda(\langle i^{(n)}\rangle)/C \le \omega(i^{(n)}, t) \le \lambda(\langle i^{(n)}\rangle)C \tag{1}$$

$t \in \tau^n\langle i^{(n)}\rangle$, $i^{(n)} \in A_{(n)}$, $n \ge 1$. The last condition we assume in this section is

(E): There exists a positive real constant, N such that

$$|\omega(i^{(n)}, w) - \omega(i^{(n)}, t)| \le \lambda(\langle i^{(n)}\rangle)|w - t|N, \quad w, t \in \langle i^{(n)}\rangle, i^{(n)} \in A_{(n)}.$$

By condition (A), denoting Σ_n the σ algebra generated by X_n, we have $V_{n\ge1}X_n = X$.

Note that the condition called (G) in [8] automatically holds in our unidimensional case. A relation stronger than condition (F) assumed in [8] holds. Indeed, using (1) one can write whatever $i^{(n)} \in A_{(n)}$ and $t, w \in \tau^n\langle i^{(n)}\rangle$

$$|v_{i^{(n)}}(w) - v_{i^{(n)}}(t)| = |\int_t^w \omega(i^{(n)})d\lambda| \le \lambda(\langle i^{(n)}\rangle)C|t - w| \tag{2}$$

Given the measures ν_1 and ν_2 on the same measurable space, the order relation "ν_1 is equivalent to ν_2" will be denoted $\nu_1 \sim \nu_2$.

Suppose that the conditions $(A), \ldots, (E)$ hold. Then: i) τ is ergodic wrt λ. ii), There exists an unique probability on $(\mathcal{B}, \Sigma)$ absolutely continuous wrt λ preserved by τ; we denote it μ. iii) Actually $\mu \sim \lambda$; let $d\mu/d\lambda =: h$.

These properties are also evidentiated in [1], [2], [9]. Since they hold, $(\mathcal{B}, \Sigma, \tau, \mu)$ is a measurable dynamical system (see [2], [8], [9]).

The class of fibred systems we deal with is delimited by conditions $(A), \ldots, (E)$ and by the requirements: $1 = \dim \mathcal{B}$; $1 = \lambda(\mathcal{B})$; v_i are, for a $n \in \mathbb{N}_*$, C^n diffeomorphisms, $\langle i \rangle \leftrightarrow \tau\langle i \rangle$. From now on we suppose all these conditions and requirements satisfied. We shall denote (E_s) where $s \in \mathbb{N}_*$ the assumption we introduce in addition in next section.

The expression used in [2] of Perron-Frobenius operators is valid also for that associated with τ under λ, denoted here $\mathcal{U}$. Let $J \in \mathcal{H}$; we denote $[n, u] := \{i^{(n)} \in A_{(n)}, u \in \tau^n \langle i^{(n)} \rangle\}$, $n \in \mathbb{N}_*$. By propositions 3 and 4 from [8] we have $[n, u] = [n, w] \neq \emptyset$, $n \geq 1$, $t \in J$. Hence we may denote $[n, u] =: [n, J]$, $J \ni u$.

For any $f \in L_1(\mathcal{B})$ (see definition in Appendix) we have

$$\mathcal{U}^n f(x) \equiv \Sigma_{i^{(n)} \in [n,x]} f(v_{i^{(n)}}(x)) \omega(i^{(n)}, x), a.e. \text{ in } \mathcal{B} \tag{3}$$

$n \geq 1$. Also $\mathcal{U}^n f \equiv \{(\mathcal{U}^n f)_{|J}, J \in \mathcal{H}\}$.

Then summing (1), last relation, we obtain, whatever $n \geq 1$

$$\mathcal{U}^n 1(x) \equiv \Sigma_{i^{(n)} \in [n,x]} \omega(i^{(n)}, x) \leq C, \quad a.e. \text{ in } \mathcal{B}.$$

By the relation

$$\mathcal{U}_1 f(x) \equiv h(x) \int f d\lambda, \quad x \in \mathcal{B}, f \in L_1(\mathcal{B})$$

we define on $L_1(\mathcal{B})$ a linear bounded operator.

Remark Assume that $\mathcal{B} = [0, 1] =: I$ and that a dual conformly to [7] algorithm (i.e. a dual system $(\mathcal{B}^\#, \Sigma^\#, \tau^\#)$) exists. Consider arbitrary $w \in I$; we associate with τ a sequence of I-valued random variables on (I, Σ, μ), $(s_n^*)_{n \in \mathbb{N}}$ defined as $s_n^w :\equiv v_{\widetilde{a^{(n)}}}(w)$, $n \geq 1$; $s_0^w \equiv w$. Assume that a kernel $\bar{\alpha}$ as defined in [7] also exists. Then $(s_n^w)_{n \geq 0}$ is a discrete Markov process on (I, Σ, μ) having $\widetilde{\mathcal{U}}$ as transition operator, where $\widetilde{\mathcal{U}}$ is the Perron-Frobenius operator rest μ. I.e. $\widetilde{\mathcal{U}}$ is the linear application $L^1(\mathcal{B}) \to L^1(\mathcal{B})$ defined as $\widetilde{\mathcal{U}}\varphi \equiv \mathcal{U}h\varphi/h$. $\square$

If, given $f \in \mathcal{C}(\Delta)$, where Δ is a subinterval of $\mathcal{B}$ one can extend $f^{(s)}$ at $\bar{\Delta}$, then we denote this extension by $f \mid_{\bar{\Delta}}^{(s)}$. Clearly $f \mid_{\bar{\Delta}}^{(s)}$ means left (resp. right) derivative $f^{(s)}()$ in end points of $\bar{\Delta}$; its restriction at Δ will be denoted $f \mid^{(s)}$. By Lemma 3.1.13, [6], $f_{|\bar{\Delta}}$ exists if f is Lipschitz on Δ, and $L(f, \Delta) = L(f, \bar{\Delta})$. As special case $\omega(i^{(n)})$, $i^{(n)} \in A_{(n)}$ have continuous extensions, finite, on $\overline{\tau^n\langle i^{(n)}\rangle}$ and on $\bar{J}$, $J \in \mathcal{H}$; we denote $f \mid^{(s)}$ the function defined as $f \mid^{(s)} (a) \equiv f^{(s)} \mid_J (a)$, $a \in J$, $J \in \mathcal{H}$.

The next proposition is Theorem 6, [2] which is valid for Markov transition operators associated to a transformation τ of a fibred system in the class we deal with in this paper in particular. In its statement $\mathcal{G}$ denotes either the space $\mathcal{L}$ or the space $\mathcal{C}$ (definitions in Appendix) in assertions valid for each of these spaces.

Theorem 1. *i. The Perron-Frobenius operator $\mathcal{U}$ is an endomorphism of normed space $\mathcal{G}$. ii. $\mathcal{U}$ is ergodic and aperiodic operator on $\mathcal{G}$. iii. The only eigenvalue σ of $\mathcal{U}$ which has $|\sigma| = 1$ is 1 and has algebraic multiplicity 1; the corresponding eigenvectors are scalar multiples of h.* $\square$

We shall use in the sequel that since $\mathcal{U}$ is aperiodic rest $\mathcal{G}$ there exists $\theta < 1$ such that

$$|||\mathcal{U}^n - \mathcal{U}_1||| = \mathcal{O}(\theta^n), \quad n \in \mathbb{N} \tag{4}$$

i.e. $\mathcal{U}_1 = \lim_{n\to\infty} \mathcal{U}^n$ in the norm $||| \cdot |||$.

Clearly the spectra of operators $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ on $\mathcal{G}$ are identical and if f_1 is an eigenvector coresponding to σ_1 of $\mathcal{U}$, f_1/h is an eigenvector corresponding to σ_1 of $\widetilde{\mathcal{U}}$.

In the special case considered above, Theorem 1 [3] yields in terms of conditional expectation $\mathbb{E}_\lambda$ rest λ

$$|||\mathbb{E}_\lambda(f(s_n^*)|s_0^w) - h\int f d\lambda||| = \mathcal{O}(\theta^n)|||f|||, \ n \in \mathbb{N}, f \in \mathcal{G}.$$

2. Conditions on higher order derivatives of τ and derivability of invariant density.

In this section we state the condition (E_s), $s \in \mathbb{N}$ on derivatives of $\omega(i^{(n)}, u)$, $u \in \tau^n\langle i^{(n)}\rangle$ under which the density h belongs to $\mathcal{L}^s$. We then prove, by Theorem 2 this appartenence.

We denote (E_s), $s \in \mathbb{N}$ the condition: $\omega(i)$, $i \in A$ have $(s+1)$th continuous derivative with finite extension on $\overline{\tau}\langle i\rangle$ and there exists a constant $\mathcal{B}_{(s)} > 0$, such that whatever $s \geq \bar{s} \geq 0$,

$$|\omega^{(\bar{s})}(i^{(n)}, t) - \omega^{(\bar{s})}(i^{(n)}, w)| \leq |w - t|\lambda(\langle i^{(n)}\rangle)\mathcal{B}_{(s)} \tag{5}$$

$t, w \in \tau^n\langle i^{(n)}\rangle$.

Evidently, condition (E_s), where $s \geq 1$ implies conditions $(E_{\bar{s}})$, $\bar{s} \leq s$. If $\bar{s} \equiv 0$ condition (E_s) becomes condition (E) plus an assumption on derivatives of $\omega(i)$. Whenever is fulfilled (E_s) we denote $\max(\mathcal{B}_{(s)}, C, N) =: \mathcal{B}$. Hence $\mathcal{B} \geq 1$.

Since condition (F') from [1] is valid for fibred systems we deal with in this paper, we may apply here all propositions proved in [1] and [2]. Note also that (2) has the same form as (5) and that the assumptions on $v(i)$ and $\omega(i)$ given in Section 1 imply that $v^{(1)}_{i^{(n)}}$ does not change its sign in $\tau^n\langle i^{(n)}\rangle$, whatever $i^{(n)} \in A_{(n)}$.

Lemma 2. *Assume that condition (E_s) holds. Then*

i. $$(1/\lambda(\langle i^{(n)}\rangle)) \cdot |\omega^{(\bar{s})}(i^{(n)}, t)| \leq \mathcal{B},\ \bar{s} \leq s,\ t \in \tau^n\langle i^{(n)}\rangle, i^{(n)} \in \mathbb{N}_* \tag{6}$$

ii. Whatever $l \in \mathbb{N}$, the restriction $(\mathcal{U}^l 1)\ |^{(n)}\ J$ is dominated by $\mathcal{B}$, $J \in \mathcal{H}$ we have

$$|(\mathcal{U}^l 1)|^{(\bar{s})} J\ |\leq \Sigma_{i^{(l)} \in [l,J]}\ |\ (\omega(i^{(l)}))^{(\bar{s})}\ |\leq \mathcal{B} \tag{7}$$

$\bar{s} \leq s$; hence $\mathcal{U}^l 1\ |_J$ is s times termwise differentiable in J.

Proof. i. Given an arbitrary cylinder $\langle i^{(n)}\rangle$ points the integer $t, w \in \tau^n\langle i^{(n)}\rangle$, $t \neq w$ and $\bar{s} \leq s$, by (5) we have

$$\frac{1}{|t - w|}|\omega^{(\bar{s})}(i^{(n)}, t) - \omega^{(\bar{s})}(i^{(n)}, w)| \leq \lambda(\langle i^{(n)}\rangle)\mathcal{B} \tag{8}$$

Hence there exists $\widetilde{t} \in [t, w]$ (or resp. $[w, t]$) such that

$$|\omega^{(\bar{s}+1)}(i^{(m)}, \widetilde{t})| \leq \lambda(\langle i^{(m)}\rangle) \cdot \mathcal{B} \tag{9}$$

Since (8) holds whatever $t, w \in \tau^n\langle i^{(n)}\rangle$ and since $\omega^{(s+1)}(i^{(n)})$ finitely exists in $\tau^m\langle i^{(m)}\rangle$, we obtain (9) with every $w \in \tau^l\langle i^{(m)}\rangle$ replacing $\widetilde{t}$. Hence we have

$$|(\omega(i^{(m)}, w))^{(\bar{s})}| \leq \lambda(\langle i^{(m)}\rangle)\mathcal{B}, \; s \leq \bar{s} \leq 1+n \tag{10}$$

whatever $w \in \tau^m\langle i^{(m)}\rangle$, $i^{(m)} \in A_{(m)}$.

ii. Summing (10) over $i^{(m)} \in [m, w]$, where $w \in J$ (hence $J \subset \tau^m\langle i^{(m)}\rangle$), since $\Sigma_{i^{(m)}\in[m,J]}\lambda(\langle i^{(m)}\rangle) \leq 1$, one gets the last inequality appearing in (7) (with n replacing l).

We denote Y the well known from Analysis implication concerning the impact on a converging in some point series (resp. sequence) of functions f_n of the uniform convergence of series (resp. sequence) of derivatives f'_n and the equality of derivative of sum (limit) of former with sum (limit) of latter.

We deduce iteratively using Y and the summability of family $\{\omega(i^{(m)}) \mid^{(s)}, i^{(m)} \in A_{(m)}\}$ that uniformly in $t \in J$ one has, punctually on $\tau^m\langle i^{(m)}\rangle$

$$\begin{aligned} &[\Sigma_{i^{(m)}\in[m,t]}\omega^{(\bar{s}-1)}(i^{(m)}, t)]^{(1)} = ((\mathcal{U}^m 1)^{(\bar{s}-1)}(t))^{(1)} = \\ &(\mathcal{U}^m 1)^{(\bar{s})}(t) = \Sigma_{i^{(m)}\in[m,t]}\omega^{(\bar{s})}(i^{(m)}, t) \end{aligned} \tag{11}$$

$m \geq \bar{s}$. Hence $\mathcal{U}^m 1 \mid_J$ is s times termwise differentiable. □

Theorem 3. *Suppose condition (E_r) holds. Then $h = d\mu/d\lambda \in \mathcal{L}^r$.*

Proof. By the above lemma one has whatever $J \in \mathcal{H}$

$$\left\{(\mathcal{U}^l 1)|_J^{(s)}\right\}_{l\geq 1} \subset \mathcal{L}(\bar{J}), \; 1 \leq s \leq r, \tag{12}$$

since $L((\mathcal{U}^l 1)^{(s)}, J) \equiv (\mathcal{U}^l 1)|_{\bar{J}}^{(s+1)}$, $s \leq r$. Hence the family of maps appearing in (12) is, whatever $s \leq r$, bounded in the norm $|||\cdot|||$ of $\mathcal{L}$. Proposition A2 from [1] can be applied; hence there exists a (sub)sequence with powers $l(n)$, $n \in \mathbb{N}$ say, $(\mathcal{U}^{l(n)}1 \mid^{(s)})_{n\in\mathbb{N}}$ converging to a map $\bar{\alpha} \in \mathcal{L}$ in the norm $|\cdot|$.

Consider the family from (12) with $1 = s$. By [1] Proposition A3, there exists $\{l_1(n)\}_n \subset \mathbb{N}$ and $g_1 \in \mathcal{L}(J)$ such that as $n \to \infty$

$$|\mathcal{U}^{l_1(n)}1|^{(1)}J - g_1| \to 0. \tag{13}$$

Having in view that, by Theorem 1, $\lim_{n\to\infty}|h_{|J} - \mathcal{U}^{l_1(n)}1_{|J}| = 0$ we deduce by Y that

$$g_1 = (h \mid^{(1)} J) \tag{13'}$$

Hence

$$|h|_{\bar{J}}^{(1)} - (\mathcal{U}^{l_1(n)}1)_{\bar{J}}^{(1)}|_{\bar{J}} \to 0 \text{ as } n \to \infty \tag{13''}$$

There exists, by [1] (again proposition [1] A3) a subsequence, $(\mathcal{U}^{l_2(n)}1)^{(2)}|_J$ say, where $l_2(n) \underset{n\to\infty}{\to} 0$ and $l_2(n) \subset l_1(n)$ converging to a map $g_2 \in \mathcal{L}(\bar{J})$. Hence (13) with 1 replaced by 2 holds. Using Y and (13′) we deduce

$$(h|_{\bar{J}}^{(1)})^{(1)} = (g_1)^{(1)} = g_2 = h|^{(2)\bar{J}} \tag{14}$$

Clearly the above (diagonal) procedure can be continued with $\bar{n} = 3, \ldots, s$. One obtains

$$|h|_{\bar{J}}^{(\bar{s})} - (\mathcal{U}^{l_{\bar{s}}(n)}1)|_{\bar{J}}^{(\bar{s})}|_{\bar{J}} \to 0 \text{ as } n \to \infty, \quad 1 \le \bar{s} \le s,$$

where $h|_{\bar{J}}^{(\bar{s})} \in \mathcal{L}(\bar{J})$, $1 \le \bar{s} \le s$. Since these relations hold whatever cell J, we also have $h \in \mathcal{L}^s$.

Remark. We even have, when (E_s) holds, $h \in \mathcal{C}^{s+1}$ as proved in [3].

3. Superior bounds for derivatives of iterates of Perron-Frobenius operator $\mathcal{U}$

One considers the Perron-Frobenius operator acting on $\mathcal{L}^s$ and $\mathcal{C}^s$. The crucial result of paper is the next lemma. The main theorem, involving the operators $V^{(s,n)}$ with approximation role results then easily.

We denote the "segment" of integers $\{n, n+1, \ldots, n+l\}$ by $|n, n+l|$, whatever $l, n \in \mathbb{N}_+$.

Lemma 4. *Let condition (E_s) be satisfied. Fix $J \in \mathcal{H}$, $\varphi \in \mathcal{L}^s$. Then whatever $i^{(m)} \in [m, J]$ one has*

$$(\omega(i^{(m)})\varphi(v_{i^{(m)}}))^{(\bar{s})} \equiv \omega^{(\bar{s})}(i^{(m)}) \cdot \varphi(v_{i^{(m)}}) + \theta(\bar{s}, i^{(m)}), \text{ any } \bar{s} \in |1, s|$$

punctually on J, where θ (which is a rest, depending on φ) is defined by

$$\sum_{\bar{n}=1}^{\bar{s}} \omega^{(\bar{s}-\bar{n})}(i^{(m)}, a) \cdot (\varphi(v_{i^{(m)}}(a)))^{(\bar{n})} \langle {\bar{s} \atop \bar{n}} \rangle = \theta(a, \bar{s}, i^{(m)}), \quad a \in J$$

and satisfies for any $\bar{s} \in |2, s|$, $a \in J$

$$|\theta(a, \bar{s}, i^{(m)})| \leq \hat{\sigma}(m)\lambda(\langle i^{(m)}\rangle)\mathcal{B}(\bar{s})(\|\varphi^{(1)}\| + \ldots + \|\varphi^{(\bar{s})}\|),$$

where

$$\mathcal{B}(r) := \mathcal{B}^{1+r} \exp\{(\bar{s}+2)(\bar{s}+1) \log \sqrt{2} + \bar{s} \log \sqrt{\frac{\bar{s}!}{\bar{s}+1}}\}, 0 \leq r \leq \bar{s}.$$

Proof. Let $\bar{s} \leq s$; one has $L(\varphi(v_{i^{(m)}})^{(\bar{s})}) < \infty$. Hence $\varphi(v_{i^{(m)}}) \mid^{(\bar{s}+1)}$ exists a.e.. Moreover, if we denote $\widetilde{\Omega}$ the set included in $\mathcal{B}$ and containing all points a where for all $i^{(m)} \in [m, a]$, $(\varphi(v_{i^{(m)}}(a))) \mid^{(\bar{s}+1)}$ exist whatever $m \geq 1$, we have $\lambda(\widetilde{\Omega}) = 1$.

The inequalities stated rely on condition (E_s) and are uniform rest $m \in \mathbb{N}$ and $J \in \mathcal{H}$. We begin by deducing the relations (16-18) for $i \in [1, a]$ instead of $i^{(m)} \in [m, a]$ and we do not write the point argument a (supposed $\in J$); hence we may also suppress the symbol "$|_J$".

Like in [3] we use the relation

$$\frac{2^{\bar{s}+1}}{(\bar{s}+1)^{1/2}} \geq \langle \begin{matrix} \bar{s} \\ \bar{n} \end{matrix} \rangle, \text{ any } \bar{n} \leq \bar{s}, \bar{s} > 1. \tag{15}$$

Using (10) and Leibnitz's formula, for $i \in [1, a]$, $a \in \widetilde{\Omega} \cap J$, $1 \leq \bar{s} \leq 1 + s$ we have

$$\begin{aligned} |(\varphi(v_i))^{(\bar{s})}| &\equiv |(\omega(i)\varphi^{(1)}(v_i))^{(\bar{s}-1)}| \equiv |\textstyle\sum_{\bar{n}=0}^{\bar{s}-1} \omega^{(\bar{s}-\bar{n}-1)}(i)(\varphi^{(1)}(v_i))^{(\bar{n})}\langle \begin{matrix} \bar{s}-1 \\ \bar{n} \end{matrix} \rangle| \leq \\ &\mathcal{B}\lambda(\langle i\rangle)\{|\varphi^{(1)}(v_i)| + \sum_{\bar{n}=1}^{\bar{s}-1} |(\varphi^{(1)}(v_i))^{(\bar{n})}\langle \begin{matrix} \bar{s}-1 \\ \bar{n} \end{matrix} \rangle|\} \leq \\ &\mathcal{B}\lambda(\langle i\rangle)\{|\varphi^{(1)}(v_i)| + \sum_{\bar{n}=1}^{\bar{s}-1} |(\varphi^{(1)}(v_i))^{(\bar{n})}|(1/\sqrt{\bar{s}}) \cdot 2^{\bar{s}}.\} \end{aligned} \tag{16}$$

Then we apply anew the same procedure for summands from last sum. Denoting $\frac{d^2}{dw^2}\varphi(w) = \varphi''(w)$, $w \in \langle i\rangle$ in next alineates (and using $\lambda(\langle i\rangle) \leq 1$)

we have

$$\begin{aligned}
&\sum_{\bar{n}=1}^{\bar{s}-1}|(\varphi^{(1)}(v_i))^{(\bar{n})}| = \sum_{\bar{n}=1}^{\bar{s}-1}|(\varphi''(v_i))^{(\bar{n}-1)}\omega(i)| \leq \\
&\quad \sum_{\bar{n}=0}^{\bar{s}-1}\sum_{\bar{\bar{n}}=0}^{\bar{n}-1}|\omega^{(\bar{n}-\bar{\bar{n}}-1)}(i)(\varphi''(v_i))^{(\bar{\bar{n}})}|\langle{\bar{n}-1 \atop \bar{\bar{n}}}\rangle \leq \\
&\quad \mathcal{B}\lambda(\langle i\rangle)[\sum_{\bar{n}=1}^{\bar{s}-1}\sum_{\bar{\bar{n}}=0}^{\bar{n}-1}|(\varphi''(v_i))^{(\bar{\bar{n}})}|]\cdot(1/\sqrt{\bar{s}-1})\cdot 2^{\bar{s}-1} \leq \\
&\quad \mathcal{B}\lambda(\langle i\rangle)[|\varphi''(v_i)| + \sum_{\bar{n}=1}^{\bar{s}-2}|(\varphi''(v_i))^{(\bar{n})}|]\cdot(1/\sqrt{\bar{s}-1})\cdot 2^{\bar{s}-1}
\end{aligned} \tag{16'}$$

(16) continues with (in formulas bellow $\bar{s}\geq 3$)

$$\leq \mathcal{B}^{1+1}\lambda(\langle i\rangle)[|\varphi^{(1)}(v_i)| + |\varphi''(v_i)| + \sum_{\bar{n}=1}^{\bar{s}-2}|(\varphi^{(s)}(v_i))^{(\bar{n})}|]\sqrt{\frac{\bar{s}-1}{\bar{s}}}\cdot 2^{\bar{s}+(\bar{s}-1)}$$

Similarly with the last estimation we get iteratively ($\bar{s}-3$ steps more: we write $(\varphi''(v_i))^{(\bar{n})} = (\varphi^{(3)}(v_i))^{(\bar{n}-1)}\omega(i)$, Leibnitz's formula, etc, ...)

$$\begin{aligned}
&\leq \mathcal{B}^{1+2}\lambda(\langle i\rangle)[|\varphi^{(1)}(v_i)| + \ldots + |\varphi^{(3)}(v_i)| + \sum_{\bar{n}=1}^{\bar{s}-1}\sum_{\bar{\bar{n}}=0}^{\bar{n}-1}|(\varphi^{(3)}(v_i))^{(\bar{n})}|]\sqrt{\frac{\bar{s}-1}{(\bar{s}-2)^{\bar{s}}}}\cdot 2^{3\bar{s}-3} \leq \\
&\mathcal{B}^{3}\lambda(\langle i\rangle)[|\varphi^{(1)}(v_i)| + \ldots + |\varphi^{(3)}(v_i)| + \sum_{\bar{n}=1}^{\bar{s}-3}|(\varphi^{(3)}(v_i))^{(\bar{n})}|]\sqrt{\frac{(\bar{s}-1)(\bar{s}-2)}{\bar{s}}}\cdot 2^{3\bar{s}-3}
\end{aligned}$$

....

$$\begin{aligned}
&\leq \lambda(\langle i\rangle)\mathcal{B}^{\bar{s}}[|\varphi^{(1)}(v_i)| + \ldots + |\varphi^{(\bar{s}-1)}(v_i)| + \sum_{\bar{n}=1}^{\bar{s}+(1-\bar{s})}\sum_{\bar{\bar{n}}=0}^{\bar{n}-1}|(\varphi^{(\bar{s})}(v_i))^{(\bar{n})}|] \\
&\qquad \times\sqrt{\frac{(\bar{s}-1)(\bar{s}-2)\ldots 1}{\bar{s}}}\cdot 2^{3\bar{s}-3+\ldots+\bar{s}-(\bar{s}-1)}
\end{aligned}$$

In conclusion if $a\notin\widetilde{\Omega}$, $a\in J$, $i\in[1,a]$, $\bar{s}\leq 1+s$ we have

$$|(\varphi(v_i(a)))^{(\bar{s})}| \leq \mathcal{B}(\bar{s}-1)\lambda(\langle i\rangle)[|\varphi^{(1)}(v_i(a))| + \ldots + |\varphi^{(\bar{s}-1)}(v_i(a))| + |\varphi^{(\bar{s})}(v_i(a))|] \tag{16''}$$

By Leibnitz's formula again, we have

$$(\omega(i)\varphi(v_i))^{(\bar{s})} \equiv \sum_{\bar{n}=0}^{\bar{s}-1} \omega^{(n)}(i)(\varphi(v_i))^{(\bar{s}-\bar{n})}\langle \begin{matrix} \bar{s} \\ \bar{n} \end{matrix} \rangle \tag{17}$$

Recording that $\mathcal{B}(\bar{s}-1) \geq \mathcal{B}(r)$, $r \leq \bar{s}-1$ and using (6) we deduce

$$\begin{aligned}
&|\sum_{\bar{n}=1}^{\bar{s}} \omega^{(\bar{s}-\bar{n})}(i)(\varphi(v_i))^{(\bar{n})}\langle \begin{matrix} \bar{s} \\ \bar{n} \end{matrix} \rangle| \leq \\
&\lambda(\langle i\rangle)\mathcal{B}\sum_{\bar{n}=1}^{\bar{s}} |(\varphi(v_i))^{(\bar{n})}| \cdot (1/\sqrt{\bar{s}-1}) \cdot 2^{\bar{s}+1} \overset{(16)}{\leq} \\
&\lambda(\langle i\rangle)(1/\sqrt{\bar{s}+1}) \cdot 2^{\bar{s}+1}\mathcal{B}\times \\
&\quad \mathcal{B}(l-1)\sum_{1\leq l\leq \bar{s}} \lambda(\langle i\rangle)[|\varphi^{(1)}(v_i)| + \ldots + |\varphi^{(l)}(v_i)|] \leq \\
&\lambda(\langle i\rangle)^2\mathcal{B}(\bar{s}-1).\mathcal{B}.(1/\sqrt{\bar{s}+1}).\sum_{l=1}^{\bar{s}}[|\varphi^{(1)}(v_i)| + \ldots + |\varphi^{(l)}(v_i)|]2^{\bar{s}+1} \leq \\
&\lambda(\langle i\rangle)^2\mathcal{B}(\bar{s}).[|\varphi^{(1)}(v_i)| + \ldots + |\varphi^{(\bar{s})}(v_i)|]
\end{aligned} \tag{18}$$

To write (18) we used that $(1/\sqrt{\bar{s}+1}).2^{\bar{s}+1}.\mathcal{B}(\bar{s}-1).\mathcal{B} \leq \mathcal{B}(\bar{s})$ since $\bar{s}-1 \leq \bar{s}$.

Now we observe that the relations (16)–(18) (and all relations written above) are valid with $i^{(m)} \in [m, a]$ replacing $i \in [1, a]$. Then the quantity θ as defined in the statement satisfies (in $\widetilde{\Omega}$)

$$\begin{aligned}
|\theta(a, \bar{s}, i^{(m)})| \equiv |\sum_{\bar{n}=1}^{\bar{s}} \langle \begin{matrix} \bar{s} \\ \bar{n} \end{matrix} \rangle \omega^{(\bar{s}-\bar{n})}(i^{(m)}, a)(\varphi(v_{i^{(m)}}(a))^{(\bar{s}-\bar{n})}) \leq \\
\hat{\sigma}(m)\mathcal{B}(\bar{s})(|\varphi^{(1)}(v_{i^{(m)}}(a))| + \ldots + |\varphi^{(\bar{s})}(v_{i^{(m)}}(a))|)\lambda(\langle i^{(m)}\rangle)
\end{aligned}$$

□

Assume that (E_s) holds. We denote

$$V^{(\bar{s},n)}\varphi(a) :\equiv \sum_{i^{(n)}\in[n,a]} \omega^{(\bar{s})}(i^{(n)}, a)\varphi(v_{i^{(n)}}(a)), n \geq 1, a \in \mathcal{B}, \varphi \in \mathcal{C}, 0 \leq \bar{s} \leq 1+s$$

Then $V^{(\bar{s},n)}$ is linear bounded operator on L_∞. Indeed, whatever $J \in \mathcal{H}$ using (7) to write the last relation below, we have

$$\begin{aligned} |V^{(\bar{s},n)}\varphi\,|_J\,| &\leq \sum_{i^{(n)}\in[\bar{n},J]} |\omega^{(\bar{s})}(i^{(n)})\cdot\varphi(v_{i^{(\bar{n})}})| \\ &\leq \|\varphi\|_J \sum_{i^{(n)}\in[n,J]} |\omega^{(s)}(i^{(n)})| \leq \mathcal{C}\cdot\|\varphi\|_J. \end{aligned}$$

$V^{(\bar{s},n)}$ applies $\mathcal{L}^{\bar{s}}$ (resp. $\mathcal{C}^{\bar{s}}$) into itself; $V^{(\bar{s},n)}$ is weakly sequentially compact on these spaces.

We denote, given a $\varphi \in \mathcal{C}$

$$R(a,n,\bar{s}) := \sum_{i^{(n)}\in[\bar{n},a]} \theta(a,i^{(n)},\bar{s}), \bar{s}\leq 1+s, \quad n\geq 1, \ a\in\mathcal{B}.$$

Theorem 5. *Assume hypothesis (E_s) holds. Fix $\varphi \in \mathcal{L}^s$, $\bar{s} \in |2,s|$, $n\in\mathbb{N}_*$*

Then

i. $(\mathcal{U}^n\varphi)^{(\bar{s})}(a) \equiv V^{(\bar{s},n)}\varphi(a) + R(a,\bar{n},\bar{s}), \quad a\in\mathcal{B},\ \bar{s}\leq s,$

where

$$|R(a,n,\bar{s})| \leq \hat{\sigma}(n).\mathcal{B}(\bar{s})(\|\varphi^{(1)}\| + \ldots + \|\varphi^{(s)}\|).$$

and $\mathcal{B}(\bar{s})$ are defined in the statement of preceding proposition.

ii. Whatever the order $\bar{s}$, $V^{(\bar{s},l)}h$ approximates as $l\to\infty$ the $\bar{s}$-th derivative of h, $h^{(\bar{s})}$:

$$h^{(\bar{s})} \equiv (\mathcal{U}h)^{(\bar{s})} \equiv \lim_{l\to\infty} V^{(\bar{s},l)}h \tag{19}$$

Whatever rank n, $\mathcal{B}(\bar{s})$ and $\sum_{n=1}^{\bar{s}} \|\varphi^{(n)}\|$ both increase with $\bar{s}$. In i. $\bar{s}$ and $\bar{n}$ are completely independent one from another.

iii. If $\|h^{(\bar{s})}\| > 0$, (19): is the asymptotic analogous of derivative of Gauss' equation $(\mathcal{U}^l h)^{(\bar{s})} \equiv h^{(\bar{s})}$, a.s., *$\bar{s}\leq s, l\in\mathbb{N}$.*

Proof of Theorem 5. i. As consequence of (6) and of $|||\varphi|||^{(s)} < \infty$ the relations (16), (17) and (18) are summable over $[1,J]$. Then i. follows by summing up (16), taking into account (17) and replacing $i\in[1,a]$ by $i^{(n)}\in[n,a]$.

ii. We already recorded that $h = \mathcal{U}^l h$, $\mathbb{N} \in l$ (see [3] Theorem 1). By above Theorem 3 $\mathcal{L} \ni h^{(s)}$ so that the left equality in (19) holds. By i with $h = \varphi$ we have

$$|(\mathcal{U}^l h)^{(\bar{s})} - V^{(\bar{s},l)} h| \leq \mathcal{B}(\bar{s}) \cdot [|h^{(1)}| + \ldots + |h^{(\bar{s})}|]\hat{\sigma}(l)$$

Since by condition (A), $\hat{\sigma}(l) \underset{l\to\infty}{\longrightarrow} 0$, the second equality (19) also holds in the norm $|.|_J$, $J \in \mathcal{H}$. Then it also holds in the stated form.

iii. Obvious (eventually see the Appendix).

Remarks to Theorem 5 1. As a consequence of [3] Theorem 5 and of i. above theorem we have

$$\lim_{n\to\infty} V^{(\bar{s},n)}\varphi \equiv h^{(\bar{s})} \int \varphi d\lambda, \quad \varphi \in \mathcal{L}^{\bar{s}}.$$

This does not mean that $V^{(s,n)}$ is aperiodic operator.

2. The domination constants $\mathcal{B}(\bar{s})$ whose existence allows our formulation of main result have independent rest n and a values so that the statements of above propositions 4,5 are satisfactory. In cases $s \leq 3$ one can obtain directly less domination values but then we have to mention in the statement particularisations nonessential in the sequel.

The existence and generality rest n of $\mathcal{B}(\bar{s})$ is a remarkable property of the class of (dynamical) fibred systems considered in [8]

3. One can obtain an analogous with i., Theorem 5 assertion for Perron–Frobenius operator $\widetilde{\mathcal{U}}$ rest μ. This is obvious by the identity following formula (20) below.

4. Clearly $V^{(\bar{s},n)}$ are different from Ruelle's operators usually denoted G.

4. Higher order derivatives of Perron-Frobenius operator associated with R.C.F

The aim of this section is to prove that the hypothesis of main result is consistent and that the method used in the proof of last theorem can be used for different purposes. It shows that the rather long differential formulas from previous section expressions easy to handle can be find.

The fibred system associated with the continued fraction expansion is:

$$\tau(x) \equiv \frac{1}{x} - \left[\frac{1}{x}\right], \text{a.e. } x \in (\cdot, 1] \equiv \mathcal{B}; \langle i \rangle = \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{N}.$$

where $[y]$ denotes the integer part of $y \in \mathbb{R}_+$

The conditions denoted $(A)-(E)$ in Section 1 are satisfied; in fact these conditions are set so to make as near as possible to pwmts the class of fibred systems considered.

All cylinders are proper and one has not conditions in digits. The application $\hat{\sigma}(m)$ appearing in condition (A) tends to null exponentially $\hat{\sigma}(m) \leq c\theta^m$, where θ is the "golden number" $\frac{3-\sqrt{5}}{2}$. As constant N of condition (E) we may take 2; as constant of Renyi (C), 4 (see also [7]).

The invariant measure rest τ is Gauss' measure on $[0,1]$, with density $\alpha(x) \equiv \dfrac{1}{(x+1)\log 2}$. A dual system exists it is not uniquely determined (see [7]); hence the above Remark applies in this case.

We repeat the definiton of some random variables because we use them also for the reversed label sequence $(\ldots, a_n, \ldots, a_1)$ for which precisations are required.

We suppose known the definition and first properties of continuants Q_n (see Ch. I, §1.2 of 6 for presentation and proofs).

As the definition of random variables p_t, q_t, ω_t, $m \geq 1$, $t \in \mathbb{N}_*$.

$$q_t := q_t(i^{(t)}) = Q_t(i_1, \ldots, i_t)$$

$$p_t := p_t(i^{(t)}) = Q_{t-1}(i_2, \ldots, i_t); \omega_t \equiv \frac{p_t}{q_t}, i^{(t)} \in \mathbb{N}_*^t$$

$$q_{-1} = p_0 \equiv 0, p_{-1} = q_0 \equiv 1; q_1 = i_1.$$

We note that $Q_t(i^{(t)}) = q_t = Q_t(\widetilde{i^{(t)}}) = q_t(\widetilde{i^{(t)}})$ (record that $\widetilde{i^{(t)}} := (i_t, \ldots, i_1)$, $\widetilde{i_{2,t-1}} := (i_{t-1}, \ldots, i_2)$, etc)

$$p_{t-1}(i^{(t)}) = Q_{t-1}(i_{t-1\ldots2}) = p_{t-1}(i^{(t-1)}) =$$
$$p_{t-1}(i_{(t,\ldots,2)}) = p_{t-1}(i_{(t,t-1,\ldots,2)}) = q_{t-2}(i_{t-1,2})$$
$$p_k(\widetilde{i_t}) = Q_{t-1}(i_{t-1,1}) = Q_{t-1}(i^{(t-1)}) = q_{t-1}(i^{(t-1)})$$

Since $v_{i^{(m)}}(y) \equiv \dfrac{p_m + p_{m-1}y}{q_m + q_{m-1}y}$ it is immediate that $\omega^{(s)}(i^{(m)}, y) = \dfrac{\pm(s+1)!}{(q_m + q_{m-1}y)^{s+2}}$, $s \in \mathbb{N}$ and in particular $\omega^{(s)}(i, y) \equiv \dfrac{(\pm 1)(s+1)!}{(y+i)^{s+2}}$, $y \in I$; $\omega(i^{(m)}, 0) \equiv \dfrac{1}{q_m^2}$. (the sign is + when s is even and - when it is odd

We also use that

$$\frac{q_{m-1}+p_{m-1}w}{q_m+p_m w} = v_{\widetilde{i^{(m)}}}(w);\ \alpha^{(s)}(w) = \frac{(\pm 1)s!}{(1+w)^{s+1}\log 2}, w \in I, s \geq 0;$$

$$\alpha^{(\bar{s})}(v_{i^{(m)}}(w)) = \frac{\bar{s}!(\pm 1)(q_m+p_m w)^{\bar{s}+1}}{(q_m+q_{m-1}+p_m w+p_{m-1}w)^{\bar{s}+1}}$$

The general formula (see end of Remark 1 and [6] Ch I)

$$\widetilde{\mathcal{U}}^m \chi\{\langle i^{(m)}\rangle\}(w) \equiv \omega(i^{(m)}, w)\frac{\alpha(v_{i^{(m)}}(w))}{\alpha(w)}\text{not } = \mu_w(\langle i^{(m)}\rangle)$$

yields in present case

$$\mu_w(\langle i^{(m)}\rangle) = \frac{(\pm)(1+w)v^{(1)}_{\widetilde{i^{(m)}}}(w)}{1 \pm v^{(1)}_{\widetilde{i^{(m)}}}(w)} = \frac{1+w}{(p_m w+q_m)((p_m+p_{m-1})w+q_m+q_{m-1})}$$

$w \in I$, $m \geq 1$.

The m-th power of Perron-Frobenius operator associated with τ rest the Gauss' measure is $\widetilde{\mathcal{U}}^m\varphi(w) = \sum_{i^{(m)}\in\mathbb{N}^m_*}\varphi(v_{i^{(m)}}(w))\cdot\widetilde{\mathcal{U}}^m\chi\{\langle i^{(m)}\rangle;\}(w)$

Now we are able to write succesively (compare with (16') and (18)

$$\begin{aligned}(V^{(s,m)}\alpha^{(s)}\varphi\chi\{\langle i^{(m)\rangle};\})(w) &= w^{(s)}(i^{m}, w)\cdot\varphi(v_{\widetilde{i^{(m)}}}(w))\alpha^{(s)}(v_{\widetilde{i^{(m)}}}(w)) \\ &= \frac{s(s+1)}{(1+w)}\omega^{(s-1)}wii^{(m)}(w)\varphi(v(i^{(m)}, w))\frac{\mu_w(\langle i^{(m)}\rangle)}{1+w}\alpha^{(s-1)}(v(\overline{i^{(m)}}, w)) \qquad (20)\\ &= \alpha(w)s(s+1)(V^{(s-1,m)}\alpha^{(s-1)}\varphi)\cdot\mu_w(\langle i^{(m)}\rangle).\end{aligned}$$

Hence we also have whatever $w \in I$, $i^{(m)} \in \mathbb{N}^m_*$, $s, m \geq 1$.

$$\begin{aligned}&\frac{1}{\log 2}(V^{(s,m)}\alpha^{(s)}\varphi)(w) \\ &\quad = s(s+1)(\pm 1)\textstyle\sum_{i^{(\bar{m})}}\mu_w(\langle i^{(m)}\rangle)(V^{(s-2,m)}\alpha^{(m-1)}\varphi\{i^{(m)}\}(w))\end{aligned} \qquad (21)$$

Clearly a differential equation of order m (order 2 in particular) with nonconstant coefficients whose sum is 1 can be written; we have

$$\alpha^{(m)}(0) \equiv (-1)^m/\log 2; \alpha^{(m)}(1) \equiv (-1)^m/2^{m+1}\log 2$$

and $1 = \sum_{i^{(m)}\in\mathbb{N}^m_*}\mu_w(\langle i^{(m)}\rangle)$.

By (21), in RCF case we obtain iterative rest (s, m) expressions of $V^{(s,m)}$ where s is the order of a derivative. This is not possible in general case. Moreover in present case these expressions are subconvex (even convex sometimes) combinations of summands of $\mathcal{U}$ non derivated.

Taking $2n = s$, $m = 1$, $\varphi = 1$ by (21) we obtain similarly

$$(V^{(2n,1)}\alpha^{(n)})(w) = \alpha^{(n)}(w)\sum_i(\mu_w(\langle i\rangle))^{n+1}(2n+1)!)(\pm 1). \tag{22}$$

It is seen that in particular with $n = 0$, this equation becomes

$$\alpha \equiv \mathcal{U}\alpha. \tag{23}$$

By Gauss' equation, α is the solution of an equality defining an eigenvector of $\mathcal{U}$. Regarding $V^{(\bar{s},\cdot)}$ (also other operators) appearing in $i.5$ Theorem which is sum of differential expressions (monomes) similar with (23) equations are usually written with linear (polynomial) combinations of differential monomes. Hence adopting the same point of view, (23) can be regarded as an extension concerning $\alpha^{(n)}$ of Gauss' equation concerning $V^{(2n,\cdot)}$: $\alpha^{(n)}$ is a solution of Gauss' equation involving $V^{(2n,1)}$ modulo the factor $\sum_i(\mu_w(\langle i\rangle))^{n+1}(2n+1)!(\pm 1)$ (which is $\neq 0$, $w \in I$, $m \geq 1$).

Consider now (22) divided only by $\sum_i(\mu_w(\langle i\rangle))^{n+1})$. Note that $\alpha^{(n)}$, $n \geq 1$ are proper vectors of $\Pi_n := (\pm 1)V^{2n,1} * / \sum_i \mu_{**}(\langle i\rangle))^{n+1}$ correspond to increasing in n eigenvalues (equal with $(2n+1)!$) of Π_n respectively.

Since $\lim_n \dfrac{\alpha^{(n)}}{n!} \in [0, \dfrac{1}{\log 2}]$ and (21) involves factors with small and great values together an estimation of numerical size of operators $V^{(s,\cdot)}$ is usefull.

We have

$$V^{(s,1)}\alpha^{(s+1)}(w) = \frac{1}{\log 2}(\pm 1)\sum_{i\in\mathbb{N}_*}\frac{(i+w)^{(s+2)}}{(i+1+w)^{s+2}(i+w)^{s+2}}((s+1)!)^{1+1} =$$

$$\frac{\pm 1}{\log 2}\sum_{i\geq 1}\frac{((s+1)!)^{1+1}}{(i+1+w)^{s+1}} = \left(\frac{\pm 1}{\log 2}\zeta(s+1,w) - \alpha(w)\right)((s+1)!)^{1+1}$$

where $\zeta(s+1,t) := \displaystyle\sum_{i\geq 1}\frac{1}{(i+t)^{s+1}}$ is decreasing on I.

Using that $\zeta(2,0) \equiv \dfrac{\pi^2}{6}$, $\zeta(2^2,0) = \dfrac{\pi^2}{90}$ we calculated that

$$V^{(1,1)}\alpha'' \in (2,2784; 3,7223); \quad V^{(3,1)}\alpha^{(4)} \in (0,4685; 1,19136).$$

Appendix. Notations of spaces and norms.

$\mathbb{N}_* = \{1, 2, \ldots\}$; $\mathbb{N} = \mathbb{N}_* \cup \{0\}$; $\triangle$ denotes an arbitrary interval of $\mathbb{R}^1$. We record that in Section I we denoted the restriction at $\triangle \subset \mathcal{B}$ of $f^{(s)}$ by $f\mid_{\triangle^{(s)}}$ and its continuous extension at $\bar{\triangle}$ by $f\mid_{\bar{\triangle}}^{(s)}$

Below $\triangle$ is an arbitrary interval $\subset \mathcal{B}$. We denote:

- $L(\triangle)$ the Banach space of Lebesgue integrable functions f on $\triangle$ endowed with the norm $\|f\|_1 = \int_\triangle |f| d\lambda$.

$\int_{\mathcal{B}} f d\lambda = \int f d\lambda$, $f \in L_1(\triangle)$

- $L_\infty(\triangle)$ the Banach space of real maps f on $\triangle$ finite $\|f\|_\triangle :=$ ess $\sup_{x\in\triangle} |f(x)|$. We denote $\|f\| \equiv \|f\|_{\mathcal{B}}$. The norm $\|f\|$ is in general denoted $|.|_{\infty,\triangle}$

- $\mathcal{C}(\triangle)$ the set of real bounded continuous functions on $\triangle$ with $\|f\|_\triangle := \sup_{x\in\triangle} |f(x)|$.

- $\mathcal{C}^s(\triangle)$, where $s \in \mathbb{N}_*$, the collection of real maps on $\triangle$ with continuous derivative $f^{(s)}$ of order s in $\triangle$, for which $|f\mid^{(s)}|_\triangle := \sum_{l=0}^n |f|_\triangle^{(l)} \cdot \frac{1}{l!} < \infty$

We endow with the norm $|f^{(s)}|_\triangle$.

- $\mathcal{C}^n$, $n \in \mathbb{N}$ the collection of real maps f on $\mathcal{B}$ with all restrictions at cells $|f\mid^{(n)}|_J < \infty$, any $J \in \mathcal{H}$. We endow this space with the norm $|f^{(n)}| := \max_{J\in\mathcal{H}} |f\mid^{(n)} J|$.

- $L(f, \triangle)$ the Lipschitz coefficient of continuous function f on $\triangle$; also of $f\mid_\triangle$.

- $\mathcal{L}^s(\triangle)$, $s \in \mathbb{N}$ the collection of real functions f on $\triangle$ with finite Lipschitz continuous derivative of order s on $\triangle$, $f^{(s)}$; i.e. f having $L(f^{(s)}, \triangle) < \infty$. We endow it with the norm $\|f^{(s)}\|_\triangle := |f|_\triangle^{(s)} + \frac{1}{s!} L(f^{(s)}, \triangle)$.

- $\mathcal{L}^s$ the collection of real maps f on $\mathcal{B}$ with $|||f^{(s)}||| :\equiv \max_{J\in\mathcal{H}} |||f^{(s)}|||_s$. Endowed with norm $|||f^{(s)}|||$ this is a Banach space.

- $|||f|||_\triangle := |||f|||_\triangle^\circ$; $|||f||| \equiv |||f|||^\circ$.

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