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Abstract

This is the first part of a series of papers which aim to develop an abstract group theoretic framework for the Cogalois Theory of field extensions.

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Introduction

The efforts to generalize the famous *Gauss' Quadratic Reciprocity Law* led to the theory of Abelian extensions of algebraic and p-adic number fields, known as *Class Field Theory*. This theory can be also developed in an abstract group theoretic framework, namely for arbitrary profinite groups. Since the profinite groups are precisely those

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topological groups which arise as Galois groups of Galois extensions, an *Abstract Galois Theory* for arbitrary profinite groups was developed within the *Abstract Class Field Theory* (see e.g., Neukirch [14]).

The aim of this paper is to present a dual theory we called *Abstract Cogalois Theory* to the *Abstract Galois Theory*. Roughly speaking, *Cogalois Theory* (see Albu [2]) investigates field extensions, finite or not, which possess a Cogalois correspondence. This theory is somewhat dual to the very classical *Galois Theory* dealing with field extensions possessing a Galois correspondence.

The basic concepts of Cogalois Theory, namely that of *G-Kneser* and *G-Cogalois* field extension, as well as their main properties are generalized to arbitrary profinite groups. More precisely, let Γ be an arbitrary profinite group, and let A be any subgroup of the Abelian group $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on the discrete group A. Then, one defines the concepts of *Kneser* subgroup and *Cogalois* subgroup of the group $Z^1(\Gamma, A)$ of all continuous 1-cocycles of Γ with coefficients in A, and one establish their main properties. Thus, we prove an *Abstract Kneser Criterion* for Kneser groups of cocycles, as well as a *Quasi-Purity Criterion* for Cogalois groups of cocycles.

The idea to involve the group $Z^1(\Gamma, A)$ in defining the abstract concepts mentioned above comes from the description due to Barrera-Mora, Rzedowski-Calderón, and Villa-Salvador [9], via the Hilbert's Theorem 90, of the Cogalois group $\mathrm{Cog}(E/F)$ of an arbitrary Galois extension E/F as a group of cocycles. More precisely, $\mathrm{Cog}(E/F)$ is canonically isomorphic to the group $Z^1(\mathrm{Gal}(E/F), \mu(E))$ of all continues 1-cocycles of the profinite Galois group $\mathrm{Gal}(E/F)$ of the extension E/F with coefficients in the group $\mu(E)$ of all roots of unity in E. Note that the multiplicative group $\mu(E)$ is isomorphic (in a noncanonical way) to a subgroup of the additive group $\mathbb{Q}/\mathbb{Z}$, and that the basic groups appearing in the investigation of E/F from the Cogalois Theory perspective are subgroups of $\mathrm{Cog}(E/F)$.

In this way, the above description of $\mathrm{Cog}(E/F)$ in terms of 1-cocycles naturally suggests to study the abstract setting of subgroups of groups of type $Z^1(\Gamma, A)$, with Γ an arbitrary profinite group and A any subgroup of $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on the discrete group A. Such a continuous action establishes through the evaluation map $\Gamma \times Z^1(\Gamma, A) \longrightarrow A$, $(\sigma, g) \mapsto g(\sigma)$, a Galois connection between the lattice $\mathbb{L}(Z^1(\Gamma, A))$ of all subgroups of $Z^1(\Gamma, A)$ and the lattice $\overline{\mathbb{L}}(\Gamma)$ of all closed subgroups of Γ. As the lattices above are naturally equipped with spectral (Stone) topologies on which the profinite group Γ acts continuously, this Galois connection relates them through canonical continuous Γ-equivariant maps. On the other hand, the continuous action of Γ on A endows the dual group $\widehat{Z^1(\Gamma, A)} = \mathrm{Hom}(Z^1(\Gamma, A), \mathbb{Q}/\mathbb{Z})$ with a natural structure of topological Γ-module, related to Γ through a canonical continuous cocycle $\eta : \Gamma \longrightarrow \widehat{Z^1(\Gamma, A)}$ which will play a key role in the study of Kneser groups of cocycles.

This paper is divided into four parts. Part I consists of two sections. In Section 0 we present the basic terminology and notation which will be used throughout the paper, as well as some lattice theoretical and topological preliminaries. In Section 1 we introduce and investigate Kneser groups of cocycles. The main result (Theorem 1.20) is an abstract version of the Kneser criterion [13] from the field theoretic Cogalois Theory, where the place of the primitive p-th roots of unity is taken by suitable cocycles.

The forthcoming Part II is devoted to Cogalois groups of cocycles. In Part III we introduce the concept of *Cogalois action* and provide a complete description of the category of all these actions. In Part IV we apply our general theory to retrieve the Abstract Kummer Theory and show how some basic results as well as some new results of the field theoretic Cogalois Theory can be easily obtained from our abstract approach.

0 Notation and Preliminaries

Throughout this paper Γ will denote a fixed profinite group with identity element denoted by 1, and A will always be a fixed subgroup of the Abelian group $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on A endowed with the discrete topology, i.e., A is a discrete Γ-module.

We denote by $\mathbb{N}$ the set $\{1, 2, \ldots\}$ of all positive natural numbers, by $\mathbb{P}$ the set of positive prime numbers, by $\mathbb{Z}$ the ring of all rational integers, by $\mathbb{Q}$ the field of all rational numbers, by $\mathbb{R}$ the field of all real numbers, and by $\mathbb{C}$ the field of all complex numbers. For any integers $k,\, m \in \mathbb{Z}$ we shall denote by $k \bmod m$ the congruence class $k + m\mathbb{Z}$ of k modulo m; if $n \in \mathbb{N}$ is a divisor of m, then we shall write occasionally $k + m\mathbb{Z} \bmod n$ instead of $k \bmod n$. For any ring R with identity element, R^* will denote the group of units of R. If q is a power of a prime number, then we denote by $\mathbb{F}_q$ the finite field with q elements.

For any $n \in \mathbb{N}$, $n \geqslant 2$ we denote by $\mathbb{D}_{2n}$ the dihedral group of order $2n$. The group of quaternions will be denoted by Q. Given an action of a group C on a group D, the semidirect product of C by D is denoted by $D \rtimes C$, with a suitable subscript, if necessary, to specify the action.

For any $p \in \mathbb{P}$ we denote by $\mathbb{Z}_p$ the ring of p-adic integers, by $\mathbb{Q}_p$ the field of p-adic numbers, and by $\mathbb{Z}_{p^\infty}$ the quasi-cyclic group of type p^∞, that is, the p-primary component $(\mathbb{Q}/\mathbb{Z})(p)$ of the quotient group $\mathbb{Q}/\mathbb{Z}$. Note that $\mathbb{Z}_{p^\infty} \cong \mathbb{Q}_p/\mathbb{Z}_p$.

For any $r \in \mathbb{Q}$, the coset of r in the quotient group $\mathbb{Q}/\mathbb{Z}$ will be denoted by $\widehat{r}$. The elements of Γ will be denoted by small Greek letters σ, τ, ρ, and the elements of A by a, b, c. The action of $\sigma \in \Gamma$ on $a \in A$ will be denoted by σa. The set of all elements of A invariant under the action of Γ will be denoted as usually by A^Γ.

An Abelian group C is said to be of *of bounded order* if $kC = \{0\}$ for some $k \in \mathbb{N}$; if C is of bounded order, then the *exponent* $\exp(C)$ of C is the least $n \in \mathbb{N}$ such that $nC = \{0\}$. The order of an element $x \in C$ will be denoted $\mathrm{ord}(x)$. If n is a positive integer, and D is an Abelian torsion group, then we shall use the notation $D[n] := \{\, x \in D \,|\, nx = 0\,\}$. For any $p \in \mathbb{P}$ we denote by $D(p)$ the p-primary component of D. By $\mathcal{O}_D$ we denote the set of all $n \in \mathbb{N}$ for which there exists $x \in D$ of order n, i.e., $D[n]$ has exponent n. With respect to the divisibility relation and the operations gcd and lcm, $\mathcal{O}_D$ is a distributive lattice with the least element 1. $\mathcal{O}_D$ has a last element if and only if D is a group of bounded order, and in this case, the last element of $\mathcal{O}_D$ is precisely $\exp(D)$.

For any topological group T we denote by $\mathbb{L}(T)$ the lattice of all subgroups of T, and by $\overline{\mathbb{L}}(T)$ the lattice of all closed subgroups of T. The notation $U \leqslant T$ means that U is a subgroup of T. For any $U \leqslant T$ we denote by $\mathbb{L}(T\,|\,U)$ (resp. $\overline{\mathbb{L}}(T\,|\,U)$) the lattice of all subgroups (resp. closed subgroups) of T lying over U. If $X \subseteq T$, then $\overline{X}$ will denote the closure of X, and $\langle X\rangle$ will denote the subgroup generated by X. The notation $U \lhd T$ means that U is a normal subgroup of T. For a subgroup U of T we shall denote by T/U the set $\{\, tU \,|\, t \in T \,\}$ of all left cosets of U in T. We denote by $\mathrm{Ch}(T)$ or by $\widehat{T}$ the *character group* of T, that is, the group of all continuous homomorphisms of T into the unit circle $\mathbb{U} = \{\, z \,|\, z \in \mathbb{C},\ |z| = 1 \,\}$. If S is another topological group, then $\mathrm{Hom}(S, T)$ will denote the set of all continuous group morphisms from S to T. Note that if T is a profinite group, then $\widehat{T}$ can be identified with the Abelian torsion group $\mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$.

Recall that a *crossed homomorphism* (or an 1-*cocycle*) of Γ with coefficients in A is a map $f : \Gamma \to A$ such that $f(\sigma\tau) = f(\sigma) + \sigma f(\tau)$, $\sigma, \tau \in \Gamma$; in particular, $f(1) = 0$. The set of all continuous crossed homomorphisms of Γ with coefficients in A is an Abelian group, which will be denoted by $Z^1(\Gamma, A)$. Note that, in fact, $Z^1(\Gamma, A)$ is a torsion group. Indeed, since Γ is a profinite group and A is a discrete space, a map $h : \Gamma \longrightarrow A$ is continuous if and only if h is locally constant, that is, there exists an open normal subgroup Δ (in particular, of finite index in Γ) such that h factorizes through the canonical surjection map $\Gamma \to \Gamma/\Delta$. Since A is a torsion group, it follows now that for any continuous map $h : \Gamma \longrightarrow A$ there exists an $n \in \mathbb{N}$ such that $h(\Gamma) \subseteq (1/n)\mathbb{Z}/\mathbb{Z}$, and then $nh = 0$, i.e., h has finite order.

The elements of $Z^1(\Gamma, A)$ will be denoted by f, g, h. Always G, H will denote subgroups of $Z^1(\Gamma, A)$ and Δ, Λ subgroups of Γ. To any $a \in A$ one assigns the 1-*coboundary* $f_a : \Gamma \to A$, defined by $f_a(\sigma) = \sigma a - a$, $\sigma \in \Gamma$. The set $B^1(\Gamma, A) := \{\, f_a \mid a \in A \,\}$ is a subgroup of $Z^1(\Gamma, A)$. The quotient group $Z^1(\Gamma, A)/B^1(\Gamma, A)$ is called the first cohomology group of Γ with coefficients in A, and is denoted as usually by $H^1(\Gamma, A)$.

Consider the *evaluation map*

$$\langle -, - \rangle : \Gamma \times Z^1(\Gamma, A) \longrightarrow A, \ \ \langle \sigma, h \rangle = h(\sigma).$$

For any $\Delta \leqslant \Gamma$, $G \leqslant Z^1(\Gamma, A)$, $g \in Z^1(\Gamma, A)$, and $\gamma \in \Gamma$ denote

$$\begin{aligned}
\Delta^{\perp} &= \{\, h \in Z^1(\Gamma, A) \,|\, \langle \sigma, h \rangle = 0, \ \forall\, \sigma \in \Delta \,\}, \\
G^{\perp} &= \{\, \sigma \in \Gamma \,|\, \langle \sigma, h \rangle = 0, \ \forall\, h \in G \,\}, \\
g^{\perp} &= \{\, \sigma \in \Gamma \,|\, \langle \sigma, g \rangle = 0 \,\}, \\
\gamma^{\perp} &= \{\, h \in Z^1(\Gamma, A) \,|\, \langle \gamma, h \rangle = 0 \,\}.
\end{aligned}$$

One verifies easily that $\Delta^{\perp} \leqslant Z^1(\Gamma, A)$, $G^{\perp} \leqslant \Gamma$, and $g^{\perp} = \langle\, g \,\rangle^{\perp}$. Observe that $g^{\perp}$ is the set of zeroes of the continuous map g from Γ to the discrete group A, hence it is an open subgroup of Γ. Since $G^{\perp} = \bigcap_{g \in G} g^{\perp}$, it follows that $G^{\perp} \in \overline{\mathbb{L}}(\Gamma)$.

The group $Z^1(\Gamma, A)$ is clearly a discrete left Γ-module with respect to the following action: $(\sigma h)(\tau) = \sigma h(\sigma^{-1}\tau\sigma)$, σ, $\tau \in \Gamma$, $h \in Z^1(\Gamma, A)$. If $\sigma \in \Gamma$ and $G \in \mathbb{L}(Z^1(\Gamma, A))$, then

$$(\sigma G)^{\perp} = \sigma G^{\perp} \sigma^{-1}.$$

For any $\Delta \in \mathbb{L}(\Gamma)$ one denotes by

$$\mathrm{res}^{\Gamma}_{\Delta} : Z^1(\Gamma, A) \longrightarrow Z^1(\Delta, A),\ h \mapsto h|_{\Delta},$$

the restriction map.

The next result collects together the main properties of the assignments $(-)^{\perp}$.

Proposition 0.1. *The following assertions hold.*

(1) *The maps*

$$\mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \overline{\mathbb{L}}(\Gamma),\ G \mapsto G^{\perp},$$
$$\overline{\mathbb{L}}(\Gamma) \longrightarrow \mathbb{L}(Z^1(\Gamma, A)),\ \Delta \mapsto \Delta^{\perp},$$

establish a Galois connection between the lattices $\mathbb{L}(Z^1(\Gamma, A))$ *and* $\overline{\mathbb{L}}(\Gamma)$, *i.e., they are order-reversing maps and* $X \leqslant X^{\perp\perp}$ *for any element* X *of* $\mathbb{L}(Z^1(\Gamma, A))$ *or* $\overline{\mathbb{L}}(\Gamma)$.

(2) *For any* $\Delta \in \mathbb{L}(\Gamma)$ *and* $G \in Z^1(\Gamma, A)$ *one has*

$$\Delta^{\perp} = \overline{\Delta}^{\perp} = \mathrm{Ker}\,(\mathrm{res}^{\Gamma}_{\Delta}) \quad \textit{and} \quad (\mathrm{res}^{\Gamma}_{\Delta}(G))^{\perp} = G^{\perp} \cap \Delta.$$

(3) *For any* $G_1,\ G_2 \in Z^1(\Gamma, A)$ *and* $\Delta_1,\ \Delta_2 \in \mathbb{L}(\Gamma)$ *one has*

$$(G_1 + G_2)^{\perp} = G_1^{\perp} \cap G_2^{\perp} \quad \textit{and} \quad \Delta_1^{\perp} \cap \Delta_2^{\perp} = \langle \Delta_1 \cup \Delta_2 \rangle^{\perp}.$$

Proof. The proof is straightforward, and therefore is left to the reader. □

Remarks 0.2. (1) Clearly, we have

$$\begin{aligned} 1^{\perp} &= Z^1(\Gamma, A), \\ \Gamma^{\perp} &= \{0\}, \\ 0^{\perp} &= \Gamma. \end{aligned}$$

Note that $(Z^1(\Gamma, A))^{\perp}$ is a closed normal subgroup of Γ contained in the closed normal subgroup $(B^1(\Gamma, A))^{\perp}$, the kernel of the action of Γ on A. Setting $H^1(\Gamma, A)^{\perp} = B^1(\Gamma, A)^{\perp}/Z^1(\Gamma, A)^{\perp}$, we obtain the pairing

$$H^1(\Gamma, A)^{\perp} \times H^1(\Gamma, A) \longrightarrow A$$

induced by the evaluation map. Note that the canonical continuous morphism

$$H^1(\Gamma, A)^{\perp} \longrightarrow \mathrm{Hom}(H^1(\Gamma, A), A) = \widehat{H^1(\Gamma, A)}$$

is injective, in particular, the profinite group $H^1(\Gamma, A)^{\perp}$ is Abelian. In general, the monomorphism above is not onto. For instance, taking $\Gamma = \mathbb{Z}/2\mathbb{Z}$, and considering the non-trivial action of Γ on $A = \mathbb{Z}/4\mathbb{Z}$, we obtain $H^1(\Gamma, A)^{\perp} = \{0\}$, while $\widehat{H^1(\Gamma, A)} \cong H^1(\Gamma, A) \cong \mathbb{Z}/2\mathbb{Z}$.

(2) Following the standard terminology (see e.g., Stenström [18]), the *closed elements* of the Galois connection given in Proposition 0.1 (1) are the elements X of $\mathbb{L}(Z^1(\Gamma, A))$ or $\overline{\mathbb{L}}(\Gamma)$ such that $X = X^{\perp\perp}$. Effective descriptions of such elements are given in Corollaries 1.6 and 1.10, and in Section 3, Part III.

(3) The last part of Proposition 0.1 can be reformulated by saying that the maps $(-)^{\perp}$ are semilattice anti-morphisms. One can ask when these maps are actually lattice anti-morphisms, i.e., they also satisfy the following conditions:

$$(G_1 \cap G_2)^{\perp} = \overline{\langle G_1^{\perp} \cup G_2^{\perp} \rangle} \quad \text{and} \quad (\Delta_1 \cap \Delta_2)^{\perp} = \Delta_1^{\perp} + \Delta_2^{\perp}$$

for all $G_1,\ G_2 \in Z^1(\Gamma, A)$ and $\Delta_1,\ \Delta_2 \in \mathbb{L}(\Gamma)$.

In Section 2, Part II we will discuss cases when the maps $(-)^{\perp}$ establish lattice anti-isomorphisms between certain sublattices of $\mathbb{L}(Z^1(\Gamma, A))$ and $\overline{\mathbb{L}}(\Gamma)$, while in Section 4, Part III we will see that for certain actions we called *Cogalois actions* we do obtain lattice anti-isomorphisms between $\mathbb{L}(Z^1(\Gamma, A))$ and $\overline{\mathbb{L}}(\Gamma)$. □

On the other hand, the posets $\overline{\mathbb{L}}(\Gamma)$ and $\mathbb{L}(Z^1(\Gamma, A))$ are equipped with natural topologies for which the canonical maps defining the Galois connection above are continuous.

The underlying topological space of the profinite group Γ is a *Stone space*, i.e., a (Hausdorff) compact and totally disconnected topological space. The Stone topology on Γ naturally makes the set of all closed subsets of Γ a *spectral space*, i.e., a T_0 quasi-compact topological space which has a topology basis consisting of open quasi-compact sets.

For more details concerning Stone and spectral spaces, which are duals by the Stone's Representation Theorem to boolean algebras and to (bounded) distributive lattices, respectively, the reader may consult [11], [15], and/or [10].

The set $\overline{\mathbb{L}}(\Gamma)$ becomes a spectral space as a closed subset of the spectral space of all closed subsets of the underlying Stone space of Γ. The spectral topology τ_s on $\overline{\mathbb{L}}(\Gamma)$ is defined by the basis of open quasi-compact sets $\mathcal{U}_{\Delta} = \overline{\mathbb{L}}(\Delta)$ for Δ ranging over all open subgroups of Γ. Note that $\overline{\{\Lambda\}} = \overline{\mathbb{L}}(\Gamma \,|\, \Lambda)$ for all $\Lambda \in \overline{\mathbb{L}}(\Gamma)$, so the spectral space $\overline{\mathbb{L}}(\Gamma)$ is irreducible with the generic point $\{1\}$ and with Γ as its unique closed point. As the poset $\overline{\mathbb{L}}(\Gamma)$ is the projective limit of the projective system of finite posets $\mathbb{L}(\Gamma/\Delta)$ for Δ ranging over all open normal subgroups of Γ, with natural order-preserving connecting maps, the topology τ_s is exactly the projective limit of the T_0 topologies on the finite sets $\mathbb{L}(\Gamma/\Delta)$ induced by the partial order given by inclusion.

The *Stone completion* τ_b of the spectral topology τ_s on $\overline{\mathbb{L}}(\Gamma)$, commonly called the *patch topology*, is the topology defined by the basis of open compact sets

$$\mathcal{V}_{\Delta,\Delta'} = \{\, \Lambda \in \overline{\mathbb{L}}(\Gamma) \,|\, \Lambda\Delta = \Delta' \}$$

for all pairs (Δ, Δ'), where Δ is an open normal subgroup of Γ and $\Delta' \in \overline{\mathbb{L}}(\Gamma|\Delta)$. The Stone space above is the projective limit of the finite discrete spaces $\mathbb{L}(\Gamma/\Delta)$ for Δ ranging over all open normal subgroups of Γ. For any subset $\mathcal{U}$ of $\overline{\mathbb{L}}(\Gamma)$, $\mathcal{U}$ is τ_s-open if and only if $\mathcal{U}$ is both τ_b-open and a *lower* subset of $\overline{\mathbb{L}}(\Gamma)$ (the later condition means that $\Lambda \in \mathcal{U}$ and $\Lambda' \in \overline{\mathbb{L}}(\Lambda) \Longrightarrow \Lambda' \in \mathcal{U}$).

Being the dual of the Abelian torsion group $Z^1(\Gamma, A)$, $\widehat{Z^1(\Gamma, A)}$ is an Abelian profinite group, namely the projective limit of the finite Abelian groups $\widehat{H} = \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z})$ for H ranging over all finite subgroups of $Z^1(\Gamma, A)$. Thus $\overline{\mathbb{L}}(\widehat{Z^1(\Gamma, A)})$ is naturally equipped with the topologies defined above. Consequently, by duality, we obtain the corresponding topologies on $\mathbb{L}(Z^1(\Gamma, A))$ as follows: for any finite subgroup F of $Z^1(\Gamma, A)$ and for any subgroup F' of F, consider the following subsets of $\mathbb{L}(Z^1(\Gamma, A))$

$$\mathcal{U}_F = \{\, G \in \mathbb{L}(Z^1(\Gamma, A)) \mid F \subseteq G \,\}$$

and

$$\mathcal{V}_{F,F'} = \{\, G \in \mathbb{L}(Z^1(\Gamma, A)) \mid G \cap F = F' \,\}.$$

The family $(\mathcal{U}_F)_F$ for F ranging over all finite subgroups of $Z^1(\Gamma, A)$ is a basis of open quasi-compact sets for a spectral topology τ_s on $\mathbb{L}(Z^1(\Gamma, A))$. Observe that for $H, G \in \mathbb{L}(Z^1(\Gamma, A))$, we have $H \in \overline{\{G\}} \iff H \leqslant G$, so the spectral space $\mathbb{L}(Z^1(\Gamma, A))$ is irreducible with the generic point $Z^1(\Gamma, A)$, and with $\{0\}$ as its unique closed point. As the poset $\mathbb{L}(Z^1(\Gamma, A))$ is the projective limit of the projective system of the finite posets $\mathbb{L}(F)$, for F ranging over all finite subgroups of $Z^1(\Gamma, A)$, with restrictions as connecting order-preserving maps, it follows that τ_s is exactly the projective limit of the T_0 topologies on the finite sets $\mathbb{L}(F)$ induced by the partial order opposite to inclusion.

On the other hand, the family $(\mathcal{V}_{F,F'})_{F,F'}$ for (F, F') ranging over all pairs of finite subgroups of $Z^1(\Gamma, A)$ with $F' \leqslant F$ is a basis of open compact sets for the Stone topology τ_b on $\mathbb{L}(Z^1(\Gamma, A))$. The Stone space above, which in fact is the Stone completion of its underlying spectral space, is the projective limit of the finite discrete spaces $\mathbb{L}(F)$ for F ranging over all finite subgroups of $Z^1(\Gamma, A)$. A subset $\mathcal{U}$ of $\mathbb{L}(Z^1(\Gamma, A))$ is τ_s-open if and only if $\mathcal{U}$ is both τ_b-open and an *upper* subset of $\mathbb{L}(Z^1(\Gamma, A))$ (the later condition means that $H \in \mathcal{U}$ and $H \leqslant G \Longrightarrow G \in \mathcal{U}$).

Proposition 0.3. *The following assertions hold.*

(1) *The canonical action*

$$\Gamma \times \overline{\mathbb{L}}(\Gamma) \longrightarrow \overline{\mathbb{L}}(\Gamma),\ (\sigma, \Lambda) \mapsto \sigma\Lambda\sigma^{-1},$$

is a coherent map, i.e., the inverse image of any open quasi-compact set is also open quasi-compact.

(2) *The canonical action*

$$\Gamma \times \mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \mathbb{L}(Z^1(\Gamma, A)),\ (\sigma, G) \mapsto \sigma G,$$

is a coherent map.

(3) *The map* $\overline{\mathbb{L}}(\Gamma) \longrightarrow \mathbb{L}(Z^1(\Gamma, A))$, $\Lambda \mapsto \Lambda^{\perp}$, *is a morphism in the category of spectral* Γ*-spaces with* Γ*-equivariant coherent maps as morphisms.*

(4) *The map in* (3) *is also a morphism in the category of Stone* Γ*-spaces with continuous* Γ*-equivariant maps as morphisms.*

(5) *The map* $\mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \overline{\mathbb{L}}(\Gamma)$, $G \mapsto G^{\perp}$, *is a morphism in the category of spectral* Γ*-spaces with continuous* Γ*-equivariant maps as morphisms.*

Proof. (1) For any open subgroup Δ of Γ, we have

$$\{ (\sigma, \Lambda) \in \Gamma \times \overline{\mathbb{L}}(\Gamma) \mid \sigma\Lambda\sigma^{-1} \leqslant \Delta \} = \bigcup_{\sigma \in \Delta \backslash \Gamma} (\Delta\sigma \times \mathcal{U}_{\sigma^{-1}\Delta\sigma}),$$

so the inverse image of the basic quasi-compact open set $\mathcal{U}_\Delta$ of the spectral space $\overline{\mathbb{L}}(\Gamma)$ is quasi-compact open as a finite union of basic quasi-compact open sets of the spectral product space $\Gamma \times \overline{\mathbb{L}}(\Gamma)$, and hence the canonical action above is a coherent map.

(2) For any finite subgroup F of $Z^1(\Gamma, A)$, the stabilizer $\Delta = \Gamma_F$ of F in Γ is open, and

$$\{ (\sigma, G) \in \Gamma \times \mathbb{L}(Z^1(\Gamma, A)) \mid F \leqslant \sigma G \} = \bigcup_{\sigma \in \Delta \backslash \Gamma} (\Delta\sigma \times \mathcal{U}_{\sigma^{-1}F}),$$

so the inverse image of the basic quasi-compact open set $\mathcal{U}_F$ of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$ is quasi-compact open as a finite union of basic quasi-compact open sets of the spectral product space $\Gamma \times \mathbb{L}(Z^1(\Gamma, A))$.

(3) Since $(\sigma\Lambda\sigma^{-1})^{\perp} = \sigma(\Lambda^{\perp})$ for all $\sigma \in \Gamma$ and $\Lambda \in \overline{\mathbb{L}}(\Gamma)$, it remains to show that the canonical map $\Lambda \mapsto \Lambda^{\perp}$ is coherent. Let F be a finite subgroup of $Z^1(\Gamma, A)$, so $F^{\perp}$ is an open subgroup of Γ. Then, the inverse image

$$\{ \Lambda \in \overline{\mathbb{L}}(\Gamma) \mid F \leqslant \Lambda^{\perp} \} = \{ \Lambda \in \overline{\mathbb{L}}(\Gamma) \mid \Lambda \leqslant F^{\perp} \} = \mathcal{U}_{F^{\perp}}$$

of the basic quasi-compact open set $\mathcal{U}_F$ of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$ through the canonical map above is open quasi-compact, as required.

(4) As a coherent map, the map above is clearly continuous with respect to the Stone topologies.

(5) Since $(\sigma G)^{\perp} = \sigma(G^{\perp})\sigma^{-1}$ for all $\sigma \in \Gamma$ and $G \in \mathbb{L}(Z^1(\Gamma, A))$, it remains to show that the canonical map $G \mapsto G^{\perp}$ is continuous. Given an open subgroup Δ of Γ, let $\mathcal{W} = \{ G \in \mathbb{L}(Z^1(\Gamma, A)) \mid G^{\perp} \leqslant \Delta \}$ denote the inverse image of the basic open set $\mathcal{U}_\Delta$ of the spectral space $\overline{\mathbb{L}}(\Gamma)$. We may assume that $\mathcal{W}$ is nonempty, i.e., $Z^1(\Gamma, A)^{\perp} \leqslant \Delta$, since otherwise we have nothing to prove. Let $\mathcal{W}_{\min}$ denote the (nonempty) set of all minimal members of $\mathcal{W}$ with respect to inclusion. Thus, it remains to show that all members of $\mathcal{W}_{\min}$ are finite subgroups of $Z^1(\Gamma, A)$, since by Zorn's Lemma it follows that $\mathcal{W} = \bigcup_{F \in \mathcal{W}_{\min}} \mathcal{U}_F$, so $\mathcal{W}$ is τ_s-open as a union of basic τ_s-open sets.

Assuming the contrary, let $G \in \mathcal{W}_{\min}$ be such that G is infinite. Consequently, by minimality, we deduce that $F^{\perp} \not\subseteq \Delta$ for all finite subgroups F of G. Since the family of nonempty closed subsets $F^{\perp} \backslash \Delta$ of Γ for F ranging over all finite subgroups of G has the finite intersection property, it follows by the compactness of Γ that $G^{\perp} \setminus \Delta = \bigcap_F (F^{\perp} \setminus \Delta) \neq \varnothing$, i.e., $G^{\perp} \not\subseteq \Delta$, which is a contradiction. $\square$

The natural continuous action of Γ on the profinite Abelian group $\widehat{Z^1(\Gamma, A)}$ induces a canonical continuous 1-cocycle $\eta : \Gamma \longrightarrow \widehat{Z^1(\Gamma, A)}$ we are going to define below, and which will play a key role in the rest of the paper.

First note that $\widehat{Z^1(\Gamma, A)} := \mathrm{Hom}(Z^1(\Gamma, A), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(Z^1(\Gamma, A), A)$. Indeed, for any $\varphi \in \widehat{Z^1(\Gamma, A)}$ and for any $g \in Z^1(\Gamma, A)$, we have $\varphi(g) \in (1/n)\mathbb{Z}/\mathbb{Z}$, where $n = \mathrm{ord}(g)$. On the other hand, as n is the lcm of the orders of $g(\sigma)$ for $\sigma \in \Gamma$, one easily deduces that $(1/n)\mathbb{Z}/\mathbb{Z} \subseteq A$, and hence $\varphi(g) \in A$, as required. The Abelian profinite group $\widehat{Z^1(\Gamma, A)}$ becomes a topological Γ-module via the canonical continuous action of the profinite group Γ given by

$$(\sigma\varphi)(g) = \sigma\varphi(g), \ \forall \sigma \in \Gamma, \ \varphi \in \widehat{Z^1(\Gamma, A)}, \ g \in Z^1(\Gamma, A).$$

Now, observe that $\widehat{\widehat{Z^1(\Gamma, A)}} = \mathrm{Hom}_{\Gamma}(\widehat{Z^1(\Gamma, A)}, A)$, i.e., any continuous morphism $\chi : \widehat{Z^1(\Gamma, A)} \longrightarrow \mathbb{Q}/\mathbb{Z}$ takes values in A and is also a morphism of Γ-modules. Indeed, the canonical morphism $\alpha : Z^1(\Gamma, A) \longrightarrow \widehat{\widehat{Z^1(\Gamma, A)}}$ defined by $\alpha(g)(\varphi) = \varphi(g)$ for $g \in Z^1(\Gamma, A)$ and $\varphi \in \widehat{Z^1(\Gamma, A)} = \mathrm{Hom}(Z^1(\Gamma, A), A)$ is an isomorphism by the Pontryagin Duality, and hence for $\chi \in \widehat{\widehat{Z^1(\Gamma, A)}}$, $\sigma \in \Gamma$, and $\varphi \in \widehat{Z^1(\Gamma, A)}$ we have $\chi(\sigma\varphi) = (\sigma\varphi)(\alpha^{-1}(\chi)) = \sigma\varphi(\alpha^{-1}(\chi)) = \sigma\chi(\varphi)$, as required.

For any subgroup G of $Z^1(\Gamma, A)$ set $X_G = Z^1(\Gamma, A)/G$. Then observe that $\widehat{X_G} = \mathrm{Hom}(X_G, A)$ is identified with a closed subgroup of $\widehat{Z^1(\Gamma, A)}$, stable under the action of Γ, so the quotient $\widehat{Z^1(\Gamma, A)}/\widehat{X_G} \cong \widehat{G} = \mathrm{Hom}(G, A)$ is also a topological Γ-module, and $\widehat{\widehat{G}} = \mathrm{Hom}_{\Gamma}(\widehat{G}, A)$.

Now consider the map

$$\eta : \Gamma \longrightarrow \widehat{Z^1(\Gamma, A)}, \ \eta(\sigma)(g) = \langle \sigma, g \rangle = g(\sigma), \ \sigma \in \Gamma, \ g \in Z^1(\Gamma, A),$$

and for any $G \leqslant Z^1(\Gamma, A)$, let $\eta_G : \Gamma \longrightarrow \widehat{G}$ denote the map obtained from η by composing it with the canonical epimorphism of topological Γ-modules

$$\mathrm{res}_{G}^{Z^1(\Gamma,A)} : \widehat{Z^1(\Gamma, A)} \longrightarrow \widehat{G}, \ \varphi \mapsto \varphi|_G.$$

Proposition 0.4. *For any $G \leqslant Z^1(\Gamma, A)$, the map $\eta_G : \Gamma \longrightarrow \widehat{G}$ is a continuous 1-cocycle satisfying the following universality property: for every $g \in G$ there exists a unique continuous morphism $\chi : \widehat{G} \longrightarrow A$ such that $\chi \circ \eta_G = g$.*

Proof. One easily checks that η_G is a continuous 1-cocycle, so $\chi \circ \eta_G \in Z^1(\Gamma, A)$ for all $\chi \in \widehat{\widehat{G}} = \mathrm{Hom}_{\Gamma}(\widehat{G}, A)$. Thus it is sufficient to show that the canonical morphism $\beta_G : \widehat{\widehat{G}} \longrightarrow Z^1(\Gamma, A)$, $\chi \mapsto \chi \circ \eta_G$, takes values in G and is the inverse of the canonical isomorphism $\alpha_G = \alpha|_G : G \longrightarrow \widehat{\widehat{G}}$ given by the Pontryagin Duality. The equality $\beta_G \circ \alpha_G = 1_G$ is obvious, so it remains only to check that β_G is injective. Let $\chi \in \widehat{\widehat{G}}$ be such that $\beta_G(\chi) = 0$, and let $g = \alpha_G^{-1}(\chi)$. For all $\sigma \in \Gamma$ we have

$$0 = \beta_G(\chi)(\sigma) = \chi(\eta_G(\sigma)) = \alpha_G(g)(\eta_G(\sigma)) = \eta_G(\sigma)(g) = g(\sigma),$$

so $g = 0$, and hence $\chi = \alpha_G(g) = 0$, as desired. $\square$

1 Kneser groups of cocycles

In this section we define the concept of abstract Kneser group, present the main properties of these groups, and establish the abstract version of the field theoretic Kneser Criterion [13].

Lemma 1.1. *If G is a finite subgroup of $Z^1(\Gamma, A)$, then $(\Gamma : G^{\perp}) \leqslant |G|$.*

Proof. The canonical cocycle $\eta_G : \Gamma \longrightarrow \widehat{G}$ defined by $\eta_G(\sigma)(g) = \langle \sigma, g \rangle = g(\sigma)$, $\sigma \in \Gamma$, $g \in G$, induces an injective map $\Gamma/G^{\perp} \longrightarrow \widehat{G}$, so $(\Gamma : G^{\perp}) \leqslant |\widehat{G}| = |G|$, as desired. □

Definition 1.2. *A subgroup G of $Z^1(\Gamma, A)$ is called a* Kneser group of *$Z^1(\Gamma, A)$ if the canonical continuous cocycle $\eta_G : \Gamma \longrightarrow \widehat{G}$ is onto.* □

Lemma 1.3. *A finite subgroup G of $Z^1(\Gamma, A)$ is a Kneser group of $Z^1(\Gamma, A)$ if and only if $(\Gamma : G^{\perp}) = |G|$.*

Proof. The result follows immediately from Lemma 1.1. □

We shall denote by $\mathcal{K}(\Gamma, A)$ the set of all Kneser groups of $Z^1(\Gamma, A)$, partially ordered by inclusion, with $\{0\}$ as the least element.

Lemma 1.4. *If $G \in \mathcal{K}(\Gamma, A)$, then $H \in \mathcal{K}(\Gamma, A)$ for any $H \leqslant G$; in other words, $\mathcal{K}(\Gamma, A)$ is a lower subset of the poset $\mathbb{L}(Z^1(\Gamma, A))$.*

Proof. Since η_H is obtained from η_G by composing it with the canonical epimorphism $\mathrm{res}^G_H : \widehat{G} \longrightarrow \widehat{H}$, $\varphi \mapsto \varphi|_H$, and η_G is onto by assumption, it follows that the cocycle η_H is onto too, so $H \in \mathcal{K}(\Gamma, A)$, as desired. □

Lemma 1.5. *If $G \in \mathcal{K}(\Gamma, A)$, then the map $\mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma), H \mapsto H^{\perp}$, is injective. In particular, $H = G \cap H^{\perp\perp}$ for every $H \in \mathbb{L}(G)$.*

Proof. Let H_1, $H_2 \in \mathbb{L}(G)$ be such that $H_1^{\perp} = H_2^{\perp}$. We have to show that $H_1 = H_2$. Since $(H_1 + H_2)^{\perp} = H_1^{\perp} \cap H_2^{\perp} = H_1^{\perp} = H_2^{\perp}$, we may assume from the beginning that $H_2 \leqslant H_1$. Since $G \in \mathcal{K}(\Gamma, A)$, it follows by Lemma 1.4 that $H_i \in \mathcal{K}(\Gamma, A)$, $i = 1, 2$, and hence the map $\Gamma/H_i^{\perp} \longrightarrow \widehat{H_i}$ induced by the surjective cocycle η_{H_i} is bijective for $i = 1, 2$. As $\eta_{H_2} = \mathrm{res}^{H_1}_{H_2} \circ \eta_{H_1}$ and $H_1^{\perp} = H_2^{\perp}$ by assumption, it follows that $\mathrm{res}^{H_1}_{H_2} : \widehat{H_1} \longrightarrow \widehat{H_2}$ is an isomorphism, and hence $H_1 = H_2$ by the Pontryagin Duality.

The last part of the statement is now immediate since $(G \cap H^{\perp\perp})^{\perp} = H^{\perp}$ for any $H \in \mathbb{L}(G)$. □

Corollary 1.6. *If $Z^1(\Gamma, A) \in \mathcal{K}(\Gamma, A)$, then the canonical map $\mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \overline{\mathbb{L}}(\Gamma)$ is injective, and $H = H^{\perp\perp}$ for every $H \in \mathbb{L}(Z^1(\Gamma, A))$, i.e., every $H \in \mathbb{L}(Z^1(\Gamma, A))$ is a closed element of the Galois connection described in Proposition* 0.1 (1).

The next statement shows that the property of a subgroup of $Z^1(\Gamma, A)$ being Kneser is of finitary character.

Proposition 1.7. *The following assertions are equivalent for $G \leqslant Z^1(\Gamma, A)$.*

(1) $G \in \mathcal{K}(\Gamma, A)$.

(2) $F \in \mathcal{K}(\Gamma, A)$ *for any finite subgroup* F *of* G.

Proof. By Lemma 1.4, we have only to prove that (2) $\Longrightarrow$ (1). By assumption the continuous cocycle $\eta_F : \Gamma \longrightarrow \widehat{F}$ is onto for any finite subgroup F of G. We are going to show that the continuous cocycle $\eta_G : \Gamma \longrightarrow \widehat{G}$ is also onto. Let $\varphi \in \widehat{G}$. Since the family $(\eta_F^{-1}(\varphi|_F))_F$ of nonempty closed subsets of Γ, for F ranging over all finite subgroups of G, has the finite intersection property, it follows by compactness that $S := \bigcap_F \eta_F^{-1}(\varphi|_F) \neq \varnothing$. Consequently, $\eta_G(\sigma) = \varphi$ for all $\sigma \in S$, as η_G is the projective limit of the projective system of maps $(\eta_F)_F$. Thus η_G is onto, and so $G \in \mathcal{K}(\Gamma, A)$, as desired. $\square$

Corollary 1.8. *The following assertions hold.*

(1) $\mathcal{K}(\Gamma, A)$ *is a closed subset of the spectral space* $\mathbb{L}(Z^1(\Gamma, A))$.

(2) $\mathcal{K}(\Gamma, A)$ *has a natural structure of spectral (Stone)* Γ*-space.*

(3) *For any* $G \in \mathcal{K}(\Gamma, A)$ *there exists a maximal Kneser group lying over* G.

(4) *The set* $\mathcal{K}(\Gamma, A)_{\max}$ *of all maximal Kneser subgroups of* $Z^1(\Gamma, A)$ *has a natural structure of Hausdorff* Γ*-space.*

Proof. (1) Let $G \in \mathbb{L}(Z^1(\Gamma, A)) \setminus \mathcal{K}(\Gamma, A)$. By Proposition 1.7, there exists a finite subgroup F of G such that $F \notin \mathcal{K}(\Gamma, A)$. Thus $G \in \mathcal{U}_F$ and $\mathcal{U}_F \cap \mathcal{K}(\Gamma, A) = \varnothing$, so $\mathcal{K}(\Gamma, A)$ is τ_s-closed, as desired.

(2) As a closed subset of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$, $\mathcal{K}(\Gamma, A)$ is a spectral space with respect to the induced topology, and hence also a Stone space with respect to the topology induced by the topology τ_b of $\mathbb{L}(Z^1(\Gamma, A))$. By Proposition 0.3, it remains to check that $\mathcal{K}(\Gamma, A)$ is stable under the action of Γ. Assuming that $G \in \mathcal{K}(\Gamma, A)$, i.e., the cocycle $\eta_G : \Gamma \longrightarrow \widehat{G}$ is onto, let $\sigma \in \Gamma$ and $\varphi \in \widehat{\sigma G}$. We have to show that $\varphi = \eta_{\sigma G}(\tau)$ for some $\tau \in \Gamma$ to conclude that $\eta_{\sigma G} : \Gamma \longrightarrow \widehat{\sigma G}$ is also surjective, i.e., $\sigma G \in \mathcal{K}(\Gamma, A)$. Let $\psi \in \widehat{G}$ defined by $\psi(g) = \sigma^{-1}\varphi(\sigma g)$ for all $g \in G$. Then $\psi = \eta_G(\rho)$ for some $\rho \in \Gamma$. Consequently, $\varphi(\sigma g) = \sigma\psi(g) = \sigma\eta_G(\rho)(g) = \sigma g(\rho) = (\sigma g)(\sigma\rho\sigma^{-1}) = \eta_{\sigma G}(\sigma\rho\sigma^{-1})$ for all $g \in G$, and hence $\varphi = \eta_{\sigma G}(\tau)$, where $\tau = \sigma\rho\sigma^{-1}$.

(3) follows at once by Proposition 1.7 and Zorn's lemma.

(4) Let $G_i \in \mathcal{K}(\Gamma, A)_{\max}$, $i = 1, 2$, be such that $G_1 \neq G_2$. Then, there exist $F_i \leqslant G_i$, $i = 1, 2$, such that F_1 and F_2 are both finite and $F_1 + F_2 \notin \mathcal{K}(\Gamma, A)$, since otherwise it would follow by Lemma 1.4 that any finite subgroup of $G_1 + G_2$ is Kneser; then $G_1 + G_2 \in \mathcal{K}(\Gamma, A)$ by Proposition 1.7, contrary to the maximality of the Kneser groups G_1 and G_2. For such a pair (F_1, F_2), we have $G_i \in \mathcal{U}_{F_i}$ for $i = 1, 2$, and $\mathcal{U}_{F_1} \cap \mathcal{U}_{F_2} \cap \mathcal{K}(\Gamma, A) = \varnothing$, so $\mathcal{K}(\Gamma, A)_{\max}$ is a Hausdorff space with respect to the topology induced from the spectral space $\mathcal{K}(\Gamma, A)$. Finally, note that $\mathcal{K}(\Gamma, A)_{\max}$ is stable under the continuous action of Γ on $\mathcal{K}(\Gamma, A)$. $\square$

Corollary 1.9. $Z^1(\Gamma, A^\Gamma) = \mathrm{Hom}(\Gamma, A^\Gamma) \in \mathcal{K}(\Gamma, A)$. *In particular, if the action of* Γ *on* A *is trivial, then* $Z^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A) \in \mathcal{K}(\Gamma, A)$.

Proof. By Proposition 1.7, we have to show that $G \in \mathcal{K}(\Gamma, A)$ for any finite subgroup G of $Z^1(\Gamma, A^\Gamma) = \mathrm{Hom}(\Gamma, A^\Gamma)$. For any such G it follows that $G^\perp = \bigcap_{g\in G} \mathrm{Ker}(g)$ is an open normal subgroup of Γ, the quotient $\Gamma/G^\perp$ is a finite Abelian group, and G can be embedded into $\mathrm{Hom}(\Gamma/G^\perp, A^\Gamma) \leqslant \mathrm{Hom}(\Gamma/G^\perp, \mathbb{Q}/\mathbb{Z}) = \widehat{\Gamma/G^\perp} \cong \Gamma/G^\perp$. Consequently, by Lemma 1.1, $(\Gamma : G^\perp) \leqslant |G| \leqslant (\Gamma : G^\perp)$, so $G \in \mathcal{K}(\Gamma, A)$ by Lemma 1.3. $\square$

Denote by $\mathcal{K}^+(\Gamma, A)$ the subset of $\mathcal{K}(\Gamma, A)$ consisting of all Kneser groups G which additionally are closed elements of the canonical Galois connection described in Proposition 0.1 (1), i.e., $G = G^{\perp\perp}$. The main properties of these groups are collected together in the next result.

Corollary 1.10. *The following assertions hold.*

(1) $G \in \mathcal{K}^+(\Gamma, A)$ *if and only if* $G^{\perp\perp} \in \mathcal{K}(\Gamma, A)$.

(2) $\mathcal{K}^+(\Gamma, A)$ *is a lower subset of the poset* $\mathbb{L}(Z^1(\Gamma, A))$.

(3) $G \in \mathcal{K}^+(\Gamma, A)$ *if and only if* $F \in \mathcal{K}^+(\Gamma, A)$ *for every finite subgroup* F *of* G.

(4) $\mathcal{K}^+(\Gamma, A)$ *is a closed subset of the spectral space* $\mathcal{K}(\Gamma, A)$.

(5) $\mathcal{K}^+(\Gamma, A)$ *inherits from* $\mathcal{K}(\Gamma, A)$ *a natural structure of spectral (Stone)* Γ*-space.*

Proof. (1) One implication is trivial, while the other one follows at once from Lemma 1.5.

(2) is an immediate consequence of (1) and Lemma 1.4.

(3) By (2), it remains to prove that if $F \in \mathcal{K}^+(\Gamma, A)$ for every finite subgroup F of G, then $G \in \mathcal{K}^+(\Gamma, A)$. By Proposition 1.7, $G \in \mathcal{K}(\Gamma, A)$, and hence we have only to show that $G = G^{\perp\perp}$. For any $g \in G^{\perp\perp}$, we have $G^\perp = (G^{\perp\perp})^\perp \leqslant g^\perp$, therefore, by compactness, there exists a finite subgroup F of G such that $F^\perp \leqslant g^\perp$. Consequently, $g \in g^{\perp\perp} \leqslant F^{\perp\perp} = F \leqslant G$, as desired.

(4) follows at once from (3), while (5) is a consequence of (4), Corollary 1.8, (2), and of the fact that $\sigma G^{\perp\perp} = (\sigma G)^{\perp\perp}$ for any $G \in \mathbb{L}(Z^1(\Gamma, A))$ and $\sigma \in \Gamma$. $\square$

Proposition 1.11. *Let* $G \in \mathcal{K}(\Gamma, A)$, $\Delta \in \overline{\mathbb{L}}(\Gamma)$, *and denote* $\widetilde{G} = \mathrm{res}^\Gamma_\Delta(G)$, $G' = \Delta^\perp \cap G$. *Then, the following assertions are equivalent.*

(1) $\widetilde{G} \in \mathcal{K}(\Delta, A)$.

(2) *The inclusion map* $\Delta \hookrightarrow G'^\perp$ *induces a continuous surjection* $\Delta \longrightarrow G'^\perp/G^\perp$.

(3) *The inclusion map* $\Delta \hookrightarrow G'^\perp$ *induces a homeomorphism* $\Delta/\widetilde{G}^\perp \longrightarrow G'^\perp/G^\perp$.

(4) $G'^\perp = \Delta G^\perp$.

Proof. By assumption, $G \in \mathcal{K}(\Gamma, A)$, so $G' \in \mathcal{K}(\Gamma, A)$ since G' is a subgroup of G and $\mathcal{K}(\Gamma, A)$ is a lower subset of the poset $\mathbb{L}(Z^1(\Gamma, A))$ by Lemma 1.4. Note that $\Delta \leqslant \Delta^{\perp\perp} \leqslant G'^{\perp}$ and $\widetilde{G}^{\perp} = G^{\perp} \cap \Delta$; hence, the canonical map $\Delta/\widetilde{G}^{\perp} \longrightarrow G'^{\perp}/G^{\perp}$ is injective, and so, (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4).

Observe that the morphism $\mathrm{res}^{\Gamma}_{\Delta} : Z^1(\Gamma, A) \longrightarrow Z^1(\Delta, A)$ induces an epimorphism $G \longrightarrow \widetilde{G}$ with kernel $\mathrm{Ker}\,(\mathrm{res}^{\Gamma}_{\Delta}) \cap G = \Delta^{\perp} \cap G = G'$, and hence, an isomorphism $G/G' \cong \widetilde{G}$, and, by the Pontryagin Duality, a continuous isomorphism $\widehat{G}/\widehat{\widetilde{G}} \cong \widehat{G'}$. The canonical continuous cocycle $\eta_G : \Gamma \longrightarrow \widehat{G}$ is onto by assumption, so it induces a homeomorphism of Stone spaces $G'^{\perp}/G^{\perp} \longrightarrow \widehat{\widetilde{G}}$ whose restriction to the closed subspace $\Delta/\widetilde{G}^{\perp} \cong \Delta G^{\perp}/G^{\perp}$ is the injective continuous map induced by the continuous cocycle $\eta_{\widetilde{G}} : \Delta \longrightarrow \widehat{\widetilde{G}}$. Now it follows at once that $\widetilde{G} \in \mathcal{K}(\Delta, A)$, i.e., $\eta_{\widetilde{G}}$ is onto, if and only if the embedding $\Delta/\widetilde{G}^{\perp} \hookrightarrow G'^{\perp}/G^{\perp}$ is onto too, which proves the equivalence (1) $\Longleftrightarrow$ (3). □

Corollary 1.12. *Let $G \in \mathcal{K}(\Gamma, A)$, and let $\Delta \in \overline{\mathbb{L}}(\Gamma)$ be such that $G^{\perp} \subseteq \Delta$. Then $\mathrm{res}^{\Gamma}_{\Delta}(G) \in \mathcal{K}(\Delta, A)$ if and only if $(\Delta^{\perp} \cap G)^{\perp} = \Delta$.* □

Proposition 1.13. *Let $G \leqslant Z^1(\Gamma, A)$, $\Delta \in \overline{\mathbb{L}}(\Gamma)$, $\widetilde{G} = \mathrm{res}^{\Gamma}_{\Delta}(G)$, and $H = \Delta^{\perp} \cap G$. If $\widetilde{G} \in \mathcal{K}(\Delta, A)$ and $H \in \mathcal{K}(\Gamma, A)$, then $G \in \mathcal{K}(\Gamma, A)$.*

Proof. Assuming that $\widetilde{G} \in \mathcal{K}(\Delta, A)$ and $H \in \mathcal{K}(\Gamma, A)$, we have to show that the cocycle $\eta_G : \Gamma \longrightarrow \widehat{G}$ is onto. Note that $G/H \cong \widetilde{G}$, so $\widehat{G}/\widehat{\widetilde{G}} \cong \widehat{H}$ by the Pontryagin Duality. Let $\varphi \in \widehat{G}$. As $H \in \mathcal{K}(\Gamma, A)$, there exists $\tau \in \Gamma$ such that $\eta_H(\tau) = \varphi|_H$, and hence $\psi := \varphi - \eta_G(\tau) \in \widehat{\widetilde{G}}$. Since the canonical map $\mathrm{res}^{G}_{H} : \widehat{G} \longrightarrow \widehat{H}$ is an epimorphism of topological Γ-modules, it follows that its kernel $\widehat{\widetilde{G}}$ is stable under the action of Γ, therefore $\tau^{-1}\psi \in \widehat{\widetilde{G}}$. As $\widetilde{G} \in \mathcal{K}(\Delta, A)$, there exists $\delta \in \Delta$ such that $\eta_{\widetilde{G}}(\delta) = \tau^{-1}\psi$. Setting $\sigma = \tau\delta$, it follows that for any $g \in G$,

$$\begin{aligned}\eta_G(\sigma)(g) - g(\tau) &= g(\sigma) - g(\tau) = \tau g(\delta) = \tau\eta_{\widetilde{G}}(\delta)(g|_{\Delta}) = \tau(\tau^{-1}\psi)(g|_{\Delta}) = \psi(g|_{\Delta}) \\ &= \varphi(g) - \eta_G(\tau)(g) = \varphi(g) - g(\tau),\end{aligned}$$

and hence $\varphi = \eta_G(\sigma)$, as desired. □

The next results investigate when an internal direct sum of Kneser subgroups of a given $G \leqslant Z^1(\Gamma, A)$ is also Kneser.

Proposition 1.14. *Let $G \leqslant Z^1(\Gamma, A)$, and assume that G is an internal direct sum of a finite family $(G_i)_{1\leqslant i\leqslant n}$ of finite subgroups. If $\gcd(|G_i|, |G_j|) = 1$ for all $i \neq j$ in $\{1, \ldots, n\}$, then*

$$G \in \mathcal{K}(\Gamma, A) \Longleftrightarrow G_i \in \mathcal{K}(\Gamma, A),\ \forall i,\ 1 \leqslant i \leqslant n.$$

Proof. Assume that every G_i is a Kneser group of $Z^1(\Gamma, A)$. Then,

$$|G| = \prod_{1\leqslant i\leqslant n} |G_i| = \prod_{1\leqslant i\leqslant n} (\Gamma : G_i^{\perp}).$$

Since $G^\perp \leqslant G_i^\perp$, it follows that $(\Gamma : G_i^\perp) \,|\, (\Gamma : G^\perp)$ for all $i = 1, \ldots, n$. But $(\Gamma : G_i^\perp) = |G_i|$ are mutually relatively prime by hypothesis, hence $\prod_{1 \leqslant i \leqslant n} (\Gamma : G_i^\perp) \,|\, (\Gamma : G^\perp)$, and so, $|G| \,|\, (\Gamma : G^\perp)$. On the other hand, $(\Gamma : G^\perp) \leqslant |G|$ by Lemma 1.1, which implies that $|G| = (\Gamma : G^\perp)$, i.e., G is a Kneser group.

The implication "$\Longrightarrow$" follows at once from Lemma 1.4. □

Remark 1.15. In general, an internal direct sum of two arbitrary nonzero Kneser subgroups of $Z^1(\Gamma, A)$ is not necessarily Kneser, as the following example shows. Let $\Gamma = \mathbb{D}_6 = \langle\, \sigma, \tau \,|\, \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1 \,\rangle$, and let $A = (1/3)\mathbb{Z}/\mathbb{Z}$ with the action defined by $\sigma a = -a$, $\tau a = a$ for $a \in A$. The map $Z^1(\Gamma, A) \longrightarrow A \times A$, $g \mapsto (g(\sigma), g(\tau))$ is a group isomorphism. Let $g, h \in Z^1(\Gamma, A)$ be defined by $g(\sigma) = 0$, $h(\sigma) = \widehat{1/3}$, $g(\tau) = h(\tau) = \widehat{1/3}$. Then, it is easily verified that $Z^1(\Gamma, A)$ has two independent Kneser subgroups of order 3, namely, $G := \langle g \rangle$ and $H := \langle h \rangle$, whose (internal direct) sum is not Kneser since $|\Gamma| = 6 < 9 = |G \oplus H|$. □

The next result is the *local-global principle* for Kneser groups.

Corollary 1.16. *A subgroup G of $Z^1(\Gamma, A)$ is a Kneser group if and only if any of its p-primary components $G(p)$ is a Kneser group.*

Proof. For the nontrivial implication, assume that $G(p) \in \mathcal{K}(\Gamma, A)$ for every $p \in \mathbb{P}$. By Proposition 1.7, we have to prove that any finite subgroup H of G is Kneser. Then $H(p) = G \cap G(p)$, so $H(p)$ is a Kneser group of $Z^1(\Gamma, A)$ for every $p \in \mathbb{P}$. If $\mathbb{I} := \{ p \in \mathbb{P} \,|\, H(p) \neq 0 \}$, then $H = \bigoplus_{p \in \mathbb{I}} H(p)$. Now, observe that $\mathbb{I}$ is finite and $\gcd(|H(p)|, |H(q)|) = 1$ for all $p \neq q$ in $\mathbb{I}$. Hence H is a Kneser group by Proposition 1.14. □

Corollary 1.16. can be reformulated in topological terms as follows.

Corollary 1.17. *The canonical isomorphism of spectral (Stone) Γ-spaces*

$$\mathbb{L}(Z^1(\Gamma, A)) \xrightarrow{\sim} \prod_{p \in \mathbb{P}} \mathbb{L}(Z^1(\Gamma, A(p)), \; G \mapsto (G(p))_{p \in \mathbb{P}},$$

induces by restriction the isomorphisms of spectral (Stone) Γ-spaces

$$\mathcal{K}(\Gamma, A) \xrightarrow{\sim} \prod_{p \in \mathbb{P}} \mathcal{K}(\Gamma, A(p))$$

and

$$\mathcal{K}^+(\Gamma, A) \xrightarrow{\sim} \prod_{p \in \mathbb{P}} \mathcal{K}^+(\Gamma, A(p)).$$

□

We are now going to present the main result of this section, namely an abstract version of the Kneser Criterion [13] from Field Theory. To do that, we need some basic notation which will be used in the sequel.

Let $\mathcal{N}(\Gamma, A)$ denote the τ_s-open set (possibly empty) $\mathbb{L}(Z^1(\Gamma, A)) \setminus \mathcal{K}(\Gamma, A)$ of all subgroups of $Z^1(\Gamma, A)$ which are not Kneser groups. Clearly, for any $G \in \mathcal{N}(\Gamma, A)$ there exists at least one minimal member H of $\mathcal{N}(\Gamma, A)$ such that $H \subseteq G$. By $\mathcal{N}(\Gamma, A)_{\min}$ we shall denote the set of all minimal members of $\mathcal{N}(\Gamma, A)$. By Proposition 1.7 and Corollary 1.16, if $G \in \mathcal{N}(\Gamma, A)_{\min}$, then necessarily G is a nontrivial finite p-group for some prime number p.

If p is an odd prime number and $\widehat{1/p} \in A \setminus A^{\Gamma}$, define the 1-coboundary

$$\varepsilon_p \in B^1(\Gamma, (1/p)\,\mathbb{Z}/\mathbb{Z}) \leqslant B^1(\Gamma, A)$$

by

$$\varepsilon_p(\sigma) = \sigma\,\widehat{1/p} - \widehat{1/p},\ \sigma \in \Gamma.$$

If $\widehat{1/4} \in A \setminus A^{\Gamma}$, define the map

$$\varepsilon_4' : \Gamma \longrightarrow (1/4)\,\mathbb{Z}/\mathbb{Z}$$

by

$$\varepsilon_4'(\sigma) = \begin{cases} \widehat{1/4} & \text{if } \ \sigma\,\widehat{1/4} = -\,\widehat{1/4} \\ \widehat{0} & \text{if } \ \sigma\,\widehat{1/4} = \widehat{1/4} \end{cases}$$

It is easily checked that

$$\varepsilon_4' \in Z^1(\Gamma, (1/4)\,\mathbb{Z}/\mathbb{Z}) \leqslant Z^1(\Gamma, A).$$

Observe that ε_4' has order 4 and $\varepsilon_4 := 2\,\varepsilon_4'$ is the generator of the cyclic group

$$B^1(\Gamma, (1/4)\,\mathbb{Z}/\mathbb{Z}) \leqslant \mathrm{Hom}(\Gamma, A[2])$$

of order 2.

Recall that by $\mathbb{P}$ we have denoted the set of all positive prime numbers. In the sequel we shall use the following notation:

$$\begin{aligned} \mathcal{P} &= (\mathbb{P} \setminus \{2\}) \cup \{4\}, \\ \mathcal{P}(\Gamma, A) &= \{p \in \mathcal{P} \,|\, \widehat{1/p} \in A \setminus A^{\Gamma}\}. \end{aligned}$$

We shall also use the following notation:

$$B_p = B^1(\Gamma, (1/p)\,\mathbb{Z}/\mathbb{Z}) = B^1(\Gamma, A[p]) = \langle \varepsilon_p \rangle \cong \mathbb{Z}/p\mathbb{Z} \quad \text{if} \quad 4 \neq p \in \mathcal{P}(\Gamma, A),$$
$$B_4 = \langle \varepsilon_4' \rangle \cong \mathbb{Z}/4\mathbb{Z} \ \text{ if } \ 4 \in \mathcal{P}(\Gamma, A).$$

Recall that we have denoted $\mathcal{O}_G := \{\mathrm{ord}(g) \,|\, g \in G\}$. For any $G \leqslant Z^1(\Gamma, A)$ we shall denote

$$\mu_G = \bigcup_{n \in \mathcal{O}_G} (1/n)\mathbb{Z}/\mathbb{Z}.$$

Observe that, since $\mathcal{O}_G$ is a directed set with respect to the divisibility relation, μ_G is a subgroup of A, and hence a discrete Γ-submodule of A too. One easily checks that μ_G is the subgroup $\sum_{g \in G} g(\Gamma)$ of $\mathbb{Q}/\mathbb{Z}$ generated by $\bigcup_{g \in G} g(\Gamma)$, and hence it is the smallest subgroup B of A for which $G \leqslant Z^1(\Gamma, B)$. Also note that $\mu_G(p) = \mu_{G(p)} = \bigcup_{g \in G(p)} g(\Gamma)$ for all $p \in \mathbb{P}$.

Lemma 1.18. *With the notation above, we have* $\mathcal{N}(\Gamma, A)_{\min} = \{ B_p \mid p \in \mathcal{P}(\Gamma, A) \}$.

Proof. If $4 \neq p \in \mathcal{P}(\Gamma, A)$, then $B_p^{\perp} = \varepsilon_p^{\perp} = \{\sigma \in \Gamma \mid \sigma\widehat{1/p} = \widehat{1/p}\}$ is the kernel of the (nontrivial) action of Γ on $A[p] = (1/p)\mathbb{Z}/\mathbb{Z}$, so $\Gamma/B_p^{\perp}$ is identified with a nontrivial subgroup of $(\mathbb{Z}/p\mathbb{Z})^* = \mathbb{F}_p^*$. Thus $(\Gamma : B_p^{\perp}) \mid p-1 < p = |B_p|$, and hence $B_p \in \mathcal{N}(\Gamma, A)_{\min}$.

If $4 \in \mathcal{P}(\Gamma, A)$ then $B_4^{\perp} = \varepsilon_4'^{\perp} = \{\sigma \in \Gamma \mid \sigma\widehat{1/4} = \widehat{1/4}\} = \varepsilon_4^{\perp}$ is the kernel of the (nontrivial) action of Γ on $A[4] = (1/4)\mathbb{Z}/\mathbb{Z}$, so $(\Gamma : B_4^{\perp}) = 2 < 4 = |B_4|$, and hence $B_4 \in \mathcal{N}(\Gamma, A)$. Since the unique proper subgroup of B_4, namely $B^1(\Gamma, A[4]) = \langle \varepsilon_4 \rangle \cong \mathbb{Z}/2\mathbb{Z}$, belongs to $\mathcal{K}(\Gamma, A)$ as $(\Gamma : \varepsilon_4^{\perp}) = 2 = \mathrm{ord}(\varepsilon_4)$, it follows that $B_4 \in \mathcal{N}(\Gamma, A)_{\min}$.

Thus, we proved the inclusion $\{ B_p \mid p \in \mathcal{P}(\Gamma, A) \} \subseteq \mathcal{N}(\Gamma, A)_{\min}$. To prove the opposite inclusion, let $G \in \mathcal{N}(\Gamma, A)_{\min}$. As we have already noticed, G is a nontrivial finite p-group for some $p \in \mathbb{P}$. Then $\mu_G = (1/p^n)\mathbb{Z}/\mathbb{Z}$ for some $n \geqslant 1$, $G \leqslant Z^1(\Gamma, \mu_G)$, and there exist $g \in G$ and $\sigma \in \Gamma$ such that $g(\sigma) = \widehat{1/p^n}$.

Obviously, $G \in \mathcal{N}(\Gamma, \mu_G)_{\min}$, so we may assume from the beginning that $A = \mu_G = (1/p^n)\mathbb{Z}/\mathbb{Z}$. Let $\Delta := B^1(\Gamma, A)^{\perp}$ denote the kernel of the action of Γ on A. We claim that $\Delta \subseteq G^{\perp}$, i.e., $\widetilde{G} := \mathrm{res}_{\Delta}^{\Gamma}(G) = \{0\}$. In particular, this will imply that $n \geqslant 2$ for $p = 2$, for otherwise, if $n = 1$ and $p = 2$ we would have $\Delta = \Gamma = G^{\perp}$, and hence $G \leqslant G^{\perp\perp} = \Gamma^{\perp} = \{0\}$, which is a contradiction.

Assume the contrary, i.e., $\Delta \nsubseteq G^{\perp}$. Then $\Delta^{\perp} \cap G \neq G$, and hence $\Delta^{\perp} \cap G \in \mathcal{K}(\Gamma, A)$ as $G \in \mathcal{N}(\Gamma, A)_{\min}$. On the other hand, $\widetilde{G} \leqslant Z^1(\Delta, A) = \mathrm{Hom}(\Delta, A)$, so $\widetilde{G} \in \mathcal{K}(\Delta, A)$ by Corollary 1.9, and hence $G \in \mathcal{K}(\Gamma, A)$ by Proposition 1.13, contrary to our assumption. This proves the claim that $\Delta \leqslant G^{\perp}$.

Thus G can be identified with a subgroup of $Z^1(\Gamma/\Delta, A)$, and moreover $G \in \mathcal{N}(\Gamma/\Delta, A)_{\min}$, so we may assume without loss of generality that Γ is a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^*$ acting (faithfully) by multiplication on $A := (1/p^n)\mathbb{Z}/\mathbb{Z}$, $G \in \mathcal{N}(\Gamma, A)_{\min}$, and $\mu_G = A$, i.e., $g(\tau) = \widehat{1/p^n}$ for some $g \in G$ and $\tau \in \Gamma$. Recall that $n \geqslant 1$ for $p \neq 2$, and $n \geqslant 2$ for $p = 2$.

First, note that G is cyclic of order p^n, generated by g. Indeed, assuming the contrary, it follows that the proper subgroup G' of G generated by g is a Kneser group of $Z^1(\Gamma, A)$ since $G \in \mathcal{N}(\Gamma, A)_{\min}$, so

$$p^n = |G'| = (\Gamma : G'^{\perp}) \leqslant |\Gamma| \leqslant \varphi(p^n) = p^{n-1}(p-1),$$

which is a contradiction. By the same reason it follows that the subgroup pG, properly contained in G, is a Kneser group of $Z^1(\Gamma, A)$, hence $(\Gamma : (pG)^{\perp}) = |pG| = p^{n-1}$. This implies that $p^{n-1} \mid |\Gamma|$ and $|(pG)^{\perp}| = (|\Gamma| : p^{n-1}) \mid (\varphi(p^n) : p^{n-1})$, and so, $t := |(pG)^{\perp}|$ is a divisor of $p-1$.

Recall that for any integers k and m we denote by $k \bmod m$ the congruence class $k + m\mathbb{Z}$ of k modulo m. Set

$$\Gamma' = \begin{cases} \{ k \bmod p^n \in (\mathbb{Z}/p^n\mathbb{Z})^* \mid k \in \mathbb{Z},\ k \equiv 1 \pmod{p} \} & \text{if } p \neq 2 \text{ and } n \geqslant 1, \\ \{ k \bmod p^n \in (\mathbb{Z}/2^n\mathbb{Z})^* \mid k \in \mathbb{Z},\ k \equiv 1 \pmod{4} \} & \text{if } p = 2 \text{ and } n \geqslant 2. \end{cases}$$

Using the considerations above, it follows that, if $p \neq 2$, then

$$\Gamma \cong \Gamma' \times (pG)^{\perp} \text{ is cyclic of order } p^{n-1}t, \text{ with } t \mid (p-1),$$

and if $p=2$, then $G^{\perp}=(2G)^{\perp}=\{1\}$ and

$$\Gamma=(\mathbb{Z}/2^n\mathbb{Z})^*\cong\Gamma'\times\{1 \bmod 2^n, -1 \bmod 2^n\}\cong\mathbb{Z}/2^{n-2}\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}.$$

Observe that if $\Gamma'=\{1\}$, then, for $p\neq 2$,

$$\Gamma\cong\mathbb{Z}/t\mathbb{Z} \text{ is a nontrivial subgroup of } \mathbb{F}_p^*,\ G=Z^1(\Gamma,\mathbb{F}_p)=B^1(\Gamma,\mathbb{F}_p)=B_p,$$

while, for $p=2$,

$$G=Z^1((\mathbb{Z}/4\mathbb{Z})^*,\mathbb{Z}/4\mathbb{Z})=B_4=\langle\varepsilon_4'\rangle,$$

as desired.

Now assume that $\Gamma'\neq\{1\}$, i.e., $n\geqslant 2$ for $p\neq 2$, and $n\geqslant 3$ for $p=2$. Set $\widetilde{G'}=\mathrm{res}^{\Gamma}_{\Gamma'}(G)=\langle g|_{\Gamma'}\rangle$. Note that $\widetilde{G'}^{\perp}=G^{\perp}\cap\Gamma'=\{1\}$ since $G^{\perp}\cap\Gamma'\subseteq(pG)^{\perp}\cap\Gamma'=\{1\}$ for $p\neq 2$, and $G^{\perp}=\{1\}$ for $p=2$. As $1<|\Gamma'|=(\Gamma':\widetilde{G'}^{\perp})\leqslant|\widetilde{G'}|$, it follows that $\widetilde{G'}\neq\{0\}$ and $\Gamma'^{\perp}\cap G\neq G$; hence $\Gamma'^{\perp}\cap G\in\mathcal{K}(\Gamma,A)$. By Proposition 1.13, it follows that $\widetilde{G'}\in\mathcal{N}(\Gamma',A)$, i.e., $p^{n-1}=(\Gamma':\widetilde{G'}^{\perp})<|\widetilde{G'}|\,|G|=p^n$ if $p\neq 2$, and $2^{n-2}=(\Gamma':\widetilde{G'}^{\perp})<|\widetilde{G'}|\,|G|=2^n$ if $p=2$. Consequently, for $p\neq 2$ we have $\widetilde{G'}\cong G\cong\mathbb{Z}/p^n\mathbb{Z}$, and for $p=2$ we have either $\widetilde{G'}\cong\mathbb{Z}/2^{n-1}\mathbb{Z}$ or $\widetilde{G'}\cong G\cong\mathbb{Z}/2^n\mathbb{Z}$. Thus we arrived to a contradiction since

$$Z^1(\Gamma',A)=B^1(\Gamma',A)\cong\begin{cases}\mathbb{Z}/p^{n-1}\mathbb{Z} & \text{if } p\neq 2,\\ \mathbb{Z}/2^{n-2}\mathbb{Z} & \text{if } p=2.\end{cases}$$

Indeed, let

$$\sigma=\begin{cases}(1+p) \bmod p^n & \text{if } p\neq 2\\ 5 \bmod 2^n & \text{if } p=2\end{cases}$$

be the canonical generator of the cyclic group Γ'. The injective group morphism

$$Z^1(\Gamma',A)\longrightarrow A,\ h\mapsto h(\sigma),$$

maps $Z^1(\Gamma',A)$ onto $\mathrm{Ker}(N)$ and $B^1(\Gamma',A)$ onto $T(A)$, where $N:A\longrightarrow A$ is the norm sending $a\in A=(1/p^n)\mathbb{Z}/\mathbb{Z}$ to $\widetilde{N}a$,

$$\widetilde{N}=\begin{cases}\displaystyle\sum_{i=0}^{p^{n-1}-1}(1+p)^i & \text{if } p\neq 2,\\ \displaystyle\sum_{i=0}^{2^{n-2}-1}5^i & \text{if } p=2,\end{cases}$$

and

$$T:A\longrightarrow A,\ a\mapsto\sigma a-a=\begin{cases}p\,a & \text{if } p\neq 2,\\ 4\,a & \text{if } p=2.\end{cases}$$

Now, it is easily checked by induction that the p-adic valuation of the natural number $\widetilde{N}$ is $n-1$ for $p\neq 2$ and $n-2$ for $p=2$. This implies that

$$\mathrm{Ker}(N)=T(A)=\begin{cases}pA\cong\mathbb{Z}/p^{n-1}\mathbb{Z} & \text{if } p\neq 2,\\ 4A\cong\mathbb{Z}/2^{n-2}\mathbb{Z} & \text{if } p=2,\end{cases}$$

as desired. $\square$

Remark 1.19. Lemma 1.18 provides a precise description of the open subset $\mathcal{N}(\Gamma, A)$ of the spectral space $\mathbb{L}(Z^1(\Gamma, A))$ as the union of the basic quasi-compact open sets $\mathcal{U}_{B_p}$ for p ranging over $\mathcal{P}(\Gamma, A)$. $\square$

The next statement, which is an equivalent form of Lemma 1.18, is actually an abstract version of the Kneser Criterion [13] from the field theoretic Cogalois Theory. Note that the place of the primitive p-th roots of unity ζ_p, p odd prime, from the Kneser Criterion [13] is taken in its abstract version by ε_p, while ε'_4 corresponds to $1 - \zeta_4$.

Theorem 1.20. (The Abstract Kneser Criterion). *The following assertions are equivalent for $G \leqslant Z^1(\Gamma, A)$.*

(1) *G is a Kneser group of $Z^1(\Gamma, A)$.*

(2) *$\varepsilon_p \notin G$ whenever $4 \neq p \in \mathcal{P}(\Gamma, A)$ and $\varepsilon'_4 \notin G$ whenever $4 \in \mathcal{P}(\Gamma, A)$.*

Proof. (1) $\Longrightarrow$ (2): Assume that $G \in \mathcal{K}(\Gamma, A)$. If $\varepsilon_p \in G$ for some $4 \neq p \in \mathcal{P}(\Gamma, A)$, then $B_p = \langle \varepsilon_p \rangle \leqslant G$, hence $B_p \in \mathcal{K}(\Gamma, A)$, which contradicts Lemma 1.18. Similarly, if $4 \in \mathcal{P}(\Gamma, A)$ and $\varepsilon'_4 \in G$ then $B_4 = \langle \varepsilon'_4 \rangle \leqslant G$, hence $B_4 \in \mathcal{K}(\Gamma, A)$, which again contradicts Lemma 1.18.

(2) $\Longrightarrow$ (1): Assume that $G \notin \mathcal{K}(\Gamma, A)$, i.e., $G \in \mathcal{N}(\Gamma, A)$. Then G contains a minimal member of $\mathcal{N}(\Gamma, A)$, i.e., an element of the set $\mathcal{N}(\Gamma, A)_{\min}$. To conclude, apply now Lemma 1.18. $\square$

Corollary 1.21. *The following assertions are equivalent for $G \leqslant Z^1(\Gamma, A)$.*

(1) *$G \in \mathcal{K}^+(\Gamma, A)$.*

(2) *$G^\perp \nsubseteq \varepsilon_p^\perp$ for all $p \in \mathcal{P}(\Gamma, A)$.*

Proof. By Corollary 1.10 (1), $G \in \mathcal{K}^+(\Gamma, A) \Longleftrightarrow G^{\perp\perp} \in \mathcal{K}(\Gamma, A)$. On the other hand, by Theorem 1.20, $G^{\perp\perp} \in \mathcal{K}(\Gamma, A)$ if and only if $\varepsilon_p \notin G^{\perp\perp}$ whenever $4 \neq p \in \mathcal{P}(\Gamma, A)$ and $\varepsilon'_4 \notin G^{\perp\perp}$ whenever $4 \in \mathcal{P}(\Gamma, A)$. Since $\varepsilon_4^\perp = \varepsilon_4'^\perp = B_4^\perp$ if $4 \in \mathcal{P}(\Gamma, A)$, and $\varepsilon \in G^{\perp\perp} \Longleftrightarrow G^\perp \subseteq \varepsilon^\perp$, the result follows. $\square$

Corollary 1.22. *$Z^1(\Gamma, A)$ is a Kneser group of itself if and only if $\mathcal{P}(\Gamma, A) = \varnothing$, i.e. $A[p] \subseteq A^\Gamma$ for all $p \in \mathcal{P}$.* $\square$

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