INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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Preprint nr. 9/2004

BUCUREȘTI

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February, 2004

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Abstract

This is the second part of a series of papers which aim to develop an abstract group theoretic framework for the Cogalois Theory of field extensions.

2000 *Mathematics Subject Classification*: 20E18, 12G05, 12F10, 12F99, 06A15, 06E15.

Key words and phrases: Profinite group, continuous 1-cocycle, Abstract Galois Theory, Abstract Kummer Theory, Abstract Cogalois Theory, Kneser group of cocycles, Cogalois group of cocycles, Stone space, spectral space, coherent map.

2 Cogalois groups of cocycles

In this section we define the concept of abstract Cogalois group and establish various equivalent characterizations for such groups, including a *Quasi-Purity Criterion*, an abstract version of the structure theorem for Kneser groups from the field theoretic

*The second author gratefully acknowledges partial support from the Grant 47/2002 awarded by the Romanian Academy and the Programme CERES 152/2001.

Cogalois theory, and an analogue of Theorem 1.20 (the abstract Kneser criterion) for Cogalois groups.

For a given subgroup G of $Z^1(\Gamma, A)$, the lattice $\mathbb{L}(G)$ of all subgroups of G and the lattice $\overline{\mathbb{L}}(\Gamma|G^\perp)$ of all closed subgroups of Γ lying over $G^\perp$ are related through the canonical order-reversing maps $H \mapsto H^\perp$ and $\Delta \mapsto G \cap \Delta^\perp = G \cap \mathrm{Ker}\,(\mathrm{res}^\Gamma_\Delta)$. Clearly, these two maps establish a Galois connection, which is induced by the one considered in Proposition 0.1 (1). Notice also that $\mathbb{L}(G)$ (resp. $\overline{\mathbb{L}}(\Gamma|G^\perp)$) is a closed subspace of the spectral (Stone) space $\mathbb{L}(Z^1(\Gamma, A))$ (resp. $\overline{\mathbb{L}}(\Gamma)$) and the two maps above are continuous by Proposition 0.3.

Definition 2.1. *A subgroup G of $Z^1(\Gamma, A)$ is said to be a* Cogalois group *of $Z^1(\Gamma, A)$ if it is a Kneser group of $Z^1(\Gamma, A)$ and the maps $(-)^\perp : \mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^\perp)$ and $G \cap (-)^\perp : \overline{\mathbb{L}}(\Gamma|G^\perp) \longrightarrow \mathbb{L}(G)$ are lattice anti-isomorphisms, inverse to one another.* □

Some characterizations of Cogalois groups of $Z^1(\Gamma, A)$ are given in the next result.

Proposition 2.2. *The following statements are equivalent for a Kneser group G of $Z^1(\Gamma, A)$.*

(1) $\Delta = (G \cap \Delta^\perp)^\perp$ *for every* $\Delta \in \overline{\mathbb{L}}(\Gamma|G^\perp)$.

(2) $\mathrm{res}^\Gamma_\Delta(G) \in \mathcal{K}(\Delta, A)$ *for every* $\Delta \in \overline{\mathbb{L}}(\Gamma|G^\perp)$.

(3) *The map* $\mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^\perp)$, $H \mapsto H^\perp$, *is onto.*

(4) *The map* $\overline{\mathbb{L}}(\Gamma|G^\perp) \longrightarrow \mathbb{L}(G)$, $\Delta \mapsto G \cap \Delta^\perp$, *is injective.*

(5) *The canonical maps* $\mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^\perp)$ *and* $\overline{\mathbb{L}}(\Gamma|G^\perp) \longrightarrow \mathbb{L}(G)$ *are homeomorphisms of spectral (Stone) spaces inverse to one another.*

(6) *G is a Cogalois group of $Z^1(\Gamma, A)$.*

Proof. (1) $\Longleftrightarrow$ (2) by Corollary 1.12.

(1) $\Longrightarrow$ (3): For any $\Delta \in \overline{\mathbb{L}}(\Gamma|G^\perp)$, we have $\Delta = H^\perp$, where $H = G \cap \Delta^\perp \in \mathbb{L}(G)$.

(3) $\Longrightarrow$ (4): Let $\Delta_1, \Delta_2 \in \overline{\mathbb{L}}(\Gamma|G^\perp)$ be such that $G \cap \Delta_1^\perp = G \cap \Delta_2^\perp$. By assumption, $\Delta_1 = H_1^\perp$, $\Delta_2 = H_2^\perp$ for some $H_1, H_2 \in \mathbb{L}(G)$. By Lemma 1.5, $H_1 = G \cap H_1^{\perp\perp} = G \cap \Delta_1^\perp = G \cap \Delta_2^\perp = G \cap H_2^{\perp\perp} = H_2$, and hence, $\Delta_1 = \Delta_2$, as desired.

(4) $\Longrightarrow$ (5): For any $H \in \mathbb{L}(G)$, we have $G \cap H^{\perp\perp} = H$ by Lemma 1.5, so the composition of the canonical maps $\mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^\perp) \longrightarrow \mathbb{L}(G)$ is the identity. It follows that the map $\Delta \mapsto G \cap \Delta^\perp$ is onto, and hence bijective, with inverse $H \mapsto H^\perp$.

(5) $\Longrightarrow$ (6): As order-reversing maps inverse to one another, the canonical maps above are lattice anti-isomorphisms inverse to one another, as desired.

(6) $\Longrightarrow$ (1): Let $\Delta \in \overline{\mathbb{L}}(\Gamma|G^\perp)$. Then, by assumption, there exists a unique $H \in \mathbb{L}(G)$ such that $\Delta = H^\perp$ and $H = G \cap \Delta^\perp$; hence $\Delta = (G \cap \Delta^\perp)^\perp$, as required. □

Corollary 2.3. *A subgroup G of $Z^1(\Gamma, A)$ is Cogalois if and only if $\mathrm{res}^{\Gamma}_{\Delta}(G)$ is a Kneser group of $Z^1(\Delta, A)$ for every $\Delta \in \overline{\mathbb{L}}(\Gamma|G^{\perp})$.*

In particular, $Z^1(\Gamma, A)$ is a Cogalois group of itself if and only if $Z^1(\Gamma, A)$ is a Kneser group of itself.

Proof. As $\Gamma \in \overline{\mathbb{L}}(\Gamma|G^{\perp})$ for every $G \leqslant Z^1(\Gamma, A)$, and $\mathcal{P}(\Delta, A) \subseteq \mathcal{P}(\Gamma, A)$ for all $\Delta \in \overline{\mathbb{L}}(\Gamma)$, the result follows immediately from Proposition 2.2 and Corollary 1.22. □

Definition 2.4. *A subgroup D of an Abelian group C is said to be* quasi n-pure, *where n is a given positive integer, if $C[n] \subseteq D$, or equivalently $C[n] = D[n]$. For $M \subseteq \mathbb{N}$, C is* quasi M-pure *if C is quasi n-pure for all $n \in M$.*

Recall that a well established concept in Group Theory is that of *n-purity*: a subgroup D of an Abelian group C is said to be *n-pure* if $D \cap nC = nD$. There is no connection between the concepts of n-purity and quasi n-purity; e.g., the subgroup $2\mathbb{Z}/4\mathbb{Z}$ of $\mathbb{Z}/4\mathbb{Z}$ is quasi 2-pure but not 2-pure, and any of the three subgroups of order 2 of the dihedral group $\mathbb{D}_4$ is 2-pure but not quasi 2-pure. Notice that the abstract notion of quasi n-purity goes back to the concept of n-purity from the field theoretic Cogalois Theory (see Albu [1], Albu and Nicolae [6]).

For any subgroup G of $Z^1(\Gamma, A)$ we denote $\mathcal{P}_G := \mathcal{O}_G \cap \mathcal{P}$, i.e., $\mathcal{P}_G$ is the set of those $p \in \mathcal{P}$ for which $\exp(G[p]) = p$.

The quasi $\mathcal{P}_G$-purity plays a basic role in the characterization of Cogalois groups of $Z^1(\Gamma, A)$. The next result is the abstract version of the *General Purity Criterion* [1], Theorem 2.3, from the field theoretic infinite Cogalois Theory.

Theorem 2.5. (The Quasi-Purity Criterion). *The following statements are equivalent for a subgroup G of $Z^1(\Gamma, A)$.*

(1) *G is Cogalois.*

(2) *The subgroup A^{Γ} of $A^{G^{\perp}}$ is quasi $\mathcal{P}_G$-pure.*

(3) *$G^{\perp} \nsubseteq \varepsilon_p^{\perp}$ for all $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$.*

Proof. (2) $\Longrightarrow$ (3): Let $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$. Then $\widehat{1/p} \in A \setminus A^{\Gamma}$, and hence $\widehat{1/p} \notin A^{G^{\perp}}$, as $A^{G^{\perp}}[p] = A^{\Gamma}[p]$ by hypothesis. Consequently, there exists $\sigma \in G^{\perp}$ such that $\sigma\,\widehat{1/p} \neq \widehat{1/p}$, i.e., $\sigma \notin \varepsilon_p^{\perp}$, which shows that $G^{\perp} \nsubseteq \varepsilon_p^{\perp}$, as desired.

(3) $\Longrightarrow$ (2): Let $p \in \mathcal{P}_G$. Then clearly $\widehat{1/p} \in A$. Assuming $\widehat{1/p} \in A^{\Gamma}$, we obtain that $A^{\Gamma}[p] = A^{G^{\perp}}[p] = (1/p)\mathbb{Z}/\mathbb{Z}$, as desired. Now assume that $\widehat{1/p} \notin A^{\Gamma}$. Since $G^{\perp} \nsubseteq \varepsilon_p^{\perp}$ by hypothesis, it follows that $A^{\Gamma}[p] = A^{G^{\perp}}[p] = \{0\}$ for $p \neq 4$, and $A^{\Gamma}[p] = A^{G^{\perp}}[p] = (1/2)\mathbb{Z}/\mathbb{Z}$ for $p = 4$.

(1) $\Longrightarrow$ (3): Suppose that G is Cogalois, and let $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$. Then $\widehat{1/p} \in A \setminus A^{\Gamma}$, and there exists a cocycle $h \in G$ of order p. Let $H \cong \mathbb{Z}/p\mathbb{Z}$ denote the subgroup of G generated by h. Since G is a Kneser group of $Z^1(\Gamma, A)$, $(\Gamma : H^{\perp}) = |H| = p$. Assuming that $G^{\perp} \subseteq \varepsilon_p^{\perp}$, we have to derive a contradiction. We distinguish the following two cases:

Case (i): $p \in \mathbb{P} \setminus \{2\}$. Since $G \in \mathcal{K}(\Gamma, A)$, it follows by Theorem 1.20 that $\varepsilon_p \notin G$. Setting $\alpha := h - \varepsilon_p \in Z^1(\Gamma, (1/p)\mathbb{Z}/\mathbb{Z}) \setminus G$, we deduce that $\mathrm{ord}(\alpha) = p$ and $\langle \varepsilon_p \rangle \cap \langle \alpha \rangle = \{0\}$. Consequently, again by Theorem 1.20, $\langle \alpha \rangle \in \mathcal{K}(\Gamma, A)$, and hence $(\Gamma : \alpha^{\perp}) = p$. Since $G^{\perp} \leqslant h^{\perp}$ and $G^{\perp} \leqslant \varepsilon_p^{\perp}$ by assumption, it follows that $G^{\perp} \leqslant \alpha^{\perp}$. As G is Cogalois, we deduce that $\alpha^{\perp} = (G \cap \alpha^{\perp\perp})^{\perp}$ and $|G \cap \alpha^{\perp\perp}| = (\Gamma : \alpha^{\perp}) = p$, therefore $G \cap \alpha^{\perp\perp} \cong \mathbb{Z}/p\mathbb{Z}$. Now consider the subgroup $H' := H + (G \cap \alpha^{\perp\perp})$ of G. As p is a prime number, it follows that either $H' = H \cong \mathbb{Z}/p\mathbb{Z}$ or $H' = H \oplus (G \cap \alpha^{\perp\perp}) \cong (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$. Since $H' \leqslant G \in \mathcal{K}(\Gamma, A)$, we deduce that $(\Gamma : H'^{\perp}) \in \{p, p^2\}$. This implies that $(\Gamma : \varepsilon_p^{\perp}) \,|\, p^2$ since $H'^{\perp} \leqslant h^{\perp} \cap \alpha^{\perp} \leqslant \varepsilon_p^{\perp}$. On the other hand, $\varepsilon_p^{\perp}$ is the kernel of the (nontrivial) action of Γ on $A[p] = (1/p)\mathbb{Z}/\mathbb{Z}$, and hence $2 \leqslant (\Gamma : \varepsilon_p^{\perp}) \,|\, (p-1)$, which is a contradiction.

Case (ii): $p = 4$. Let $\varepsilon_4' \in Z^1(\Gamma, A[4]) = Z^1(\Gamma, (1/4)\mathbb{Z}/\mathbb{Z})$ be the 1-cocycle defined in Section 1, and remember that $\varepsilon_4 = 2\,\varepsilon_4'$. As $\widehat{1/4} \notin A^{\Gamma}$, the action of Γ on $A[4] = (1/4)\mathbb{Z}/\mathbb{Z}$, whose kernel is $\varepsilon_4^{\perp} = {\varepsilon_4'}^{\perp}$, is nontrivial, and hence $\Gamma/\varepsilon_4^{\perp} \cong (\mathbb{Z}/4\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z}$, i.e., $(\Gamma : \varepsilon_4^{\perp}) = 2$. Since G is Cogalois and $G^{\perp} \leqslant \varepsilon_4^{\perp}$ by assumption, it follows that $\varepsilon_4^{\perp} = (G \cap \varepsilon_4^{\perp\perp})^{\perp}$ and $|G \cap \varepsilon_4^{\perp\perp}| = (\Gamma : \varepsilon_4^{\perp}) = 2$, i.e., $G \cap \varepsilon_4^{\perp\perp} \cong \mathbb{Z}/2\mathbb{Z}$. One easily checks that ε_4 is the unique element of order 2 of $\varepsilon_4^{\perp\perp}$, and hence $G \cap \varepsilon_4^{\perp\perp} = \langle \varepsilon_4 \rangle$, in particular, $\varepsilon_4 \in G$. On the other hand, since $G \in \mathcal{K}(\Gamma, A)$, it follows by Theorem 1.20 that $\varepsilon_4' \notin G$, and hence $h \notin \{\varepsilon_4', -\varepsilon_4'\}$. Set $\beta := h - \varepsilon_4'$ and $H_1 := \langle h, \varepsilon_4 \rangle \leqslant G$. Then $0 \neq \beta \notin \langle \varepsilon_4' \rangle$. Two subcases arise:

Subcase (1): $\varepsilon_4 \in H$. Then $2h = \varepsilon_4$ and $2\beta = 2h - 2\varepsilon_4' = 2h - \varepsilon_4 = 0$, i.e., $\mathrm{ord}(\beta) = 2$. By Lemma 1.1, we have $(\Gamma : \beta^{\perp}) \leqslant |\langle \beta \rangle| = 2$. Observe that $\beta^{\perp} \neq \Gamma$, for otherwise, we would have $0 \neq \beta \in \beta^{\perp\perp} = \Gamma^{\perp} = \{0\}$, which is a contradiction. Thus, $(\Gamma : \beta^{\perp}) = 2$. On the other hand, $G^{\perp} \leqslant H^{\perp} = H^{\perp} \cap \varepsilon_4^{\perp} = h^{\perp} \cap {\varepsilon_4'}^{\perp} \leqslant \beta^{\perp}$, and hence $G \cap \beta^{\perp\perp} \leqslant G \cap H^{\perp\perp} = H$, $\beta^{\perp} = (G \cap \beta^{\perp\perp})^{\perp}$, and $|G \cap \beta^{\perp\perp}| = (\Gamma : \beta^{\perp}) = 2$, as G is Cogalois. Since $\langle \varepsilon_4 \rangle$ is the unique subgroup of order 2 of $H \cong \mathbb{Z}/4\mathbb{Z}$, it follows that $G \cap \beta^{\perp\perp} = \langle \varepsilon_4 \rangle$. Therefore $\beta \in (\beta^{\perp})^{\perp} = ((G \cap \beta^{\perp\perp})^{\perp})^{\perp} = \varepsilon_4^{\perp\perp}$, so $\beta = \varepsilon_4$ since $\mathrm{ord}(\beta) = 2$ and ε_4 is the unique element of order 2 contained in $\varepsilon_4^{\perp\perp}$. In particular, $\beta \in G$, and hence $\varepsilon_4' = h - \beta \in G$, which is a contradiction.

Subcase (2): $\varepsilon_4 \notin H$. Then $H_1 = H \oplus \langle \varepsilon_4 \rangle \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Since $2\beta = 2h - \varepsilon_4 \neq 0$ and $4\beta = 0$, it follows that $\mathrm{ord}(\beta) = 4$. But $\varepsilon_4' \notin \langle \beta \rangle$, so $\langle \beta \rangle \in \mathcal{K}(\Gamma, A)$ by Theorem 1.20, and then, $(\Gamma : \beta^{\perp}) = 4$. Since $H_1 \leqslant G$, $G^{\perp} \leqslant H_1^{\perp} = h^{\perp} \cap \varepsilon_4^{\perp} = h^{\perp} \cap {\varepsilon_4'}^{\perp} \leqslant \beta^{\perp}$, and G is Cogalois, it follows that $H_2 := G \cap \beta^{\perp\perp} \leqslant G \cap H_1^{\perp\perp} = H_1$, $H_2^{\perp} = \beta^{\perp}$, and $|H_2| = (\Gamma : \beta^{\perp}) = 4$. Thus, H_2 is a subgroup of order 4 of H_1. Setting $H_3 = H_2 + \langle \varepsilon_4 \rangle$, we deduce that $H_3^{\perp} = H_2^{\perp} \cap \varepsilon_4^{\perp} = \beta^{\perp} \cap {\varepsilon_4'}^{\perp} \leqslant h^{\perp} \cap \varepsilon_4^{\perp} = H_1^{\perp} \leqslant H_2^{\perp} \cap \varepsilon_4^{\perp}$, so $H_3^{\perp} = H_1^{\perp}$, and hence $H_3 = H_1$ by Lemma 1.5, as $G \in \mathcal{K}(\Gamma, A)$ and $H_1 + H_3 \leqslant G$. Since $H_1 = H \oplus \langle \varepsilon_4 \rangle \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $|H_2| = 4$, we deduce that $H_2 \cong \mathbb{Z}/4\mathbb{Z}$, and hence either $H_2 = H$ or $H_2 = \langle h - \varepsilon_4 \rangle$. Assuming that $H_2 = H$, it follows that $(h - \varepsilon_4')^{\perp} = \beta^{\perp} = H_2^{\perp} = H^{\perp} = h^{\perp}$. Therefore $H^{\perp} \leqslant {\varepsilon_4'}^{\perp} = \varepsilon_4^{\perp}$, and so $\varepsilon_4 \in G \cap \varepsilon_4^{\perp\perp} \leqslant G \cap H^{\perp\perp} = H$, which is a contradiction. Thus, it remains only to consider the case $H_2 = \langle h - \varepsilon_4 \rangle$. Then $(h - \varepsilon_4')^{\perp} = \beta^{\perp} = H_2^{\perp} = (h - \varepsilon_4)^{\perp}$. Replacing β with $h + \varepsilon_4'$ and proceeding as above, we finally obtain that $(h + \varepsilon_4')^{\perp} = (h - \varepsilon_4)^{\perp} = (h - \varepsilon_4')^{\perp}$, and hence $\Gamma \setminus \varepsilon_4^{\perp} \subseteq h^{\perp}$, as one easily checks. On the other hand, since $(\Gamma : \varepsilon_4^{\perp}) = 2$, it follows that $\Gamma = \varepsilon_4^\perp \cup \sigma\, \varepsilon_4^\perp$ for some (for all) $\sigma \in \Gamma \setminus \varepsilon_4^\perp$. Consequently, for every $\tau \in \varepsilon_4^\perp$ and $\sigma \in \Gamma \setminus \varepsilon_4^\perp$ we have $0 = h(\sigma\tau) = h(\sigma) + \sigma h(\tau) = \sigma h(\tau)$, and hence $\varepsilon_4^\perp \leqslant h^\perp$. Thus $h^\perp = \Gamma$, i.e., $h = 0$, which is a contradiction.

(3) $\Longrightarrow$ (1): Using Corollary 2.3, we have to show that $\widetilde{G} := \mathrm{res}_\Delta^\Gamma(G) \in \mathcal{K}(\Delta, A)$ for every $\Delta \in \overline{\mathbb{L}}(\Gamma | G^\perp)$. Assuming the contrary, it follows by Theorem 1.20 that there exist $\Delta \in \overline{\mathbb{L}}(\Gamma | G^\perp)$ and $p \in \mathcal{P}(\Delta, A) \subseteq \mathcal{P}(\Gamma, A)$, i.e., $\widehat{1/p} \in A \setminus A^\Delta \subseteq A \setminus A^\Gamma$, such that $\varepsilon_p|_\Delta \in \widetilde{G}$ if $p \neq 4$ and $\varepsilon_4'|_\Delta \in \widetilde{G}$ if $p = 4$. Consequently, there exists $h \in G$ such that $h|_\Delta = \varepsilon_p|_\Delta$ if $p \neq 4$, and $h|_\Delta = \varepsilon_4'|_\Delta$ if $p = 4$. Let $n = \mathrm{ord}(h)$. Since $\mathrm{ord}(\varepsilon_p|_\Delta) = p$ for $p \neq 4$ and $\mathrm{ord}(\varepsilon_4'|_\Delta) = 4$ for $p = 4$, as $\widehat{1/p} \in A \setminus A^\Delta$, it follows that $p \,|\, n$, and hence $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$. On the other hand, $G^\perp \leqslant h^\perp \cap \Delta \leqslant \varepsilon_p^\perp$, contrary to our hypothesis. □

We denote by $\mathcal{C}(\Gamma, A)$ the poset of all Cogalois groups of $Z^1(\Gamma, A)$ and by $\mathcal{C}^+(\Gamma, A)$ its subset consisting of all Cogalois groups G which additionally are closed elements of the canonical Galois connection described in Proposition 0.1 (1), i.e., $G = G^{\perp\perp}$. Remember that $\mathcal{K}(\Gamma, A)$ denotes the poset of all Kneser groups of $Z^1(\Gamma, A)$ and $\mathcal{K}^+(\Gamma, A) = \{\, G \in \mathcal{K}(\Gamma, A) \,|\, G = G^{\perp\perp} \,\}$.

Corollary 2.6. $\mathcal{C}^+(\Gamma, A) = \mathcal{K}^+(\Gamma, A)$.

Proof. Apply Theorem 2.5 and Corollary 1.21 □

Set $\mathcal{M}(\Gamma, A) := \mathcal{K}(\Gamma, A) \setminus \mathcal{C}(\Gamma, A)$. Obviously $\mathcal{C}(\Gamma, A)$, $\mathcal{C}^+(\Gamma, A)$, and $\mathcal{M}(\Gamma, A)$ are stable under the action of Γ.

Corollary 2.7. *$\mathcal{C}(\Gamma, A)$ is a closed subset of the spectral space $\mathcal{K}(\Gamma, A)$. In particular, $\mathcal{C}(\Gamma, A)$ has a natural structure of spectral (Stone) Γ-space, and $\mathcal{C}^+(\Gamma, A)$ is a closed Γ-subspace.*

Proof. By Theorem 2.5, $\mathcal{M}(\Gamma, A) = \bigcup_{p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)} \{\, G \in \mathcal{K}(\Gamma, A) \,|\, G^\perp \leqslant \varepsilon_p^\perp \,\}$ is the inverse image trough the canonical continuous map $\mathcal{K}(\Gamma, A) \longrightarrow \overline{\mathbb{L}}(\Gamma)$, $G \mapsto G^\perp$, of the union $\bigcup_{p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)} \mathcal{U}_{\varepsilon_p^\perp}$ of basic open sets of the spectral space $\overline{\mathbb{L}}(\Gamma)$, and hence $\mathcal{M}(\Gamma, A)$ is an open subset of the spectral space $\mathcal{K}(\Gamma, A)$. Consequently, $\mathcal{C}(\Gamma, A)$ is closed, as desired. □

Corollary 2.8. *The following assertions hold.*

(1) *$\mathcal{C}(\Gamma, A)$ is a lower subset of the poset $\mathcal{K}(\Gamma, A)$.*

(2) *The property of a subgroup G of $Z^1(\Gamma, A)$ being Cogalois is of finitary character, i.e., $G \in \mathcal{C}(\Gamma, A)$ if and only if $F \in \mathcal{C}(\Gamma, A)$ for all finite subgroups F of G.*

(3) *For any $G \in \mathcal{C}(\Gamma, A)$ there exists a maximal Cogalois group lying over G.*

(4) *The set $\mathcal{C}(\Gamma, A)_{\max}$ of all maximal Cogalois subgroups of $Z^1(\Gamma, A)$ has a natural structure of Hausdorff Γ-space.*

Proof. (1) For any $G \in \mathcal{C}(\Gamma, A)$, the closure $\overline{\{G\}} = \mathbb{L}(G)$ in the spectral space $\mathcal{K}(\Gamma, A)$ is contained in $\mathcal{C}(\Gamma, A)$ since the latter set is closed by Corollary 2.7. Thus $H \in \mathcal{C}(\Gamma, A)$ whenever $H \leqslant G$, as desired.

(2) According to the definition of the spectral topology on $\mathbb{L}(Z^1(\Gamma, A))$, for any $G \in \mathbb{L}(Z^1(\Gamma, A))$, $\mathbb{L}(G)$ is the closure of the subset of $\mathbb{L}(Z^1(\Gamma, A))$ consisting of all finite subgroups of G, so (2) follows at once from Corollary 2.7.

(3) and (4) follow in a similar way as the assertions (3) and (4) of Corollary 1.8. □

Corollary 2.9. *Let p be an odd prime number, and let G be a p-subgroup of $Z^1(\Gamma, A)$. Then G is Cogalois if and only if G is Kneser.*

Proof. By Corollaries 1.7 and 2.8 (2), we may assume that the p-group G is finite. Assume that G is Kneser and prove that G is Cogalois with the aid of Theorem 2.5. Of course, we may assume that $p \in \mathcal{P}(\Gamma, A)$, for otherwise we have nothing to prove. As we have already seen at the beginning of the proof of Lemma 1.18, the index $(\Gamma : \varepsilon_p^{\perp})$ is a divisor $\neq 1$ of $p-1$, in particular it is prime to p. Since the p-group G is Kneser, it follows that $(\Gamma : G^{\perp}) = |G|$ is a power of p, and hence $G^{\perp} \not\subseteq \varepsilon_p^{\perp}$, as desired. □

Remarks 2.10. (1) Corollary 2.9 may fail for $p = 2$. Indeed the simplest example of a Kneser non-Cogalois 2-group is the one corresponding to the action of type D_4 or D_8 (see Definition 2.16 and Lemma 2.17).

(2) In contrast with the property of Kneser groups given in Corollary 1.16, the condition that all p-primary components of G are Cogalois, is in general not sufficient to ensure G being Cogalois. To see that, observe that the group corresponding to the action of type D_{pr} is Kneser but not Cogalois, and has all its primary components Cogalois (see again Definition 2.16 and Lemma 2.17). □

The next theorem essentially shows that a subgroup $G \leqslant Z^1(\Gamma, A)$ is Cogalois if and only if G has a prescribed structure, and is the abstract version of the structure theorem [1, Theorem 4.3] for Kneser groups from the field theoretic infinite Cogalois Theory.

For any subgroup G of $Z^1(\Gamma, A)$ and for any prime number p, denote

$$\widetilde{G}_p = \begin{cases} G^{\perp\perp}(p) & \text{if either } p \in \mathcal{P}_G, \text{ or } p = 2 \text{ and } 4 \in \mathcal{P}_G, \\ G^{\perp\perp}[2] & \text{if } p = 2,\ 4 \notin \mathcal{P}_G, \text{ and } G[2] \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\widetilde{G} = \bigoplus_{p \in \mathbb{P}} \widetilde{G}_p .$$

Now, consider the subgroup

$$\mu_G = \bigcup_{n \in \mathcal{O}_G} (1/n)\mathbb{Z}/\mathbb{Z} = \sum_{h \in G} h(\Gamma) = \bigoplus_{p \in \mathbb{P}} \Big(\bigcup_{h \in G(p)} h(\Gamma) \Big)$$

of A, and let $Z^1(\Gamma\,|\,G^\perp,\mu_G) = G^{\perp\perp} \cap Z^1(\Gamma,\mu_G)$ denote the subgroup of $Z^1(\Gamma, A)$ consisting of those cocycles which are trivial on $G^\perp$ and take values in μ_G. Clearly,

$$G \leqslant Z^1(\Gamma\,|\,G^\perp,\mu_G) \leqslant \widetilde{G} \leqslant G^{\perp\perp},$$

which implies that

$$G^\perp = Z^1(\Gamma\,|\,G^\perp,\mu_G)^\perp = \widetilde{G}^\perp.$$

Notice also that

$$\mathcal{P}_G = \mathcal{P}_{Z^1(\Gamma\,|\,G^\perp,\mu_G)} = \mathcal{P}_{\widetilde{G}}\,.$$

Theorem 2.11. *With the notation above, the following assertions are equivalent for a Kneser group G of $Z^1(\Gamma, A)$.*

(1) *G is Cogalois.*

(2) *$G = Z^1(\Gamma\,|\,G^\perp,\mu_G)$.*

(3) *$G = \widetilde{G}$.*

Proof. (1) $\Longrightarrow$ (3): If G is Cogalois, then $\widetilde{G}$ is also Cogalois by Theorem 2.5 since $\mathcal{P}_G = \mathcal{P}_{\widetilde{G}}$ and $G^\perp = \widetilde{G}^\perp$. In particular, $\widetilde{G} \in \mathcal{K}(\Gamma, A)$, and hence $G = \widetilde{G} \cap G^{\perp\perp} = \widetilde{G}$, by Lemma 1.5, as desired.

(3) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (1): Assume that $G = Z^1(\Gamma\,|\,G^\perp,\mu_G)$ and G is not Cogalois. Then, by Theorem 2.5, there exists $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$ such that $G^\perp \subseteq \varepsilon_p^\perp$. Therefore, $\varepsilon_p \in Z^1(\Gamma\,|\,G^\perp,\mu_G) = G$ for $p \neq 4$, and $\varepsilon_4' \in Z^1(\Gamma\,|\,G^\perp,\mu_G) = G$ for $p = 4$. By Theorem 1.20, we deduce that G is not a Kneser group, contrary to our hypothesis. $\square$

Recall that by $\mathcal{C}(\Gamma, A)$ we have denoted the Γ-poset of all Cogalois groups of $Z^1(\Gamma, A)$; this set also has a natural structure of spectral (Stone) Γ-space by Corollary 2.7.

Corollary 2.12. *For any G, $H \in \mathcal{C}(\Gamma, A)$ we have $H \leqslant G$ if and only if $G^\perp \leqslant H^\perp$. In particular, the map $\mathcal{C}(\Gamma, A) \longrightarrow \mathbb{L}(\Gamma)$, $G \mapsto G^\perp$, is coherent and injective.*

Proof. Let G, $H \in \mathcal{C}(\Gamma, A)$ be such that $G^\perp \leqslant H^\perp$, and prove that $H \leqslant G$. By the definition of the groups $\widetilde{G}$ and $\widetilde{H}$, and using Theorem 2.11, it suffices to show that $\mathcal{P}_H \subseteq \mathcal{P}_G$ and $H[2] \neq \{0\} \Longrightarrow G[2] \neq \{0\}$. Let $p \in \mathcal{P}_H \cup \{2\}$ and $h \in H$ be such that $\mathrm{ord}(h) = p$. Since $H \in \mathcal{C}(\Gamma, A)$, we have $(\Gamma : h^\perp) = p$, and moreover, there exists only one proper subgroup (of index 2) lying over $h^\perp$ if $p = 4$. Since $G \in \mathcal{C}(\Gamma, A)$ and $G^\perp \leqslant H^\perp \leqslant h^\perp$, it follows that $G \cap h^{\perp\perp}$ is a cyclic subgroup of G of order p, and hence either $p \in \mathcal{P}_G$ or $p = 2$ and $G[2] \neq \{0\}$, as desired.

The injectivity of the canonical map $\mathcal{C}(\Gamma, A) \longrightarrow \mathbb{L}(\Gamma)$ is now obvious, so it remains only to show that it is coherent. Let Δ be an open subgroup of Γ, and denote by $\mathcal{W} = \{\, G \in \mathcal{C}(\Gamma, A)\,|\, G^\perp \leqslant \Delta \,\}$ the inverse image through the map considered above of the basic quasi-compact open set $\mathcal{U}_\Delta$ of the spectral space $\mathbb{L}(\Gamma)$. We have to show that $\mathcal{W}$ is also open quasi-compact. We may assume that $\mathcal{W} \neq \varnothing$, since otherwise we have nothing to prove. For any $G \in \mathcal{W} \subseteq \mathcal{C}(\Gamma, A)$, it follows that $(G \cap \Delta^{\perp})^{\perp} = \Delta$ and $G \cap \Delta^{\perp}$ is a finite subgroup of G of order $(\Gamma : \Delta)$, in particular, it belongs to $\mathcal{W}$. As the canonical map above is injective, it follows that $F := G \cap \Delta^{\perp}$ does not depend on the choice of $G \in \mathcal{W}$. Consequently, $W = \mathcal{U}_F \cap \mathcal{C}(\Gamma, A)$ is a basic quasi-compact open set of the spectral space $\mathcal{C}(\Gamma, A)$, as desired. □

Remarks 2.13. (1) An alternative proof of the first part of Corollary 2.12 can be done using the following fact: if G is Cogalois, then the order/index-preserving map $U \mapsto U^{\perp}$ maps bijectively the cyclic subgroups of G (which are the only finite subgroups U of the torsion Abelian group G for which the lattice $\mathbb{L}(U)$ is distributive) onto the open subgroups Δ of Γ lying over $G^{\perp}$ for which the lattice $\mathbb{L}(\Gamma \,|\, \Delta)$ is distributive. In particular, $\mathcal{O}_G$ consists of those positive integers n for which there exists an open subgroup Δ of Γ lying over $G^{\perp}$ such that $(\Gamma : \Delta) = n$ and the lattice $\mathbb{L}(\Gamma \,|\, \Delta)$ is distributive.

(2) By Corollary 2.12, $\mathcal{C}(\Gamma, A)$ is identified through the injective coherent map $G \mapsto G^{\perp}$ with a closed subspace of the spectral (Stone) space $\overline{\mathbb{L}}(\Gamma | Z^1(\Gamma, A)^{\perp})$, which is stable under the coherent action of Γ by conjugation. □

Corollary 2.14. *The following assertions are equivalent for* $G \in \mathcal{C}(\Gamma, A)$.

(1) G *is stable under the action of* Γ, *i.e.*, G *is a* Γ*-submodule of* $Z^1(\Gamma, A)$.

(2) $G^{\perp} \lhd \Gamma$.

(3) $\mu_G^{G^{\perp}} = \mu_G$.

Proof. (1) $\Longrightarrow$ (2) holds for any $G \leqslant Z^1(\Gamma, A)$ since $(\sigma G)^{\perp} = \sigma G^{\perp} \sigma^{-1}$ for all $\sigma \in \Gamma$.

(2) $\Longrightarrow$ (3): As $\mu_G = \sum_{g \in G} g(\Gamma)$, we have only to show that $\sigma g(\tau) = g(\tau)$ for all $g \in G$, $\sigma \in G^{\perp}$, $\tau \in \Gamma$. Since, by assumption, $G^{\perp} \lhd \Gamma$, we have $\tau^{-1} \sigma \tau \in G^{\perp}$, so $0 = g(\tau^{-1} \sigma \tau) = \tau^{-1}(\sigma g(\tau) - g(\tau))$, and hence $\sigma g(\tau) = g(\tau)$, as desired. Note that the implication (2) $\Longrightarrow$ (3) holds for any $G \leqslant Z^1(\Gamma, A)$.

(3) $\Longrightarrow$ (1): Let $g \in G$, $\tau \in \Gamma$, and prove that $\tau g \in G$. Since $G = Z^1(\Gamma \,|\, G^{\perp}, \mu_G)$ by Theorem 2.10, we have to show that $\tau g|_{G^{\perp}} = 0$ and $(\tau g)(\Gamma) \subseteq \mu_G$. From the hypothesis it follows that $(\tau g)(\sigma) = \tau g(\tau^{-1} \sigma \tau) = \sigma g(\tau) - g(\tau) = 0$ for any $\sigma \in G^{\perp}$, as desired. Note that the latter condition holds in general since any subgroup of A, in particular μ_G, is stable under the action of Γ. □

Corollary 2.15. *If* $G \in \mathcal{C}(\Gamma, A)$ *is a* Γ*-submodule of* $Z^1(\Gamma, A)$, *then*

$$G \cong Z^1(\Gamma / G^{\perp}, \mu_G).$$

Proof. Since G is Cogalois, we have $G = Z^1(\Gamma \,|\, G^{\perp}, \mu_G)$ by Theorem 2.11, and since G is a Γ-submodule of $Z^1(\Gamma, A)$, we have $G^{\perp} \lhd \Gamma$ by Corollary 2.14. To conclude, observe that $Z^1(\Gamma \,|\, G^{\perp}, \mu_G) \cong Z^1(\Gamma / G^{\perp}, \mu_G)$. □

According to Lemma 1.18, the Kneser groups are precisely those subgroups of $Z^1(\Gamma, A)$ which do not contain some particular cyclic groups, namely the minimal subgroups B_p which are not Kneser, $p \in \mathcal{P}(\Gamma, A)$. Using Corollary 2.8 we are going to present a similar characterization for Cogalois groups. To do that we will first describe effectively the minimal subgroups of $Z^1(\Gamma, A)$ which are Kneser but not Cogalois. A special class of actions which are introduced below plays a major role in this description.

Definition 2.16. *Let Γ be a finite group, and let A be a finite subgroup of $\mathbb{Q}/\mathbb{Z}$ on which the group Γ acts. One says that the action of Γ on A, or the Γ-module A, is*

(1) *of type* D_4 *if* $\Gamma = \mathbb{D}_4 = \langle \sigma, \tau \,|\, \sigma^2 = \tau^2 = (\sigma\tau)^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $A = (1/4)\mathbb{Z}/\mathbb{Z}$, *and* $\sigma\,\widehat{1/4} = -\,\widehat{1/4}$, $\tau\,\widehat{1/4} = \widehat{1/4}$.

(2) *of type* D_8 *if* $\Gamma = \mathbb{D}_8 = \langle \sigma, \tau \,|\, \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1 \rangle \cong \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, $A = (1/4)\mathbb{Z}/\mathbb{Z}$, *and* $\sigma\,\widehat{1/4} = -\,\widehat{1/4}$, $\tau\,\widehat{1/4} = \widehat{1/4}$.

(3) *of type* D_{pr} *if* $\Gamma = \langle \sigma, \tau \,|\, \sigma^r = \tau^p = \sigma\tau\sigma^{-1}\tau^{-u} = 1 \rangle \cong \mathbb{Z}/p\mathbb{Z} \rtimes_u \mathbb{Z}/r\mathbb{Z}$, $A = (1/pr)\mathbb{Z}/\mathbb{Z}$, *and* $\sigma\widehat{1/pr} = u\widehat{1/pr}$, $\tau\widehat{1/pr} = \widehat{1/pr}$, *where* $p \in \mathbb{P}$, $p > 2$, $r \in \mathbb{N}$, $r > 1$, $r \,|\, (p-1)$, *and* $u \in (\mathbb{Z}/pr\mathbb{Z})^*$ *is such that the order of* $u \bmod p$ *in* $(\mathbb{Z}/p\mathbb{Z})^*$ *is* r *and* $u \bmod l = 1 \bmod l$ *for all* $l \in \mathcal{P}$, $l \,|\, r$. □

Remember that by $\mathcal{M}(\Gamma, A)$ we have denoted the τ_s-open set (possibly empty) $\mathcal{K}(\Gamma, A) \setminus \mathcal{C}(\Gamma, A)$ of all Kneser groups of $Z^1(\Gamma, A)$ which are not Cogalois groups. Clearly, for any $G \in \mathcal{M}(\Gamma, A)$ there exists at least one minimal member H of $\mathcal{M}(\Gamma, A)$ such that $H \subseteq G$. By $\mathcal{M}(\Gamma, A)_{\min}$ we shall denote the set of all minimal members of $\mathcal{M}(\Gamma, A)$, and call them *minimal Kneser non-Cogalois groups*. Observe that whenever $G \in \mathcal{M}(\Gamma, A)_{\min}$, then necessarily G is a nontrivial finite group according to Corollary 2.8 (2).

Lemma 2.17. *The following conditions are equivalent for $G \in \mathcal{K}(\Gamma, A)$.*

(1) $G \in \mathcal{M}(\Gamma, A)_{min}$.

(2) $G^{\perp} \lhd \Gamma$ *and the action of* $\Gamma/G^{\perp}$ *on* μ_G *is one of the types* D_4, D_8, *or* D_{pr} *defined above.*

Proof. (1) $\Longrightarrow$ (2): First assume that $G \in \mathcal{M}(\Gamma, A)_{\min}$. Then, as was observed above, G is finite. Since G is not Cogalois, it follows by Theorem 2.5 that there exists $p \in \mathcal{P}(\Gamma, A) \cap \mathcal{P}_G$ such that $G^{\perp} \subseteq \varepsilon_p^{\perp}$. Assume p is minimal with the property above, and let H be a cyclic subgroup of G of order p. Since G is Kneser, its subgroup H is also Kneser, and hence $(\Gamma : H^{\perp}) = |H| = p$, in particular, $H \neq B_p$. We distinguish the following two cases:

Case (i): $p = 4$. We are going to show that $G^{\perp} \lhd \Gamma$ and the action of $\Gamma/G^{\perp}$ on μ_G is either of type D_4 or of type D_8. Two subcases arise:

Subcase (1): $\varepsilon_4 \in H$. As $H \cong \mathbb{Z}/4\mathbb{Z}$ and $H^{\perp} \leqslant \varepsilon_4^{\perp}$, H is not Cogalois by Theorem 2.5, so by the minimality of G we have $G = H \cong \mathbb{Z}/4\mathbb{Z}$ and $\mu_G = (1/4)\mathbb{Z}/\mathbb{Z}$. Since

$\sigma g - g \in B^1(\Gamma, \mu_G) = \langle \varepsilon_4 \rangle \leqslant G$ for all $\sigma \in \Gamma$, $g \in G$, it follows that G is stable under the action of Γ, therefore $G^\perp \lhd \Gamma$ and $G \leqslant Z^1(\Gamma/G^\perp, \mu_G)$. As the Kneser non-Cogalois group G is cyclic of order 4, it follows that $\Gamma/G^\perp \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the action of $\Gamma/G^\perp$ on μ_G is of type D_4.

Subcase (2): $\varepsilon_4 \notin H$. First, show that $\varepsilon_4 \in G$. Since G is Kneser, it follows that $G(2)^\perp \leqslant \varepsilon_4^\perp$, for otherwise $(G(2)^\perp : (G(2)^\perp \cap \varepsilon_4^\perp)) = (\Gamma : \varepsilon_4^\perp) = 2$, so $2\,|G(2)| = (\Gamma : (G(2)^\perp \cap \varepsilon_4^\perp)) \mid |\Gamma| = |G|$, which is a contradiction. Thus, the 2-primary component $G(2)$ is Kneser, and is not Cogalois by Theorem 2.5. Consequently, by the minimality of G, we deduce that $G = G(2)$. Since $L := \mathrm{res}^\Gamma_{\varepsilon_4^\perp}(G)$ is a 2-group as a factor of G and $4 \notin \mathcal{P}(\varepsilon_4^\perp, A)$, it follows by Theorem 2.5 that L is a Cogalois (in particular, Kneser) group of $Z^1(\varepsilon_4^\perp, A)$. Therefore, $(G \cap \varepsilon_4^{\perp\perp})^\perp = \varepsilon_4^\perp$ by Corollary 1.12, so the Kneser group $G \cap \varepsilon_4^{\perp\perp}$ of $Z^1(\Gamma, A)$ is cyclic of order 2. Since the only cocycle of order 2 belonging to $\varepsilon_4^{\perp\perp}$ is ε_4, we deduce that $\varepsilon_4 \in G$, as desired.

Consequently, by the minimality of G, we have $G = H \oplus \langle \varepsilon_4 \rangle \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mu_G = (1/4)\mathbb{Z}/4\mathbb{Z}$, and $L \cong H \cong \mathbb{Z}/4\mathbb{Z}$. Moreover, since $\varepsilon_4 \in G$, it follows as in the Subcase (1) that G is stable under the action of Γ. Therefore $G^\perp \lhd \Gamma$ and G is canonically identified with a subgroup of $Z^1(\Gamma/G^\perp, \mu_G)$. In particular, $G^\perp \lhd \varepsilon_4^\perp$, and $\varepsilon_4^\perp/G^\perp = \varepsilon_4^\perp/L^\perp \cong \mathbb{Z}/4\mathbb{Z}$ as $L \cong \mathbb{Z}/4\mathbb{Z}$ is a Cogalois group of $Z^1(\varepsilon_4^\perp, A)$. Observe that the canonical action of $H^\perp/G^\perp \cong \Gamma/\varepsilon_4^\perp \cong \mathbb{Z}/2\mathbb{Z}$ on $\varepsilon_4^\perp/G^\perp \cong \mathbb{Z}/4\mathbb{Z}$ is non-trivial, for otherwise we would have $\Gamma/G^\perp \cong G$, contrary to the fact that G is not Cogalois. Thus, $\Gamma/G^\perp \cong \varepsilon_4^\perp/G^\perp \rtimes \Gamma/\varepsilon_4^\perp \cong \mathbb{D}_8$, i.e., the action of $\Gamma/G^\perp$ on μ_G is of type D_8, as required.

Case (ii): $p \in \mathbb{P} \setminus \{2\}$. We are going to show that $G^\perp \lhd \Gamma$ and the action of $\Gamma/G^\perp$ on μ_G is of type D_{pr}, where $r := (\Gamma : \varepsilon_p^\perp)$. Let G' denote the subgroup of G consisting of all its elements of order prime to p. As G is Kneser, so is also G', and hence $(G'^\perp : G^\perp) = (G : G')$ is a power of the prime number p. Consequently, its divisor $(G'^\perp : G'^\perp \cap \varepsilon_p^\perp)$ is also a power of p. On the other hand, as $\varepsilon_p^\perp$, the kernel of the non-trivial action of Γ on $A[p]$, is normal in Γ, we have $G'^\perp \cap \varepsilon_p^\perp \lhd G'^\perp$. So, the factor group $G'^\perp/(G'^\perp \cap \varepsilon_p^\perp)$ is identified with a subgroup of the cyclic group $\Gamma/\varepsilon_p^\perp$ of order r, with $r \mid p-1$ and $(r,p) = 1$. Therefore $G'^\perp \leqslant \varepsilon_p^\perp$. Since $G' \neq G$, it follows from the minimality of G that G' is Cogalois. Thus, $K := G' \cap \varepsilon_p^{\perp\perp}$ is also Cogalois and $K^\perp = \varepsilon_p^\perp$. Moreover, K is cyclic of order r since $\Gamma/K^\perp \cong \mathbb{Z}/r\mathbb{Z}$. In particular, we have $\mu_K = (1/r)\mathbb{Z}/\mathbb{Z} \leqslant A$. As $K^\perp \lhd \Gamma$, Corollaries 2.14 and 2.15 imply that $(1/r)\mathbb{Z}/\mathbb{Z} \leqslant A^{\varepsilon_p^\perp}$ and $K \cong Z^1(\Gamma/\varepsilon_p^\perp, (1/r)\mathbb{Z}/\mathbb{Z})$.

From the minimality condition satisfied by G it follows that $G = H \oplus K \cong \mathbb{Z}/pr\mathbb{Z}$ and $\mu_G = (1/pr)\mathbb{Z}/\mathbb{Z}$. Since $K^\perp = \varepsilon_p^\perp \lhd \Gamma$ and $((\Gamma : H^\perp), (\Gamma : K^\perp)) = (p, r) = 1$, we deduce that $\Gamma = H^\perp K^\perp$ and $G^\perp = H^\perp \cap K^\perp \lhd H^\perp$. So, to conclude that $G^\perp \lhd \Gamma$ it suffices to show that $G^\perp \lhd K^\perp$. For any $\lambda \in G^\perp$, $\nu \in K^\perp$, $h \in H$ we have $h(\nu\lambda\nu^{-1}) = h(\nu) - (\nu\lambda\nu^{-1})h(\nu) = 0$ since $h(\nu) \in (1/p)\mathbb{Z}/\mathbb{Z} = A^{K^\perp}$ and $\nu\lambda\nu^{-1} \in K^\perp$. Thus $G^\perp \lhd \Gamma$, the kernel of the canonical action of $\Gamma/G^\perp$ on μ_G is $\varepsilon_4^\perp/G^\perp$, and $\Gamma/G^\perp = \varepsilon_4^\perp/G^\perp \rtimes H^\perp/G^\perp$. Let $\sigma \in H^\perp$, $\tau \in \varepsilon_p^\perp$, $u \in (\mathbb{Z}/pr\mathbb{Z})^*$ be such that $\sigma G^\perp$ is a generator of $H^\perp/G^\perp \cong \mathbb{Z}/r\mathbb{Z}$, $\tau G^\perp$ is a generator of $\varepsilon_p^\perp/G^\perp \cong \mathbb{Z}/p\mathbb{Z}$, and $\sigma\widehat{1/pr} =$

$u\widehat{1/pr}$. Clearly $\tau\widehat{1/pr} = \widehat{1/pr}$ and the order of $u \bmod p \in (\mathbb{Z}/p\mathbb{Z})^*$ is r. Moreover, $\sigma\tau\sigma^{-1} \equiv \tau^u (\bmod G^\perp)$ since $G = H \oplus K$, $h(\sigma\tau\sigma^{-1}) = \sigma h(\tau) = uh(\tau) = h(\tau^u)$ for all $h \in H$ (as $h|_{\varepsilon_p^\perp} \in \mathrm{Hom}\,(\varepsilon_p^\perp, (1/p)\mathbb{Z}/\mathbb{Z})$), and $k(\sigma\tau\sigma^{-1}) = k(\tau^u) = 0$ for all $k \in K$.

Consequently, $\Gamma/G^\perp \cong \mathbb{Z}/p\mathbb{Z} \rtimes_u \mathbb{Z}/r\mathbb{Z}$. Therefore, to conclude that the action of $\Gamma/G^\perp$ on μ_G is of type D_{pr}, it remains only to check that $u \bmod l = 1 \bmod l$ i.e., $\widehat{1/l} \in A^\Gamma$ for all $l \in \mathcal{P}$, $l \,|\, r$. Assuming the contrary, let $l \in \mathcal{P}(\Gamma, A)$ be such that $l \,|\, r$. Since $\widehat{1/r} \in A^{\varepsilon_p^\perp}$, we deduce that $G^\perp \leqslant \varepsilon_p^\perp \leqslant \varepsilon_l^\perp$. Thus $l \in \mathcal{P}(\Gamma, A) \cap \mathcal{P}_G$ and $G^\perp \leqslant \varepsilon_l^\perp$; hence $l \geqslant p$, which is a contradiction.

(2) $\Longrightarrow$ (1): Assume that $G^\perp \lhd \Gamma$ and the action of $\Gamma/G^\perp$ on μ_G is of one of the types D_4, D_8, or D_{pr}. Since G is canonically identified with a subgroup of $Z^1(\Gamma, \mu_G)$, we may assume without loss of generality that $G^\perp = \{1\}$ and $A = \mu_G$, i.e., (Γ, A) is one of the actions described in Definition 2.16. We have to show that every Kneser group $G \leqslant Z := Z^1(\Gamma, A)$ satisfying $G^\perp = \{1\}$ and $\mu_G = A$ is minimal non-Cogalois. We distinguish the following three cases:

Case (a): (Γ, A) is of type D_4. Then, the morphism $h \mapsto (h(\sigma), h(\tau))$ maps isomorphically Z onto $A \times 2A \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus $Z = \langle \varepsilon'_4 \rangle \oplus \langle \varphi \rangle$, where $\varphi \neq \varepsilon_4$ is defined by $\varphi(\sigma) = 0$, $\varphi(\tau) = \widehat{1/2}$. Notice that $G := \langle \varepsilon'_4 + \varphi \rangle \cong \mathbb{Z}/4\mathbb{Z}$ is the unique Kneser group of Z such that $\mu_G = A$, in particular $G^\perp = \{1\}$, and G is the unique Kneser non-Cogalois subgroup of Z as well.

Case (b): (Γ, A) is of type D_8. Then, the morphism $h \mapsto (h(\sigma), h(\tau))$ maps isomorphically Z onto $A \times A \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Consequently, $Z = \langle \varepsilon'_4 \rangle \oplus \langle \alpha \rangle$, where the cocycle α is defined by $\alpha(\sigma) = 0$, $\alpha(\tau) = \widehat{1/4}$. Observe that there exist only two Kneser groups G of Z such that $G^\perp = \{1\}$, i.e., $|G| = |\Gamma| = 8$, hence $\mu_G = A = (1/4)\mathbb{Z}/\mathbb{Z}$, namely $G_1 = \langle \varepsilon_4 \rangle \oplus \langle \alpha \rangle$ and $G_2 = \langle \varepsilon_4 \rangle \oplus \langle \alpha + \varepsilon'_4 \rangle$, both isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and stable under the action of Γ. They are also the only Kneser (minimal) non-Cogalois groups of Z of order 8. Notice that, on the other hand, $\langle \varepsilon'_4 + 2\alpha \rangle \cong \mathbb{Z}/4\mathbb{Z}$ is the unique Kneser non-Cogalois subgroup of order 4, the corresponding action being of type D_4.

Case (c): (Γ, A) is of type D_{pr}, where p is an odd prime number and $r \,|\, p-1$, $r > 1$. Let $u \in (\mathbb{Z}/pr\mathbb{Z})^*$ be the unit defining the action. Since $N(\sigma) = \sum_{i=0}^{r-1} u^i = 0 \bmod pr$, the morphism $h \mapsto (h(\sigma), h(\tau))$ maps isomorphically Z onto $A \times rA \cong \mathbb{Z}/pr\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Consequently, $Z = B_p \oplus \langle \alpha \rangle \oplus \langle \beta \rangle$, where the cocycles α and β are defined by $\alpha(\sigma) = \widehat{1/r}$, $\alpha(\tau) = 0$, $\beta(\sigma) = 0$, $\beta(\tau) = \widehat{1/p}$. As $\mathcal{P}(\Gamma, A) = \{p\}$, the necessary and sufficient condition for a subgroup G of Z to be Kneser is, according to Theorem 1.20, that $G \cap B_p = 0$. Consequently, G is a maximal Kneser group of $Z \Longleftrightarrow G$ is a direct summand of $B_p \Longleftrightarrow G$ is a Kneser group isomorphic to $\mathbb{Z}/pr\mathbb{Z} \Longleftrightarrow G$ is a Kneser group with $G^\perp = \{1\} \Longleftrightarrow G$ is a Kneser group with $\mu_G = A$. The only subgroups of Z satisfying the equivalent conditions above are the subgroups $G_i = \langle i\varepsilon_p + \alpha + \beta \rangle \cong \mathbb{Z}/pr\mathbb{Z}$, $i \in \mathbb{Z}/p\mathbb{Z}$. Since $\mathcal{P}(\Gamma, A) = \{p\}$ and the unique subgroup $H \leqslant G_i, i \in \mathbb{Z}/p\mathbb{Z}$, for which $p \,|\, |H|$ and $H^\perp \leqslant \varepsilon_p^\perp$ is the whole group G_i, it follows by Theorem 2.5 that the G_i's are also the only Kneser non-Cogalois subgroups of Z. Notice that, in contrast with the actions of type D_4 or D_8, the subgroups $G_i, i \in \mathbb{Z}/p\mathbb{Z}$ are not stable under the action of Γ. More precisely, Γ acts transitively on the set $\{G_i \,|\, i \in \mathbb{Z}/p\mathbb{Z}\}$ with stabilizers $\langle \tau^i \sigma \tau^{-i} \rangle \cong \mathbb{Z}/r\mathbb{Z}, i \in \mathbb{Z}/p\mathbb{Z}$. □

Corollary 2.18. *Any Kneser minimal non-Cogalois group of $Z^1(\Gamma, A)$ is isomorphic either to $\mathbb{Z}/4\mathbb{Z}$, or to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, or to $\mathbb{Z}/pr\mathbb{Z}$ for an odd prime number p and a divisor $r \neq 1$ of $p-1$.*

Proof. Let G be a Kneser minimal non-Cogalois group of $Z^1(\Gamma, A)$. By Lemma 2.17, $G^{\perp} \lhd \Gamma$ and the action of $\Gamma/G^{\perp}$ on μ_G is of one of the types D_4, D_8 or D_{pr}. The possible isomorphism types for the group G are now immediate from the proof of the implication (2) $\Longrightarrow$ (1) of Lemma 2.17. □

The next result provides an analogue of Theorem 1.20 for Cogalois groups.

Theorem 2.19. *The following statements are equivalent for a Kneser subgroup G of $Z^1(\Gamma, A)$.*

(1) *G is Cogalois.*

(2) *G contains no H for which $H^{\perp} \lhd \Gamma$ and the action of $\Gamma/H^{\perp}$ on μ_H is one of the types D_4, D_8, or D_{pr}.*

Proof. The result follows at once from Lemma 2.17 and from the following fact we already mentioned just before Lemma 2.17: for any $L \in \mathcal{M}(\Gamma, A)$ there exists at least one $K \in \mathcal{M}(\Gamma, A)_{\min}$ such that $K \subseteq L$. □

As it follows from Lemma 2.17, the fact that all the p-primary components of a subgroup G of $Z^1(\Gamma, A)$ are Cogalois does not imply that the whole group G is Cogalois. The next result provides a supplementary lattice theoretic (topological) condition which ensures such an implication, obtaining in this way a *local-global principle* for Cogalois groups.

Theorem 2.20. *Let G be a subgroup of $Z^1(\Gamma, A)$, and let*

$$\theta : \overline{\mathbb{L}}(\Gamma|G^{\perp}) \longrightarrow \prod_{p \in \mathbb{P}} \overline{\mathbb{L}}(\Gamma|G(p)^{\perp}), \ \Delta \mapsto (\overline{\langle \Delta \cup G(p)^{\perp} \rangle})_{p \in \mathbb{P}}.$$

Then, the following statements are equivalent.

(1) *G is Cogalois.*

(2) *$G(p)$ is Cogalois for all prime numbers p, and the order-preserving map θ is a lattice isomorphism.*

(3) *$G(p)$ is Cogalois for all prime numbers p, and the coherent map θ is a homeomorphism of spectral (Stone) spaces.*

(4) *G is Kneser, $G(2)$ is Cogalois, and $\Delta = \Gamma$ whenever $\Delta \in \overline{\mathbb{L}}(\Gamma|G^{\perp})$ is such that $\theta(\Delta) = \theta(\Gamma)$.*

Proof. (1) $\Longrightarrow$ (2): Assuming that G is Cogalois, we only have to prove that θ is a lattice isomorphism. As G and the $G(p)$'s are Cogalois, the canonical order-reversing maps $\varphi : \mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^{\perp})$, $\varphi_p : \mathbb{L}(G(p)) \longrightarrow \overline{\mathbb{L}}(\Gamma|G(p)^{\perp})$, $H \mapsto H^{\perp}$ are lattice anti-isomorphisms. On the other hand, since the canonical map

$$\psi : \mathbb{L}(G) \longrightarrow \prod_{p\in\mathbb{P}} \mathbb{L}(G(p)),\ H \mapsto (H(p))_{p\in\mathbb{P}}$$

is a lattice isomorphism, the composed map

$$(\prod_{p\in\mathbb{P}} \varphi_p) \circ \psi \circ \varphi^{-1} : \overline{\mathbb{L}}(\Gamma|G^{\perp}) \longrightarrow \prod_{p\in\mathbb{P}} \overline{\mathbb{L}}(\Gamma|G(p)^{\perp}),\ \Delta \mapsto ((G \cap \Delta^{\perp})(p)^{\perp})_{p\in\mathbb{P}}$$

is also a lattice isomorphism, so it remains only to check that $(\prod_{p\in\mathbb{P}} \varphi_p) \circ \psi \circ \varphi^{-1} = \theta$, i.e., $(G \cap \Delta^{\perp})(p)^{\perp} = \overline{\langle \Delta \cup G(p)^{\perp} \rangle}$ for all $p \in \mathbb{P}, \Delta \in \overline{\mathbb{L}}(\Gamma|G^{\perp})$. Now, as φ is a lattice anti-isomorphism, we deduce that

$$(G \cap \Delta^{\perp})(p)^{\perp} = ((G \cap \Delta^{\perp}) \cap G(p))^{\perp} = \overline{\langle (G \cap \Delta^{\perp})^{\perp} \cup G(p)^{\perp} \rangle} = \overline{\langle \Delta \cup G(p)^{\perp} \rangle},$$

as desired.

(2) $\Longleftrightarrow$ (3) is obvious.

(2) $\Longrightarrow$ (4) follows at once from Corollary 1.16.

(4) $\Longrightarrow$ (1): Assuming that G is Kneser but not Cogalois, we have to show that either $G(2)$ is not Cogalois or there exists $\Delta \in \overline{\mathbb{L}}(\Gamma|G^{\perp})$ such that $\Delta \neq \Gamma$ and $\theta(\Delta) = \theta(\Gamma)$. Let H be a minimal non-Cogalois subgroup of G. According to Lemma 2.17, $H^{\perp}$ is an open normal subgroup of Γ and the action of $\Gamma/H^{\perp}$ on μ_H is one of the actions described in Definition 2.16. If the action above is of type D_4 or of type D_8, then it follows that $H \leqslant G(2)$, and hence $G(2)$ is not Cogalois. So, it remains to consider only the case when the action is of type D_{pr}, where p is an odd prime number and $r \,|\, p-1$, $r \geqslant 2$. Notice that $\overline{\langle H^{\perp} \cup G(p)^{\perp} \rangle} = H^{\perp}G(p)^{\perp}$ as $H^{\perp} \lhd \Gamma$, $(\Gamma : H^{\perp}G(p)^{\perp})$ is a power of p as $G \in \mathcal{K}(\Gamma, A)$, and $(\Gamma : H(p)^{\perp}) = |H(p)| = p$ as $H(p) \leqslant G \in \mathcal{K}(\Gamma, A)$ and $H(p) \cong \mathbb{Z}/p\mathbb{Z}$ (since $H \cong \mathbb{Z}/pr\mathbb{Z}$ by Corollary 2.18 and $(p, r) = 1$). On the other hand, since $H^{\perp} \leqslant H^{\perp}G(p)^{\perp} \leqslant H(p)^{\perp} \leqslant \Gamma$ and $(\Gamma : H^{\perp}) = pr$, $r \,|\, p-1$, it follows that $H^{\perp}G(p)^{\perp} = H(p)^{\perp}$. As $\Gamma/H^{\perp} \cong \mathbb{Z}/p\mathbb{Z} \rtimes_u \mathbb{Z}/r\mathbb{Z}$ for a suitable $u \in (\mathbb{Z}/pr\mathbb{Z})^*$ by Definition 2.16, there exists an open subgroup Δ of Γ lying over $H^{\perp}$ such that $(\Gamma : \Delta) = p$ and $\Delta \neq H(p)^{\perp}$. Consequently, $\overline{\langle \Delta \cup G(p)^{\perp} \rangle} = \overline{\langle \Delta \cup H(p)^{\perp} \rangle} = \Gamma$, and, similarly, $\overline{\langle \Delta \cup G(q)^{\perp} \rangle} = \Gamma$ for any prime number $q \neq p$ since all open subgroups of Γ lying over $G(q)^{\perp}$ have q-th power indices in Γ as $G \in \mathcal{K}(\Gamma, A)$. Thus, we found a subgroup Δ of Γ with the desired properties, which finishes the proof. □

Finally, we consider the case when G is stable under the action of Γ. Then, the local-global principle for Cogalois groups has the following simple formulation.

Proposition 2.21. *The following assertions are equivalent for a Γ-submodule G of $Z^1(\Gamma, A)$.*

(1) *G is Cogalois.*

(2) *$G(p)$ is Cogalois for all prime numbers p.*

(3) *G is Kneser, and $G(2)$ is Cogalois.*

Proof. The implication (1) $\Longrightarrow$ (2) is trivial, while the implication (2) $\Longrightarrow$ (3) follows at once from Corollary 1.16.

(3) $\Longrightarrow$ (1): Assuming that the Γ-module G is Kneser but not Cogalois, we have only to show that $G(2)$ is not Cogalois. Let H be a minimal non-Cogalois subgroup of G. According to Lemma 2.17, $H^{\perp} \lhd \Gamma$ and the action of $\Gamma/H^{\perp}$ on μ_H is the one described in Definition 2.16. If the action is of type D_4 or of type D_8, then $H \leqslant G(2)$, and hence $G(2)$ is not Cogalois, as desired. Now assume that the action is of type D_{pr}. Then, as in the proof of Theorem 2.19 we deduce that $(\Gamma : H^{\perp}G(p)^{\perp}) = p$. On the other hand, $G(p)^{\perp} \lhd \Gamma$ since $G(p)$ is a Γ-submodule of G. Hence $H^{\perp}G(p)^{\perp} \lhd \Gamma$, and so, $\mathbb{Z}/p\mathbb{Z}$ is a quotient of $\Gamma/H^{\perp} \cong \mathbb{Z}/p\mathbb{Z} \rtimes_u \mathbb{Z}/r\mathbb{Z}$, which is a contradiction. □

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