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by

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S - spectral decompositions I

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In this paper we study some types of spectral decompositions, residual by some way, having as main methods the restrictions and quotients of decomposable operators. The starting point of this was the intention to give an answer to an open problem from decomposable operators' theory, which was formulated in the 6th chapter of the monograph belonging to I. Colojoară and C. Foiaş named "Theory of Generalised Spectral Operators". The restriction and the quotient of a decomposable operator to a spectral maximal space are they decomposable or not? In other words, is a decomposable operator strongly decomposable or not? E. I. Albrecht built up an example of a decomposable operator that is not strongly decomposable. Still, if the spectrum of T has a dimension of 1 (more exactly $\sigma(T) \in C$, see [15] Def. 5) then decomposability implies strongly decomposability ([15] theorem 6). Generally the restriction and the quotient of a decomposable operator T are S-decomposable operators (for a spectral maximal space Y of T), where S is a part of the frontier of $\sigma(T \mid Y)$, $S = \delta\sigma(T \mid Y) \cap \sigma(T)$. The definition of the S-decomposability is close to the one of the decomposability, only the open covering of the spectrum is replaced with a S-covering (see [76] or the definition from the preliminaries). Most of the decomposable operators' properties are also true for the S-decomposable ones, of course in adjusted, specific form. One must notice that the properties of the commuters have no correspondent. The paper contains three chapters. In the first one we study the restrictions and quotients of a decomposable (strongly decomposable or spectral) operator to an invariant subspace, particularly a spectral maximal space. We emphasise properties of the (topological) dimension of various parts of the spectrum, where the sets having a dimension of 0 play an important role. In the second chapter we give the properties of S-decomposable operators: structure theorems of spectral maximal spaces, S-decomposability conditions, properties linked to direct sums, functional calculus with analytic functions, S-spectral capacities etc. The third chapter contains some conclusions on multidimensional spectral theory. We try to generalise for operators systems some results obtained for a single operator. We study the restrictions and quotients of spectral and decomposable systems of operators, and in the last paragraph we define the systems analogue of the residual single extendibility.

0. PRELIMINARIES

In this paper we will use several notations and definitions from the specialised literature, that will be also remind it here. Let X be a Banach space and let $B(X)$ be the algebra of all bounded linear operators of X. If $T \in B(X)$ and Y is a linear (closed) invariant subspace of T (meaning $TY \subset Y$), we shall denote by $T \mid Y$ the restriction of T to Y, and by $\dot{T}$ the operator induced by T in the quotient space $\dot{X} = X / Y$. In all that follows by subspace of X we mean a closed linear variety of X. We shall also denote by $\sigma(T)$ the spectrum of T, $R(\lambda, T)$ the resolving of T and by $\mathbf{C}$ the complex plan.

DEFINITION 0.1. A subspace $Y \subset X$ is called *spectral maximal space* of $T \in B(X)$ if Y is invariant of T and for any other subspace Z also invariant of T such that $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ we have $Z \subset Y$ [48], [37].

An operator $T \in B(X)$ is *decomposable* if for any finite open covering $\{G_i\}_1^n$ of the spectrum $\sigma(T)$ there exists a system of spectral maximal spaces of T $\{Y_i\}_1^n$ such that $\sigma(T \mid Y_i) \subset G_i$ ($i = 1,2,\dots,n$) and $X = \sum_{i=1}^{n} Y_i$ [48] [37]. An operator $T \in B(X)$ is strongly decomposable if $T \mid Y$ is decomposable for any spectral maximal space Y of T [2].

We call that $T \in B(X)$ has a *single analytic extension* if for any analytic function $f : \omega \to X$ (where $\omega \subset \mathbf{C}$ is an open set), solution of the equation

$$(\lambda I - T) f(\lambda) \equiv 0,$$

is by any means identical null [46], [45]. The single analytic extendibility provides the possibility to attach each element x of space X a set from the complex plan $\mathbf{C}$ in which outside the demeanor of some entities defined by the operator becomes controllable. A point λ_0 is belonging to the local resolving $\rho_T(x)$ of $x \in X$ if in its neighbourhood there exists a single analytic function $x(\lambda)$, which verifies the identity

$$(\lambda i - T) x(\lambda) \equiv x.$$

The set $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$ is named the spectrum of x regarding T. Obviously we have $\sigma(x) \supset \sigma(T)$ (where $\rho(T) = \mathbf{C} \setminus \sigma(T)$ is the resolving set of T), hence $\sigma_T(x) \subset \sigma(T)$. One denotes

$$X_T(F) = \{x \in X, \sigma_T(x) \subset F\}.$$

THEOREM 0.1. *If $X_F(T)$ is closed, then it is a spectral maximal space of T and* $\sigma(T \mid X_T(F)) \subset F$ [38].

In order to study the restrictions and quotients of an operator, as also to define and analyse the properties of S-decomposable operators we need several notions from the residual spectral decompositions theory, brought up by F. H. Vasilescu in [76], [77], [80].

DEFINITION 0.2. A family of open sets $\{G_S\} \cup \{G_i\}_1^n$ of the complex plan $\mathbf{C}$ will be said to be an S-covering of the compact set $\sigma \subset \mathbf{C}$ ($S \subset \mathbf{C}$ also compact) if $G_S \cup \left(\bigcup_{i=1}^{n} G_i \right) \supset \sigma \cup S$ and $\overline{G} \cap S = \varnothing$ ($i = 1,2,\dots,n$) [76].

DEFINITION 0.3. An open set $\Omega \subset \mathbf{C}$ is of *analytical singleness* for $T \in B(X)$, if for any open set $\omega \subset \Omega$ and any analytic function $f_0 : \omega \to X$ verifying the equation

$$(\lambda I - T) f_0(\lambda) \equiv 0,$$

it follows that $f_0(\lambda)\equiv 0$ in ω [76]. One shows that for any $T\in B(X)$ there exists a (single) largest open set Ω_T of *analytic singleness* [76]. We shall call *analytic residuum* and we shall denote by S_T the set $S_T=\mathbf{C}\setminus\Omega_T$. From the definition results that in any of its points S_T has either the (topological) dimension 2 or $\dim S_T=-1$ (meaning $S_T=\varnothing$: $\operatorname{Int} S_T=\varnothing$ implies $S_T=\varnothing$). The case $S_T=\varnothing$ fits the single analytic extendibility [76].

If $x\in X$, a point $\lambda\in\delta_T(x)$ if, in a proximity V_λ of λ, there exists at least an analytic function $f_x(\mu)$ (called *T*-associated with *x*) such that $(\mu I-T)f_x(\mu)\equiv x$ for $\mu\in V$. We shall put $\gamma_T(x)=\mathbf{C}\setminus\delta_T(x)$, $\rho_T(x)=\delta_T(x)\cap\Omega_T$, $\sigma(x)=C\setminus\rho_T(x)=$ $=\gamma_T(x)\cup S_T$ and $X_T(F)=\{x\in X,\sigma_T(x)\subset F\}$. When $S_T=\varnothing$, we have $\sigma_T(x)=\gamma_T(x)$ and a single analytic function $x(\lambda)$, *T*-associated with *x*, for any $x\in X$ exists in $\rho_T(x)=\delta_T x$.

THEOREM 0.2. *If $T\in B(X)$, $S_T\neq 0$ and $X_T(F)$ is closed for $F\subset\mathbf{C}$ closed, $F\supset S$, then $X_T(F)$ is a spectral maximal space of T and $\sigma(T\mid X_T(F))\subset F$ ([76], propositions 2.4., 3.4.).*

DEFINITION 0.4. Let $T\in B(X)$ and let $S\subset\mathbf{C}$ compact. We shall say that *T* is *S-decomposable* if for any finite open *S*-covering of $\sigma(T)$ $\{G_S\}\cup\{G_i\}_1^n$ there exists the system of spectral maximal spaces of *T* $\{Y_S\}\cup\{Y_i\}_1^n$ such that:

(i) $\sigma(T\mid Y_S)\subset G_S$, $\sigma(T\mid Y_i)\subset G_i$ $(1\leq i\leq n)$;

(ii) $X=Y_S+\sum_{i=1}^n Y_i$.

If condition (ii) is replaced by

(ii') $Z=Z\cap Y_S+\sum_{i=1}^n Z\cap Y_i$,

where *Z* is any spectral maximal space of *T*, then we shall say that *T* is *strongly S-decomposable*, and when the same condition (ii) is replaced with the weaker one

(ii'') $X=\overline{Y_S+\sum_{i=1}^n Y_i}$,

we shall say that *T* is *weakly S-decomposable*.

If in the definition of *S*-decomposability, Y_S is not necessarily a spectral maximal space of *T* and $\sigma(T\mid Y_S)=\widetilde{G}_S$ (if $A\subset\mathbf{C}$ is bounded, we denote $\widetilde{A}=\mathbf{C}\setminus D_\infty$, where D_∞ is the unbounded component of $\mathbf{C}\setminus A$), then we say that $T\in\mathrm{D}_S$.

An operator $T\in B(X)$ is named *(m,S)-decomposable* if in the definition of *S*-decomposability we consider the *S*-covering composed out of exactly $m+1$ sets, that is $\{G_S\}\cup\{G_i\}_{i=1}^m$. If $m=1$, we have $(1,S)$-decomposability and we shall prove that a $(1,S)$-decomposable operator is *S*-decomposable.

Since in this paper we use quite much the (topological) dimension theory, we shall give several definition and examples. We will be interested in the dimensions of the sets from the complex plan $\mathbf{C}$ or $\mathbf{C}^n$; we use [13], [65], [67], [66].

DEFINITION 0.5. Let X be a separable metric space. The symbol $\dim pX$ means the dimension of X in the point p. The following three conditions define this condition through induction:

1) $\dim X = -1$ means $X = \varnothing$;

2) if $X \neq \varnothing$, $\dim X$ is the superior edge of $\dim pX$ for any $p \in X$;

3) $\dim pX \leq n+1$ if there exist any open neighbourhoods of p, no matter how small, such that their frontiers be of a dimension less than or equal to n.

Condition 3) can be composed like this:

3') $\dim pX \leq n+1$ means that in the family of the vicinities of p there exists an open one which frontier has a dimension less or equal to n.

By definition, a (nevoid) space has the *dimension 0* if for any of its points there exist neighbourhoods no matter how small having a void frontier. Thus, for example, the space of the rational numbers on the axis has the dimension 0: each interval with irrational extremities is a vicinity for the numbers contained within and has a void frontier (the frontier contains no rational numbers). Same for the set of irrational numbers, and generally speaking any frontier set of the real axis has the dimension 0. The space of real numbers have the dimension less or equal to 1 since the frontier of an interval is formed out of two points and this set has the dimension 0. Analogously the plan has a dimension less or equal to 2 (since the circle is of a dimension less or equal to 1) and generally speaking the Cartesian space E^n has a dimension less or equal to n. The proof of the fact that $\dim E^n = n$ is not elementary. The dimension of a space's set is never bigger than the one of the space itself.

THEOREM 0.3. *The necessary and sufficient condition for a subset N of R^n to have the dimension n is for N to contain a nevoid open subset in R^n ([13], lemma 1.2.),* $\operatorname{Int} N \neq \varnothing$.

We remind that a space is named *totally disconnected* if any of its components reduces to a single point. One proves that a space X locally compact has the dimension 0 if and only if it is totally disconnected.

The model space having a dimension of 0 is Cantor's set (discontinuum). Any space having a dimension of 0 is topologically contained in Cantor's discontinuum. A characterisation of the spaces having a dimension of 0 (that sometimes is considered as a definition [66] XIX, 1) will be useful to us:

THEOREM 0.4. *A nevoid space X has a dimension of 0 if to any finite open covering $X = G_0 \cup G_1 \cup \ldots \cup G_m$ corresponds a closed covering $X = F_0 \cup F_1 \cup \ldots \cup F_m$ such that $F_i \subset G_i$, $F_i \cap F_j = \varnothing$ ($i \neq j, i, j = 1,2,\ldots,m$). The sets F_i are therefore closed-open.*

From 0.4 theorem results that the frontier of a set contained in the plan has a dimension less or equal than 1, and for a set from the axis the dimension is less or equal to 0.

Being given a compact set L in a plan having a dimension of 1, is the frontier of any compact subset $L_1 \subset L$ (in the relative topology of L) of dimension 0 or not? There are examples of sets having a dimension 1 for which the answer is negative. For example the set Γ is defined as follows:

$$\begin{cases} \left\{\left(x, \sin\frac{1}{x}\right), x \in (0,1]\right\} = M, \\ \{(0,y) \quad ,-1 \le y \le 1\} = F. \end{cases}$$

We have $\Gamma = M \cup F$ compact, $F \subset \Gamma$ compact and the frontier of F, $\partial F = \partial M = F$, $\dim \partial F = 1$.

Also the "fan" set ([66], XVIII):

$$\Gamma_1 : \begin{cases} \{(x, 1-nx), x > 0, 1-nx > 0, n = 1,2,\ldots\} = M_1, \\ \{(0,y), 0 \le y \le 1\} = F_1, \end{cases},$$

$\Gamma_1 = M_1 \cup F_1$, F_1 closed, $\partial F_1 = = \partial M_1 = F_1$, $\dim F_1 = 1$.

For the decomposable operators it seems that not only the compact sets with dimensions 1 (spectrum of operators or parts of them) that have a good demeanour as mentioned above imply a great interest.

Definition 0.6. We shall denote by C the class of all compact sets $\sigma \subset \mathbf{C}$ with $\dim \sigma \le 1$, and moreover meeting the property that for any closed subset $\sigma_1 \subset \sigma$ we have $\dim \partial\sigma_1 = 0$ ($\partial\sigma_1$ is the frontier of σ_1 in the topology of σ).

The family C isn't void: any interval or finite reunion of intervals on the real axis belongs to class C ; any set from the plan that is homomorphic with $[0,1]$ belongs to class C ; finite reunions of sets from C belong to C , the disk $\{|\lambda| = 1\}$ belongs to C . Let us remark that the countable reunion of sets from C may not belong to C . Example: the set $L = [0,1] \cup \bigcup_{n=1}^{\infty} \left\{(x,y) \mid x \in [0,1], y = \frac{1}{n}\right\}$ doesn't belong to C . The sets Γ, Γ_1, L are not locally connexe and probable there exists an association between the sets that do not belong to C and the sets that are not locally connexe.

Finally we remind the property: if X and Y are two separable metric spaces, then $\dim(X \times Y) \le \dim X + \dim Y$. In the following we will remind a few ideas that will be necessary in the third chapter of multidimensional spectral theory.

Let E^n be the external algebra, generated by the undefined $\sigma = (s_1, s_2, \ldots, s_n)$, over the body of complex numbers $\mathbf{C}$ ([87] p. 183). The algebra E^n is the complex algebra with the e identity satisfying the relation $s_i \wedge s_j = -s_j \wedge s_i$, where by $(s_i, s_j) \to s_i \wedge s_j$ we denote multiplication in E^n. The E^n algebra is gradual and $E^n = \sum_{p=0}^{\infty} \oplus E_p^n$, where $E_p^n \wedge E_q^n \subset E_{po+q}^n$, E_p^n is generated by the elements $s_{j_1} \wedge s_{j_2} \wedge \ldots \wedge s_{j_p}$ $(1 \le j_1 \le j_2 \le \ldots \le j_p \le n)$. We take $E_0^n \approx \mathbf{C}$, whare elements of E_0^n represent identity multiply. Also, $E_n^n \approx \mathbf{C}$, the base of E_n^n, that is constitute from the single element $s_1 \wedge s_2 \wedge \ldots \wedge s_n$, and $E_p^n = (0)$, $p > n$.

Let now X be a Banach space, $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ a system of commuting operators, and let A be an operators algebra which centre contains the system

$a_1, a_2, \ldots, a_n$. We shall denote (see [58], chapter I) $\Lambda^p(\sigma, X) = E_p^n(X) = X \otimes_e E_p^n$. $\Lambda^p[\sigma, X]$ can be viewed as a module over any A algebra, having the above property. By writing xk for $x \otimes k$, $x \in X$, $k \in E^n$ we notice that $\Lambda^p[\sigma, X]$, the space of all external forms having a p degree in s and coefficients from X is create of elements like

$$\psi = \sum_{1 \le j_1 < j_2 < \ldots < j_p \le n} x_{j_1 \ldots j_p} s_{j_1} \wedge s_{j_2} \wedge \ldots \wedge s_{j_p} \quad (x_{j_1 j_2 \ldots j_p} \in X).$$

We have $\Lambda^0[\sigma, X] = \Lambda^n[\sigma, X] = X$ and we put $\Lambda^p[\sigma, X] = 0$ for $p < 0$ or $p > n$.

Through the spectrum of a system $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ one comprehends, generally speaking, the complementary in $\mathbf{C}^n$ of the set with the elements $z = (z_1, z_2, \ldots, z_n) \subset \mathbf{C}^n$ with the property that the system $z - a = (z_1 - a_1, z_2 - a_2, \ldots, z_n - a_n)$ is not singular on X. From the sense given to the notion of nonsingularity one can obtain several notions of spectrum. We shall be interested in the spectrum instituted by Taylor [70] that seems to have more benefits against the classical ones.

The nonsingularity of $z - a$ according to J. L. Taylor's means the accuracy of a certain series created using the space and the operators. This series is a variation, adjusted for more operators of the elementary series

$$0 \to X \xrightarrow{z-a} X \to 0$$

that - in case of a single operator - shows the property of $z - a$ of being simultaneous injectiv and surjectiv

There are two types of series used to define the nonsingularity of an operators system, a complex of Koszul [88] chains or a complex of cochains very much resembling a complex of differential forms. Both can be described in terms of exterior algebra, the natural duality existing between the two complexes makes them been simultaneously exactly and define the same idea of nonsingularity.

The shared spatial base of the two complexes is represented by the spaces $\Lambda^p[\sigma, X]$, the two ones make a difference only through the link (frontier) operators, respectively the cofrontier.

If $1 \le p \le n$, we shall denote by

$$\delta_p = \delta_p(a) : \Lambda^p[\sigma, X] \to \Lambda^{p-1}[\sigma, X]$$

the operator defined by

$$\delta_p\left(xs_{j_1} \wedge \ldots \wedge s_{j_p}\right) = \sum_{i=1}^{p} (-1)^{i-1} xs_{j_1} \wedge \ldots \wedge \hat{s}_{j_i} \wedge \ldots \wedge s_{j_p}$$

and

$$\delta_p\left(\sum_{1 \le j_1 < \ldots < j_p \le n} x_{j_1 \ldots j_p} s_{j_1} \wedge \ldots \wedge s_{j_p} \right) = \sum_{1 \le j_1 < \ldots < j_p \le n} \delta_p\left(x_{j_1 \ldots j_p} s_{j_1} \wedge \ldots \wedge s_{j_p}\right)$$

where the circumflex stroke marks the absence of the letter above there is placed; if $p < 0$ or $p > n$, we put $\delta_p = 0$. We shall also denote by

$$\delta_p = \delta^p(a) : \Lambda^p[\sigma, X] \to \Lambda^{p+1}[\sigma, X]$$

the homomorphism that acts upon a form from $\Lambda^p[\sigma, X]$ through the exterior multiplication on the left side with $a_1 s_1 + \ldots + a_n s_n$ (when $p<0$ or $p>n$ we put $\delta_p = 0$). The fact that the system $a = (a_1, a_2, \ldots, a_n)$ is commutative assure the verification of the relations $\delta_p \delta_{p+1} = 0$ and $\delta^{p+1}\delta^p = 0$, $p \in \mathbf{Z}$. The chains complex constitute from modules $\Lambda^p[\sigma, X]$ and the frontier operators δ_p is named the Koszul complex associated to the system $a = (a_1, a_2, \ldots, a_n)$ and is noted $E(X, a)$. The complex of chains represented by the modules $\Lambda^p[\sigma, X]$ and the cofrontier operators δ^p, $p \in \mathbf{Z}$ will be mark as $F(X, a)$. Hence we have

$$E(X,a): 0 \to X = \Lambda^n[\sigma, X] \xrightarrow{\delta_n} \Lambda^{n-1}[\sigma, X] \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} \Lambda^2[\sigma, X] \xrightarrow{\delta_2} \Lambda^1[\sigma, X] \xrightarrow{\delta_1} \Lambda^0[\sigma, X] = X \to 0$$

and

$$F(X,a): 0 \to X = \Lambda^0[\sigma, X] \xrightarrow{\delta^0} \Lambda^1[\sigma, X] \xrightarrow{\delta^1} \Lambda^2[\sigma, X] \xrightarrow{\delta^2} \ldots \xrightarrow{\delta^{n-2}} \Lambda^{n-1}[\sigma, X] \xrightarrow{\delta^{n-1}} \Lambda^n[\sigma, X] = X \to 0$$

Generally speaking the two series aren't accurate. The omology modules are marking the incorrectness

$$H_p(X,a) = \operatorname{Ker}(\delta_{p+1} : \Lambda^{p+1} \to \Lambda^p) / \operatorname{Im}(\delta_p : \Lambda^p \to \Lambda^{p-1})$$

for $E(X, a)$ and the coomology modules are marking it for $F(X, a)$,

$$H^p(X,a) = \operatorname{Ker}(\delta^p : \Lambda^p \to \Lambda^{p+1}) / \operatorname{Im}(\delta^{p-1} : \Lambda^{p-1} \to \Lambda^p).$$

One can easily verify that the two complexes $E(X, a)$ and $F(X, a)$ are equal regarding accuracy [58].

Definition 0.7. The system $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ is said to be nonsingular, if it is precisely the complex $E(X, a)$ or equivalent, the complex $F(X, a)$. The set of those elements $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$ for which the system $z - a = (z_1 - a_1, \ldots, z_n - a_n)$ is nonsingular on *X* is said to be the resolving set of *a* on *X* and is denoted by $r(a, X)$. The complementary in $\mathbf{C}^n$ of this set, $\mathbf{C}^n \setminus r(a, X)$, is said to be the spectrum of *a* on *X* and is denoted by $\sigma(a, X)$.

We shall use the following function spaces defined on an open set $U \subset \mathbf{C}^n$ and taking values in a complex Banach space X :B (U, X) – the space of continuous functions admitting (regarding distributions) continuous partial derivatives regarded to $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$ ([71], §2); $\mathrm{B}_0(U, X)$ – a subspace of B (U, X) consisted of the functions with a compact support; $\mathrm{C}^\infty(U, X)$ – the space of continuous functions admitting partial derivatives of any rank; $\mathrm{C}_0^\infty(U, X)$ – a subspace of $\mathrm{C}^\infty(U, X)$ consisted of the functions with a compact support; U (U, X) – the space of analytic functions on *U*. We will permanently use the fact that B (U, X)=$\mathrm{C}^\infty(U, X)$ [82].

If $U \subset \mathbf{C}^n$ is open, F is one of the function spaces described above and $\sigma = (s_1, s_2, \ldots, s_n)$ a indeterminate system, then we shall denote by α the operator that acts upon an exterior form ψ in the indeterminate $\sigma = (s_1, s_2, \ldots, s_n)$ with coefficients in F , $\psi \in \Lambda^p[\sigma, \mathrm{F}\ ]$, according to the relation

$$(\alpha\psi)(z) = [(z_1 - a_1)s_1 + (z_2 - a_2)s_2 + \ldots + (z_n - a_n)s_n] \wedge \psi(z)$$

and we shall denote by $\alpha \oplus \bar{\partial}$ the operator that acts similarly upon the exterior forms $\psi \in \Lambda^p[\sigma \cup d\bar{z}, \mathrm{F}\ ]$ in the indeterminate s and $d\bar{z}$ with coefficients in F :

$$((\alpha \oplus \bar{\partial})\psi)(z) = \left[(z_1 - a_1)s_1 + (z_2 - a_2)s_2 \ldots + (z_n - a_n)s_n + \frac{\partial}{\partial \bar{z}_1} d\bar{z} + \ldots + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n\right] \wedge \psi(z)$$

DEFINITION 0.8. *The resolving analytic set* of x regarding $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ is the set of those elements like $z = (z_1, z_2, \ldots, z_n) \in \mathbf{C}^n$ so that there exists a closed V neighbourhood of z and n functions analytic on V with values in X, $f_1, f_2, \ldots, f_n$, satisfying the identity $x = (\zeta_1 - a_1)f_1(\zeta) + \ldots + (\zeta_n - a_n)f_n(\zeta)$, $\zeta \in V$. The complementary of this set in $\mathbf{C}^n$ is said to be the analytic spectrum of x regarding a. We shall denote them by $\rho(a, x)$ respectively $\sigma(a, x)$.

DEFINITION 0.9. *The resolving set* of x regarding $a = (a_1, a_2, \ldots, a_n)$, denoted by $r(a, x)$, is the reunion of all open sets V having the property that there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathrm{C}^\infty(V, X)]$ satisfying the equality

$$x(s_1 \wedge s_2 \wedge \ldots \wedge s_n) = \left[(z_1 - a_1)s_1 + \ldots + (z_n - a_n)s_n + \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \ldots + \frac{\partial}{\partial \bar{z}_n} d\bar{z}_n\right] \wedge \psi(z)$$

The complementary of this set (in $\mathbf{C}^n$) is said to be *the spectrum* of x regarding a, $sp(a, x) = \mathbf{C}^n \setminus r(a, x)$.

In order to obtain a global solution ψ for the equation $sx = (\alpha \oplus \bar{\partial})\psi$, it is necessary that the system satisfies a condition similar to the property of single analytic extension in the case of a single operator namely:

DEFINITION 0.10. We shall say that the system $a = (a_1, a_2, \ldots, a_n)$ verifies the coomologic property (L) if $H^{n-1}(\mathrm{C}^\infty(G, X), \alpha \oplus \bar{\partial}) = 0$ for any open set $G \subset \mathbf{C}^n$. In this case we denote: $X_{[a]}(F) = \{x, x \in X, sp(a, x) \subset F\}$ ($F \subset \mathbf{C}^n$ closed), $X_{[a]}(F) = \{x, x \in X, \sigma(a, x) \subset F\}$.

DEFINITION 0.11. Let X be a Banach space, let $\mathrm{S}(X)$ be the family of the closed linear subspaces of X, let $S \subset \mathbf{C}^n$ be a compact set and let F_S be the family of closed sets $F \subset \mathbf{C}^n$ that have the property: either $F \cap S = \varnothing$ or $F \supset S$.

We shall call *S-spectral capacity* an application $\mathrm{E} : \mathrm{F}_S \to \mathrm{S}(X)$ that meets the properties:

1°. $\mathrm{E}(\varnothing) = \{0\}$, $\mathrm{E}(\mathbf{C}^n) = X$;

2°. $\mathrm{E}\left(\bigcap_{i=1}^{\infty} F_i\right) = \bigcap_{i=1}^{\infty} \mathrm{E}(F_i)$ for any series $\{F_i\}_{i \in N} \subset \mathrm{F}_S$;

3°. for any open finite S-covering $\{G_S\} \cup \{G_j\}_{j=1}^{m}$ of $\mathbf{C}^n$ we have

$$X = \mathrm{E}(\overline{G_S}) + \sum_{j=1}^{m} \mathrm{E}(\overline{G_j}).$$

A commuting system of operators $a=(a_1,a_2,...,a_n)\subset B(X)$ is said to be *S-decomposable* if there exists a spectral *S*-capacity such that

4°. $a_j\mathsf{E}(F)\subset\mathsf{E}(F)$ for any $F\in\mathsf{F}_S$ and for any *j*;

5°. $\sigma(a,\mathsf{E}(F))\subset F$ for any $F\in\mathsf{F}_S$.

In case $S=\varnothing$, the spectral *S*-capacity is said to be a spectral capacity, and the system is decomposable. We must notice that for systems of operators having $n\geq 2$, one doesn't know whether the definition of the *S*-decomposability (and of the decomposability) given for an operator (see [37]) is equivalent with definition 0.11. or not.

RESTRICTIONS AND QUOTIENTS OF DECOMPOSABLE OPERATORS

1.1. RESTRICTIONS AND QUOTIENS OF OPERATORS. RELATIONS BETWEEN THEIR SPECTRUM

We shall start with a paragraph that contains results regarding the relations between the spectrum of an operator and the spectrum of the restrictions and quotients regarding an invariant subspace, the corresponding relations between the local spectrum and the analytic residues as well as some results on particular invariant subspaces.

1.1.1. Proposition. *Let $T\in B(X)$, let Y be an invariant subspace of T and let $\dot{T}$ be the operator induced by T on the quotient space $\dot{X}=X/Y$. Then we have:*

(i) $S_{\dot{T}}\subset S_T\cup\sigma(T\mid Y)$,

(ii) $S_T\subset S_{\dot{T}}\cup\sigma(T\mid Y)$,

(iii) $S_{T\mid Y}\subset S_T\cap\sigma(T\mid Y)$,

(iv) $\gamma_{\dot{T}}(x)\subset\gamma_T(x)\subset\gamma_{\dot{T}}(\dot{x})\cup\sigma(T\mid Y)$,

(v) $\sigma_{\dot{T}}(\dot{x})\subset\sigma_T(x)\cup\sigma(T\mid Y)$; $\sigma_T(x)\subset\sigma_{\dot{T}}(\dot{x})\cup\sigma(T\mid Y)$ $(x\in\dot{x})$.

Proof. Proof of (i) is given in 2.7. [80]. Let now $\omega\subset\Omega_{\dot{T}}\cap\rho(T\mid Y)$ be an open set and let $x(\lambda):\omega\to X$ be an analytic function such that for $\lambda\in\omega$ we have

$$(\lambda I-T)x(\lambda)\equiv 0.$$

Then, for $\lambda\in\omega$, we have

$$(\lambda I-\dot{T})\dot{\overline{x(\lambda)}}\equiv\dot{0},$$

hence $\dot{\overline{x(\lambda)}}=\dot{0}$, consequently $x(\lambda)\in Y$. It will follow that $(\lambda I-T)x(\lambda)=(\lambda I-T\mid Y)x(\lambda)$ and $(\lambda I-T\mid Y)^{-1}(\lambda I-T\mid Y)x(\lambda)=x(\lambda)\equiv 0$, hence

$$S_T\subset S_{\dot{T}}\cup\sigma(T\mid Y).$$

Inclusion (iii) is obvious. We shall verify (iv). Let $f_x:\omega\to X$ be an analytic function ($\omega\subset\delta_T(x)$ is an open set) such that $(\lambda I-T)f_x(\lambda)=x$. Then $(\lambda I-\dot{T})\dot{\overline{f_x(\lambda)}}=\dot{x}$ with $\dot{\overline{f_x(\lambda)}}$ analytic on ω, hence $\delta_T(x)\subset\delta_{\dot{T}}(\dot{x})$ and $\gamma_{\dot{T}}(\dot{x})\subset\gamma_T(x)$. In order to verify the inclusion $\gamma_T(x)\subset\gamma_{\dot{T}}(\dot{x})\cup\sigma(T\mid Y)$, let $\dot{f}:\omega\to\dot{X}$ be an analytic function; then, for

$\lambda \in \omega$, there exists a neighbourhood ω_λ of λ, $\omega_\lambda \subset \omega$ and an analytic function $f : \omega_\lambda \to X$ such that $\dot{\overline{f(\mu)}} = \dot{f}(\mu)$ for $\mu \in \omega_\lambda$ (see [79] lemma 2.1.). If $\lambda_0 \in \omega \subset \delta_{\dot{T}}(\dot{x}) \cap \rho(T \mid Y)$ and

$$\left(\lambda I - \dot{T}\right)\dot{f}_{\dot{x}}(\lambda) = \dot{x}$$

with $\dot{f}_{\dot{x}}$ analytic on ω (ω open), then there exist the analytic functions $f_x(\lambda)$ and $y(\lambda)$ on $\omega_1 \subset \omega$ such that

$$(\lambda I - T)f_x(\lambda) = x + y(\lambda),$$

where $x \in \dot{x}$, $\dot{\overline{f_x(\lambda)}} = \dot{f}_{\dot{x}}(\lambda)$ and $y(\lambda) \in Y$. By denoting $Z(\lambda) = (\lambda I - T \mid Y)^{-1} y(\lambda)$ we obtain

$$(\lambda i - T)(f_x(\lambda) - Z(\lambda)) = x,$$

hence $\lambda_0 \in \delta_T(x)$ and $\gamma_T(x) \subset \gamma_{\dot{T}}(\dot{x}) \cup \sigma(T \mid Y)$. One also obtains $\sigma_{\dot{T}}(\dot{x}) = \gamma_{\dot{T}}(\dot{x}) \cup$ $\cup S_{\dot{T}} \subset \gamma_T(x) \cup S_T \cup \sigma(T \mid Y) = \sigma_T(x) \cup \sigma(T \mid Y)$ and $\sigma_T(x) = \gamma_T(x) \cup S_T \subset \gamma_{\dot{T}}(\dot{x}) \cup S_{\dot{T}} \cup$ $\cup \sigma(T \mid Y) = \sigma_{\dot{T}}(\dot{x}) \cup \sigma(T \mid Y)$, that is (v).

1.1.2. Proposition. *If $T \in B(X)$ and Y is an invariant subspace of T, then we have:*

(i) $\sigma(T) \subset \sigma(\dot{T}) \cup \sigma(T \mid Y)$,

(ii) $\sigma(\dot{T}) \subset \sigma(T) \cup \sigma(T \mid Y)$,

(iii) $\sigma(T \mid Y) \subset \sigma(T) \cup \sigma(\dot{T})$.

Proof. The inclusions result from lemma 1.2. [70], but we shall furthermore give an elementary proof. (i) and (ii) result easily from the equality

$$\sigma(T) = \bigcup_{x \in X} \sigma_T(x)$$

([76] corollary 1) and from the preceding proposition. We shall verify (iii). Let us take $\lambda \in \rho(T) \cap \rho(\dot{T})$; if $R(\lambda, T) \not\subset Y$, then there exists $y_0 \in Y$ such that

$$R(\lambda, T)y_0 = z_0 \notin Y.$$

In other words $\dot{y}_0 = \left(\lambda I - \dot{T}\right)\dot{z}_0 = \dot{0}$. Since $\lambda \in \rho(\dot{T})$, it results that $\dot{z}_0 = \dot{0}$ and hence $z_0 \in Y$; contradiction! Consequently $R(\lambda, T)Y \subset Y$ and

$$\sigma(T \mid Y) \subset \sigma(T) \cup \sigma(\dot{T}).$$

1.1.3. Corollary. *Considering the premises from the preceding proposition, by denoting by D_∞ the unbounded component of $\rho(T)$ and by D_n $(n \in N)$ the bounded components, we have:*

1°. $D_\infty \cap \sigma(\dot{T}) = \varnothing$,

2°. $D_n \subset \sigma(\dot{T})$

if and only if $D_n \subset \sigma(T \mid Y)$ (meaning if and only if there exists $\lambda_0 \in D_n$ such that $R(\lambda_0, T)Y \not\subset Y$).

Proof. It results from the preceding proposition and from proposition 4.11. [41].

1.1.4. COROLLARY. *Considering the same premises as in proposition 1.1.2. we have*

(i) $\sigma(T)\setminus\sigma(T\mid Y)=\sigma(\dot{T})\setminus\sigma(T\mid Y)$,
(ii) $\sigma(T)\setminus\sigma(\dot{T})=\sigma(T\mid Y)\setminus\sigma(\dot{T})$,
(iii) $\sigma(\dot{T})\setminus\sigma(T)=\sigma(T\mid Y)\setminus\sigma(T)$.

If we denote $\sigma_1=\sigma(T)$, $\sigma_2=\sigma(\dot{T})$, $\sigma_3=\sigma(T\mid Y)$, *then*

$$\bigcup_{i=1}^{3}\sigma_i=\bigcap_{\substack{i,j=1\\ i\neq f}}^{3}(\sigma_i\cup\sigma_j).$$

Proof. It results from proposition 1.1.2.

1.1.5. *Remark.* If $\operatorname{Int}(\sigma(T)\cap\sigma(T\mid Y))=\varnothing$ (or in other words $\dim(\sigma(\dot{T})\cap\sigma(T\mid Y))\leq 1$), and T has the single valued extension property then $\dot{T}$ also has the single valued extension property; indeed, by proposition 2.7. [80] we have $S_{\dot{T}}\subset$ $\subset\sigma(T\mid Y)\cup S_T=\sigma(T\mid Y)$, whence $S_{\dot{T}}\subset\sigma(T\mid Y)\cap\sigma(\dot{T})$, meaning $\operatorname{Int}S_{\dot{T}}=\varnothing$; consequently $S_{\dot{T}}=\varnothing$ and $\dot{T}$ has the single valued extension property.

1.1.6. Proposition. *Let $T\in B(X)$ an let Y be an invariant subspace of T. Then the equality*

$$\sigma(T)=\sigma(\dot{T})\cup\sigma(T\mid Y)$$

is true in each of the following cases:

(1) $\sigma(T\mid Y)\subset\sigma(T)$,
(2) $\rho(T)$ *is connected,*
(3) $\sigma(T\mid Y)\cap\sigma(\dot{T})=\varnothing$,
(4) $\dot{T}$ *has the single valued extension property.*

Proof. (1) If $\sigma(T\mid Y)\subset\sigma(T)$, then, according to proposition 1.1.2., one obtains $\sigma(\dot{T})\subset\sigma(T\mid Y)\cup\sigma(T)=\sigma(T)$, hence $\sigma(T\mid Y)\cup\sigma(\dot{T})\subset\sigma(T)$ and accordingly $\sigma(\dot{T})\cup\sigma(T\mid Y)=\sigma(T)$. (2) $\rho(T)$ connected, it results $D_\infty=\rho(T)$, $D_n=\varnothing$ ($n\in N$, see 4.11. [41] and corollary 1.1.3.), hence $\sigma(T\mid Y)\subset\sigma(T)$. (3) When $\sigma(T\mid Y)\cap\sigma(\dot{T})=\varnothing$ we shall have again $\sigma(T\mid Y)\subset\sigma(T)$, because otherwise there will exist a bounded component $D_n\neq 0$ of $\rho(T)$ with $D_n\subset\sigma(T\mid Y)$ and according to corollary 1.1.3. we also have $D_n\subset\sigma(\dot{T})$, $D_n\subset\sigma(T\mid Y)\cap\sigma(\dot{T})=\varnothing$, contradiction. (4) If $S_{\dot{T}}=\varnothing$, from $\gamma_{\dot{T}}(\dot{x})=\sigma_{\dot{T}}(\dot{x})$ and $\sigma_{\dot{T}}(\dot{x})\subset\gamma_T(x)\subset\gamma_T(x)\cup S_T=\sigma_T(x)$ (proposition 1.1.2.) one obtains

$$\sigma(\dot{T})=\bigcup_{\dot{x}\in\dot{X}}\sigma_{\dot{T}}(\dot{x})\subset\bigcup_{x\in X}\sigma_T(x)=\sigma(T)$$

hence $\sigma(\dot{T})\subset\sigma(T)$, meaning $\sigma(T)=\sigma(\dot{T})\cup\sigma(T\mid Y)$.

1.1.7. *Remark.* Let $T\in B(X)$ and let Y be an invariant subspace of T. From the ones above it follows that if $\sigma(T\mid Y)\not\subset\sigma(T)$, then $\sigma(\dot{T})\not\subset\sigma(T)$ and consequently $S_{\dot{T}}\neq\varnothing$; it results that if T is decomposable and $\sigma(T\mid Y)\not\subset\sigma(T)$, then $\dot{T}$ hasn't the single valued extension property (particularly $\dot{T}$ is not decomposable). Hence we have the possibility to obtain operators that don't have the single valued extension property through factorisations with subspaces that are not σ-stabile for T (meaning invariant of T and $\sigma(T \mid Y) \subset \sigma(T)$).

1.1.8. Definition. Let $T \in B(X)$; a subspace $Y \subset X$ is said to be *T-absorbing* if for any $x \in Y$, the equation $(\lambda I - T)y = x$ has solutions y only in Y for any $\lambda \in \sigma(T \mid Y)$. We remind that any spectral maximal space Z of T is T-absorbing ([76] definition 3.2. and proposition 3.1.).

1.1.19. Proposition. *Let $T \in B(X)$ and let Y be an invariant subspace of T, T-absorbing with $\sigma(T \mid Y) \supset S_T$ (particularly $S_T = \emptyset$). Then $\dot{T}$, the operator induced by T on $\dot{X} = X / Y$ has the single valued extension property ($S_{\dot{T}} = \emptyset$).*

Proof. We have $S_{\dot{T}} \subset S_T \subset \sigma(T \mid Y)$ (proposition 1.1.1.) hence it will be suffice to prove that $\operatorname{Int}\sigma(T \mid Y) \subset \Omega_{\dot{T}}$. Let $G \subset \operatorname{Int}\sigma(T \mid Y)$ be an open set and let $\dot{f}(\lambda)$ be an analytic function on G such that $(\lambda I - \dot{T})\dot{f}(\lambda) = \dot{0}$ for $\lambda \in G$. Then there exists an open set $G_1 \subset G$ and an analytic function $f(\lambda)$ on G_1 so that $\dot{\overline{f(\lambda)}} = \dot{f}(\lambda)$ and $(\lambda I - T)f(\lambda) = y(\lambda)$ with $y(\lambda) \in Y$ ([79] lemma 2.1.). Since Y is T-absorbing and $\lambda \in G_1 \subset \sigma(T \mid Y)$, one obtains $\dot{f}(\lambda) \in Y$, $\dot{f}(\lambda) = \dot{0}$ on G_1, meaning $\dot{f}(\lambda) = \dot{0}$; it follows $S_{\dot{T}} = \emptyset$ and $\dot{T}$ has the single valued extension property.

1.1.10. Corollary. *Let $T \in B(X)$ with the single valued extension property and let Y be a spectral maximal space of T. Then $\dot{T}$ has the single valued extension property.*
The corollary above was observed by Şt. Frunză.

1.1.11. Proposition. *Let $T \in B(X)$ and let Y be an T-absorbing, invariant subspace of T. Then Y is σ-stabile for T and*

$$\sigma(T) = \sigma(T \mid Y) \cup \sigma(\dot{T}).$$

Proof. If $D_n (n \in N)$ is a connected component of $\rho(T)$ and $\lambda_0 \in D_n \subset c(T \mid Y)$, then $R(\lambda_0, T)Y \not\subset Y$ ([41], theorem 4.11.), hence $(\lambda_0 I - T)^{-1} y = z \notin Y$ for at least a single y from Y; but Y is T-absorbing and hence $y = (\lambda_0 I - T)z$ implies $z \in Y$; contradiction! Consequently $D_n = \emptyset$, $\sigma(T \mid Y) \subset \sigma(T)$ and $\sigma(T) = \sigma(T \mid Y) \cup \sigma(\dot{T})$.

1.1.12. Lemma. *If $T \in B(X)$ has the single valued extension property and $X = Y_1 + Y_2 + \ldots + Y_n$ where Y_i ($i = 1, 2, \ldots, n$) are spectral maximal spaces of T, then*

$$\sigma(T) = \bigcup_{i=1}^{n} \sigma(T \mid Y_i).$$

Proof. We have

$$\sigma(T) = \bigcup_{x \in X} \sigma_T(x)$$

and

$$\sigma(T \mid Y_i) = \bigcup_{x \in Y_i} \sigma_{T|Y_i}(x) = \bigcup_{x \in Y_i} \sigma_T(x)$$

since $\sigma_{T|Y_i}(x) = \sigma_T(x)$ if $x \in Y_i$ ([37], I, 3.5.), hence

$$\sigma(T)=\bigcup_{x\in X}\sigma_T(x)\subset\bigcup_{i=1}^{n}\left(\bigcup_{y_i\in Y_i}\sigma_T(y_i)\right)=\bigcup_{i=1}^{n}\sigma(T\mid Y_i)\subset\sigma(T).$$

1.1.13. LEMMA. *Let X be a Banach space, and let X_1, X_2 two linear closed subspaces of X such that $X_1\cap X_2=\{0\}$ and X_1+X_2 is closed. If $Y_i\subset X_i$ $(i=1,2)$ are two linear closed subspaces, then Y_1+Y_2 is also closed*

Proof. Indeed, if $y_n\in Y_1+Y_2$, $y_n\to y$ then $y_n=y_n^1+y_n^2$, $y_n^i\in Y_i$, $(i=1,2)$; since X_1+X_2 is closed, by the closed graph it follows $y_n^i\to y^i\in Y_i$ $(i=1,2)$ hence $y\in Y_1+Y_2$.

1.1.14. COROLLARY. *If T is decomposable and Y_1, Y_2 are two invariant subspaces of T such that*

$$\sigma(T\mid Y_1)\cap\sigma(T\mid Y_2)=\varnothing$$

then Y_1+Y_2 is closed.

Proof. We have $Y_1\subset X_T(\sigma(T\mid Y_1))$, $Y_2\subset X_T(\sigma(T\mid Y_2))$, $X_T(\sigma(T\mid Y_1))+$ $+X_T(\sigma(T\mid Y_2))=X_T(\sigma(T\mid Y_1)\cup\sigma(T\mid Y_2))$ ([4], 2.3.); the last space being closed, it follows by the preceding lemma that Y_1+Y_2 is closed.

1.1.15. *Remark.* Considering the premises of the preceding corollary, if we denote $\dot{X}=X/Y_1$ and $\varphi: X\to\dot{X}$ the canonical application, it follows that Y_2 can be identified with $\dot{Y}_2=\varphi(Y_2)$ since Y_2 and $\dot{Y}_2$ are (topologically) isomorphic, and $T\mid Y_2$ and $T\mid\dot{Y}_2$ are similar and $\sigma(\dot{T}_2\mid\dot{Y}_2)=\sigma(T\mid Y_2)$.

1.1.16. LEMMA. *Let $T\in B(X)$, let Y be an invariant subspace of T and $\dot{X}=X/Y$, where T, φ are same as above. If $\dot{Z}$ is an invariant subspace of $\dot{T}$ with $\sigma(\dot{T}\mid\dot{Z})\cap\sigma(T\mid Y)=\varnothing$, then one can find an invariant subspace of T, Z (topologically) isomorphic with $\dot{Z}=\varphi(Z)$ and $\sigma(T\mid Z)=\sigma(\dot{T}\mid\dot{Z})$.*

Proof. Since $\varphi^{-1}(\dot{Z})/Y=\dot{Z}$ and according to proposition 1.1.6. it follows that

$$\sigma(T\mid\varphi^{-1}(\dot{Z}))=\sigma(\dot{T}\mid\dot{Z})\cup\sigma(T\mid Y).$$

We also have $\varphi^{-1}(\dot{Z})=\dot{Z}+Y'$, where $\sigma(T\mid Z)=\sigma(\dot{T}\mid\dot{Z})$, $\sigma(T\mid Y)=\sigma(T\mid Y')$. Since Y' is a spectral maximal space of $T\mid\varphi^{-1}(\dot{Z})$ ([37], I.3.10.), it follows that $Y\subset Y'$. But on the other hand

$$\sigma(\dot{T}\mid\varphi(Y'))\subset\sigma(T\mid Y),$$

hence $\sigma(\dot{T}\mid\varphi(Y))\cap\sigma(\dot{T}\mid\dot{Z})=\varnothing$, whence $\varphi(Y')=\{\dot{0}\}$ and $Y'=Y$. Our affirmation follows now from lemma 1.1.13. and by the preceding remark.

1.1.17. COROLLARY. *If in preceding lemma T is decomposable and $\dot{Z}$ is a spectral maximal space of $\dot{T}$, then Z is also a spectrd maximal space of T*

Proof. Let W be an invariant subspace of T such that $\sigma(T\mid W)\subset\sigma(T\mid Z)=$ $=\sigma(\dot{T}\mid\dot{Z})$. Since $\sigma(T\mid W)\cap\sigma(T\mid Y)=\varnothing$, from remark 1.1.15. it follows that $\sigma(\dot{T}\mid\varphi(W))=\sigma(T\mid W)$, hence $\varphi(W)\subset\dot{Z}$ meaning $W\subset Z$.

1.1.18. *Remark.* (a) Let $T \in B(X)$ and let Y_1, Y_2 be two invariant subspaces of T such that

$$\sigma(T \mid Y_1) \cap \sigma(T \mid Y_2) = \varnothing .$$

Then we have $Y_1 \cap Y_2 = \{0\}$ in each of the following cases: 1°) T has the single valued extension property (particularly T is decomposable); 2°) Y_1, Y_2 are T-absorbing, invariant subspaces of T (particularly Y_1, Y_2 are spectral maximal spaces for T); 3°) $\sigma(T \mid Y_1) \subset D_\infty^2$ and $\sigma(T \mid Y_2) \subset D_\infty^1$, where D_∞^i is the unbounded component of the resolving $\rho(T \mid Y_i)$ ($i = 1,2$). Indeed, if $S_T = \varnothing$, we have

$$Y_1 \cap Y_2 \subset X_T(\sigma(T \mid Y_1)) \cap X_T(\sigma(T \mid Y_2)) =$$
$$= X_T(\sigma(T \mid Y_1) \cap \sigma(T \mid Y_2)) = X_T(\varnothing) = \{0\};$$

2°) results by the fact that the intersection of two T-absorbing subspaces Y_1, Y_2 is a $T \mid Y_i$- absorbing subspace ($i = 1,2$). Indeed, let $(\lambda I - T \mid Y_1)y = x$ with $x \in Y_1 \cap Y_2$ and $y \in Y_1$, hence $(\lambda I - T)y = x$; since $x \in Y_2$ and Y_2 is T-absorbing, it follows that $y \in Y_2$, $y \in Y_1 \cap Y_2$ and consequently $Y_1 \cap Y_2$ is $T \mid Y_1$-absorbing. In accordance with 2.19. [21], if Y is a T-absorbing, invariant subspace of T, then $\sigma(T \mid Y) \subset \sigma(t)$, hence

$$\sigma(T \mid Y_1 \cap Y_2) \subset \sigma(T \mid Y_1) \cap \sigma(T \mid Y_2) = \varnothing ,$$

whence $Y_1 \cap Y_2 = \{0\}$. For 3°) we notice that $Y_1 \cap Y_2$ is an invariant subspace of $T \mid Y_1$ and $T \mid Y_2$ and according to proposition 5.4.11. [41] we have

$$\sigma(T \mid Y_1 \cap Y_2) \subset \mathbf{C}D_\infty^1 \cap \mathbf{C}D_\infty^2 = \varnothing .$$

(b) If T is decomposable and Y_1, Y_2 are two spectral maximal spaces of T such that $Y_1 \cap Y_2 = \{0\}$, then $\dim(\sigma(T \mid Y_1) \cap \sigma(T \mid Y_2)) \le 1$; when $\sigma(T)$ is on a curve, then $\dim(\sigma(T \mid Y_1) \cap \sigma(T \mid Y_2)) \le 0$. It follows by lemma II.4.3. [37].

1.1.19. THEOREM. *Let $T \in B(X)$ be a decomposable operator, let Y be an invariant subspace of T, let $\dot{T}$ be the operator induced by T in the quotient space $\dot{X} = X / Y$ and let $\varphi : X \to \dot{X}$ be the canonical map. Then for any closed set $F \subset \mathbf{C}$ such that*

$$F \supset \sigma(T \mid Y) \text{ or } F \cap \sigma(T \mid Y) = \varnothing ,$$

we can say that $\varphi(X_T(F))$ is a spectral maximal space for $\dot{T}$. Conversely, if $\dot{Z}$ is a spectral maximal space of $\dot{T}$ such that

$$\sigma(\dot{T} \mid \dot{Z}) \cap S = \varnothing \text{ or } \sigma(\dot{T} \mid \dot{Z}) \supset S$$

(where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$), then there exists a spectral maximal space Z of T such that $\varphi(Z) = \dot{Z}$.

Proof. First of all we assume that $F \supset \sigma(T \mid Y)$. Then $Y \subset X_T(F)$ and according to proposition 1.1.1., $\gamma_{\dot{T}}(\dot{x}) \subset \gamma_T(x) = \sigma_T(x) \subset \sigma_{\dot{T}}(\dot{x}) \cup \sigma(T \mid Y)$ and $S_{\dot{T}} \subset \sigma(T \mid Y) \cap \sigma(\dot{T})$, hence

$$\varphi(X_T(F)) \subset \dot{X}_{\dot{T}}(F) \subset \varphi(X_T(F \cup \sigma(T \mid Y))) = \varphi(X_T(F)).$$

Consequently $\dot{X}_{\dot{T}}(F)$ is closed and it is a spectral maximal space of $\dot{T}$ (proposition 3.4. [76]). Let now $F\cap\sigma(T\mid Y)=\varnothing$. Then $\sigma(T\mid X_T(F))\cap\sigma(T\mid Y)=\varnothing$ and by corollary 1.1.14. and remark 1.1.15. it follows that $\varphi(X_T(F))$ is closed and

$$\sigma(\dot{T})\mid\varphi(X_T(F))=\sigma(T\mid X_T(F)).$$

If $\dot{W}$ is an invariant subspace of $\dot{T}$ such that

$$\sigma(\dot{T}\mid\dot{W})\subset\sigma(\dot{T}\mid\varphi(X_T(F)))$$

then, according to lemma 1.1.16., there exists an invariant subspace for *T*, *W* with $\varphi(W)=\dot{W}$ and $\sigma(T\mid W)=\sigma(\dot{T}\mid\dot{W})$, hence $W\subset X_T(F)$; consequently $\dot{W}\subset\varphi(X_T(F))$, meaning $\varphi(X_T(F))$ is a spectral maximal space of $\dot{T}$. Conversely, let $\dot{Z}$ be a spectral maximal space of $\dot{T}$. If $\sigma(\dot{T}\mid\dot{Z})\cap S=\varnothing$ (hence $\sigma(\dot{T}\mid\dot{Z})\cap\sigma(T\mid Y)=\varnothing$), then, according to corollary 1.1.17., there exists a spectral maximal space of *T*, *Z* such that $\varphi(Z)=\dot{Z}$. When $\sigma(\dot{T}\mid\dot{Z})\supset S$, we have

$$\varphi\left(X_T\left(\sigma(\dot{T}\mid\dot{Z})\cup\sigma(T\mid Y)\right)\right)=\dot{X}_{\dot{T}}\left(\sigma(\dot{T}\mid\dot{Z})\cup\sigma(T\mid Y)\right)=$$
$$=\dot{X}_{\dot{T}}\left(\sigma(\dot{T}\mid\dot{Z})\right)=\dot{Z}$$

and for $Z=X_T\left(\sigma(\dot{T}\mid\dot{Z})\cup\sigma(T\mid Y)\right)$.

1.1.20. COROLLARY. *Considering the premises of the preceding theorem, we have for any closed $F\supset\sigma(T\mid Y)$ the equality*

$$\varphi(X_T(F))=\dot{X}_{\dot{T}}(F).$$

If $S_{\dot{T}}=\varnothing$, we also have the above equality in case $F\cap\sigma(T\mid Y)=\varnothing$.

Proof. If $F\supset\sigma(T\mid Y)$ the equality is verified during the preceding proof. Let now $F\cap\sigma(T\mid Y)=\varnothing$ and $S_{\dot{T}}=\varnothing$. It follows:

$$\varphi\left(X_T(F\cup\sigma(T\mid Y))\right)=\varphi(X_T(F))\dot{+}\varphi\left(X_T(\sigma(T\mid Y))\right)=$$
$$=\dot{X}_{\dot{T}}(F\cup\sigma(T\mid Y))=\dot{X}_{\dot{T}}(F)\dot{+}\dot{X}_{\dot{T}}(\sigma(T\mid Y))$$

(see [4], 2.3.) and since $\varphi\left(X_T(\sigma(T\mid Y))\right)=\dot{X}_{\dot{T}}(\sigma(T\mid Y))=\dot{0}$ we have $\varphi(X_T(F))=\dot{X}_{\dot{T}}(F)$.

1.1.21. *Remark.* From the proof of theorem 1.1.19. and corollary 1.1.20. one can see that those stay true if *T* is believed to be only two-decomposable.

1.1.22. COROLLARY. *Let $T\in B(X)$ be a spectral (scalar) operator [respectively $\mathcal{U}$-scalar, generalised spectral (scalar)] and let Y be an invariant subspace of T. If $\dot{Z}$ is a spectral maximal space of $\dot{T}$ such that*

$$\sigma(\dot{T}\mid\dot{Z})\cap\sigma(T\mid Y)=\varnothing.$$

Then $\dot{T}\mid\dot{Z}$ is a spectral (scalar) [respectively $\mathcal{U}$-scalar, generalised spectral (scalar)] operator.

Proof. According to theorem 1.1.19., there exists a spectral maximal space of *T*, *Z* such that

$$\sigma(T\mid Y)\cap\sigma(T\mid Y)=\varnothing,$$

$\dot{Z}=\varphi(Z)$ ($\varphi: X\to\dot{X}=X/Y$ is the canonical map) and

$$\sigma(\dot{T}\mid\dot{Z})=\sigma(T\mid Z).$$

Since $Z \cap Y \subset X_T(\sigma(T \mid Z)) \cap X_T(\sigma(T \mid Y)) = X_T(\sigma(T \mid Z) \cap \sigma(T \mid Y)) = X_T(\varnothing) = \{0\}$, the map $U = \varphi \mid Z$ is bijective, hence U is bicountinuous (according to the closed graph theorem). On the other hand, $T \mid Z$ and $\dot{T} \mid \dot{Z}$ are similar (one can write $T \mid Z = U^{-1}(\dot{T} \mid \dot{Z})U$) hence $\dot{T} \mid \dot{Z}$ is spectral (scalar) [respectively $\mathfrak{U}$-scalar, generalised spectral (scalar)].

1.2. RESTRICTIONS AND QUOTIENTS OF DECOMPOSABLE OPERATORS

We shall continue with a paragraph that refers to the restrictions and quotients of a decomposable (strongly decomposable or spectral) operator regarding an invariant subspace, and we shall study particularly the case in which the invariant subspace is a spectral maximal space for the operator. Thus we try to give answer to an open problem asserted in [37], 6.5. For operators with spectra belonging to the class C one can prove that decomposability implies strongly decomposability. We also study the particular case in which $\dim S = 0$, where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$.

1.2.1. Theorem. *Let $T \in B(X)$ be a decomposable operator and let Y be an invariant subspace for T. Then $\dot{T}$, the operator induced by T in the quotient space $\dot{X} = X / Y$ is a S-decomposable operator, where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$.*

Proof. Let $\{G_S\} \cup \{G_i\}_{i=1}^n$ be an open finite S-covering of $\sigma(\dot{T})$. If we put $G'_S = G_S \cup \rho(T)$ and $G'_i = G_i \cup \rho(T \mid Y)$ we shall obtain an open finite covering $\{G'_S\} \cup \{G'_i\}_1^n$ of $\sigma(T)$ (since $\sigma(T) \subset \sigma(T \mid Y) \cup \sigma(\dot{T})$). Then there exists a system $\{Y_S\} \cup \{Y_i\}_1^n$ of spectral maximal spaces of T such that

$$\sigma(T \mid Y_S) \subset G'_S,$$
$$\sigma(T \mid Y_i) \subset G'_i \ (i = 1,2,\dots,n)$$

and

$$X = Y_S + \sum_{i=1}^n Y_i .$$

But from lemma 1.1.12. it follows that

$$\sigma(T) = \sigma(T \mid Y_S) \cup \left(\bigcup_{i=1}^n \sigma(T \mid Y_i) \right)$$

and since $\sigma(T \mid Y_i) \cap \sigma(T \mid Y) \subset G'_i \cap \sigma(T \mid Y) = \varnothing$ ($i = 1,2,\dots,n$) we shall obtain

$$\sigma(T \mid Y) \cap \sigma(T) \subset \sigma(T \mid Y_S),$$

hence

$$Y \subset X_T(\sigma(T \mid Y)) \subset X_T(\sigma(T \mid Y_S)) = Y_S .$$

According to the theorem 1.1.19., $\dot{Y}_S = \varphi(Y_S)$ and $Y'_i = \varphi(Y_i)$ are spectral maximal spaces of $\dot{T}$ and we have that $\dot{Y}_S = \varphi(X_T(\sigma(T \mid Y_S))) = \dot{X}_{\dot{T}}(\sigma(T \mid Y_S) \cup \sigma T \mid Y)$, hence

$$\sigma(\dot{T} \mid \dot{Y}_S) \subset (\sigma(T \mid Y_S) \cup \sigma(T \mid Y)) \cap \sigma(\dot{T}) =$$
$$= (\sigma(T \mid Y_S) \cap \sigma(\dot{T})) \cup S \subset (G'_S \cap \sigma(\dot{T})) \cup S \subset G_S$$

and

$$\sigma(\dot{T} \mid \dot{Y}_i) = \sigma(T \mid Y_i) \subset G'_i \subset G_i \ (i = 1,2,\ldots,n).$$

Finally, we also have

$$\dot{X} = \dot{Y}_S + \sum_{i=1}^{n} \dot{Y}_i ,$$

hence $\dot{T}$ is S-decomposable.

1.2.2. Lemma. *Let $T \in B(X)$, and let Y be an invariant subspace of T and let $\dot{T}$ be the operator induced by T in the quotient space $\dot{X} = X / Y$. If T and $\dot{T}$ have the single valued extension property, then*

$$Y_1 = X_T\left(\sigma(T \mid Y) \setminus \sigma(\dot{T})\right) \subset Y .$$

Proof. If $x \in Y_1$, we have $\sigma_T(x) \subset \sigma(T \mid Y) \setminus \sigma(\dot{T})$ and $\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x) \cap \sigma(\dot{T}) \subset$ $\subset (\sigma(T \mid Y) \setminus \sigma(\dot{T})) \cap \sigma(\dot{T}) = \varnothing$, hence $\dot{x} = \dot{0}$ and consequently $x \in Y$.

1.2.3. Proposition. *Let $T \in B(X)$ be a decomposable operator, let Y be an invariant subspace of T such that $S_{\dot{T}} = \varnothing$ and $\sigma = \sigma(T \mid Y) \setminus \sigma(\dot{T}) \neq \varnothing$. Then we have $T \mid Y \in \mathrm{D}_S(Y)$, where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$.*

Proof. If $\sigma \neq \varnothing$, then $X_T(\sigma) \neq \{0\}$ (where σ is an open set in $\sigma(T)$; see lemma II.1.2.[37]). According to the corollary 1.1.4. we have

$$\sigma(T) \setminus \sigma(\dot{T}) = \sigma(T \mid Y) \setminus \sigma(\dot{T}) = \sigma$$

and by the preceding lemma it follows that $X_T(\sigma) \subset Y$. Let $\{G_S\} \cup \{G_i\}_1^n$ be an open finite S-covering of $\sigma(T \mid Y)$. By denoting $G'_i = G_i \cap \rho(\dot{T})$ and $G'_S = G_S \cup \rho(T \mid Y)$ one obtains an open finite covering $\{G'_S\} \cup \{G_i\}_1^n$ of $\sigma(T)$. Since T is decomposable, there exists a system of spectral maximal spaces $\{Y_S\} \cup \{Y_i\}_1^n$ of T such that

$$\sigma(T \mid Y_S) \subset G'_S,\ \sigma(T \mid Y_i) \subset G'_i \ (i = 1,2,\ldots,n),$$

$$X = Y_S + \sum_{i=1}^{n} Y_i .$$

But $\sigma(T \mid Y_i) \subset G'_i \cap \sigma(T) = G_i \cap \rho(\dot{T}) \cap \sigma(T) = G_i \cap \sigma \subset \sigma$, hence $Y_i \subset X_T(\sigma) \subset Y$ ($i = 1,2,\ldots,n$). If $x \in Y$, then

$$x = y_S + y_1 + \ldots + y_n$$

with $y_S \in Y_S$, $y_i \in Y_i$ ($i = 1,2,\ldots,n$), hence $y_S = x - (y_1 + y_2 + \ldots + y_n) \in Y$. It follows that

$$Y = Y'_S + \sum_{i=1}^{n} Y_i ,$$

where $Y'_S = Y_S \cap Y$ and $\sigma(T \mid Y'_S) \subset \widetilde{G}_S$, hence $T \mid Y \in \mathrm{D}_S(Y)$.

1.2.4. Corollary. *Having the same premises as in the preceding proposition, the restriction $T \mid Y$ is a S-residual decomposable operator.*

Proof. It follows by the definition of the class $\mathsf{D}_S(Y)$.

1.2.5. Corollary. *If $T \in B(X)$ is a decomposable operator and Y is a spectral maximal space of T, then both $T \mid Y$ and $\dot{T}$ are S-decomposable operators, where $S = \partial\sigma(T \mid Y) \cap \sigma(\dot{T})$, $\operatorname{Int} S = \varnothing$ and $S_{\dot{T}} = \varnothing$ (∂A is the frontier of A).*

Proof. It follows by theorem 1.2.1., by the proof of proposition 1.2.3. (because in this case $Y_S' = Y_S \cap Y$ is a spectral maximal space of T and $\sigma(T \mid Y_S') \subset G_S$) and by applying the relation $\sigma(\dot{T}) = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$ ([2], 1.4.) as well as the remark 1.1.5.

1.2.6. Proposition. *If $T \in B(X)$ is a S-decomposable operator and $S_T = \varnothing$, then $X_T(F)$ is closed for any F closed such that $F \supset S$ or $F \cap S = \varnothing$.*

Proof. It is identical with the one of theorem II.1.5. [37] since the demeanour of the operator in this case is resembling the one of a two-decomposable operator.

1.2.7. *Remarks.* (a) Considering the premises of the preceding proposition, if S_1 is a separated part of S, then $X_T(F)$ is closed for any $F \supset S_1$ closed such that $F \cap (S \setminus S_1) = \varnothing$. Indeed, $X_T(F \cup (S \setminus S_1))$ is closed and we have the equality

$$X_T(F \cup (S \setminus S_1)) = X_T(F) + X_T(S \setminus S_1),$$

whence it follows that $X_T(F)$ is closed (see [4], 2.3.). (b) If $T \in B(X)$ is S-decomposable and

$$S = S_1 \cup S_2 \cup \ldots \cup S_p,$$

where $S_i \cap S_j = \varnothing$ for $i \neq j$ ($i, j = 1,2,\ldots,p$), then in corollary 1.2.5. we can choose the S-covering such that

$$Y_S = Y_S^1 \oplus Y_S^2 \oplus \ldots \oplus Y_S^p$$

and

$$T \mid Y_S = (T \mid Y_S^1) \oplus (T \mid Y_S^2) \oplus \ldots \oplus (T \mid Y_S^p),$$

where the sum is (topologically) direct, and $Y_S^1, Y_S^2, \ldots, Y_S^p$ are spectral maximal spaces of T. Indeed, since $S_1, S_2, \ldots, S_p$ are separated parts, we will be able to choose $G_S = G_S^1 \cup G_S^2 \cup \ldots \cup G_S^p$ such that $G_S^i \cap G_S^j = \varnothing$, $G_S^1, G_S^2, \ldots, G_S^p$ open and $S_i \subset G_S^i$ ($i = 1,2,\ldots,p$). Since $\sigma(T \mid Y_S) \subset G_S^1 \cup G_S^2 \cup \ldots \cup G_S^p$ we have

$$\sigma(T \mid Y_S) = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_p \quad (\sigma_i \subset G_S^i)$$

σ_i compact and separated. Our assertion results now by the theorem of decomposition after the separated parts of the spectrum ([60], theorem 5.11.1.).

1.2.8. Definition. We remind that a topological space $W \neq \varnothing$ has the dimension 0 (or is totally disconnected) if for any finite covering $G_1 \cup G_2 \cup \ldots \cup G_n = W$ of W, there exists a finite closed covering $F_1 \cup F_2 \cup \ldots \cup F_n = W$ of W such that $F_i \subset G_i$, $F_i \cap F_j = \varnothing$ for $i \neq j$ ($i, j = 1,2,\ldots,n$). We also remind that a subset N of R^n has the dimension n if and only if $\operatorname{Int} N \neq \varnothing$ in R^n (see [67], §20, I and [13] I, theorem 4.4.).

1.2.9. PROPOSITION. *Let $T \in B(X)$ be a S-decomposable operator with $S_T = \varnothing$ and let S_1 be a separated part of S, with $\dim S_1 = 0$. Then T is S'-decomposable, where $S' = S \setminus S_1$.*

Proof. Let $\{G_{S'}\} \cup \{G_i\}_1^n$ be an open finite S'-covering of $\sigma(T)$ with $G_{S'} \cap S_1 = \varnothing$. We put $G_i' = G_i \cap \mathbf{C}S_1$. Since $\{G_i\}_1^n$ is a covering of S_1 and $\dim S_1 = 0$, there exists an open covering $\{G_i''\}_1^n$ of S_1 such that $G_i'' \subset G_i$ and $G_i'' \cap G_j'' = \varnothing$ ($i \neq j, i, j = 1,2,\dots,n$) (lemma 6.1. §6 [13]). If

$$G_{S_1} = \bigcup_{i=1}^{n} G_i'' \text{ and } G_S = G_{S_1} \cup G_{S'},$$

we observe that $\{G_S\} \cup \{G_i\}_1^n$ is an open *S*-covering of $\sigma(T)$ and since S' and S_1 are separated, we can choose G_{S_1}, G_S, such that $G_{S_1} \cap G_{S'} = \varnothing$. Let $\{Y_S\} \cup \{Y_i'\}_1^n$ be the corresponding system of spectral maximal spaces such that

$$\sigma(T \mid Y_S) \subset G_S,$$
$$\sigma(T \mid Y_i') \subset G_i' \ (i = 1,2,\dots,n)$$

and

$$X = Y_S + \sum_{i=1}^{n} Y_i'.$$

We also have $Y_S = X_T(\sigma(T \mid Y_S))$ and $Y_i' = X_T(\sigma(T \mid Y_i'))$. But, according to remark 1.2.7. we can write

$$Y_S = X_T(\sigma') + X_T(\sigma_1) + \dots + X_T(\sigma_n),$$

where $\sigma' \cap \sigma_i = \varnothing$, $\sigma_i \cap \sigma_j = \varnothing$ ($i \neq j, i, j = 1,2,\dots,n$) and

$$\sigma(T \mid Y_S) = \sigma' \cup \sigma_1 \cup \dots \cup \sigma_n$$

with $\sigma' \subset G_{S'}$, $\sigma_i \subset G_i''$. Consequently

$$X = X_T(\sigma') + [X_T(\sigma_1) + \dots + X_T(\sigma_n)] + [Y_1' + \dots + Y_n'].$$

By denoting $Y_{S'} = X_T(\sigma')$, $Y_i = X_T[\sigma(T \mid Y_i') \cup \sigma_i]$ one obtains

$$\sigma(T \mid Y_{S'}) \subset G_{S'},\ \sigma(T \mid Y_i) \subset G_i \ (1 \leq i \leq n)$$

and

$$X = Y_{S'} + \sum_{i=1}^{n} Y_i,$$

hence T is S'-decomposable.

1.2.10. THEOREM. *Let $T \in B(X)$ be a S-decomposable operator such that $\dim S = 0$. Then T is decomposable.*

Proof. In the preceding proposition we take $S_1 = S$, hence $S' = S \setminus S_1 = \varnothing$ and T is $\varnothing$-decomposable, meaning that it is a decomposable operator.

1.2.11. COROLLARY. *If $T \in B(X)$ is decomposable and Y is an invariant subspace for T such that*

$$\dim\left(\sigma(T \mid Y) \cap \sigma(\dot{T})\right) = 0,$$

then $\dot{T}$ is decomposable. When Y is a spectral maximal space of T, both $T \mid Y$ and $\dot{T}$ are decomposable operators.

Proof. It follows by the preceding theorem and corollary 1.2.5.

1.2.12. Definition. We shall denote by C the class of all closed sets $\sigma \subset \mathbf{C}$ with $\dim \sigma \leq 1$ and having moreover the property that for any open subset $\sigma_1 \subset \sigma$ we have $\dim \partial\sigma_1 \leq 0$ ($\partial\sigma_1$ is the frontier of σ_1 in the relative topology of σ).

1.2.13. Theorem. *Let $T \in B(X)$ be a decomposable operator with $\sigma(T) \in$ C . Then T is strongly decomposable.*

Proof. The case $\dim\sigma(T) = 0$ is contained in [37]. Hence we only have to analyse the case $\dim\sigma(T) = 1$. Let Y be a spectral maximal space of T and $S = \partial\sigma(T \mid Y) \cap \sigma(\dot{T})$. Since $\sigma(T) \in$ C it follows that $\dim \partial\sigma(T \mid Y) = 0$. But from the formula $\sigma(\dot{T}) = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$ it follows that $\sigma(T \mid Y) \cap \sigma(\dot{T}) = \partial\sigma(T \mid Y) \cap \sigma(\dot{T})$, and corollary 1.2.11. yields that $T \mid Y$ is decomposable, hence T is strongly decomposable.

1.2.14. Corollary. *Let T be a decomposable operator with $\sigma(T) \in$ C . Then T^* is a strongly decomposable operator.*

Proof. It follows from the preceding theorem and from the corollary 3.3. [75].

1.2.15. Corollary. *If T is a decomposable operator with a real spectrum (or having its spectrum on a curve), then T is strongly decomposable.*

Remark. Corollary 1.2.15. was previously observed by C. Foiaş and C. Apostol.

1.2.16. Corollary. *Let $T \in B(X)$ be a decomposable operator with $\sigma(T) \in$ C and Y a spectral maximal space of T. Then $\dot{T}$ is strongly decomposable.*

Proof. Since T is strongly decomposable, by theorem 1.8. [2] it follows that $\dot{T}$ is strongly decomposable.

1.2.17. Corollary. *Let $T \in B(X)$ be a 2-decomposable operator with $\sigma(T) \in$ C . Then T, T^*, T^{**}, ... are strongly decomposable. If X is reflexive then T^* is strongly decomposable if and only if T is 2-decomposable.*

Proof. If T is 2-decomposable and $\sigma(T) \in$ C , then by the proof of proposition 1.2.9. there follows that T is strongly 2-decomposable, meaning it is strongly decomposable. From theorem 1.2.13 it results that T^*, T^{**}, ... are strongly decomposable.

1.2.18. Corollary. *Let $T \in B(X)$ a decomposable operator and let Y be a subspace invariant to T such that*

$$\dim \sigma(T \mid Y) = 0$$

Then $\dot{T}$ is decomposable.

Proof. There follows by corollary 1.2.11., since we have $\dim(\sigma(T \mid Y) \cap \sigma(\dot{T})) = 0$.

1.2.19. PROPOSITION. *Let $T \in B(X)$ be a decomposable operator and let Y be an invariant subspace for T such that*

$$\dim(\sigma(T \mid Y) \cap \sigma(\dot{T})) = 0.$$

Then $T \mid Y$ admits the following spectral decomposition: for any open covering $\{G_i\}_1^n$ of $\sigma(T \mid Y)$ with simple connected sets, there exist the subspaces $\{Y_i\}_1^n$, invariant for T, such that $\sigma(T \mid Y_i) \subset G_i$ and $X = \sum_{i=1}^{n} Y_i$.

Proof. Let $\{G_i\}_1^n$ be a finite open covering of $\sigma(T \mid Y)$ with simple connected sets. Let us set $G_i' = G_i \cap \rho(\dot{T})$; then $\{G_i'\}_1^n$ is a covering of $\sigma(T \mid Y) \setminus S$ (where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$). Since $\{G_i\}_1^n$ is also a covering of S and $\dim S = 0$, there exists an open covering $\{G_i^S\}_1^n$ of S such that $G_i^S \subset G_i$, $G_i^S \cap G_j^S = \varnothing$ for $i \neq j$ $(i, j = 1,2,\ldots,n)$ (see lemma 6.1., [13]). By putting $\bigcup_{i=1}^{n} G_i^S = G_S$, it is obvious that $\{G_S\} \cup \{G_i'\}_1^n$ is a S-covering of $\sigma(T \mid Y)$. By proposition 1.2.3. and remark 1.2.7. we obtain

$$Y = (Y_1 + Y_2 + \ldots + Y_n) + (Y_S^1 + Y_S^2 + \ldots + Y_S^n),$$

where Y_i $(i = 1,2,\ldots,n)$ are spectral maximal spaces for T (hence for $T \mid Y$ also) with $\sigma(T \mid Y_i) \subset G_i'$ $(i = 1,2,\ldots,n)$, and Y_S^i $(i = 1,2,\ldots,n)$ are only invariant subspaces for T and $\sigma(T \mid Y_S^i) \subset \widetilde{G}_S^i$ $(Y_S^i \subset Y)$. Let us set

$$X_i = X_T(\sigma(T \mid Y_i) \cup \sigma(T \mid Y_S^i))$$

and let us notice that X_i are spectral maximal spaces for T and $\sigma(T \mid X_i) \subset G_i' \cup \widetilde{G}_S^i \subset G_i$, $Y_i \subset X_i$, $Y_S^i \subset X_i$ $(i = 1,2,\ldots,n)$. Since

$$Y = \sum_{i=1}^{n} (Y_i + Y_S^i) \subset \sum_{i=1}^{n} X_i \cap Y \subset Y$$

and $\sigma((T \mid Y) \mid X_i \cap Y) \subset G_i$, our proof is over.

1.2.20. COROLLARY. *Let $T \in B(X)$ be a decomposable operator and let Y be a T-absorbing subspace invariant for T, so that $\dim(\sigma(T \mid Y) \cap \sigma(\dot{T})) = 0$; then $T \mid Y$ is decomposable.*

Proof. It follows from the preceding proposition, since in this case we can consider the covering $\{G_i\}_1^n$ of $\sigma(T \mid Y)$ with open arbitrary sets (not necessarily simple connected) and by the fact the intersection of two T-absorbing subspaces invariant for T, Y_1, Y_2, is $T \mid Y_i$-absorbing (see remark 1.1.18.); we also use the result from [24], where one proves that in the definition of decomposability the spectral maximal spaces can be replaced with T-absorbing subspaces (also see 2.4.11.).

1.3. SPECTRUM-SETS. THE SPECTRUM'S DECOMPOSITION AFTER ITS PARTS

During this paragraph we shall define the spectrum-sets for an T operator, which are in some way generalisations of sets-spectra; they are not separated parts of the spectrum any more but compact subsets of it, which are the spectrum of some restrictions of the operator. We shall further emphasise the subsets of the spectrum of a decomposable operator having size of 2 or 1, and we shall study the restrictions and quotients related to the corresponding subspaces.

1.3.1. Definition. Let $T \in B(X)$ and let $\sigma \subset \sigma(T)$ be a compact set. σ is a *set-spectrum* for T if there exists a invariant subspace Y for T such that

$$\sigma(T \mid Y) = \sigma .$$

1.3.2. Proposition. *Let $T \in B(X)$ be a decomposable operator and $\sigma \subset \sigma(T)$ such that $\sigma = \overline{\operatorname{Int}\sigma}$ (in the topology of $\sigma(T)$). Then σ and $\sigma' = \overline{\sigma(T)\setminus\sigma}$ are sets-spectra for T and*

$$\sigma(T \mid X_T(\sigma)) = \sigma ,\ \sigma(T \mid X_T(\sigma')) = \sigma' .$$

Proof. According to theorem 1.3.8. [37] we have $\sigma(T \mid X_T(\sigma)) \subset \sigma$. It will suffice to prove that $\sigma(T \mid X_T(\sigma)) \supset \operatorname{Int}\sigma$ (the interior is considered in the relative topology of $\sigma(T)$). Let $\lambda_0 \in \operatorname{Int}\sigma$; then there exists a disk $\delta = \{\lambda : |\lambda - \lambda_0| < \rho\}$ such that $\delta \cap \sigma(T) \subset \operatorname{Int}\sigma$. We shall put

$$d_1 = \left\{\lambda : \lambda \in \sigma(T), |\lambda - \lambda_0| < \frac{1}{2}\right\},$$

$$G_0 = \left\{\lambda : |\lambda - \lambda_0| < \frac{3}{4}\rho\right\},$$

$$G_1 = \left\{\lambda : |\lambda - \lambda_0| < \frac{5}{8}\rho\right\}.$$

Consequently $G_0 \cup G_1 \supset \sigma(T)$, $G_1 \cap d_1 = \varnothing$. If Y_0, Y_1 are the corresponding spectral maximal spaces of T such that

$$\sigma(T \mid Y_0) \supset G_0 ,\ \sigma(T \mid Y_1) \subset G_1 ,\ X = Y_0 + Y_1 ,$$

then from the equality

$$\sigma(T) = \sigma(T \mid Y_0) \cup \sigma(T \mid Y_1)$$

and since $\sigma(T \mid Y_1) \cap d_1 = \varnothing$, we have

$$d_1 \subset \sigma(T \mid Y_0) \subset G_0 \cap \sigma(T) \subset \delta \cap \sigma(T) \subset \operatorname{Int}\sigma \subset \sigma$$

It will follow

$$Y_0 = X_T(\sigma(T \mid Y_0)) \subset X_T(\sigma),$$

consequently $\lambda_0 \in d_1 \subset \sigma(T \mid Y_0) \subset \sigma(T \mid X_T(\sigma))$, meaning $\sigma \subset \sigma(T \mid X_T(\sigma))$. Since $\sigma' = \overline{\operatorname{Int}\sigma'}$ (in $\sigma(T)$) we shall also have $\sigma(T \mid X_T(\sigma')) = \sigma'$.

1.3.3. Corolarry. *Let $T \in B(X)$ be a decomposable operator and let Y be a spectral maximal space of T. Then there exists a spectrd maximal space Y_1 of T such that*

$$\sigma(T \mid Y_1) = \sigma(\dot{T})$$

where $\dot{T}$ is the operator induced by T in $\dot{X} = X / Y$.

Proof. From the equality $\sigma(\dot{T}) = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$ [2] and by the preceding proposition it follows that $\sigma = \sigma(\dot{T})$ is a set-spectrum of T, hence $Y_1 = X_T(\sigma)$ and $\sigma = \sigma(T \mid X_T(\sigma))$.

1.3.4. *Remarks.* (a) From proposition 1.3.2. results that an operator $T \in B(X)$ is decomposable if and only if for any open and finite covering $\{G_i\}_1^n$ of $\sigma(T)$, where $G_i \subset \sigma(T)$, G_i is open in $\sigma(T)$, there exists a system of spectral maximal spaces of T $\{Y_i\}_1^n$ such that

$$\sigma(T \mid Y_i) = \overline{G_i}, \; X = \sum_{i=1}^{n} Y_i .$$

Indeed, if $\{G_i'\}_1^n$ is a open covering of $\sigma(T)$, then $\overline{G_i} = \overline{G_i' \cap \sigma(T)}$ is a set-spectrum for T and $Y_i = X_T(\overline{G_i})$ (T is supposed to be decomposable). Conversely, it is obvious. (b) Let W be an arbitrary subset of X and

$$\sigma = \overline{\bigcup_{x \in W} \sigma_T(x)};$$

then σ is a set-spectrum for T if T is decomposable. Indeed, we have $\sigma(T \mid X_T(\sigma)) \subset \sigma$ and

$$\sigma(T \mid Y_T(\sigma)) = \overline{\bigcup_{x \in X_T(\sigma)} \sigma_T(x)} \supset \overline{\bigcup_{x \in W} \sigma_T(x)} = \sigma .$$

If T is a spectral operator and $\sigma = \overline{\operatorname{Int}\sigma}$ (in the topology of $\sigma(T)$), then $\sigma(T \mid E(\sigma)X) = \sigma$, where E is the spectral measure of T; also, if $\sigma = \overline{\bigcup_{x \in W} \sigma_T(x)}$ we have $\sigma(T \mid E(\sigma)X) = \sigma$, where W is an arbitrary subset of X.

1.3.5. Definition. Let Y, Y_1 be two invariant subspaces of $T \in B(X)$. Y_1 will be said to be the *spectral complement* (related to T) of Y if $\sigma(T \mid Y_1) = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$. Y, Y_1 will be said to be *spectrally conjugated* (related to T), if each is the spectral complement of the other.

1.3.6. Proposition. *If $T \in B(X)$ is decomposable, then any Y subspace invariant for T admits a spectral complement Y_1 (related to T); there exists a single spectral complement Y_1 of Y (related to T) which is moreover a spectral maximal space of T.*

Proof. Let $\sigma = \overline{\sigma(T) \setminus \sigma(T \mid Y)}$. If $\sigma = \varnothing$ then $Y_1 = \{0\}$. When $\sigma \neq \varnothing$, we shall put $Y_1 = X_T(\sigma)$. Since σ is a set-spectrum for T and $\sigma(T) \setminus \sigma(T \mid Y)$ is open in $\sigma(T)$, we have $Y_1 \neq \{0\}$ and $\sigma(T \mid Y_1) = \sigma$. Obviously, Y_1 is the only spectral complement of Y (related to T) which is also a spectral maximal space of T.

1.3.7. *Remark.* Let $T \in B(X)$ be a decomposable operator and let Y be a invariant subspace for T so that $\sigma_1 = \overline{\sigma(T) \setminus \sigma(T \mid Y)} \neq \varnothing$ and $\sigma_2 = \overline{\sigma(T) \setminus \sigma_1}$; then $Y_1 = X_T(\sigma_1)$ and $Y_2 = X_T(\sigma_2)$ are spectrally conjugated (related to T). When Y is a spectral maximal space of T and $\overline{\operatorname{Int} \sigma(T \mid Y)} = \sigma(T \mid Y)$ (in the topology of $\sigma(T)$), then we have $Y = Y_2$.

1.3.8. Proposition. *Let $T \in B(X)$ be a decomposable operator having the following property*

$$\sigma_1 = \overline{\sigma(T) \setminus \overline{\operatorname{Int} \sigma(T)}} \in \mathrm{C}$$

and let us put $\sigma = \overline{\operatorname{Int} \sigma(T)}$, $Y = X_T(\sigma)$, $Y_1 = X_T(\sigma_1)$, $\dot{X} = X / Y$, $\tilde{X} = X / Y_1$ and denote by $\dot{T}$, $\tilde{T}$ the operators induced by T in $\dot{X}$, $\tilde{X}$. Then $T \mid Y$, $\tilde{T}$ are the decomposable operators, and $T \mid Y_1$, $\dot{T}$ are strongly decomposable; we also have

$$\sigma(T \mid Y) = \sigma(\tilde{T}) \text{ and } \sigma(T \mid Y_1) = \sigma(\dot{T}).$$

Proof. σ and σ_1 are spectrum-sets for T and we have $\sigma(T \mid Y) = \sigma = \sigma(\tilde{T})$, $\sigma(T \mid Y_1) = \ = \sigma_1 = \sigma(\dot{T})$ (see corollary 1.3.3. and [2], 1.4.). We have $\sigma \cap \sigma_1 = \partial(\sigma \cup \sigma_1)$ in the topology of σ_1 since $\partial(\sigma \cap \sigma_1) = (\sigma \cap \sigma_1) \cap \overline{(\sigma_1 \setminus (\sigma_1 \cap \sigma))} = (\sigma \cap \sigma_1) \cap \overline{(\sigma_1 \setminus \sigma)} =$
$= (\sigma_1 \cap \sigma) \cap \overline{(\overline{\sigma(T) \cap \complement\sigma} \cap \complement\sigma)} = (\sigma_1 \cap \sigma) \cap ((\sigma(T) \cap \complement\sigma) \cap \complement\sigma) = (\sigma_1 \cap \sigma) \cap \overline{(\sigma(T) \setminus \sigma)} =$
$= \sigma_1 \cap \sigma$ (we used the definition of the frontier in the relative topology of σ_1 and the fact that if A is open then $\overline{A \cap \overline{X}} = \overline{A \cap X}$; (see [68], ex. 1, §8). Since $\sigma_1 \in \mathrm{C}$, it follows that

$$\dim(\sigma \cap \sigma_1) = \dim \partial(\sigma \cap \sigma_1) = 0$$

hence from corollaries 1.2.20. and 1.3.3. we obtains that $T \mid Y$, $T \mid Y_1$, $\tilde{T}$, $\dot{T}$ are decomposable; but $\sigma(T \mid Y_1) = \sigma(\dot{T}) \in \mathrm{C}$ and according to theorem 1.2.13., $T \mid Y_1$ and $\dot{T}$ are strongly decomposable operators.

1.3.9. Corollary. *With the same conditions as in the preceding proposition, if Y is a spectral maximal space of T such that $\sigma(T \mid Y) \supset \operatorname{Int} \sigma(T)$ (particularly $Y = X_T(F)$ with $F \supset \operatorname{Int} \sigma(T)$ closed) and $Y_1 = X_T\left(\overline{\sigma(T) \setminus \sigma(T \mid Y)}\right)$, then, by using the symbols fixed* *above, we have that $T \mid Y$, $\widetilde{T}$ are decomposable and $T \mid Y_1$, $\dot{T}$ are strongly decomposable.*

Proof. Let $\sigma = \overline{\operatorname{Int}\sigma(T)}$ and $\sigma_1 = \overline{\sigma(T) \setminus \sigma}$. If $F \supset \operatorname{Int}\sigma(T)$, $F \subset \sigma(T)$ closed, then $X_T(\sigma) \subset X_T(F)$ and

$$\sigma = \sigma(T \mid X_T(\sigma)) \subset \sigma(T \mid X_T(F)).$$

If $\sigma(T \mid Y) \supset \operatorname{Int}\sigma(T)$, we have once again

$$\dim\left(\sigma(T \mid Y) \cap \sigma(\dot{T})\right) = \dim\left(\sigma(T \mid Y_1) \cap \sigma(T)\right) = 0$$

(since $\sigma \subset \sigma(T \mid Y)$ and the frontier of $\overline{\sigma(T) \setminus \sigma(T \mid Y)} = \sigma(T \mid Y_1)$ has dimension 0 in the topology of σ_1) and the proof is the same with the one of the proposition above.

1.3.10. Lemma. Let $T \in B(X)$ be a decomposable operator such that $\sigma_1 = \overline{\sigma(T) \setminus \overline{\operatorname{Int}\sigma(T)}} \in \mathrm{C}$ and let Y be a spectral maximal space of T. Then $T \mid Y$ is a S-decomposable operator, where

$$S = \sigma(T \mid Y) \cap \overline{\operatorname{Int}\sigma(T)}.$$

Proof. We shall put $\sigma = \overline{\operatorname{Int}\sigma(T)}$ and $X_T(\sigma(T \mid Y) \cup \sigma) = Z$. If $\sigma(T \mid Y) \subset \sigma$, then $S = \sigma(T \mid Y)$ and our affirmation is obvious. Let now $\sigma(T \mid Y) \not\subset \sigma$ and $\{G_S\} \cup \{G_i\}_1^n$ a bounded and open S-covering of $\sigma(T \mid Y)$. $\sigma(T \mid Y) \cup \sigma$ is a set-spectrum for T (see proposition 1.3.2. and remark 1.3.4.) hence

$$\sigma(T \mid Z) = \sigma(T \mid Y) \cup \sigma.$$

If we take $G'_S = G_S \cup \rho(T \mid Y)$ and $G'_i = G_i \cap \complement\sigma$ $(i = 1,2,\ldots,n)$ then $\{G'_S\} \cup \{G'_i\}_1^n$ is a bounded and open covering of $\sigma(T \mid Z)$. According to corollary 1.3.9., the operator $T \mid Z$ is decomposable. Let $\{Z_S\} \cup \{Z_i\}_1^n$ be the corresponding system of spectral maximal spaces of $T \mid Z$ such that

$$\sigma(T \mid Z_S) \subset G'_S,\ \sigma(T \mid Z_i) \subset G'_i\ (i = 1,2,\ldots,n),$$

$$Z = Z_S + \sum_{i=1}^{n} Z_i .$$

Since $\sigma(T \mid Z_i) \subset (G_i \cap \complement\sigma) \cap (\sigma(T \mid Y) \cup \sigma) \subset \sigma(T \mid Y)$, one obtains $Z_i \subset Y$ $(i = 1,2,\ldots,n)$. If $x \in Y$, then

$$x = y_S + y_1 + y_2 + \ldots + y_n$$

where $y_S \in Z_S$, $y_i \in Z_i \subset Y$ $(i = 1,2,\ldots,n)$ hence $y_S = x - (y_1 + \ldots + y_n) \in Y$; consequently

$$Y = Z_S \cap Y + \sum_{i=1}^{n} Z_i ,$$

$$\sigma(T \mid Z_S \cap Y) \subset G'_S \cap \sigma(T \mid Y) \subset G_S ,$$

$$\sigma(T \mid Z_i) \subset G_i \cap \complement\sigma \subset G_i$$

hence $T \mid Y$ is an S-decomposable operator.

1.3.11. THEOREM. *Let $T \in B(X)$ be a decomposable operator having the following property:* $\sigma_1 = \overline{\sigma(T) \setminus \operatorname{Int} \sigma(T)} \in \mathrm{C}$. *Then T is strongly S-decomposable where* $S = \overline{\operatorname{Int} \sigma(T)}$.

Proof. Let $\{G_S\} \cup \{G_i\}_1^n$ be a bounded open S-covering of $\sigma(T)$ and let $\{H_S\} \cup \{H_i\}_1^n$ be another open S-covering of $\sigma(T)$ such that $\overline{H}_S \subset G_S$, $\overline{H}_i \subset G_i$ ($i = 1,2,\ldots,n$). If $\{Y_S\} \cup \{Y_i\}_1^n$ is the corresponding system of spectral maximal spaces of T such that

$$\sigma(T \mid Y_S) \subset \overline{H}_S \subset G_S,$$
$$\sigma(T \mid Y_i) \subset \overline{H}_i \subset G_i \ (i = 1,2,\ldots,n),$$
$$X = Y_S + \sum_{i=1}^{n} Y_i,$$

then we have $Z_S = X_T(\overline{H}_S) \supset Y_S$, $Z_i = X_T(\overline{H}_i) \supset Y_i$, $\sigma(T \mid Z_S) \subset \overline{H}_S$, $\sigma(T \mid Z_i) \subset \overline{H}_i$ ($i = 1,2,\ldots,n$). But $T \mid Y$ is S_1-decomposable (where $S_1 = S \cap \sigma(T \mid Y)$) for any spectral maximal space Y of T, according to the preceding lemma. Since $\{\overline{H}_S\} \cup \{H_i\}_1^n$ is also a S-covering for $\sigma(T \mid Y)$, let $\{X_{S_1}\} \cup \{X_i\}_1^n$ be the corresponding system of spectral maximal spaces of $T \mid Y$. From the inclusions

$$\sigma(T \mid X_{S_1}) \subset \sigma(T \mid Y) \cap \overline{H}_S,$$
$$\sigma(T \mid X_i) \subset \sigma(T \mid Y) \cap \overline{H}_i,$$

one obtains

$$X_{S_1} = Y_{T|Y}(\sigma(T \mid X_{S1})) \subset X_T(\sigma(T \mid Y) \cap \overline{H}_S) \subset X_T(\overline{H}_S) = Z_S,$$
$$X_i = Y_{T|Y}(\sigma(T \mid X_i)) \subset Z_i \ (i = 1,2,\ldots,n).$$

Consequently $X_{S_1} \subset Y \cap Z_S$, $X_i \subset Y \cap Z_i$, so from the equality

$$Y = X_{S_1} + \sum_{i=1}^{n} X_i$$

it follows

$$Y = Y \cap Z_S + \sum_{i=1}^{n} Y \cap Z_i,$$

meaning that T is strongly S-decomposable.

1.3.12. PROPOSITION. *With the same conditions as in proposition 1.3.8. if we denote by $\dot{T}^*$ and $\widetilde{T}^*$ the operators induced by T^* in $X^* = X^* / Y^{\perp}$ and $\widetilde{X}^* = X^* / Y_1^{\perp}$ we shall have:* (a) *$\dot{T}^*$ and $T^* \mid Y_1^{\perp}$ are decomposable;* (b) *T^* and $T^* \mid Y^{\perp}$ are strongly decomposable.*

Proof. If T is decomposable, then T^* is 2-decomposable ([54] theorem 2.3) hence decomposable [86], and since $X^*/Y^\perp = Y^*$, $X^*/Y_1 = Y_1^\perp$, $(X/Y)^* = Y^\perp$, $(X/Y_1)^* = Y_1^\perp$ (see [45], I, 2.4.18.) it follows easily that $\dot{T}^*$ and T_Y^* are similar ($T_Y = T \mid Y, T_{Y_1} = T \mid Y_1$); also, $T^* \mid Y^\perp$ with $(\dot{T})^*$, $\widetilde{T}^*$ with $T_{Y_1}^*$ and $T^* \mid Y_1^\perp$ with $(\widetilde{T})^*$. From the proposition 1.3.8. and the corollary 1.3.9 it follows now our assertion.

1.3.13. Proposition. *Let $T \in B(X)$ be a strongly decomposable operator and Y a subspace invariant to T. Then $\dot{T}$ is strongly S-decomposable where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$.*

Proof. Since $\dot{T}$ is S-decomposable (according to theorem 1.2.1.) we have left to prove the equality

$$\dot{Z} = \dot{Z} \cap \dot{Y}_S + \dot{Z} \cap \dot{Y}_1 + \ldots + \dot{Z} \cap \dot{Y}_n$$

for any spectral maximal space $\dot{Z}$ of $\dot{T}$. But from the proof of theorem 1.2.1., keeping the symbols, it follows that $\dot{Y}_S, \dot{Y}_1, \ldots, \dot{Y}_n$ are the images of the spectral maximal spaces of $T, Y_S, Y_1, \ldots, Y_n$ through the canonical map. T being strongly decomposable we have

$$Z = Z \cap Y_S + Z \cap Y_1 + \ldots + Z \cap Y_n$$

for any spectral maximal space Z of T. We may suppose that $G_i \cap \sigma(T \mid Y) = \varnothing$ ($i = 1,2,\ldots,n$). Let

$$Z_1 = X_T\left(\sigma(\dot{T} \mid \dot{Z}) \cup \sigma(T \mid Y)\right).$$

Theorem 1.1.19. and corollary 1.1.20. yield that $\dot{Z}_1 = \dot{X}_{\dot{T}}\left(\sigma(\dot{T} \mid \dot{Z}) \cup \sigma(T \mid Y)\right)$ is a spectral maximal space of $\dot{T}$ and $\sigma(\dot{T} \mid \dot{Z}_1) \subset \sigma(\dot{T} \mid \dot{Z}) \cup \sigma(T \mid Y)$ (see [76], proposition 2.4. and 3.4.). Since

$$Z_1 = Z_1 \cap Y_S + Z_1 \cap Y_1 + \ldots + Z_1 \cap Y_n$$

we shall have

$$\dot{Z}_1 = \dot{Z}_1 \cap \dot{Y}_S + \dot{Z}_1 \cap \dot{Y}_1 + \ldots + \dot{Z}_1 \cap \dot{Y}_n .$$

But $\dot{Z} \subset \dot{Z}_1$ and from the inclusions $\sigma(\dot{T} \mid \dot{Z}_1 \cap \dot{Y}_i) \subset \sigma(\dot{T} \mid \dot{Z}_1) \cap \sigma(\dot{T} \mid \dot{Y}_i) \subset \left(\sigma(\dot{T} \mid \dot{Z}) \cup \sigma(T \mid Y)\right) \cap G_i \subset \sigma(\dot{T} \mid \dot{Z})$ it follows $\dot{Z}_1 \cap \dot{Y}_i \subset \dot{Z}$ ($i = 1,2,\ldots,n$). If $\dot{z} \in \dot{Z}$, then $\dot{z} = \dot{x}_S + \dot{x}_1 + \ldots + \dot{x}_n$ with $\dot{x}_S \in \dot{Y}_S$ and $\dot{x}_i \in \dot{Y}_i \cap \dot{Z}$ ($i = 1,2,\ldots,n$) (since $\dot{Y}_i \cap \dot{Z}_i = \dot{Y}_i \cap \dot{Z}$) hence

$$\dot{x}_S = \dot{z} - (\dot{x}_1 + \dot{x}_2 + \ldots + \dot{x}_n) \in \dot{Z},$$

whence it follows that

$$\dot{Z} = \dot{Z} \cap \dot{Y}_S + \dot{Z} \cap \dot{Y}_1 + \ldots + \dot{Z} \cap \dot{Y}_n .$$

1.3.14. Corollary. *Considering the circumstances from the preceding proposition, if $\dot{Z}$ is a spectral maximal space of $\dot{T}$, then $\dot{T} \mid \dot{Z}$ is S_1-decomposable, where*

$$S_1 = S \cap \sigma(T \mid Y).$$

Proof. We have $S \cap \sigma(\dot{T} \mid \dot{Z}) = \sigma(T \mid Y) \cap \sigma(\dot{T} \mid \dot{Z})$ and from the preceding proposition it results that $\dot{T} \mid \dot{Z}$ is S_1-decomposable.

1.3.15. Lemma. *Let $T \in B(X)$ be a strongly S-decomposable operator and let Y be a spectral maximal space of T. Then $T \mid Y$ is S_1-decomposable, where $S_1 = S \cap \sigma(T \mid Y)$.*

Proof. Let $\{G_{S_1}\} \cup \{G_i\}_1^n$ be an open, bounded S_1-covering of $\sigma(T \mid Y)$; by putting $G_S = G_{S_1} \cup \rho(T \mid Y)$, it follows that $\{G_S\} \cup \{G_i\}_1^n$ is an open S-covering of $\sigma(T)$. If $\{Y_i\}_1^n \cup \{Y_S\}$ is the corresponding system of spectral maximal spaces of T, then

$$Y = Y \cap Y_S + \sum_{i=1}^{n} Y \cap Y_i,$$

$$\sigma(T \mid Y \cap Y_S) \subset \sigma(T \mid Y) \cap (G_{S_1} \cup \rho(T \mid Y)) = G_{S_1},$$

$$\sigma(T \mid Y \cap Y_1) \subset \sigma(T \mid Y) \cap G_i \subset G_i \ (i = 1,2,\ldots,n),$$

therefore $T \mid Y$ is S_1-decomposable.

1.3.16. Theorem. *Let $T \in B(X)$ be a strongly S-decomposable operator such that*

$$\dim S = 0.$$

Then T is strongly decomposable.

Proof. Let Y be a spectral maximal space of T. According to the preceding lemma there follows that $T \mid Y$ is S_1-decomposable where $S_1 = S \cap \sigma(T \mid Y)$. Hence $\dim S_1 = 0$ and according to theorem 1.2.10., $T \mid Y$ is decomposable, hence T is strongly decomposable.

1.3.17. Corollary. *Let $T \in B(X)$ be a strongly decomposable operator and let Y be an invariant subspace for T so that*

$$\dim(\sigma(T \mid Y) \cap \sigma(\dot{T})) = 0$$

where $\dot{T}$ is the operator induced by T in the quotient space $\dot{X} = X / Y$. Then $\dot{T}$ is a strongly decomposable operator.

Proof. It easily follows from the preceding proposition and from proposition 1.2.13.

1.3.18. Corollary. *Let $T \in B(X)$ be a strongly decomposable operator and let Y be an invariant subspace to T so that*

$$\dim \sigma(T \mid Y) = 0.$$

Then $\dot{T}$ is strongly decomposable.

Proof. It easily follows from the preceding corollary.

1.3.19. Proposition. *Let H be a Hilbert space and let $T \in B(H)$ be a strongly decomposable operator. If Y is an invariant subspace for T and T^*, and*

$\dim(\sigma(T \mid Y) \cap \sigma(T \mid Y^{\perp})) = 0$ *(especially* $\dim \sigma(T \mid Y) = 0$*), then* $T \mid Y$ *and* $T \mid Y^{\perp}$ *are strongly decomposable.*

Proof. It follows from corollary 1.3.17.

1.4. RESIDUAL SPECTRAL MEASURES

In this paragraph we shall introduce a spectral measure, residual in some sense, and we shall demonstrate that the restrictions and quotients of spectral operators admit such a spectral measure (spectral S-measure). We shall further emphasise some properties of the operators that admit residual spectral measures.

1.4.1. Definition. Let X be a Banach space, and let $B(X)$ the algebra of all linear bounded operators on X, let P_X be the set of the projectors of X and B_S be the family of all Borelian sets B of the complex plan $\mathbf{C}$ that have the property that $B \cap S = \varnothing$ or $B \supset S$, where S is a compact set of $\mathbf{C}$; an application $E_S : \mathsf{B}_S \to \mathsf{P}_X$ will be said to be a S- *spectral measure* if:

1°. $E_S(B_1 \cap B_2) = E_S(B_1)E_S(B_2)$, $(B_1, B_2 \in \mathsf{B}_S)$,

2°. $E_S\left(\bigcup_{i=1}^{\infty} B_n\right)x = \sum_{i=1}^{\infty} E_S(B_n)x$, $(B_n \in \mathsf{B}_S, B_n \cap B_m = \varnothing, n \neq m)$,

3°. $E_S(\mathbf{C}) = I$,

4°. $\sup_{B \in \mathsf{B}_S} \|E_S(B)\| < \infty$.

An operator $T \in B(X)$ will be said to be *S-spectral* if there exists a S-spectral measure such that $TE_S(B) = E_S(B)T$ and $\sigma(T \mid E_S(B)X) \subset \overline{B}$ $(B \in \mathsf{B}_S)$.

1.4.2. *Remark.* A T operator is S-spectral if and only if it is a direct sum $T = T_1 \oplus T_2$, where T_1 is spectral and $\sigma(T_2) \subset S$. Indeed, if T is S-spectral, then one easily verifies that the map $E : \mathsf{B} \to \mathsf{P}_X$ (where $\mathsf{B} = \mathsf{B}_{\varnothing}$) defined by $E(B) = E_S(B \cap \complement S)$ is a spectral measure for $T_1 = T \mid E_S(\complement S)X$ $(B \in \mathsf{B})$, hence $T = T_1 \oplus T_2$ where $T_2 = T \mid E_S(S)X$ and $\sigma(T \mid E_S(S)X) \subset S$. Conversely, if $T_1 \in B(X_1)$ is spectral, and $T_2 \in B(X_2)$ with $\sigma(T_2) \not\supset \sigma(T_1)$, T_2 not spectral, by putting $S = \sigma(T_2)$, $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, the map $E_S : \mathsf{B}_S \to \mathsf{P}_X$ defined by the equalities $E_S(B) = E(B) \oplus 0$ if $B \cap S = \varnothing$ and $E_S(B) = E(B) \oplus I_2$ if $B \supset S$ $(B \in \mathsf{B}_S)$ is a S-spectral measure of T (E is the spectral measure of T_1, and I_2 is the identical operator in X_2).

1.4.3. PROPOSITION. *Let $T \in B(X)$ a spectral (scalar) operator having the spectral measure E, let Y be a linear, closed invariant to T subspace, $\dot{T}$ is the operator induced by T in $\dot{X} = X / Y$ and $\varphi : X \to \dot{X}$ the canonical application. Then $\dot{T} = \dot{T}_1 \oplus \dot{T}_2$, where $\dot{T}_1 = \dot{T} \mid \varphi(E(\sigma')X)$ is spectral (scalar), $\dot{T}_2 = \dot{T} \mid \varphi(E(\sigma)X)$, $\sigma = \sigma(T \mid Y)$, $\sigma' = \sigma(\dot{T}) \setminus \sigma(T \mid Y)$ and $\sigma(\dot{T}_2) \subset S = \sigma(T \mid Y) \cap \sigma(\dot{T})$.*

Proof. The operator $T \mid E(\sigma')X$ is spectral (scalar) ([45], III, XV, 16) and since $Y \subset E(\sigma)X = X_T(\sigma)$, we have $Y \cap E(\sigma')X = \{0\}$. But $E(\sigma')X + Y = E(\sigma')X \oplus Y$ ($E(\sigma')X + Y$ being closed; see lemma 1.1.13.) so $\sigma(E(\sigma')X)$ can be identified with $E(\sigma')X$, and $\dot{T}_1$ with $T \mid E(\sigma')X$, meaning $\dot{T}_1$ is spectral (scalar). There is easy to verify that $\varphi(X_T(\sigma)) = \dot{X}_{\dot{T}}(\sigma) = \dot{X}_{\dot{T}}(S)$ (corollary 1.1.20.) consequently $\sigma(\dot{T} \mid \varphi(X_T(\sigma))) \subset S$.

1.4.4. PROPOSITION. *Let $T \in B(X)$ be spectral (scalar) and let Y be an invariant subspace to T with $X_T(\sigma) \subset Y$ (where $\sigma = \sigma(T \mid Y) \setminus \sigma(\dot{T})$): let also $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$ and $T_Y = T \mid Y$. Then $T_Y \mid E(\sigma)Y$ and $T_Y \mid \overline{X_T(\sigma)}$ are spectral (scalar), and $T_Y = (T_Y E(\sigma)Y) \oplus (T_Y \mid E(S)Y)$ where $\sigma(T_Y \mid E(S)Y) \subset S \cap \sigma(T_Y)$.*

Proof. σ being open in $\sigma(T)$, there exists a growing series of open sets $(\sigma_n)_{n \in N}$ with $\sigma = \bigcup_{n \in N} \sigma_n$; from the continuity of the measures $E(\cdot)x$ it results that $E(\sigma) = \lim_{n \to \infty} E(\sigma_n)$, therefore $E(\sigma_n)X = X_T(\sigma_n) \subset X_T(\sigma)$ ([41], V, 1.9.) implies $E(\sigma)X \subset \overline{X_T(\sigma)} \subset Y$. The subspaces $E(\sigma)X$ and $\overline{X_T(\sigma)}$ are invariant to T and the spectral measure E, so $T_Y \mid E(\sigma)Y$ and $T_Y \mid \overline{X_T(\sigma)}$ are spectral (scalar). From $Y \subset X_T(\sigma(T \mid Y)) = E(\sigma(T \mid Y))X$ it follows that $Y = E(\sigma(T \mid Y))Y = E(\sigma)Y + E(S)Y$; hence Y is invariant to $E(\sigma)$ and $E(S)$; consequently $E(\sigma) \mid Y$ and $E(S) \mid Y$ are projectors in Y, $E(\sigma)Y$ and $E(S)Y$ are closed, and $Y = E(\sigma)Y + E(S)Y$. We also obtain that $\sigma(T_Y \mid E(S)Y) \subset \sigma(T \mid E(S)X) \cap \sigma(T \mid Y) \subset \subset \widetilde{S} \cap \sigma(T \mid Y)$.

1.4.5. THEOREM. *Let $T \in B(X)$ be a spectral operator and let Y be a subspace invariant for T such that $X_T(\sigma) \subset Y$ (where $\sigma = \sigma(T \mid Y) \setminus \sigma(\dot{T})$) and $S = \widetilde{S}$, where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$. Then $T \mid Y$ and $\dot{T}$ are S-spectral operators.*

Proof. There follows by the preceding propositions.

1.4.6. COROLLARY. Let $T \in B(X)$ be a spectral (scalar) operator and let Y be an invariant subspace for T so that $\dim(\sigma(T \mid Y) \cap \sigma(\dot{T})) = 0$. Then $T \mid Y$ and $\dot{T}$ are spectral (scalar).

Proof. From $\dim(\sigma(T \mid Y) \cap \sigma(\dot{T})) = 0$ it follows that $S_T = \varnothing$ (remark 1.1.5.), $X_T(\sigma(T \mid Y) \setminus \sigma(\dot{T})) \subset Y$ and according the preceding proposition, we have $Y = Y_1 \oplus Y_2$, where $Y_1 = E(\sigma)X = E(\sigma)Y$ and $Y_2 = E(S)Y$ ($\sigma = \sigma(T \mid Y) \setminus \sigma(\dot{T}), S = \sigma(T \mid Y) \cap \sigma(\dot{T})$).

Obviously, Y_1 is invariant to the spectral measure E. But $\sigma(T \mid Y_2) \subset \sigma(T \mid E(S)X) \subset S$ (since $\complement S$ is connected and $S = \widetilde{S}$) therefore Y_2 is also invariant to E and according to theorem V.4.4. [41] $T \mid Y$ and $\dot{T}$ are spectral (scalar).

1.4.7. Corollary. *Let H be a Hilbert space and $T \in B(H)$ a normal operator. If Y is an invariant subspace to T so that $\dim S = 0$, where $S = \sigma(T \mid Y) \cap \sigma(\dot{T})$, then $T \mid Y$ and $T \mid H \ominus Y$ are normal.*

Proof. From the preceding corollary it results that Y is invariant to the spectral measure E of T, hence Y is also invariant to T^*.

Remark. Corollary 1.4.6. is a generalisation of the result obtained in [44] which states that the restriction $T \mid Y$ of a spectral operator to an invariant subspace Y to T is a spectral operator if $\sigma(T \mid Y)$ is totally disconnected (meaning $\sigma(T \mid Y) = 0$).

1.4.8. Proposition. *Let $T \in B(X)$ be a subscalar operator and $\widetilde{T} \in B(\widetilde{X})$ the minimal scalar extension of T. Then T is S-scalar, where $S = \sigma(T) \cap \sigma(\dot{\widetilde{T}})$, $\dot{\widetilde{T}}$ being the operator induced by T in the quotient space $\dot{X} = X / X$.*

Proof. It is known that a subscalar operator is the restriction of a scalar operator to an invariant subspace for the operator. The assertion follows by proposition 1.4.4.

1.4.9. Proposition. *Let H be a Hilbert space and $T \in B(H)$ a subnormal operator; if we denote by $\widetilde{T} \in B(\widetilde{X})$ the minimal extension of T, then $T = T_1 \oplus T_2$, where T_1 is normal, $\sigma(T_1) \subset \sigma(T) \setminus \sigma(\dot{\widetilde{T}})$, $\sigma(T_2) \subset \sigma(T) \cap \sigma(\dot{\widetilde{T}})$, $\dot{\widetilde{T}}$ being the operator induced by T in the quotient space $\dot{H} = \widetilde{H} / H$.*

Proof. Same as for proposition 1.4.4.

1.4.10. *Remark.* Let $T \in B(X)$ (or $T \in B(H)$) be a subscalar operator (respectively subnormal) and $\widetilde{T}$ the scalar minimal extension of T (respectively a scalar extension of T) such that $\dim\left(\sigma(T) \cap \sigma(\dot{\widetilde{T}})\right) = 0$, where $\dot{\widetilde{T}}$ is the operator induced by T in the quotient space $\dot{\widetilde{X}} = X / X$ (respectively $\dot{H} = \widetilde{H} / H$). Then T is scalar (respectively normal). It results from propositions 1.4.8., 1.4.9. and corollaries 1.4.6., 1.4.7.

1.4.11. Proposition. *Let $T \in B(X)$ be a S-spectral operator. Then T is a strongly S-decomposable $S_T \subset S$.*

Proof. It follows from the fact that a spectral operator is strongly decomposable and from remark 1.4.2.

1.4.12. *Remark.* If $T \in B(X)$ is S-spectral it will be enough to take $S \subset \sigma(T)$. Indeed, by remark 1.4.2. it follows that $T = T_1 \oplus T_2$, where $\sigma(T_2) \subset S$, and T_1 is spectral. We have $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, hence $\sigma(T_2) \subset \sigma(T)$ and $\sigma(T_2) \subset \sigma(T) \cap S = S'$. But the direct sum of a spectral operator T_1 with another operator T_2 is an operator, $T = T_1 \oplus T_2$, S_1-spectral where $S_1 = \sigma(T_2) = \sigma(T \mid E(S)X) \subset S \cap \sigma(T)$.

1.4.13. Proposition. *Let $T \in B(X)$ be S-spectral; then the support of the spectral S-measure of T is $\sigma(T)$, and $E_S(\sigma(T)) = I$.*

Proof. According to remark 1.4.2. we have $T = T_1 \oplus T_2$ where $T_1 = T \mid E_S(\mathbf{C}S)X$ is spectral, and $T_2 = T \mid E_S(S)X$ with $\sigma(T_2) \subset S$. Let $Y_1 = E_S(\mathbf{C}S)X$, $Y_2 = E_S(S)X$; we obviously have $X = Y_1 \oplus Y_2$ and $T = T_1 \oplus T_2$. Since T_1 is spectral with the spectral measure E defined by the equality $E(B) = E_S(B \cap \mathbf{C}S)$ for any $B \subset \mathbf{C}$ borelian, the support of the spectral measure E is $\sigma(T_1)$. But $E_S(S) = I_2$ is the identical operator in Y_2 (we have $E_S(S)y = E_S(S)E_S(S)x = E_S(S)x = y$ for any $y \in E_S(S)X$), hence $I = E_S(C) = E_S(CS) + E_S(S) = E(\sigma(T_1)) \oplus I_2 = I_1 \oplus I_2 = E_S(\sigma(T_1) \cup S) = E_S(\sigma(T))$, consequently the support of E_S is $\sigma(T)$. One can also verify directly, as in proof of proposition 1.5. [41]. Accordingly supp $E_S = (T)$.

1.4.14. Proposition. *Let $T \in B(X)$ be a S-spectral operator, let E_S be one of its S-spectral measures and w a proper value of T. If $F \subset \mathbf{C}$ is a closed set from B_S and $w \notin S \cup F$, then, for any proper vector x corresponding to w, we have*

$$E_S(F)x = 0,\ E_S(\{w\})x = x,$$

when $w \in S$, then

$$E_S(F)x = 0,\ E_S(F_S)x = x$$

for any F, F_S closed such that $F \cap S = \varnothing$, $F_S \supset S$.

Proof. One proceeds as for spectral operators. For the first case, since $w \notin \sigma(T \mid E_S(F)X)$ we have

$$\begin{aligned} E_S(F)x &= R(w, T \mid E_S(F)X) \cdot (w - T)E_S(F)x = \\ &= R(w, T \mid E_S(F)X)E_S(F) \cdot (w - T)x = 0. \end{aligned}$$

By putting

$$F_n = \left\{\lambda, |\lambda - w| \geq \frac{1}{n}\right\},$$

and from $E_S(F_n)x = 0$ (for $n = K$ big enough such that $F_K \supset S$) it follows

$$(I - E_S(\{w\}))x = E_S(\mathbf{C} \setminus \{w\})x = E_S\left(\bigcup_{n=K}^{\infty} F_n\right)x = \lim_{n\to\infty} E_S(F_n)x = 0,$$

hence $E_S(\{w\})x = x$. One verifies the same in the second case, when $w \in S$.

Remark. From the preceding proposition it results again very easy that $S_T \subset S$.

1.4.15. Theorem. *Let $T \in B(X)$ be a S-spectral operator and let E_S be one of its S-spectral measures. Then for any closed $F \subset \mathbf{C}$ such that $F \supset S$ we have*

$$E_S(F)X = X_T(F)$$

Proof. Since $\sigma(T \mid E_S(F)X) \subset F$ we evidently have

$$E_S(F)X \subset X_T(F).$$

Let us verify the inverse inclusion. Let $x \in X_T(F)$, hence $\rho_T(x) \supset \mathbf{C} \setminus F$. Let σ be closed (compact), $\sigma \cap F = \varnothing$. Let us prove that $E(\sigma)x = 0$. We consider a admissible system Γ of simple Jordan curves that contains in "the exterior" σ and leaves in "the interior" the set F, hence $\Gamma \subset \mathbf{C} \setminus F \subset \rho_T(x)$. If $x(\lambda)$ is the analytic function defined on $\rho_T(x)$ such that $x = (\lambda I - T)x(\lambda)$, then

$$\int_\Gamma x(\lambda) d\lambda = 0.$$

Hence we shall be allowed to write:

$$\begin{aligned} E_S(\sigma)x &= \frac{1}{2\pi i}\int_{\|T\|+1} R(\lambda, T)E_S(\sigma)x d\lambda = \\ &= \frac{1}{2\pi i}\int_{\|T\|+1} R(\lambda, T \mid E_S(\sigma)X)E(\sigma)x d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma R(\lambda, T \mid E_S(\sigma)X)E(\sigma)x d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma E_S(\sigma)R(\lambda, T \mid E_S(\sigma)X)x d\lambda = \\ &= \frac{1}{2\pi i}\int_\Gamma E_S(\sigma)x(\lambda) d\lambda = E_S(\sigma)\frac{1}{2\pi i}\int_\Gamma x(\lambda) d\lambda = 0. \end{aligned}$$

The set $\mathbf{C} \setminus F$ being open we have $\mathbf{C} \setminus F = \bigcup_{n \in \mathrm{N}} \sigma_n$ with σ_n closed $\sigma_n \subset \sigma_{n+1}$ (σ_n can be replaced with the compact sets $\sigma_n \cap \sigma(T)$), consequently

$$(I - E_S(F))x = E_S(\mathbf{C} \setminus F)x = E_S\left(\bigcup_{n \to \infty} \sigma_n\right)x = \lim_{n \to \infty} E_S(\sigma_n)x = 0$$

hence $x = E_S(F)x \in E_S(F)X$, whence

$$X_T(F) \subset E_S(F)X.$$

1.4.16. Corollary. *Let $T \in B(X)$ be a S-spectral operator and let E_S be one of its spectral capacities. Then the map E defined by the equality*

$$\mathrm{E}(F) = E_S(F)X \text{ for } F \in \mathrm{F}_S$$

is the spectral S-capacity of the strongly S-decomposable operator T.

Proof. T is strongly S-decomposable hence it admits a spectral S-capacity E which is unique (see theorem 2.5.5.); from the preceding theorem there follows that $E_S(F)X = X_T(F)$ if $F \supset S$ and $E_S(F)X = Y_F$ if $F \cap S = \varnothing$, where $E_S(F \cup S)X = E_S(F)X \oplus E_S(S)X = X_T(F \cup S) = Y_F \oplus X_T(S)$. In theorem 2.5.5. and corollary 2.5.6. there is proved that the spectral S-capacity of a strongly S-decomposable operator is given by the equalities $\mathsf{E}(F)=X_T(F)$ for $F\supset S$ and $\mathsf{E}(F)=Y_F$ for $F\cap S=\varnothing$, where Y_F is given by $X_T(F\cap S)=Y_F\oplus X_T(S)$.

1.4.17. *Remarks.* a). From the preceding theorem and corollary it follows that if T is S-spectral then $E_S(F)X$ is a spectral maximal space for T hence a subspace of X, ultrainvariant to T for any $F\in\mathsf{F}_S$. b). Considering the conditions and the proof of the preceding corollary it results that for $F\subset\mathbf{C}$ closed such that $F\cap S=\varnothing$ we have $E_S(F)X=Y_F$, where Y_F is the spectral maximal space given by the equality $E_S(F\cup S)X=X_T(F\cup S)=Y_F\oplus X_T(S)$ and $\sigma(T\mid Y_F)\subset F$.

1.4.18. Proposition. *Let $T\in B(X)$ be a S-spectral operator and let E_S be one of its S-spectral measures. Then for any operator $A\in B(X)$ interchangeable with T we have*

$$AE_S(B)=E_S(B)A$$

for any $B\in\mathsf{B}_S$.

Proof. The standard procedure is applied, observing that it is enough to verify only for $B\in\mathsf{B}_S$, $B\supset S$, since $T_1=T\mid E(\mathbf{C}\setminus S)X$ is spectral (remark 1.4.2.). Let σ be closed, $\sigma\in\mathsf{B}_S$. Because $E(\sigma)X$ is an ultrainvariant subspace to T it follows that

$$AE_S(\sigma)=E_S(\sigma)AE_S(\sigma)$$

Let F and $F_1\in\mathsf{B}_S$ closed, $F\cap F_1=\varnothing$, $F\supset S$. So, using the preceding equality, we obtain

$$E_S(F)AE(F_1)=E_S(F\cap F_1)AE_S(F_1)=E_S(\varnothing)AE_S(F_1)=0.$$

The set $\mathbf{C}\setminus F$ being open, there exists a growing series of sets $(\sigma_n)_{n\in\mathrm{N}}$ closed such that $\mathbf{C}\setminus F=\bigcup_{n\in\mathrm{N}}\sigma_n$. Hence

$$E_S(F)AE_S(\mathbf{C}\setminus F)=\lim_{n\to\infty}E_S(F)AE_S(\sigma_n)=0$$

whence

$$E_S(F)A=E_S(F)AE(C)=E_S(F)AE_S(F)+E_S(F)AE_S(C\setminus F)=$$
$$=E_S(F)AE_S(F)=AE_S(F)\text{q.e.d.}$$

1.4.19. Theorem. *Any S-spectral operator $T\in B(X)$ has a single spectral S-measure E_S.*

Proof. Let E_S, E'_S be two spectral S-measures of T. From the exchange of E_S and E'_S with T, it follows that

$$E_S(B)=E'_S(B_1)E_S(B)$$

for any $B,B_1\in\mathsf{B}_S$. We also have $E_S(F)X=X_T(F)=E'_S(F)$ for F closed, $F\supset S$ and $E_S(F_1)X=Y_{F_1}=E'_S(F_1)X$ for F_1 closed, $F_1\cap S=\varnothing$, where Y_{F_1} is given by the equality

$Y_{F_1} \oplus X_T(S) = X_T(F_1 \cup S)$ (remark 1.4.17.). One knows that if $P, Q \in B(X)$ are two projectors, then $QP = P$ is equivalent with $PX \subset QX$ (Lemma 1.12. [41]).
Consequently

$$E_S(F)E_S'(F) = E_S'(F),\ E_S'(F)E_S(F) = E_S(F)$$

and similarly for F_1, hence $E_S(F) = E_S'(F)$ for any closed $F \in \mathrm{B}_S$. According to the regularity of the measures $\langle E_S(\cdot)x, x^*\rangle$ and $\langle E_S'(\cdot)x, x^*\rangle$ it results that $E_S(B) = E_S'(B)$ for any $B \in \mathrm{B}_S$.

1.4.20. Corollary. *If $T \in B(X)$ is a S-spectral operator, then T can be written uniquely as follows*

$$T = T_1 \oplus T_2$$

where $T_1 = T \mid E_S(\mathbf{C} \setminus S)X$, and $T_2 = T \mid E_S(S)X$ has the spectrum $\sigma(T_2) \subset S$.

Proof. There follows by the preceding theorem and by remark 1.4.2.

1.4.21. Proposition. *Let $T \in B(X)$ be a S-spectral operator such that*

$$\dim S = 0.$$

Then T is a U-scalar operator; if, moreover, T is the restriction to a subspace invariant to a spectral operator, then T itself is spectral.

Proof. It results from the fact that an operator T with $\dim \sigma(T) = 0$ is U-scalar, also a spectral operator is U-scalar [37]. The second assertion results from 4.13. [41].

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