INSTITUTUL DE MATEMATICĂ
AL ACADEMIEI ROMÂNE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

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by

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Preprint nr. 2/2005

BUCUREȘTI

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January, 2005

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S - spectral decompositions II

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CHAPTER II

S-DECOMPOSABLE OPERATORS

This paper is devoted to the study of the *S*-decomposable operators defined in the introduction (see [21], [16]). First, we show some structural properties of spectral maximal spaces of the *S*-decomposable operators. Then, we shall present the behavior of these operators at direct sums, at projections, at separate parts of the spectrum, at the Riesz-Dunfort functional calculus and at the quasinilpotent equivalence. We will also give proof of an important structural theorem of spectral maximal spaces, generalising the following from [53] and [59]. We shall define and study the spectral *s*-capacities, and give several *s*-decomposability criteria. We shall further study the restrictions and the *S*-decomposable operators' quotients.

2.1. THE STRUCTURE OF SPECTRAL MAXIMAL SPACES OF *S*-DECOMPOSABLE OPERATORS

In this paragraph we shall generalise the corresponding follows from [37], [48], obtained for decomposable operators. The main result will be that $X_T(F)$ is a spectral maximal space for any $F \subset S \subset S_T$, F closed.

2.1.1. LEMMA. *Let $T \in B(X)$ be a S-decomposable operator, and let G be an open set such that:*

$$G \cap (\sigma(T) \setminus S) = \varnothing$$

then there exists a maximal spectral space $Y \neq \{0\}$ of T such that $\sigma(T/Y) \subset G$. If $\dim S \leq 1$ *and* $G \cap \operatorname{Int} \sigma(T) \neq \varnothing$ *(G being an open set), then there exists a maximal spectral space $Y \neq \{0\}$ of T such that $\sigma(T/Y) \subset G$.*

Proof. Let G_S be an open set such that:

$$S \subset G_S \not\supset \sigma(T)$$

and

$$G_S \cup G \supset \sigma(T).$$

T being S-decomposable, there exists a sistem of spectral maximal spaces Y_S, Y from T such that:

$$\sigma(T \mid Y_S) \subset G_S,\ \sigma(T \mid Y) \subset G$$

and

$$X = X_S + Y.$$

If $Y = \{0\}$, we have $Y_S = X$ and $\sigma(T \mid Y_S) = \sigma(T) \subset G_S$, contradiction, hence $Y \neq \{0\}$. When $\dim S \leq 1$ and $G \cap \operatorname{Int} \sigma(T) \neq \varnothing$ it follows that $G \cap (\sigma(T \mid Y) \setminus S) \neq \varnothing$, consequently $Y \neq \{0\}$.

2.1.2. THEOREM. *If* $T \in B(X)$ *is S-decomposable where* $\dim S \leq 1$, *then*

$$\sigma_p^0(T) = \sigma_r^0(T) = \varnothing \text{ (see [37], 1.3.6.),}$$

T has the single-valued extension property ($S_T = \varnothing$) *and* $\sigma(T) = \sigma_l(T)$. *If* $S_T \neq \varnothing$, *then* $S_T \subset S$ *and* $\dim S = 2$.

Proof. If $\sigma_p^0(T) = \varnothing$, let G be a component of $\sigma_p^0(T)$. Then, by proposition 1.3.7. [37], there doesn't exist any spectral maximal space $Y \neq \{0\}$ of T such that

$$\sigma(T \mid Y) \subset G;$$

by the preceding lemma, $G \cap \sigma(T) = \varnothing$, therefore $G \cap \sigma_p^0(T) = \varnothing$ which is impossible (since $G \subset \sigma_p^0(T) \subset \operatorname{Int} \sigma(T)$). Same for $\sigma_r(T)$.

Consequently

$$\sigma_p^0(T) = \sigma_r^0(T) = \varnothing$$

since $S_T = \overline{\sigma_p^0(T)}$, and $\sigma_r^0(T) = \sigma(T) \setminus \sigma_l(T)$, we have $S_T = \varnothing$ (meaning that T has the single-valued extension property) and

$$\sigma(T) = \sigma_l(T).$$

Now let $S_T \neq \varnothing$. In order to verify the inclusion $S_T \subset S$ it will suffice to verify that $\sigma_p^0(T) \subset S$. Suppose that $\sigma_p^0(T) \not\subset S$; then there exists a component G_0 of $\sigma_p^0(T)$ such that:

$$G_0 \not\subset S \text{ and } G_0 \cap (\sigma(T) \setminus S) \neq \varnothing.$$

By the preceding lemma there follows that there exists a spectral maximal space Y_0 of T, $Y_0 \neq \{0\}$ such that:

$$\sigma(T \mid Y_0) \subset G_0;$$

contradicts proposition 1.3.7. [37], consequently $S_T \subset S$. But $S_T \neq \varnothing$ implies $\dim S = 2$ (we have $\operatorname{Int} S_T \neq \varnothing$) hence $\operatorname{Int} S \neq 0$.

2.1.3. THEOREM. *Let $T \in B(X)$ be a s-decomposable operator and let $F \subset \mathbb{C}$ be a closed set such that*

$$S \subset F \subset \sigma(T).$$

Then $X_T(F)$ is a spectral maximal space of T and

$$\sigma(T \mid X_T(F)) \subset F.$$

Conversely, for any spectral maximal space Y of T such that $\sigma(T \mid Y) \supset S$ we have

$$Y = X_T(\sigma(T \mid Y)).$$

Proof. Let $F \subset \sigma(T)$ be closed such that $S \subset F$ ($S_T \subset S \subset F$) and let G_S, H be two open sets satisfying conditions $G_S \supset F$, $H \cap F = \varnothing$ and $G_S \cup H \supset \sigma(T)$. We shall put

$$G_1 = G_S,\ G_2 = H.$$

Let $\{Y_i\}_1^2$ be a corresponding system of spectral maximal spaces of T such that:

$$\sigma(T \mid Y_i) \subset G_i \ \ (i = 1,2)$$

and

$$X = Y_1 + Y_2.$$

If $x \in X_T(F)$, then $x = y_1 + y_2$, $y_i \in Y_i$ $(i = 1,2)$ and $\sigma_T(x) \subset F$; for $\lambda \in \rho_T(x)$ $x(\lambda)$ has meaning and

$$(\lambda I - T)x(\lambda) = x$$

hence for $\lambda \in \complement F \cap \rho(T \mid Y_2)$ we have

$$(\lambda I - T)(R(\lambda, T \mid Y_2)y_2 - x(\lambda)) = y_2 - x = -y_1,$$

from which it follows that $\lambda \in \rho_T(y_1)$. But $\lambda \notin S \supset S_T$, consequently $\lambda \in \delta_T(y_1) \cap \Omega_T = \rho_T(y_1)$ and from this it derive that

$$\sigma_T(y_1) \subset F \cup \sigma(T \mid Y_2) \subset F \cup \overline{G}_2$$

therefore

$$\complement F \cap \complement \overline{G}_2 \subset \rho_T(y_1).$$

Let now Γ be a bounded system of simple closed curves surrounding F and included in $\complement F \cap \complement \overline{G}_2$. For $\lambda \in \Gamma$ we have

$$y_1(\lambda) = -R(\lambda, T \mid Y_2)y_2 + x(\lambda), \text{ Hence}$$

$$\frac{1}{2\pi i}\int_\Gamma y_1(\lambda)d\lambda = -\frac{1}{2\pi i}\int_\Gamma R(\lambda, T \mid Y_2)y_2\, d\lambda + \frac{1}{2\pi i}\int_\Gamma x(\lambda)d\lambda.$$

The spectral maximal space Y_1 of T being T-absorbing ([76], proposition 3.1.), if $y_1 \in Y_1$, then $y_1(\lambda) \in Y_1$ for $\lambda \in \rho_T(y_1)$ and since $\sigma(T \mid Y_2)$ is "outside" Γ we obtain

$$\frac{1}{2\pi i}\int_\Gamma y_1(\lambda)d\lambda \in Y_1,\ \frac{1}{2\pi i}\int_\Gamma R(\lambda, T \mid Y_2)y_2\, d\lambda = 0.$$

Consequently

$$x=\frac{1}{2\pi i}\int_{|\lambda|=\|T\|+1} R(\lambda,T)x\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma} x(\lambda)d\lambda=\frac{1}{2\pi i}\int_{\Gamma} y_1(\lambda)d\lambda\in Y_1,$$

thus

$$X_T(F)\subset\bigcap_{G_1\supset F} Y=Z.$$

By other means, if $z\in\mathbf{Z}$ then from the inclusions

$$\gamma_T(z)\subset\gamma_{T|Y_1}(z)\subset\sigma(T\,|\,Y_1)\subset G_1$$

it follows that

$$\sigma_T(z)=\gamma_T(z)\cup S_T\subset\bigcap_{G_1\supset F} G_1=F_1$$

hence $z\in X_T(F)$ and $Z\subset X_T(F)$; so we conclude that

$$X_T(F)=\bigcap_{G_1\supset F} Y_1,$$

from where it follows that $X_T(F)$ is closed. By proposition 3.4. [76], $X_T(F)$ is a spectral maximal space of T and $\sigma(T\,|\,X_T(F))\subset F$. Conversely, if Y is a spectral maximal space of T such that $\sigma(T\,|\,Y)\supset S$, then according to those proved before we obtain that

$$\sigma(T\,|\,X_T(\sigma(T\,|\,Y)))\subset\sigma(T\,|\,Y)$$

hence

$$X_T(\sigma(T\,|\,Y))\subset Y.$$

But from the evident inclusion $Y\subset X_T(\sigma(T\,|\,Y))$ one finally obtains

$$Y=X_T(\sigma(T\,|\,Y)).$$

At this moment the theorem is completely proved. When T has the single-valued extension property ($S_T=\varnothing$) we have the following

2.1.4. Corollary. *Let $T\in B(X)$ a s-decomposable operator with $S_T=\varnothing$ and let $F\in\mathbf{C}$ be such that either $S\cap F=\varnothing$ or $F\supset S_1$ and $F\cap(S\setminus S_1)=\varnothing$, where S_1 is a separated part of S. Then $X_T(F)$ is a spectral maximal space of T and $\sigma(T\,|\,X_T(F))\subset F$. Conversely, if Y is a spectral maximal space of T such that $\sigma(T\,|\,Y)=F$ and F has one of the two properties above, then $Y=X_T(\sigma(T\,|\,Y))$.*

Proof. If $F\cap S=\varnothing$ ($F\subset\sigma(T)$ closed), by the preceding theorem $X_T(S)$ and $X_T(F\cup S)$ are spectral maximal spaces of T and

$$X_T(F\cup S)=X_T(F)+X_T(S),$$

whence it follows that $X_T(F)$ is also a spectral maximal space for T (see [4], proposition 4.9) and $\sigma(T(X_T(F)))\subset F$.

If

$$S = S_1 \cup (S \setminus S_1),$$

where S_1 is a separated part of S and $F \supset S_1$, $F \cap (S \setminus S_1) = \varnothing$, then

$$X_T(F \cup (S \setminus S_1)) = X_T(F) + X_T(S \setminus S_1);$$

therefore $X_T(F)$ is again a spectral maximal space of T. The final part of the corollary results identically as in the preceding theorem namely from the evident inclusions $Y \subset X_T(\sigma(T \mid Y))$ and $\sigma(T \mid X_T(\sigma(T \mid Y))) \subset \sigma(T \mid Y)$.

2.1.5. Proposition. *Let $T \in B(X)$ a S-decomposable operator and S_1 a separated part of S with* $\dim S_1 = 0$. *Then T is S'-decomposable where $S' = S \setminus S_1$.*

Proof. The case $S_T = \varnothing$ has been proved in proposition 1.2.9. Keeping the notations from the proposition 1.2.9. prove, we will obtain the spectral maximal spaces $\{Y_S\} \cup \{Y'\}_1^n$ of T such that $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_i') \subset G_i'$ $(i = 1,2,\dots,n)$

and

$$X = Y_S + Y_1' + Y_2' + \dots + Y_n'.$$

But $Y_S = Y_{\sigma'} + Y_{\sigma_1} + Y_{\sigma_2} + \dots + Y_{\sigma_n}$, where $\sigma(T \mid Y_S) = \sigma' \cup \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_n$, $\sigma(T \mid Y_{\sigma'}) = \sigma'$, $\sigma(T \mid Y_{\sigma_i}) = \sigma_i$ $(i = 1,2,\dots,n)$. $Y_{\sigma'}, Y_{\sigma_i}$ being spectral maximal spaces of T, and $\sigma' \subset G_{S'}$, $\sigma_i \subset G_i' \subset G_i$. Let $\hat{\sigma}_i = \sigma_i \cup \sigma(T \mid Y_i')$. Since $\hat{\sigma}_i \cap S' = \varnothing$, we have $X_T(S' \cup \hat{\sigma}_i) = X_T(S') + Y_{\hat{\sigma}_i}$, where $Y_{\hat{\sigma}_i}$ are spectral maximal spaces of T, $\sigma(T \mid Y_{\hat{\sigma}_i}) \subset \hat{\sigma}_i \subset G_i$ $(i = 1,2,\dots,n)$. We have $Y_i' + Y_{\sigma_i} \subset Y_{\hat{\sigma}_i}$ and $X_T(S') + Y_{\sigma'} \subset X_T(\sigma' \cup S') = Y_{S'}$, therefore $X = Y_{S'} + Y_{\hat{\sigma}_1} + \dots + Y_{\hat{\sigma}_n}$, and T is S'-decomposable.

2.1.6. *Remark.* Let $T \in B(X)$ be a S-decomposable operator and $S_1 \subset S$ the closing of the set of S's points in which S has the dimension 0, $\dim S_1 = 0$ and thus that $S' = S \setminus S_1$ be closed (and thus separated from S_1); then from the preceding proposition it follows that T is S'-decomposable.

2.1.7 Proposition. *Let $T_\alpha \in (X_\alpha)$ $(\alpha = 1,2)$ and let $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$. If $Y \subset X_1 \oplus X_2$ is a spectral maximal space of $T_1 \oplus T_2$, then $Y = Y_1 \oplus Y_2$, where Y_1, Y_2 are spectral maximal spaces of T_1 respectively T_2.*

Proof. Let P_1 and P_2 be the corresponding projections: $X_1 = P_1(X_1 \oplus X_2)$, $X_2 = P_2(X_1 \oplus X_2)$. It is easy to verify that P_1 and P_2 switch with $T_1 \oplus T_2$ and since Y is ultrainvariant at $T_1 \oplus T_2$, it follows that Y is invariant to P_1 and P_2. By putting $Y_1 = P_1 Y$ and $Y_2 = P_2 Y$, we have $Y_1 \subset Y$, $Y_2 \subset Y$, $Y_1 \oplus Y_2 \subset Y$, P_1 and P_2 also being projections in the Banach space Y, Y_1, Y_2 closed. If $y \in Y$, then $y = P_1 y \oplus P_2 Y \in Y_1 \oplus Y_2$, so $Y = Y_1 \oplus Y_2$. Let Z_α $(\alpha = 1,2)$ two invariant at T subspace such that

$$\sigma(T_\alpha \mid Z_\alpha) \subset \sigma(T_\alpha \mid Y_\alpha) \ (\alpha = 1,2).$$

Then $Z = Z_1 \oplus Z_2$ is an (closed) invariant subspace at $T_1 \oplus T_2$ and

$$\sigma(T_1 \oplus T_2 \mid Z_1 \oplus Z_2) \subset \sigma(T_1 \oplus T_2 \mid Y_1 \oplus Y_2),$$

hence $Z_1 \oplus Z_2 \subset Y_1 \oplus Y_2$. From this inclusion it obviously follows that

$$Z_1 \subset Y_1,\ Z_2 \subset Y_2$$

consequently Y_1 and Y_2 are spectral maximal spaces of T_1, respectively T_2.

2.2. DIRECT SUMS AND RIESZ-DUNFORD FUNCTIONAL CALCULUS WITH *S*-DECOMPOSABLE OPERATORS

In the beginning of paragraph 2 we give a simple *S*-decomposability criterion that greatly simplifies the subsequent proofs. We prove there that the direct sum of two operators is $S = S_1 \cup S_2$-decomposable if and only if each operator is S_α-decomposable ($\alpha = 1,2$). Particularly when $P \in B(X)$ is a projection and *T* is *S*-decomposable there is proved that $T \mid PX$ is S_1-decomposable (where $S_1 = S \cap \sigma(T \mid PX)$). We further study the demeanour of the *S*-decomposable operators in the functional calculus with analytic functions and at quasinilpotent equivalence.

2.2.1. Definition. Let $T \in B(X)$ and let $S \subset \mathbf{C}$ be a compact set. *T* is said to satisfy *condition* α_S if $X_T(F)$ is closed for any closed $F \supset S$. *T* is also said to satisfy *condition* β_S if for any finite and open *S*-covering $\{G_S\} \cup \{G_i\}_1^n$ of $\sigma(T)$ and for any $x \in X$ we have

$$x = x_S + x_1 + x_2 + \ldots + x_n,$$

where

$$\gamma_T(x_S) \subset G_S,\ \gamma_T(x_i) \subset G_i\ (i = 1,2,\ldots,n).$$

2.2.2 Lemma. *An operator $T \in B(X)$ is S-decomposable if and only if T meets conditions α_S and β_S.*

Proof. Since $\overline{G}_i \cap S = \varnothing$ we have

$$X_T(\overline{G}_i \cup S) = Y_i \oplus Y_S,$$

where $X_T(\overline{G}_i \cup S)$, Y_i and Y_S are spectral maximal spaces of *T* (see [76], propositions 2.4. and 3.4.); also, if Y is a spectral maximal space of T we have $\gamma_T(x) \subset \gamma_{T|Y}(x) \subset \sigma(T \mid Y)$ for any $x \in Y$. Considering these remarks, our assertion is obvious.

2.2.3 Theorem. *Let $T_\alpha \in B(X_\alpha)$ ($\alpha = 1,2$) and let $S = S_1 \cup S_2$; then if T_α is S_α-decomposable ($\alpha = 1,2$), $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ is S-decomposable.*

Proof. From the equalities

$$X_{1_{T_1}}(F) \oplus X_{2_{T_2}}(F) = (X_1 \oplus X_2)_{T_1 \oplus T_2}(F)\ (F \supset S),$$

$$\gamma_{T_1\oplus T_2}(x_1\oplus x_2)=\gamma_{T_1}(x_1)\cup\gamma_{T_2}(x_2)\ (x_\alpha\in X_\alpha,\alpha=1,2),$$

$$\left(x^1_{S_1}\oplus x^2_{S_2}\right)+\sum_{i=1}^{n}\left(x^1_i\oplus x^2_i\right)=\left(x^1_{S_1}+\sum_{i=1}^{n}x^1_i\right)\oplus\left(x^2_{S_2}+\sum_{i=1}^{n}x^2_i\right)$$

it follows that if T_1 and T_2 meet conditions α_{S_1}, α_{S_2} and β_{S_1}, β_{S_2}, then $T_1\oplus T_2$, meets conditions α_S and β_S.

2.2.4 Proposition. *Let $T_1\oplus T_2\in B(X_1\oplus X_2)$ a S-decomposable operator; then T_α $(\alpha=1,2)$ are S-decomposable operators where $S_\alpha=S\cap\sigma(T_\alpha\mid X_\alpha)$ $(\alpha=1,2)$.*

Proof. Let $F\supset S_1$ closed; we shall be allowed to write

$$X_{1_{T_1}}(F\cup S)\oplus X_{2_{T_2}}(F\cup S)=(X_1\oplus X_2)_{T_1\oplus T_2}(F\cup S)$$

and since $T_1\oplus T_2$ is S-decomposable, also using proposition 2.1.7. it follows that $X_{1_{T_1}}(F\cup S)$ is closed, hence $X_{1_{T_1}}(F)=X_{1_{T_1}}(F\cup S)$ is closed. Similarly, we verify that $X_{2_{T_2}}(F)$ is closed for any closed $F\supset S_2$. Hence T_1 and T_2 meet conditions α_{S_1} and α_{S_2}. The fact that T_1 and T_2 satisfy conditions β_{S_1} and β_{S_1} is proved same as for the preceding proposition.

2.2.5. Theorem. *Let $T_\alpha\in B(X_\alpha)$ $(\alpha=1,2)$, let S be compact and let $S_\alpha=S\cap\sigma(T_\alpha)$ $(\alpha=1,2)$ $(S=S_1\cup S_2\subset\sigma(T_1)\cup\sigma(T_2))$. Then T_α $(\alpha=1,2)$ are S_α-decomposable operators if and only if $T_1\oplus T_2$ is S-decomposable.*

Proof. There follows from the preceding assertions.

2.2.6. Corollary. *The operators $T_\alpha\in B(X_\alpha)$ $(\alpha=1,2)$ are decomposable if and only if $T_1\oplus T_2$ is decomposable.*

Proof. There follows either from the preceding theorem, or directly from lemma 2.2.2, because T_1 and T_2 satisfy conditions α_S and β_S (with $S=\varnothing$) if and only if $T_1\oplus T_2$ meets conditions α_S and β_S.

2.2.7. Propositon. *Let $T\in B(X)$ be a S-decomposable operator and let $P\in B(X)$ such that $P^2=P$ and $PT=TP$. Then $T\mid PX$ is a S_1-decomposable operator, where $S_1=S\cap\sigma(T)$.*

Proof. We have $X=Y_1+Y_2$, where $Y_1=PX$ and $Y_2=(I-P)X$ $(Y_1\cap Y_2=\{0\})$, hence in accordance with proposition 2.2.4 $T\mid PX$ is S_1-decomposable.

2.2.8. Corollary. *Let $T\in B(X)$ be a S-decomposable operator, and also a separated part of $\sigma(T)$. Then $T\mid E(\sigma,T)X$ is a S_1-decomposable operator, where $S_1=S\cap\sigma$ and*

$$E(\sigma,T)=\frac{1}{2\pi i}\int_\Gamma R(\lambda,T)d\lambda,$$

Γ being a Jordan closed curves system surrounding σ and separating sets σ and $\sigma'=\sigma(T)\setminus\sigma$.

Proof. There follows by the preceding proposition.

From now on we shall put

$$f(T)=\frac{1}{2\pi i}\int f(\lambda)R(\lambda,T)d\lambda \text{ ([45], I, VII. 3.9).}$$

2.2.9. Proposition. *Let $T\in B(X)$ be a S-decomposable operator and let $f:G\to \mathbf{C}$ ($G\supset\sigma(T)$, G open and connected) be an analytic function, injective on $\sigma(T)$. Then $f(T)$ is S_1-decomposable, where $S_1=f(S)$.*

Proof. Let $F\subset\sigma(f(T))$ closed, $F\supset S_1$; from the relations

$$S_{f(T)}=f(S_T)\subset f(S)=S_1\subset f(\sigma(T))=\sigma(f(T))$$

and

$$f^{-1}(F)\supset f^{-1}(S_1)=S$$

it follows that

$$X_{f(T)}(F)=X_T\left(f^{-1}(F)\right)$$

is closed (see [77] theorems 2.1,2.4), therefore $f(T)$ meets condition α_{S_1}. If $\{G_{S_1}\}\cup\{G_i\}_1^n$ is an open and finite S_1-covering of $\sigma(F(T))$, then $\{f^{-1}(G_{S_1})\}\cup\{f^{-1}(G_i)\}_1^n$ is a S-covering of $\sigma(T)$. From the equality

$$\gamma_{f(T)}(x)=f(\gamma_T(x))\ (x\in X)\ [77]$$

it will follow that $f(T)$ also meets condition β_{S_1}, therefore $f(T)$ is a S_1-decomposable operator.

2.2.10. Corollary. *Let $T\in B(X)$ be a S-decomposable operator and let $f:G\to\mathbf{C}$ ($G\supset\sigma(T)$, G open) be an analytic injective function on each $\sigma_i=G_i\cap\sigma(T)$, where G_i is connected component of G. Then $f(T)$ is $f(S)$-decomposable.*

Proof. $\sigma(T)$ being connected, there exists a finite number of connected components G_i of G which cross $\sigma(T)$, let these be $G_1,\dots,G_n$. The sets σ_i are separated parts of $\sigma(T)$ and

$$\sigma(T)=\sigma_1\cup\sigma_2\cup\ldots\cup\sigma_n;$$

hence

$$X=\bigoplus_{i=1}^{n}E(\sigma_i,T)X$$

and

$$T=\bigoplus_{i=1}^{n}(T\mid E(\sigma_i,T)X).$$

Since

$$f(T)=\bigoplus_{i=1}^{n}f(T)E(\sigma_i,T)X=\bigoplus_{i=1}^{n}f(T\mid E(\sigma_i,T)X)$$

by propositions 2.2.5. and 2.2.9. and corollary 2.2.8. it follows that $f(T)$ is $f(S)$-decomposable.

2.2.11. PROPOSITION. *Let $T \in B(X)$ and let f be an analytic function such that there exists $G \supset \sigma(T)$, G open and f injective on G. Then, if $f(T)$ is S_1-decomposable, T is S-decomposable, where $S = f^{-1}(S_1) \cap \sigma(T)$.*

Proof. Since $S_{f(T)} = f(S_T)$ and $X_T(f^{-1}(F)) = X_{f(T)}(F)$ (theorems 2.1., 2.4. [77]) with $F \supset S_1 \supset S_{f(T)}$, we conclude that for any closed set $F' \supset S \supset S_T$ (using the fact that *f* is injective on $\sigma(T)$), $F' \subset \sigma(T)$, there exists a closed *F* such that $F = f(F')$ and $F' = f^{-1}(F)$; therefore $X_T(F') = X_{f(T)}(F)$ is closed and T meets condition α_S. If $\{G_S\} \cup \{G_i\}_1^n$ is an open S-covering of $\sigma(T)$, we can choose G_S, $G_i \subset G$ ($i = 1,2,...,n$), and then from $G_i \cap S = \varnothing$ and $f(G_i \cap S) = f(G_i) \cap f(S) = \varnothing$ (*f* is injective on *G*), as well as from the fact that $f(G_S) \supset f(S) = S_1$ it follows that $\{f(G_S)\} \cup \{f(G_i)\}_1^n$ is an open S_1-covering of $\sigma(f(T))$. But $f(T)$ meets condition β_{S_1}, hence for any $x \in X$ we have

$$x = x_{S_1} + x_1 + x_2 + ... + x_n,$$

where $\gamma_{f(T)}(x_{S_1}) \subset f(G_S)$, $\gamma_{f(T)}(x_1) \subset G_i$; but since $\gamma_T(x_{S_1}) = f^{-1}(\gamma_{f(T)}(x_{S_1}))$, $\gamma_T(x_i) = f^{-1}(\gamma_{f(T)}(x_i))$ it follows that T also meets condition β_S, therefore T is S-decomposable.

2.2.12. COROLLARY. *Let $T \in B(X)$ with $\sigma(T)$ contained in an angle $\varphi < \frac{2\pi}{k}$ (having vertex in the origin) where k is a integer positive number. Then T is S-decomposable if and only if T^k is S_1-decomposable, where $S_1 = S_k$ ($S^k = \{\lambda_1 \in \mathbf{C}; \lambda_1 = \lambda^k, \lambda \in S\}$).*

Proof. There follows by propositions 2.2.10. and 2.2.11.

2.2.13. PROPOSITION. *Let $T_1, T_2 \in B(X)$. If T_1 is S-decomposable, with $S_{T_1} = \varnothing$ (meaning that T_1 has the single-valued extension property; particularly* $\dim S \leq 1$*) and T_1, T_2 are quasinilpotent equivalent [38], then T_2 is also decomposable.*

Proof. If T_1, T_2 are quasinilpotent equivalent, then

$$\sigma(T_1) = \sigma(T_2),\ \sigma_{T_1}(x) = \sigma_{T_2}(x)\ (x \in X, S_{T_1} = S_{T_2} = \varnothing)$$

and

$$X_{T_1}(F) = X_{T_2}(F)$$

for any closed $F \subset \mathbf{C}$ [38] therefore T_2 also meets conditions α_S and β_S, that is T_2 is also S-decomposable.

2.2.14. *Remark.* If T_1 is S-decomposable ($S_{T_1} = \varnothing$) and S is minimal, meaning that there doesn't exist any compact subset $S_1 \subset S$, $S_1 \neq S$, such that T_1 is S_1-decomposable, then in the proposition above S is also minimal for T_2 in this sense. Indeed, supposing that there would exist $S_1 \subset S$ compact, $S_1 \neq S$, such that T_2 is S_1-decomposable, then by the preceding assertion, T_1 would also be S_1-decomposable, contradiction.

2.2.15. COROLLARY. *Let $T \in B(X)$ be a S-decomposable operator with $S_T = \varnothing$ and let Q a generalised nilpotent operator which commutes with T. Then $T+Q$ is $S_{\overline{\alpha}}$-decomposable.*

Proof. It follows from the preceding assertion.

2.2.16. PROPOSITION. *Let $T_1, T_2 \in B(X)$ with $S_{T_1}, S_{T_2} \neq \varnothing$. If T_1 is quasinilpotent equivalent with T_2, then*

$$\gamma_{T_1}(x) = \gamma_{T_2}(x)$$

for any $x \in X$.

Proof. Let $\lambda \in \delta_{T_1}(x)$; then there exists an analytic function $x_1(\lambda)$ defined on a neighbourhood $\omega \in \lambda$ such that $(\lambda I - T_1)x_1(\lambda) = x$ for any $\lambda \in \omega$. When proving theorem 1.2.4. [37] there is proved that if T_1 is quasinilpotent equivalent with T_2 and $x_1(\lambda)$ verifies the condition above on ω, then

$$x_2(\lambda) = \sum_{n=0}^{\infty} (-1)^n (T_2 - T_1)^{[n]} \frac{x_1^{(n)}(\lambda)}{n!}$$

is absolutely and uniformly convergent on every compact $K \subset \omega$, therefore it is analytic on ω and moreover it verifies the equality

$$(\lambda I - T_2)x_2(\lambda) = (\lambda I - T_1)x_1(\lambda) = x$$

for any $\lambda \in \omega$. Consequently $\delta_{T_1}(x) \subset \delta_{T_2}(x)$; analogously, one verifies the inclusion $\delta_{T_2}(x) \subset \delta_{T_1}(x)$, hence $\gamma_{T_1}(x) = \gamma_{T_2}(x)$.

2.2.17. PROPOSITION. *Let $T_1, T_2 \in B(X)$ with $S_{T_1} = S_{T_2}$, and let T_1, T_2 be quasinilpotent equivalent. Then, if T_1 is S-decomposable T_2 is also S-decomposable.*

Proof. From the equality $S_{T_1} = S_{T_2}$ and from the preceding proposition it follows that for $F \supset S \supset S_{T_2}$ closed, we have

$$X_{T_2}(F) = X_{T_1}(F),$$

hence condition α_S is also met by T_2. From the equality $\gamma_{T_2}(x) = \gamma_{T_1}(x)$ ($x \in X$) it follows that T_2 also meets condition β_S, therefore in accordance with lemma 2.2.2. T_2 is S-decomposable.

2.3. A BISHOP PROPERTY FOR *S*-DECOMPOSABLE OPERATORS

We will prove that spectral maximal spaces of the S-decomposable operators can be analogously characterised with the ones of the $F \supset S$ u-scalar operators [53] and decomposable [50] namely that $X_T(F) = N_c(T, F)$ for any closed $F \supset S$. We remind that the definition of $N_c(T, F)$ was inspired by Bishop's definition of $N(T, F)$ [82].

2.3.1. Definition. Let $T \in B(X)$ and let $F \subset \mathbf{C}$ be a compact set. We denote by $N_c(T, F)$ the set of all $x \in X$ for which we have the property: for all $\varepsilon > 0$ and $K \subset \mathbf{C} \setminus F$ compact, there exists an analytic function defined on a neighbourhood of K verifying the inequality:

$$\|x - (\lambda I - T) f(\lambda)\| < \varepsilon, \ \lambda \in K.$$

2.3.2. Lemma. *Let $T \in B(X)$ and let $X = \sum_{i=1}^{n} Y_i$ where Y_i are σ-stable subspaces for T (meaning Y_i are invariant subspaces for T and $\sigma(T \mid Y_i) \subset \sigma(T)$ [83]. Then*

$$\sigma(T) = S_T \cup \bigcup_{i=1}^{n} \sigma(T \mid Y).$$

Proof. Obviously, we have

$$S_T \cup \bigcup_{i=1}^{n} \sigma(T \mid Y) \subset \sigma(T).$$

Since $x = y_1 + y_2 + \ldots + y_n$ with $y_i \in Y_i$ $(i = 1, 2, \ldots, n)$ and $\gamma_T(x) \subset \bigcup_{i=1}^{n} \gamma_T(y_i)$, $\gamma_T(y_i) \subset \gamma_{T|Y_i}(y_i) \subset \sigma(T \mid Y_i)$, it will follow [76] that

$$\sigma(T) = S_T \cup \bigcup_{x \in X} \gamma_T(x) \subset S_T \cup \bigcup_{x = y_1 + y_2 + \ldots + y_n} \left(\bigcup_{i=1}^{n} \gamma_T(y_i) \right) \subset S_T \cup \bigcup_{i=1}^{n} \sigma(T \mid Y_i).$$

2.3.3. Lemma. *Let $T \in B(X)$ be a S-decomposable operator and let $\sigma \subset \sigma(T)$ compact such that $\sigma \cap S = \varnothing$ and $\sigma = \overline{\operatorname{Int} \sigma}$ (in the topology of $\sigma(T)$). Then there exists a spectral maximal space Y of T with $\sigma(T \mid Y) = \sigma$ (that is σ is set-spectrum of T (see definition 1.3.1.)).*

Proof. It is similarly carried out as for decomposable operators. We have

$$X_T(\sigma \cup S) = Y_\sigma \oplus X_T(S),$$

where Y is spectral maximal space of T and $\sigma(T \mid Y_\sigma) \subset \sigma$. It will be enough to prove that

$$\operatorname{Int} \sigma \subset \sigma(T \mid Y_\sigma)$$

($\operatorname{Int} \sigma$ in the topology of $\sigma(T)$). Let $\lambda_0 \in \operatorname{Int} \sigma$; then there exists a disk $\delta = \{\lambda; \lambda \in \mathbf{C}, |\lambda - \lambda_0| < \rho\}$ such that $\delta \cap \sigma(T) = \operatorname{Int} \sigma$. We put

$$\delta_1 = \left\{\lambda; \lambda \in \sigma(T), |\lambda - \lambda_0| < \frac{\rho}{2}\right\},$$

$$G_0 = \left\{\lambda; \lambda \in \mathbf{C}, |\lambda - \lambda_0| < \frac{3}{4}\rho\right\},$$

$$G_S = \left\{\lambda; \lambda \in \mathbf{C}, |\lambda - \lambda_0| < \frac{5}{8}\rho\right\}.$$

It follows that $G_0 \cup G_S \supset \sigma(T)$ and $G_S \cap \delta_1 = \varnothing$. If Y_S, Y_0 are corresponding spectral maximal spaces of T such that

$$\sigma(T \mid Y_S) \subset G_S,\ \sigma(T \mid Y_0) \subset G_0 \text{ and } X = Y_S + Y_0,$$

then, by the preceding lemma it follows that

$$\sigma(T) = S \cup \sigma(T \mid Y_S) \cup \sigma(T \mid Y_0)$$

and since $\delta_1 \cap (\sigma(T \mid Y_S) \cup S) = \varnothing$, we have

$$\delta_1 \subset \sigma(T \mid Y_0) \subset G_0 \cap \sigma(T) \subset \delta \cap \sigma(T) \subset \operatorname{Int}\sigma \subset \sigma.$$

Consequently $Y_0 \subset X_T(\sigma \cup S) = Y_\sigma \oplus X_T(S)$, whence $Y_0 \subset Y_\sigma$ and $\sigma(T \mid Y_0) \subset \sigma(T \mid Y_\sigma)$; one obtains $\lambda_0 \in \delta_1 \subset \sigma(T \mid Y_0) \subset \sigma(T \mid Y_\sigma)$ that is $\sigma \subset \sigma(T \mid Y_\sigma)$.

2.3.4. *Remark.* If $T \in B(X)$ is S-decomposable and also has the property of the single-valued extension, then, for any compact σ, $\sigma \subset \sigma(T)$ which has the following properties: $\overline{\operatorname{Int}\sigma} = \sigma$ (in the topology of $\sigma(T)$) and there exists a separated part S_1 of S such that $\sigma \supset S_1$, $\sigma \cap (S \setminus S_1) = \varnothing$, we have the property $\sigma = \sigma(T \mid X_T(\sigma))$, hence σ is a set-spectrum for T. If T has not the property of the single-valued extension and $\sigma \supset S \supset S_T$, $\overline{\operatorname{Int}\sigma} = \sigma$, it is possible that σ is no more a set-spectrum for T, more exactly $\sigma(T \mid X_T(\sigma)) \neq \sigma$ (but $\sigma(T \mid X_T(\sigma)) \subset \sigma$); this occurs because for decomposable operators with $S_T \neq \varnothing$ it is possible that we do not have $\sigma(T) = \sigma(T \mid Y_S) \cup \left(\bigcup_{i=1}^{n} \sigma(T \mid Y_i)\right)$.

2.3.5. Proposition. *Let $T \in B(X)$ and let Y_1, Y_2 be two spectral maximal spaces of T. Then $Y_1 \cap Y_2$ is a spectral maximal space of T, hence Y_1, Y_2 are reciprocal σ-stabile.*

Proof. According to proposition 3.1. [76] a spectral maximal space of T is T-absorbing. Let us verify that the intersection $Y_1 \cap Y_2$ of two T-absorbing subspaces is a $T \mid Y_i$-absorbing subspace ($i = 1,2$). Indeed, if $(\lambda I - T)y = x$, where $x \in Y_i$, then $y \in Y_i$; for $\lambda \in \rho(T \mid Y_i)$ we have $y = R(\lambda, T \mid Y_i)x \in Y_i$, and for $\lambda \in \sigma(T \mid Y_i)$ there follows by the fact that Y_i is T-absorbing. Let now $(\lambda I - T \mid Y_1)y = x$ with $x \in Y_1 \cap Y_2$ and $y \in Y_1$; then $(\lambda I - T)y = x$ and since $x \in Y_1 \cap Y_2$ and both Y_1 and Y_2 are T-absorbing it follows that $y \in Y_1 \cap Y_2$, meaning $Y_1 \cap Y_2$ is $T \mid Y_1$-absorbing. Same for $T \mid Y_2$. If Y is an invariant subspace to T, T-absorbing, then $\sigma(T \mid Y) \subset \sigma(T)$. Indeed, in that case Y is invariant to solvent $(\lambda I - T)^{-1}$ (from the equality $(\lambda I - T)^{-1} y = z$ with $y \in Y$, we obtain

$y=(\lambda I-T)z$ hence $z\in Y$) and we have $(\lambda I-T\mid Y)^{-1}=(\lambda I-T)^{-1}\mid Y$ for $\lambda\in\rho(T)$) hence $\rho(T)\subset\rho(T\mid Y)$. Consequently we have $\sigma(T\mid Y_1\cap Y_2)\subset\sigma(T\mid Y_1)\cap\sigma(T\mid Y_2)$. Let Z be an invariant subspace to T such that $\sigma(T\mid Z)\subset\sigma(T\mid Y_1\cap Y_2)\subset\sigma(T\mid Y_1)\cap\sigma(T\mid Y_2)$. It follows $Z\subset Y_1\cap Y_2$.

2.3.6 Proposition. *Let $T\in B(X)$ be a S-decomposable operator and let Y be a spectral maximal space of T such that $\sigma(T\mid Y)\cap S=\varnothing$ or $\sigma(T\mid Y)\supset S$. Then*

$$\sigma(\dot{T})=\overline{\sigma(T)\setminus\sigma(T\mid Y)},$$

where $\dot{T}$ is the operator induced by T in the quotient space $\dot{X}=X/Y$.

Proof. We adapt the proof given at [2] for decomposable operators. From the equality $\sigma(T)=\sigma(\dot{T})\cup\sigma(T\mid Y)$ one can notice that only the following inclusion is left to be proved

$$\sigma(\dot{T})\subset\overline{\sigma(T)\setminus\sigma(T\mid Y)}.$$

Let $\sigma(T\mid Y)\cap S=\varnothing$. If $\lambda\in\sigma(\dot{T})\setminus\overline{\sigma(T)\setminus\sigma(T\mid Y)}$, let G_1, G_2 be two open sets such that $\lambda\notin G_S\supset\overline{\sigma(T)\setminus\sigma(T\mid Y)}$, $G_1\cap\overline{\sigma(T)\setminus\sigma(T\mid Y)}=\varnothing$, $G_1\cap G_S\supset\sigma(T)$. By setting the corresponding spectral maximal spaces to Y_1, Y_S we have

$$\sigma(T\mid Y_1)\subset G_1\cap\sigma(T)\subset\sigma(T\mid Y),$$

hence $Y_1\subset Y$. Let $\dot{x}\in\dot{X}$ such that $(\lambda I-\dot{T})\dot{x}=\dot{0}$, and $x\in\dot{x}$. Since $x=y_1+y_S$, with $y_1\in Y_1$, $y_S\in Y_S$, it follows that $(\lambda I-T)x=y$, with $y\in Y$, from wich $(\lambda I-T)x_S=(\lambda I-T\mid Y_S)x_S=y-(\lambda I-T)y_1\in Y$, hence $(\lambda I-T)x_S\in Y\cap Y_S$. In accordance with proposition 2.3.5., $Y\cap Y_S$ is a spectral maximal space of T therefore also a spectral maximal space of $T\mid Y_S$ ([4], 1.4.2.(ii)). But $Y\cap Y_S$ is ultrainvariant to $T\mid Y_S$ and since $\lambda\notin\sigma(T\mid Y_S)$ we obtain $x_S=R(\lambda,T\mid Y_S)(\lambda I-T\mid Y_S)x_S\in Y\cap Y_S$, hence $x\in Y$, $\dot{x}=\dot{0}$. Consequently $\lambda I-\dot{T}$ is injective. Let now $y=y_1+y_S\in\dot{y}$, $y_1\in Y_1$, $y_S\in Y_S$, where $\dot{y}=\dot{y}_S$; $\lambda\notin\sigma(T\mid Y_S)$, $x\in Y_S$ and $(\lambda I-T)x=y_S$ it follows $(\lambda I-\dot{T})\dot{x}=\dot{y}$ ($x=(\lambda I-T\mid Y_S)^{-1}y_S$) hence $\lambda I-\dot{T}$ is surjective. We came to a contradiction with the initial assumption that $\lambda\in\sigma(\dot{T})$ and the assertion is proved. Same for the case $\sigma(T\mid Y)\supset S$.

2.3.7. Proposition. *Let $T\in B(X)$ a S-decomposable operator and let $\{f_n\}_n$ be a series of analytic functions defined on an open set $G\subset\mathbf{C}$, with $G\cap S=\varnothing$ and $f_n(\lambda)\in X$ such that for $n\to\infty$*

$$(\lambda I-T)f_n(\lambda)\to 0$$

uniformly on every compact $\subset G$. Then for $n\to\infty$ we also have

$$f_n(\lambda)\to 0$$

uniformly on every compact $\subset G$.

Proof. Without restricting the generality, we can suppose $G=\{\lambda\in\mathbf{C},|\lambda|<R\}$, $R>0$ and moreover that (1) remains uniform on G. Let us proof that (2) is true uniformly on $G_0=\{\lambda\in\mathbf{C},|\lambda|<R_0\}$, $0<R_0<R$. Let ρ_1, ρ_2 that verify $R_0<\rho_1$, $\rho_1<\rho_2<R$ and let us set $H_i=\{|\lambda|<\rho_i\}$ ($i=1,2$). If F is closed and $F\cap S=\varnothing$ we will set by Y_F the spectral maximal space of T such that

$$X_T(F\cup S)=Y_F+X_T(S),$$

and $\sigma(T\mid Y_F)\subset F$. Let $\dot{X}=X/Y_{\overline{H}_2\setminus H_1}$ be a quotient space, and $\dot{T}$ be the operator induced by T in $\dot{X}$. Since $\sigma=\sigma(T)\cap(H_2\setminus H_1)$ is a set-spectrum for T (lemma 2.2.2.) we have $\sigma(T\mid Y_{\overline{H}_2\setminus H_1})=\sigma$ and obviously $Y_{\overline{H}_2\setminus H_1}=Y_\sigma$. In accordance with proposition 2.2.5.,

$$\sigma(\dot{T})=\overline{\sigma(T)\setminus\sigma(T\mid Y)}$$

and we obtain that $(H_2\setminus\overline{H}_1)\cap\sigma(\dot{T})=\varnothing$, hence

$$\sigma(\dot{T})\subset\overline{H}_1\cup(\mathbf{C}\setminus H_2).$$

But (1) implies

$$(\lambda I-\dot{T})\dot{\overline{f_n(\lambda)}}\to\dot{0}$$

uniformly on G. For $\lambda\in H_2\setminus\overline{H}_1$, by (3) and (4) it follows

$$R(\lambda;\dot{T})(\lambda I-\dot{T})\dot{\overline{f_n(\lambda)}}=\dot{\overline{f_n(\lambda)}}\to\dot{0} \qquad (3)$$

and in accordance with the principle of the maximum (4)

$$\dot{\overline{f_n(\lambda)}}\to\dot{0}\ \text{uniformly on}\ \overline{H}_2.$$

The proof further continues as in [50]; however we sketch the proof on the hole. Let

$$f_n(\lambda)=\sum_{K=0}^{\infty}a_{n^K}\lambda^K\quad(n=1,2,\dots)$$

the series development of $f_n(\lambda)$ which is convergent in G. By Cauchy's inequalities it follows

$$\dot{a}_{n^K}=\frac{\max\{\|\dot{\overline{f_n(\lambda)}}\|,\ \lambda\in\overline{H}_1\}}{\rho_1^K}=\frac{\varepsilon_n}{\rho_1^K}.$$

In accordance with the definition of the norm in $\dot{X}$, for any n and K there exists $A_{n^K}\in X$ such that

$$\dot{A}_{n^K}=\dot{a}_{n^K}\ \text{and}\ \|A_{n^K}\|\le\|\dot{a}_{n^K}\|+\frac{1}{2^n\cdot\rho_1^K}. \qquad (6)$$

Let $F_n(\lambda)=\sum_{K=0}^{\infty}A_{n^K}\lambda^K$; then

$$\|F_n(\lambda)\| \le \sum_{K=0}^{\infty}\|A_{nK}\|\cdot|\lambda|^K \le \sum_{K=0}^{\infty}\left(\frac{\varepsilon_n}{\rho_1^K}+\frac{1}{2^n\cdot\rho_1^K}\right)\cdot|\lambda|^K = \left(\varepsilon_n+\frac{1}{2^n}\right)\frac{1}{1-\frac{|\lambda|}{\rho_1}}$$

if $\lambda \in G_0$ ($|\lambda| < R_0 < \rho_1$), hence

$$\|F_n(\lambda)\| \le \left(\varepsilon_n+\frac{1}{2^n}\right)\frac{\rho_1}{\rho_1-R_0}$$

for G_0. Also, we have

$$\varphi_n(\lambda) = f_n(\lambda) - F_n(\lambda) \in Y_{\overline{H}_2-H_1} \tag{7}$$

Analogously, there exists $\widetilde{\rho}_1$, $\widetilde{\rho}_2$ such that $R_0 < \widetilde{\rho}_1 < \widetilde{\rho}_2 < \rho_1$ and an analytic function $\widetilde{F}_n(\lambda)$ defined on a neighbourhood of G_0 and verifying on $\overline{G}_0$ the inequality

$$\|\widetilde{F}_n(\lambda)\| \le \left(\widetilde{\varepsilon}_n+\frac{1}{2^n}\right)\frac{\rho_1}{\rho_1-\rho_0}$$

(where $\widetilde{\varepsilon}_n$ is obtained same way as ε_n); we also have

$$\widetilde{\varphi}_n(\lambda) = f_n(\lambda) - F_n(\lambda) \in Y_{\overline{\widetilde{H}}_2\setminus\widetilde{H}_1}. \tag{9}$$

It will follow that

$$\|\varphi_n(\lambda)-\widetilde{\varphi}_n(\lambda)\| = \|F_n(\lambda)-\widetilde{F}_n(\lambda)\| \le \left(\varepsilon_n+\widetilde{\varepsilon}_n+\frac{1}{2^{n-1}}\right)\cdot\frac{R_1}{R_1-R_0} \tag{10}$$

for $\lambda \in G_0$. But $\overline{H}_2 \setminus H_1$ and $\overline{\widetilde{H}}_2 \setminus \widetilde{H}_1$ are compact and disjunct, therefore by proposition 2.2.4. and remark 1.1.18. we have

$$Y_{(\overline{H}_2\setminus H_1)\cup(\overline{\widetilde{H}}_2\setminus\widetilde{H}_1)} = Y_{\overline{H}_2\setminus H_1} \oplus Y_{\overline{\widetilde{H}}\setminus\widetilde{H}_1}.$$

Consequently there exists a constant N such that

$$\|x\| + \|\widetilde{x}\| \le N\cdot\|x+\widetilde{x}\|$$

for $x \in Y_{\overline{H}_2\setminus H_1}$, $\widetilde{x} \in Y_{\overline{\widetilde{H}}_2\setminus\widetilde{H}_1}$. (11)

From (11) it follows that

$$\|\varphi_n(\lambda)\| \le N\cdot\left(\varepsilon_n+\widetilde{\varepsilon}_n+\frac{1}{2^{n-1}}\right)\frac{R_1}{R_1-R_0}$$

for $\lambda \in G_0$; finally, (8), (9) and (11) yield

$$\|f_n(\lambda)\| \le (N+1)\left(\varepsilon_n+\widetilde{\varepsilon}_n+\frac{1}{2^{n-1}}\right)\frac{R_1}{R_1-R_0}$$

for $\lambda \in \overline{G}_0$, from where, in accordance with (5), $\varepsilon_n+\widetilde{\varepsilon}_n \to 0$ when $\frac{1}{2^{n-1}} \to 0$ and the proof is over.

2.3.8. Theorem. *Let $T \in B(X)$ be a S-decomposable operator. Then for any closed F, $F \subset \mathbf{C}$, $F \supset S$ we have*

$$X_T(F) = N_c(T;F).$$

Proof. This is verify as in [50]. Let $x \in N_c(T;F)$ and G an open subset included in $\mathbf{C} \setminus F$ such that $\overline{G}$ is compact and also included in $\mathbf{C} \setminus F$. For $\varepsilon = \frac{1}{n}$ let $f_n(\lambda)$ be an analytic function taking values in X defined on a neighbourhood of $\overline{G}$ and verifying the inequality

$$\|x - (\lambda I - T)f_n(\lambda)\| < \frac{1}{n} \quad (1)$$

for all $\lambda \in \overline{G}$. The existence of such a function is given by the definition of $N_c(T;F)$. Let now K be an arbitrary compact set included in G. If $\{f_n(\lambda)\}_{n=1}^{\infty}$ was not uniformly convergent on K, then there would exist $\varepsilon > 0$ and the series $\{\lambda_i\} \subset K$, $n_1 < m_1 < n_2 < m_2 < \ldots$ such that $\|f_{mj}(\lambda_j) - f_{nnj}(\lambda_j)\| \geq \varepsilon$. Setting $g_j(\lambda) = f_{mj}(\lambda) - f_{nj}(\lambda)$ and using (1) as well as the preceding proposition we can obtain an obvious contradiction. Therefore it follows that $\{f_n\}_{n=1}^{\infty}$ uniformly converge (in X) on every compact $K \subset G$. For $\lambda \in G$ we put $f(\lambda) = \lim_{n \to \infty} f_n(\lambda)$. Then $f(\lambda)$ is analytic on G and (in accordance with (1)) it verifies the equation $(\lambda I - T)f(\lambda) = x$ in G, therefore $x \in X_T(\mathbf{C} \setminus G)$ whence $x \in \bigcap X_T(\mathbf{C} \setminus G) = X_T(\bigcap(\mathbf{C} \setminus G))$, the intersection being considered for all G, G compact, $\overline{G} \subset \mathbf{C} \setminus F$ compact; hence $F = \bigcap(\mathbf{C} \setminus G)$. As conclusion we will present several results regarding T-absorbing families or spectral maximal subspaces for the $T \in B(X)$ operator, which will prove to be useful.

2.3.9. PROPOSITION. *Let $T \in B(X)$ and let $\{Y_\alpha\}_{\alpha \in A}$ be an arbitrary family of T-absorbing subspaces which are invariant to T. Then $y = \bigcap_{\alpha \in A} Y_\alpha$ is $(T \mid Y_a)$-absorbing for any index $\alpha \in A$ and*

$$\sigma(T \mid Y) \subset \bigcap_{\alpha \in A} \sigma(T \mid Y_a).$$

Proof. Let $\beta \in A$ fixed; obviously, if Y_α is a T-absorbing subspace and $(\lambda I - T)x = y \in Y_\alpha$ then $x \in Y_\alpha$. Let now $(\lambda I - T \mid Y_\beta)y_\beta = x \in Y$ ($y_\beta \in Y_\beta$); therefore $x \in Y_\alpha$ for all $\alpha \in A$. Since all Y_α are T-absorbing, it follows that $y_\beta \in Y_\alpha$ for all $\alpha \in A$, therefore $y_\beta \in Y$ and consequently Y is a $(T \mid Y_\beta)$-absorbing subspace. Since β is arbitrary from A it follows that Y is a $(T \mid Y_\alpha)$-absorbing subspace for all indexes $\alpha \in A$. Proving proposition 2.3.5. it verifies that if Y is a T-absorbing subspace, then $\sigma(T \mid Y) \subset \sigma(T)$, consequently $\sigma(T \mid Y) \subset \bigcap_{\alpha \in A} \sigma(T \mid Y_\alpha)$ (where now $Y = \bigcap_{\alpha \in A} Y_\alpha$).

2.3.10. Proposition. *Let $T \in B(X)$ and let $\{Y_a\}_{\alpha\in A}$ be a family of spectral maximal spaces of T. Then $Y = \bigcap_{\alpha\in A} Y_\alpha$ is a spectral maximal space of T.*

Proof. By the preceding proposition it follows that Y is a $(T \mid Y_\alpha)$-absorbing for each index $\alpha \in A$ (since a spectral maximal space of T is T-absorbing [76]), hence $\sigma(T \mid Y) \subset \bigcap_{\alpha\in A} \sigma(T \mid Y_\alpha)$. If Z is invariant for T subspace with $\sigma(T \mid Z) \subset \sigma(T \mid Y)$, then $\sigma(T \mid Z) \subset \sigma(T \mid Y_\alpha)$ for all indexes $\alpha \in A$, hence $Z \subset Y_\alpha$ for any $\alpha \in A$, therefore $Z \subset Y$.

2.3.11. Corollary. *The family of T-absorbing, invariant subspaces (particularly the family of spectral maximal spaces of an operator T) is formed out of reciprocal σ-stabile for T subspaces.*

Proof. It follows easily by previous propositions.

2.4. *S*-DECOMPOSABILITY CONDITIONS FOR AN OPERATOR

We shall further give several *S*-decomposability criteria for an operator. We will also show that spectral maximal spaces from the *S*-decomposability definition (particularly the one of the decomposability) can be replaced with reciprocal σ-stabile subspaces or invariant, *T*-absorbing subspaces. Also, there is generalised for $(1, S)$-decomposable operators the result obtained in [84] for *2*-decomposable operators: an operator T is 2-decomposable if and only if $X_T(F)$ is closed and $\sigma(\dot{T}) = \overline{\sigma(T) \setminus \sigma(T \mid X_T(F))}$, where $\dot{T}$ is the operator induced by T in $\dot{X} = X / X_T(F)$, $F \subset \mathbf{C}$ arbitrary and closed.

2.4.1. Proposition. *Let $T \in B(X)$, and let $S \subset \sigma(T)$ be a compact set such that $S_T \subset S$ ([76] Def. 2.2.). Then the following conditions are equivalent:*

a) T is $(1, S)$-decomposable

b) $X_T(F)$ is closed for any $F \supset S$ closed and

$$\sigma(T \mid X / Y_{\overline{G}}) = \overline{\sigma(T) \setminus \sigma(T \mid Y_{\overline{G}})}$$

where G is open in $\sigma(T)$ such that $\overline{G} \cap S = \varnothing$, and $Y_{\overline{G}}$ is the spectral maximal space of T defined by the equality

$$X_T(S \cup \overline{G}) = X_T(S) \oplus Y_{\overline{G}}.$$

Proof. The fact that a) implies b) follows by proposition 2.3.5. and by the fact that $\overline{G}$ is a set-spectrum for T (lemma 2.3.3.). Let us prove that b) implies a). We first notice that for any open G, $\overline{G} \cap S = \varnothing$, there exists a spectral maximal space $Y_{\overline{G}}$ defined by the equality

$$X_T(S \cup \overline{G}) = X_T(S) \oplus Y_{\overline{G}},$$

where . Indeed, since $X_T(F)$ is closed for any closed $F \supset S$, by proposition 3.4. [76] there follows that $X_T(S \cup \overline{G})$ is spectral maximal space for T and

$$\sigma(T \mid X_T(S \cup \overline{G})) \subset S \cup \overline{G}.$$

In accordance with theorem of decomposition by separated parts of the spectrum there follows that $X_T(S \cup \overline{G}) = X_T(S) \oplus Y_{\overline{G}}$, where $Y_{\overline{G}}$ is a spectral maximal space of $T \mid X_T(S \cup \overline{G})$ also of T, and $\sigma(T \mid Y_{\overline{G}}) \subset \overline{G}$. We further notice that the equality

$$\sigma(T \mid X / Y_{\overline{G}}) = \sigma(T) \setminus \sigma(T \mid Y_{\overline{G}})$$

is equivalent with the inclusion

$$\sigma(T \mid X / Y_{\overline{G}}) \subset \sigma(T) \setminus G.$$

This follows by the equalities

$$\sigma(T) = \sigma(T \mid Y_{\overline{G}}) \cup \sigma(T \mid X / Y_{\overline{G}}),$$

$$\operatorname{Int} \sigma(\mathrm{T} \mid \mathrm{Y}_{\overline{G}}) = \sigma(T) \setminus \overline{(\sigma(T) \setminus \sigma(T \mid Y_{\overline{G}}))},$$

since

$$\sigma(T \mid X / Y_{\overline{G}}) \subset \sigma(T) \setminus \operatorname{Int} \sigma(\mathrm{T} \mid \mathrm{Y}_{\overline{G}}) = \sigma(T) \setminus G = \sigma(T) \setminus (\sigma(T) \setminus \overline{\sigma(T) \setminus \sigma(T \mid Y_{\overline{G}})}) =$$
$$= \sigma(T) \setminus \sigma(T \mid Y_{\overline{G}}) \subset \sigma(T \mid X / Y_{\overline{G}})$$

Let $\{G_S, G\}$ be an open and finite S-covering of $\sigma(T)$ and let us put $H = G_S \cap G$; then

$$\sigma(T \mid X / Y_{\overline{H}}) \subset \sigma(T) \setminus (G_S \cap G) = (\sigma(T) \setminus G_S) \cup (\sigma(T) \setminus G).$$

Since $\sigma(T) \setminus G_S$ and $\sigma(T) \setminus G$ have a void intersection, it follows that $X / Y_{\overline{H}} = Z_S \oplus Z$, where $\sigma(T \mid Z_S) \subset \sigma(T) \setminus G_S$ and $\sigma(T \mid Z) \subset \sigma(T) \setminus G$. If φ is the canonical map defined from X to $X / Y_{\overline{H}}$ then

$$X = \varphi^{-1}(X / Y_{\overline{H}}) = \varphi^{-1}(Z_S \oplus Z) = \varphi^{-1}(Z_S) + \varphi^{-1}(Z).$$

But $Z_S = \varphi^{-1}(Z_S) / Y_{\overline{H}}$, $\varphi^{-1}(Z) / Y_{\overline{H}}$ hence

$$\sigma(T \mid \varphi^{-1}(Z_S)) = \sigma(Z \mid Z_S) \cup \sigma(T \mid Y_{\overline{H}}) \subset \overline{H} \cup (\sigma(T) \setminus G_S) \subset G,$$

$$\sigma(T \mid \varphi^{-1}(Z)) = \sigma(T \mid Z) \cup \sigma(T \mid Y_{\overline{H}}) \subset (\sigma(T) \setminus G) \cup \overline{H} \subset G_S,$$

meaning T is a $(1, S)$-decomposable.

2.4.2 Definition. A family of linear (closed) subspaces of X, $\Sigma = \{Y_i\}_{i \in I}$ is said to be *reciprocal* σ*-stabile* for $T \in B(X)$, if each Y_i is invariant for T and moreover

$$\sigma(T \mid Y_i \cap Y_j) \subset \sigma(T \mid Y_i) \cap \sigma(T \mid Y_j)$$

for any $i, j \in I$. We say that $T \in B(X)$ verify the (D_S) property if there exists a reciprocal σ-stabile for T subspaces family Σ such that for any open $(1, S)$-covering of $\sigma(T)$ there exist subspaces $\{Y_S, Y\} \subset \Sigma$ such that $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_G) \subset G$ and $X = Y_S + Y$ (we remind that $G_S \supset S$, $\overline{G} \cap S = \varnothing$).

2.4.3. LEMMA. *If $T \in B(X)$ verifies property (D_S) then $S \supset S_T$.*

Proof. Let $F \subset \sigma(T)$ closed, $F \cap S = \varnothing$, G_S, G open such that $G \supset F$, $G \cap S = \varnothing$, $G_S \subset S$ and $G_S \cup G \supset \sigma(T)$; if $x \in X$ and $\lambda \in F$ such that $(\lambda I - T)x = 0$, then $x \in Y$, where $X = Y_S + Y$, $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y) \subset G$. Indeed, we have $x = y + y_S$ with $y \in Y$, $y_S \in Y_S$ and

$$y'_S = (\lambda I - T)y_S = -(\lambda I - T)y \in Y \cap Y_S .$$

But $\lambda \in F$ implies $\lambda \notin G_S$ and $\lambda \notin G_S \cap G$, hence

$$\lambda \notin (T \mid Y \cap Y_S) \subset \sigma(T \mid Y) \cap \sigma(T \mid Y_S) \subset G_S \cap G,$$

consequently

$$y''_S = (\lambda I - T \mid Y \cap Y_S)^{-1} y'_S \in Y .$$

Since

$$y''_S - y_S \in Y_S \text{ and } \lambda \notin \sigma(T \mid Y_S) \text{ one obtains}$$

$$(\lambda I - T)(y''_S - y_S) = (\lambda I - T \mid Y_S \cap Y)(\lambda I - T \mid Y_S \cap Y)^{-1} y'_S - y'_S = 0,$$

hence $y_S = y''_S \in Y$, that is $x \in Y$. Let now $x : H \to X$ be an analytic function such that $(\lambda I - T)x(\lambda) = 0$ (*H* open, $H \cap S = \varnothing$; we can suppose that *H* is connected). Let also, δ and δ' two closed disjunct disks contained in *H*. Accordingly the above, taking into account that $F = \delta$ and $F = \delta'$, $\{G_S, G\}$ and $\{G'_S, G'\}$, the $(1, S)$-coverings of $\sigma(T)$, it follows $\{Y_S, Y\}$ and $\{Y'_S, Y'\}$, the corresponding subspaces of these $(1, S)$-coverings and we shall have $x(\lambda) \in Y$ for any $\lambda \in \delta$, and $x(\lambda) \in Y'$ for any $\lambda \in \delta'$. From analyticity we have $x(\lambda) \in Y$ for any $\lambda \in H$, hence $x(\lambda) \in Y \cap Y'$. But $\sigma(T \mid Y \cap Y') \subset G \cap G'$ and since $\delta \cap \delta' = \varnothing$, we are allowed to choose *G*, G', such that $G \cap G' = \varnothing$, whence it results that $Y \cap Y' = \{0\}$; consequently $x(\lambda) \equiv 0$ on *H* hence $S \supset S_T$.

We remind the next proposition which was proved in [51].

2.4.4. PROPOSITION. *Let X be a Banach space, and let Y_1, Y_2 be two linear (closed) subspaces such that $X = Y_1 + Y_2$ and $f : G \to X$ (G open) an analytic function. Then for any $\lambda \in G$ there exists a neighbourhood of λ $H \subset G$ and two analytic functions $g_i : H \to Y_i$ ($i = 1,2$) such that $f(\mu) = g_1(\mu) + g_2(\mu)$ for $\mu \in H$.*

2.4.5. THEOREM. *Let $T \in B(X)$ and let $S \supset S_T$ compact. If T has the property that for any open $(1, S)$-covering of $\sigma(T)$, $\{G_S, G\}$, there exists the subspaces reciprocal σ-stabile for T such that $X = Y_S + Y$ and $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y) \subset G$, then T is a $(1, S)$-decomposable operator.*

Proof. It is enough to prove that $X_T(F)$ is closed for any closed $F \supset S$ (see lemma 2.2.2.). Let $G_1 = G_S \supset F$ and G_2 open such that $\overline{G_2} \cap F = \varnothing$, $G_1 \cup G_2 \supset \sigma(T)$. It will exist the system of reciprocal σ-stabile for *T* subspaces $\{Y_1, Y_2\}$ such that

$$X = Y_1 + Y_2,\ \sigma(T \mid Y_i) \subset G_i,\ (i = 1,2).$$

If $x \in X_T(F)$ we shall prove that $x \in Y_i$. We have $x = y_1 + y_2$ with $y_i \in Y_i$ $(i = 1,2)$ and

$$\rho_T(x) \cap \delta_T(y_2) = (\delta_T(x) \cap \Omega_T) \cap (\Omega_T \cap \delta_T(y_2)) = \rho_T(x) \cap \rho_T(y_2),$$

hence for $\lambda \in \rho_T(x) \cap \rho_T(y_2)$ we can write

$$y_1(\lambda) = x(\lambda) - y_2(\lambda). \tag{1}$$

But $\gamma_T(y_2) \subset \overline{G_2}$, hence $\mathbf{C} \setminus \overline{G_2} \subset \delta_T(y_2)$, whence $G' = \rho_T(x) \cap (\mathbf{C} \setminus \overline{G_2}) \subset$ $\subset \rho_T(x) \cap \delta_T(y_2) = \rho_T(x) \cap \rho_T(y_2)$; it follows that equality (1) takes place on G'. Let us verify that $y_1(\lambda) \in Y_1$. We shall apply proposition 2.4.4. when $y_1 : G' \to X$. For a fixed $\lambda \in G'$ we obtain a neighbourhood $H \subset G'$ of λ and the analytic functions $g_i : H \to Y_i$ $(i = 1,2)$ such that

$$y_1(\mu) = g_1(\mu) + g_2(\mu) \text{ for } \mu \in H. \tag{2}$$

Applying the operator $\mu I - T$ to equality (2) we will obtain $Y_1 \in y_1 - (\mu I - T)g_1(\mu) =$ $= (\mu I - T)g_2(\mu) \in Y_2$, hence $h(\mu) = (\mu I - T),\ g_2(\mu) \in Y_1 \cap Y_2$.

Then

$$k(\mu) = ((\mu I - T)(Y_1 \cap Y_2))^{-1} h(\mu)$$

is an analytic function on H taking values in $Y_1 \cap Y_2$, $k(\mu)$ having sense since the following inclusions take place

$$\lambda \in H \subset G' \subset \mathbf{C} \setminus G_2 \subset \rho(T \mid Y_2) \subset \rho(T \mid Y_1) \cup \rho(T \mid Y_2) \subset \rho(T \mid Y_1 \cap Y_2).$$

From the equality $(\mu I - T)(k(\mu) - g_2(\mu)) = 0$ $(\mu \in G' \cap \Omega_T)$ it follows that $k(\mu) = g_2(\mu) \in Y_1 \cap Y_2 \subset Y_1$ for any $\mu \in H$, hence

$$y_1(\mu) = g_1(\mu) + g_2(\mu) \in Y_1.$$

Observing that G' is "exterior" to $\sigma(T \mid Y_2) \subset \overline{G_2}$, let Γ be a system of simple curves closed in G', surrounding $\sigma_T(x)$; Γ being "exterior" to $\sigma(T \mid Y_2)$ it follows

$$\frac{1}{2\pi i} \int_\Gamma y_2(\lambda) d\lambda = \frac{1}{2\pi i} \int_\Gamma ((\lambda I - T) \mid Y_2)^{-1} y_2 \, d\lambda = 0$$

hence

$$\frac{1}{2\pi i} \int_\Gamma y_1(\lambda) d\lambda = \frac{1}{2\pi i} \int_\Gamma x(\lambda) d\lambda - \frac{1}{2\pi i} \int_\Gamma y_2(\lambda) d\lambda =$$

$$= \frac{1}{2\pi i} \int_{|\lambda| = \|T\| + 1} x(\lambda) d\lambda = \frac{1}{2\pi i} \int_{|\lambda| = \|T\| + 1} (\lambda I - T)^{-1} x = x$$

Whence it results that $x \in \bigcup_{G_1 \supset F} Y_1$ hence $X_T(F) \subset \bigcap_{G_1 \supset F} Y_1$; we have $\sigma_T(x) = \gamma_T(x) \cup S_T \subset$ $\subset \sigma(T \mid Y_1) \cup S_T \subset \sigma(T \mid Y_1) S \subset G_1$ for any $G_1 \supset F$ open, hence $\sigma_T(x) \subset \bigcap_{G_1 \supset F} G_1 = F$, that is $\bigcap_{G_1 \supset F} Y_1 \subset X_T(F)$. Consequently

$$X_T(F) = \bigcap_{G_1 \supset F} Y_1 ,$$

whence we deduce that $X_T(F)$ is closed, q.e.d.

2.4.6. Corollary. *Let $T \in B(X)$ verifying property (D_S); then T is a $(1,S)$-decomposable operator.*

Proof. There follows by lemma 2.4.3. and theorem 2.4.5.

2.4.7. Corollary. *Let $T \in B(X)$ and let $S \supset S_T$, $S \subset \sigma(T)$ compact. If for any S-covering $\{G_S\} \cup \{G_i\}_1^n$ of $\sigma(T)$ there exists the family $\{Y_S\} \cup \{Y_i\}_1^n$ of subspaces reciprocal σ-stabile for T such that $X = Y_S + \sum_{i=1}^{n} Y_i$, $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_i) \subset G_i$ $(i = 1,2,\ldots,n)$, then T is S-decomposable.*

2.4.8. Corollary. *Let $T \in B(X)$ and let Σ be a family of subspaces reciprocal σ-stabile for T. If for any S-covering $\{G_S\} \cup \{G_i\}_1^n$ of $\sigma(T)$ there exists the subspaces $\{Y_S\} \cup \{Y_i\}_1^n \subset \Sigma$ with the following properties $X = Y_S + \sum_{i=1}^{n} Y_i$, $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_i) \subset G_i$ $(i = 1,2,\ldots,n)$, then T is S-decomposable.*

Proof. Both corollaries follow easily by lemma 2.4.3., theorem 2.4.5. and lemma 2.2.2.

2.4.9. Theorem. *Let $T \in B(X)$ and let $S \subset \sigma(T)$ compact. If for any open, T-absorbing, S-covering $\{G_S\} \cup \{G_i\}_1^n$ of T, having the properties $X = Y_S + \sum_{i=1}^{n} Y_i$ and $\sigma(T \mid Y_S) \subset G_S$, $\sigma(T \mid Y_i) \subset G_i$ $(i = 1,2,\ldots,n)$, then T is S-decomposable.*

Proof. By corollary 2.3.11. there follows that a family of *T*-absorbing, invariant for *T* subspaces is reciprocal σ-stabile; the theorem follows by the preceding corollary.

2.4.10. Theorem. *Let $T \in B(X)$ and let Σ be a family of subspaces reciprocal σ-stabile for T. If for any open covering $\{G_1, G_2\}$ of $\sigma(T)$ there exist the subspaces $\{Y_1, Y_2\} \subset \Sigma$ having the properties $X = Y_1 + Y_2$, $\sigma(T \mid Y_i) \subset G_i$ $(i = 1,2)$ then T is decomposable.*

Proof. There follows by corollary 2.4.8. and by the fact that a 2-decomposable operator is decomposable (which was recently obtained in [86]).

2.4.11. Proposition. *Let $T \in B(X)$. If for any open covering $\{G_1, G_2\}$ of $\sigma(T)$ there exists the invariant T-absorbing subspaces $\{Y_1, Y_2\}$ of T such that $X = Y_1 + Y_2$, $\sigma(T \mid Y_i) \subset G_i$ $(i = 1,2)$ then T is decomposable.*

Proof. There follows by the preceding theorem and by proposition 2.3.9.

2.4.12. COROLLARY. *Let $T \in B(X)$ having property of the single-valued extension. If for any open covering $\{G_1, G_2\}$ of $\sigma(T)$ there exists the subspaces reciprocal σ-stabile for T, $\{Y_1, Y_2\}$ such that $X = Y_1 + Y_2$, $\sigma(T \mid Y_i) \subset G_i$ $(i = 1,2)$ then T is decomposable.*

Proof. There follows by theorem 2.4.10.; since we supposed $S_T = \varnothing$, one requires no more that Y_1, Y_2 belong to a larger reciprocal σ-stabile subspaces family.

2.5. SPECTRAL *S*-CAPACITIES

During this paragraph we shall generalise the concept of spectral capacity [4], [51], by defining the spectral *S*-capacities and show that an operator is *S*-decomposable if and only if it admits a spectral *S*-capacity.

2.5.1. DEFINITION. Let F_S be the family of all closed sets F of the complex plan **C** which have the following property: either $F \cap S = \varnothing$ or $F \supset S$, where *S* is a compact fixed set of **C**; if *X* is a Banach space, denote by $\mathsf{E}(X)$ the family of all (closed and linear) subspaces of *X*.

a) An map $\mathsf{E} : \mathsf{F}_S \to \mathsf{S}(X)$ which verifies the following properties:

(i) $\mathsf{E}(\varnothing) = \{0\}$, $\mathsf{E}(\mathbf{C}) = X$;

(ii) $\mathsf{E}\left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{i=1}^{\infty} \mathsf{E}(F_n)$, where $F_n \in \mathsf{F}_S$ $(n = 1,2,\ldots)$

(iii) if $\{G_S\} \cup \{G_i\}_1^n$ is an open *S*-covering of **C** then

$$X = \mathsf{E}(\overline{G}_S) + \sum_{i=1}^{m} \mathsf{E}(\overline{G}_i).$$

b) By definition an operator $T \in B(X)$ is said to admit a spectral *S*-capacity E if for each $F \in \mathsf{F}_S$ we have

(iv) $T\mathsf{E}(F) \subset \mathsf{E}(F)$;

(v) $\sigma(T \mid \mathsf{E}(F)) \subset F$.

Remark. By condition (ii) there follows that $F_1, F_2 \in \mathsf{F}_S$ and $F_1 \subset F_2$ implies $\mathsf{E}(F_1) \subset \mathsf{E}(F_2)$. Indeed, if $F_1 \subset F_2$ then $F_1 \cap F_2 = F_1$, hence $\mathsf{E}(F_1) = \mathsf{E}(F_1 \cap F_2) = \mathsf{E}(F_1) \cap$ $\cap \mathsf{E}(F_2) \subset \mathsf{E}(F_2)$.

2.5.2. THEOREM. *If $T \in B(X)$ is S-decomposable, then T admits a spectral S-capacity* E.

Proof. Let $F \in \mathsf{F}_S$ with $F \cap S = \varnothing$. Then $X_T(F \cup S)$ is a spectral maximal space of *T* and

$$X_T(F \cup S) = Y_F \oplus X_T(S),$$

where Y_F is also a spectral maximal space of T and $\sigma(T \mid Y_F) \subset F$. Indeed, we have $X_T(F \cup S) = Y_F \oplus Y_S$ because $\sigma(T \mid X_T(F \cup S)) = \sigma_F \cup \sigma_S$, with $\sigma_F \subset F$, $\sigma_S \subset S$, where $\sigma_F = \sigma(T \mid Y_F)$, $\sigma_S = \sigma(T \mid Y_S)$, $\sigma_F \cap \sigma_S = \varnothing$; obviously, Y_F and Y_S are spectral maximal spaces of $T \mid X_T(F \cup S)$ hence of T also, and $Y_S = X_T(S)$. We shall set $\mathcal{E}(F) = Y_F$ if $F \cap S = \varnothing$ and $\mathcal{E}(F) = X_T(F)$ if $F \supset S$. Let us verify that $\mathcal{E}$ thus defined is a spectral S-capacity of T. First, we have

$$\mathcal{E}(\varnothing) = \{0\}, \mathcal{E}(\mathbb{C}) = X$$

since $\varnothing \cap S = \varnothing$ there follows $\sigma(T \mid Y_\varnothing) \subset \varnothing$ and $\mathcal{E}(\mathbb{C}) = X_T(\mathbb{C}) = X_T(\sigma(T)) = X$. Conditions (iii) and (iv) are obviously met, we only have to verify condition (ii). One can easily prove that $F_1, F_2 \in \mathcal{F}_S$ and $F_1 \subset F_2$ implies $\mathcal{E}(F_1) \subset \mathcal{E}(F_2)$; for $F_1, F_2 \supset S$ it is evident that $X_T(F_1) \subset X_T(F_2)$, and from the inclusion $Y_{F_1} + X_T(S) = X_T(F_1 \cup S) \subset X_T(F_2 \cup S) =$ $= Y_{F_2} \oplus X_T(S)$ it results that $Y_{F_1} \subset Y_{F_2}$ $(F_1, F_2 \in \mathcal{F}_S, F_i \cap S = \varnothing, i = 1,2)$. Let $F_i \subset \mathcal{F}_S$, $F_i \supset S$ $(i = 1,2,\ldots)$; we have $\bigcap_{i=1}^{\infty} F_i \subset S$ and hence from the equality

$$X_T\left(\bigcap_{i=1}^{\infty} F_i\right) = \bigcap_{i=1}^{\infty} X_T(F_i)$$

we obtain

$$\mathcal{E}\left(\bigcap_{i=1}^{\infty} F_i\right) = \bigcap_{i=1}^{\infty} \mathcal{E}(F_i).$$

When $F_i \in \mathcal{F}_S$ and $F_i \cap S = \varnothing$ $(i = 1,2,\ldots)$, then $\bigcap_{i=1}^{\infty} F_i \subset F_i$ $(i = 1,2,\ldots)$ implies

$$Y_{\bigcap_{i=1}^{\infty} F_i} \subset Y_{F_i},\ Y_{\bigcap_{i=1}^{\infty} F_i} \subset \bigcap_{i=1}^{\infty} Y_{F_i};$$

but $Y = \bigcap_{i=1}^{\infty} Y_{F_i}$ is a spectral maximal space of T and $\sigma(T \mid Y) \subset \bigcap_{i=1}^{\infty} F_i$, hence we have

$$Y \subset X_T\left(\left(\bigcap_{i=1}^{\infty} Fi\right) \cup S\right) = Y_{\bigcap_{i=1}^{\infty} F_i} \oplus X_T(S)$$

whence $Y \subset Y_{\bigcap_{i=1}^{\infty} F_i}$, that is $\mathcal{E}\left(\bigcap_{i=1}^{\infty} F_i\right) = \bigcap_{i=1}^{\infty} \mathcal{E}(F_i)$. Let now $F, F_S \in \mathcal{F}_S$ with $F \cap S = \varnothing$ and $F_S \supset S$; then, obviously, $Y_{F \cap F_S} \subset Y_F \cap Y_{F_S}$; from $\sigma(T \mid Y_F \cap Y_{F_S}) \subset F \cap F_S$ it results that $Y_F \cap Y_{F_S} \subset X_T((F \cap F_S) \cup S) = Y_{F \cap F_S} \oplus X_T(S)$ and hence $Y_F \cap Y_{F_S} \subset Y_{F \cap F_S}$.

Threrefore

$$\mathcal{E}(F \cap F_S) = \mathcal{E}(F) \cap \mathcal{E}(F_S).$$

Finally, if $F_i \in \mathsf{F}_S$ are arbitrary ($i=1,2,\ldots$), by putting $F_i' = F_i$ if $F_i \cap S = \varnothing$ and $F_i'' = F_i$ if $F_i \supset S$ ($i=1,2,\ldots$), we obtain

$$\mathsf{E}\left(\bigcap_{i=1}^{\infty} F_i\right) = \mathsf{E}\left(\left(\bigcap_{j=1}^{\infty} F_j'\right) \cap \left(\bigcap_{i=1}^{\infty} F_i''\right)\right) = \mathsf{E}\left(\bigcap_{j=1}^{\infty} F_j'\right) \cap \mathsf{E}\left(\bigcap_{i=1}^{\infty} F_i''\right) =$$

$$= \left(\bigcap_{j=1}^{\infty} \mathsf{E}\left(F_j'\right)\right) \cap \left(\bigcap_{i=1}^{\infty} \mathsf{E}\left(F_i''\right)\right) = \bigcap_{i=1}^{\infty} \mathsf{E}\left(F_i\right)$$

2.5.3. Proposition. *If $F \to \mathsf{E}(F)$ is a spectral S-capacity for T, then $\mathsf{E}(F)$ is a spectral maximal space of T; more exactly, $Y \subset \mathsf{E}(S)$, $TY \subset Y$, $\sigma(T \mid Y) \subset F$ implies $Y \subset \mathsf{E}(F)$.*

The proof will be given in chapter III in a more general case for operators systems (also see [17]).

2.5.4. Theorem. *If $T \in B(X)$ admits a spectral S-capacity E, then T is S-decomposable.*

Proof. It follows by the preceding proposition and property (iii) of the S-capacity definition.

2.5.5. Theorem. *An operator $T \in B(X)$ is S-decomposable if and only if it admits a spectral S-capacity E.*

Proof. There follows by theorems 2.5.2. and 2.5.5.

2.5.6. Corollary. *If $T \in B(X)$ admits a spectral S-capacity E, then this S-capacity is single-determined, $S \supset S_T$ and for any closed $F \supset \sigma(T)$, $F \supset S$ we have*

$$\mathsf{E}(F) = X_T(F).$$

Proof. Let E^* be another capacity of T; then $\sigma(T \mid \mathsf{E}^*(F)) \subset F$ implies $\mathsf{E}^*(F) \subset \mathsf{E}(F)$ and identically $\mathsf{E}(F) \subset \mathsf{E}^*(F)$, hence E is single-determined. Since T is S-decomposable $S \supset S_T$. The inclusion $\sigma(T \mid \mathsf{E}(F)) \subset \mathsf{E}$ implies $\mathsf{E}(F) \subset X_T(F)$. But $X_T(F)$ being closed and $\sigma(T \mid X_T(F)) \subset F$ we also have $X_T(F) \subset \mathsf{E}(F)$, hence $\mathsf{E}(F) = X_T(F)$.

2.5.7. *Remarks.* (a). If $T \in B(X)$ is S-decomposable and $F \subset \mathbb{C}$, $F \cap S = \varnothing$, by proof of theorem 2.5.2. there follows that $\mathsf{E}(F) = Y_F$, where Y_F is the spectral maximal space of T given by the equality $Y_F \oplus X_T(S) = X_T(F \cup S) = \mathsf{E}(F \cup S)$, E being the spectral capacity of T. (b). Let $T \in B(X)$ be a S-decomposable operator; then it will suffice to take $S \subset \sigma(T)$. Indeed, if E is the spectral S-capacity of T and $S^* = S \cap \sigma(T)$, one easily verifies that application E^* defined by the equalities $\mathsf{E}^*(F) = \mathsf{E}(F \cup S)$ for $F \supset S^*$ and $\mathsf{E}^*(F) = \mathsf{E}(F \cap \sigma(T))$ if $F \cap S^* = \varnothing$ is a spectral S-capacity of T.

2.5.8. DEFINITION. We denote by suppE, and call the *support of the spectral S-capacity*, the set

$$\text{suppE} = \bigcap_{\mathrm{E}(F)=X} F .$$

2.5.9. PROPOSITION. *If $T \in B(X)$ is S-decomposable and* E *is its spectral S-capacity, then*

$$\text{suppE} = \sigma(T).$$

Proof. By the preceding remark we have $S \subset \sigma(T)$. If $F \supset \sigma(T) \supset S$, then $\mathrm{E}(F) = X_T(F) = X_T(F \cap \sigma(T)) = X_T(\sigma(T)) = X$, hence $\bigcap_{\mathrm{E}(F)=X} F \supset \sigma(T)$; but $\mathrm{E}(\sigma(T)) = X_T(\sigma(T)) = X$ hence also $\bigcap_{\mathrm{E}(F)=X} F \subset \sigma(T)$.

2.6. RESTRICTIONS AND QUOTIENTS OF THE *S*-DECOMPOSABLE OPERATORS

The following paragraph is devoted to the study of the restrictions and quotients of the *S*-decomposable operators and strongly *S*-decomposable operators. One can notice that the class of the *S*-decomposable operators is somehow closed regarding restrictions and quotients: the restriction or quotient of an *S*-decomposable (or strongly *S*-decomposable operator) is also a S'-decomposable (or strongly S'-decomposable) operator, where S' is generally speaking another compact set than *S*.

2.6.1. DEFINITION. $T \in B(X)$ is said to satisfy *strongly condition* β_S if for any spectral maximal space *Y* of *T*, the restriction $T \mid Y$ satisfies condition β_{S_1} (see definition 2.2.1.), where $S_1 = S \cap \sigma(T \mid Y)$, meaning if for any open S_1-covering of $\sigma(T \mid Y)$, $\{G_{S_1}\} \cup \{G_i\}_1^n$ we have for any $x \in Y$, $x = y_{S_1} + y_1 + \ldots + y_n$ with $y_{S_1}, y_i \in Y$ $(i = 1,2,\ldots,n)$ and $\gamma_{T|Y}(y_{S_1}) \subset G_{S_1}$, $\gamma_{T|Y}(y_i) \subset G_i$.

2.6.2. PROPOSITION. *An operator $T \in B(X)$ is strongly S-decomposable if and only if it satisfies condition α_S and strongly condition β_S.*

Proof. Let *T* strongly *S*-decomposable; then obviously *T* satisfies condition α_S and strongly condition β_S. Conversely, let $T \in B(X)$ and $\mathrm{H} = \{H_S\} \cap \{H_i\}_1^n$ be two open *S*-coverings of $\sigma(T)$ such that $\overline{H}_S \subset G_S$, $\overline{H}_i \subset G_i$ $(i = 1,2,\ldots,n)$. If *Y* is an arbitrary spectral maximal space of *T*, then G and H are also S_1 coverings of $\sigma(T \mid Y)$; consequently, if $x \in Y$, then $x = y_{S_1} + y_1 + \ldots + y_n$ with $y_{S_1}, y_i \in Y$ $(i = 1,2,\ldots,n)$ and $\gamma_{T|Y}(y_{S_1}) \subset H_S \cap \sigma(T \mid Y)$, $\gamma_{T|Y}(y_i) \subset H_i \cap \sigma(T \mid Y)$ $(i = 1,2,\ldots,n)$. Since *Y* is *T*-absorbing we have $\gamma_T(y_{S_1})=\gamma_{T|Y}(y_{S_1})$, $\gamma_T(y_i)=\gamma_{T|Y}(y_i)$ and hence $y_i \in X_T(\overline{H}_i \cap S)=Y_i \oplus Y_S^i$, where Y_i, Y_S are spectral maximal spaces of T with $\sigma(T \mid Y_i) \subset \overline{H}_i \subset G_i$ and $Y_S^i \subset Y_S = X_T(\overline{H}_S) \ni Y_{S_1}$; it follows that

$$Y = Y \cap Y_S + Y \cap Y_1 + \ldots + Y \cap Y_n$$

and hence T is strongly S-decomposable.

2.6.3. Corollary. *An operator $T \in B(X)$ is strongly S-decomposable if and only if $T \mid Y$ is S_1-decomposable for any spectral maximal space Y of T, where $S_1 = S \cap \sigma(T \mid Y)$.*

Proof. If $T \mid Y$ is S_1-decomposable for any spectral maximal space Y of T, then T satisfies strongly condition β_S and by the preceding proposition it follows that T is strongly S-decomposable. Conversely, it is obvious.

2.6.4. Proposition. *If $T \in B(X)$ is strongly S-decomposable, then for any spectral maximal space Y of T, $T \mid Y$ is a strongly S_1-decomposable operator, where $S_1 = S \cap \sigma(T \mid Y)$.*

Proof. If $\{G_{S_1}\} \cup \{G_i\}_1^n$ is an open S_1-covering of $\sigma(T \mid Y)$ and $H_S = G_{S_1} \cup \rho(T \mid Y)$, $H_i = G_i \cap \complement S$, then $\{H_S\} \cup \{H_i\}_1^n$ is a S-coverage of $\sigma(T)$. Let $\{Y_S\} \cup \{Y_i\}_1^n$ be the system of spectral maximal spaces of T and

$$Z_{S_1} = Y_S \cap Y,\ Z_i = Y_i \cap Y \ (i = 1,2,\ldots,n).$$

If Z is another spectral maximal space of $T \mid Y$, then Z is also a spectral maximal space for T and

$$Y \cap Z = (Y \cap Z) \cap Y_S + \sum_{i=1}^{n} (Y \cap Z) \cap Y_i = Z_{S_1} \cap Z + \sum_{i=1}^{n} (Z_i \cap Z),$$

and $\sigma(T \mid Z_i) \subset G_i$, $\sigma(T \mid Z_{S_1}) \subset G_{S_1}$; hence $T \mid Y$ is strongly S_1-decomposable.

2.6.5. Corollary. *Let $T \in B(X)$ be a strongly S-decomposable operator and Y a spectral maximal space of T such that $\sigma(T \mid Y) \cap S = \varnothing$; then $T \mid Y$ is strongly decomposable.*

Proof. By the preceding proposition it follows that $T \mid Y$ is strongly S_1-decomposable with $S_1 = \sigma(T \mid Y) \cap S = \varnothing$, therefore strongly decomposable.

2.6.6. Lemma. *If $T \in B(X)$ is a strongly S-decomposable operator and Y, Z are two spectral maximal spaces of T such that $Y \supset Z$ and $\sigma(T \mid Y) \supset S$ or $\sigma(T \mid Z) \cap S = \varnothing$ then*

$$\sigma(\overline{\dot{T} \mid Z}) = \overline{\sigma(T \mid Z) \setminus \sigma(T \mid Y)},$$

where $\overline{\dot{T} \mid Z}$ is the operator induced by $T \mid Y$ in Y / Z.

Proof. It follows by proposition 2.3.6. and the preceding corollary.

2.6.7. PROPOSITION. *Let* $T \in B(X)$ *a S-decomposable operator and* Y *a spectral maximal space of T. Then* $T \mid Y$ *is* S_1*-decomposable, where* $S_1 = (S \cup \sigma(\dot{T})) \cap \sigma(T \mid Y)$ *and* $\dot{T}$ *is the operator induced by T in* $\dot{X} = X / Y$.

Proof. We have $\sigma(T) = \sigma(\dot{T}) \cup \sigma(T \mid Y)$. Let $\{G_{S_1}\} \cup \{G_i'\}_1^n$ be a S_1-covering of $\sigma(T \mid Y)$ and $G_i = G_i' \cap \rho(\dot{T})$, $G_S = G_{S_1} \cup \rho(T \mid Y)$; then $\{G_S\} \cup \{G_i\}_1^n$ is a S-coverage of $\sigma(T)$. Let $\{Y_S\} \cup \{Y_i\}_1^n$ be the system of spectral maximal spaces of T such that

$$\sigma(T \mid Y_S) \subset G_S,\ \sigma(T \mid Y_i) \subset G_i\ (i = 1,2,\ldots,n)$$

and

$$X = Y_S + \sum_{i=1}^{n} Y_i .$$

Form the inclusions

$$\sigma(T \mid Y_i) \subset G_i \cap (\sigma(\dot{T}) \cup \sigma(T \mid Y)) = G_i \cap \sigma(T \mid Y) \subset \sigma(T \mid Y)$$

we have that $Y_i \subset Y$ $(i = 1,2,\ldots,n)$. If $x \in Y$, then

$$x = y_S + y_1 + \ldots y_n$$

where $y_S \in Y_S$, $y_i \in Y_i \subset Y$, hence

$$y_S = x - (y_1 + y_2 + \ldots + y_n) \in Y .$$

Consequently

$$Y = Y_{S_1} + Y_1 + \ldots + Y_n$$

where $Y_{S_1} = Y_S \cap Y$, hence $T \mid Y$ is S_1-decomposable.

2.6.8. COROLLARY. *Let* $T \in B(X)$ *a S-decomposable operator and Y a spectral maximal space of T such that* $\sigma(T \mid Y) \cap S = \varnothing$ *or* $\sigma(T \mid Y) \supset S$. *Then* $T \mid Y$ *is* S_1*-decomposable, where* $S_1 = \sigma(\dot{T}) \cap \partial\sigma(T \mid Y)$ *and* $\dim S_1 \le 1$.

Proof. There follows by the preceding proposition and by lemma 2.6.6.

2.6.9. COROLLARY. *Let* $T \in B(X)$ *a S-decomposable operator with* $S_T = \varnothing$ *and Y a spectral maximal space of T. Then* $\dot{T}$ *is* S_1*-decomposable, where* $S_1 = \sigma(\dot{T}) \cap (S \cup \sigma(T \mid Y))$; *if* $S \subset \sigma(T \mid Y)$ *or* $S \cap \sigma(T \mid Y) = \varnothing$, *then* $S_1 = \partial\sigma(T \mid Y) \cap \sigma(\dot{T})$ *or* $S_1 = (\partial\sigma(T \mid Y) \cap \sigma(\dot{T})) \cup S$, *where* $\dot{T}$ *is the operator induced by T in* $\dot{X} = X / Y$.

Proof. It will be enough to prove that $\dot{T}$ is S_1'-decomposable, where $S_1' = S \cup \sigma(T \mid Y)$ (see 2.5.7.). Let F be a closed set such that $F \supset S \cup \sigma(T \mid Y)$; then by proposition 1.1.1. it follows that

$$\overline{\dot{X_T(F)}} = \dot{X}_{\dot{T}}(F)$$

Indeed, if $x \in X_T(F)$, $\sigma_T(x) \subset F$, $\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x) \cup \sigma(T \mid Y) \subset F$ and hence $\dot{x} \in \dot{X}_{\dot{T}}(F)$; conversely, if $\dot{x} \in \dot{X}_{\dot{T}}(F)$, then $\sigma_T(x) \subset \sigma_{\dot{T}}(\dot{x}) \cup \sigma(T \mid Y) \subset F$ and $x \in X_T(F)$, hence $\dot{x} \in \overline{X_T(F)}^{\boldsymbol{\cdot}}$. Since $X_T(F) \supset Y$ the subspace $\overline{X_T(F)}^{\boldsymbol{\cdot}}$ is closed, therefore $\dot{X}_{\dot{T}}(F)$ is also closed, meaning $\dot{T}$ satisfies condition α_{S_1}. From the inclusion $\gamma_{\dot{T}}(\dot{x}) \subset \gamma_T(x)$ and by the fact that any S_1'-covering of $\sigma(\dot{T})$ is a *S*-covering of $\sigma(T)$ there follows that $\dot{T}$ is S_1'-decomposable, and therefore also S_1-decomposable. If $\sigma(T \mid Y) \supset S$, then by proposition 2.3.6. there follows that $S_1 = \partial\sigma(T \mid Y) \cap \sigma(\dot{T})$. When $S \cap \sigma(T \mid Y) = \varnothing$ we obviously have $S_1 = (\partial\sigma(T \mid Y) \cap \sigma(\dot{T})) \cup S$.

2.6.11. Corollary. *Let $T \in B(X)$ be a S-decomposable operator with $\sigma(T) \in$C and let Y be a spectral maximal space of T such that $\sigma(T \mid Y) \supset S$. Then $\dot{T}$ is strongly decomposable.*

Proof. There follows by the preceding proposition and theorem 1.2.13.

2.6.12. Lemma. *Let $T \in B(X)$ be a strongly S-decomposable operator and Y a spectral maximal space of T with $\sigma(T \mid Y) \supset S$. If $\dot{Z}$ is spectral maximal space of $\dot{T}$ ($\dot{T}$ being the operator induced by T in $\dot{X} = X / Y$), then $Z = \varphi^{-1}(\dot{Z})$ is a spectral maximal space of T, where $\varphi : X \to \dot{X}$ is the canonical map.*

Proof. We have $S_{\dot{T}} = \varnothing$ (see 1.1.9.). If $Z \supset Y$ and Z is an invariant to T linear (closed) subspace of X, Y is also a spectral maximal space of $T \mid Z$ (see 1.2 [2]) hence $S \subset \sigma(T \mid Y) \subset \sigma(T \mid Z)$, that is $X_T(\sigma(T \mid Z)) \supset Y$ is a spectral maximal space of *T*. By lemma 2.6.6. there follows

$$\sigma\left(\overline{T \mid X_T(\sigma(T \mid Z))}^{\boldsymbol{\cdot}}\right) = \overline{\sigma(T \mid X_t(\sigma(T \mid Z))) \setminus \sigma(T \mid Y)}.$$

But $\sigma(T \mid X_T(\sigma(T \mid Z))) \subset \sigma(T \mid Z)$ and $\sigma(T \mid Z) = \sigma(\overline{T \mid Z}^{\boldsymbol{\cdot}}) \cup \sigma(T \mid Y)$ (Y being a spectral maximal space of $T \mid Z$ hence

$$\sigma\left(\overline{T \mid X_T(Gs(T \mid Z))}^{\boldsymbol{\cdot}}\right) = \overline{\left(\left(\sigma(\overline{T \mid Z}^{\boldsymbol{\cdot}}) \cup \sigma(T \mid Y)\right) \setminus \sigma(T \mid Y)\right)} \subset \sigma(\overline{T \mid Z}^{\boldsymbol{\cdot}})$$

From the equalities $\overline{T \mid X_T(\sigma(T \mid Z))}^{\boldsymbol{\cdot}} = \dot{T} \mid \overline{X_T(\sigma(T \mid Z))}^{\boldsymbol{\cdot}}$, $\overline{T \mid Z}^{\boldsymbol{\cdot}} = \dot{T} \mid \dot{Z}$ one obtains $\varphi(X_T(T \mid Z)) \subset \varphi(Z)$, hence $X_T(\sigma(T \mid Z)) \subset Z$; consequently $Z = X_T(\sigma(T \mid Z))$, meaning *Z* is a spectral maximal space of *T*.

2.6.13. Theorem. *Let $T \in B(X)$ be a strongly S-decomposable operator and Y a spectral maximal space of T with $\sigma(T \mid Y) \supset S$. Then $\dot{T}$ is a strongly S_1-decomposable operator, where $S_1 = S \cap \sigma(\dot{T})$, and $\dot{T}$ is the operator induced by T in $\dot{X} = X / Y$.*

Proof. Let $\{G_{S_1}\} \cup \{G_i\}_1^n$ be an open S_1-covering of $\sigma(\dot{T})$ and $G_S = G_{S_1} \cup \rho(\dot{T})$; we can suppose that $G_i \cap S = \varnothing$ ($i = 1,2,\dots,n$). Then $\{G_S\} \cup \{G_i\}_1^n$ is a *S*-covering of $\sigma(T)$. Let $\{Y_S\} \cup \{Y_i\}_1^n$ be the corresponding system of spectral maximal spaces of *T* such that

$$\sigma(T \mid Y_S) \subset G_S,\ \sigma(T \mid Y_i) \subset G_i,\ (i = 1,2,\dots,n)$$

and

$$X = Y_S + \sum_{i=1}^{n} Y_i .$$

We shall set $\sigma_S = \sigma(T \mid Y_S) \cup \sigma(T \mid Y)$, $\sigma_i = \sigma(T \mid Y_i) \cup \sigma(T \mid Y)$ $(i = 1,2,\dots,n)$; $Z_S = X_T(\sigma_S)$ $(i = 1,2,\dots,n), Z_i = X_T(\sigma_i)$ $(i = 1,2,\dots,n)$ are spectral maximal spaces of T (we have $\sigma_S \supset S$, $\sigma_i \supset S$, see theorem 2.1.3.) and $Y \subset Z_S$, $Y \subset Y_i$. Consequently $\dot{Z}_S$, $\dot{Z}_i$ are spectral maximal spaces of $\dot{T}$ ([4], 3.2.) and by lemma 2.6.6. one obtains

$$\sigma(\dot{T} \mid \dot{Z}_S) = \sigma(\dot{\overline{T \mid Z_S}}) = \overline{\sigma(T \mid Z_S) \setminus \sigma(T \mid Y)}$$
$$\overline{(\sigma(T \mid Y_S) \cup \sigma(T \mid Y)) \setminus \sigma(T \mid Y)} \subset \sigma(T \mid Y_S) \subset G_S,$$

and analogously

$$\sigma(\dot{T} \mid \dot{Z}_i) = \sigma(\overline{T \mid Z_i}) \subset \sigma(T \mid Y_i) \subset G_i \ (i = 1,2,\dots,n).$$

If $\dot{Z}$ is an arbitrary spectral maximal space of $\dot{T}$, then $Z = \varphi^{-1}(\dot{Z})$ is a spectral maximal space of T (where φ is the canonical map; see preceding lemma) hence

$$Y_S \cap Z + Y_1 \cap Z + \dots + Y_n \cap Z = Z .$$

But from the inclusions $\dot{Y}_S \subset \dot{Z}_S$, $\dot{Y}_i \subset \dot{Z}_i$, $\varphi(Y_S \cap Z) \subset \dot{Y}_S \cap \dot{Z}$, $\varphi(Y_i \cap Z) \subset \dot{Y}_i \cap \dot{Z}$ $(i = 1,2,\dots,n)$ it results

$$\dot{Z} = \varphi(Y_S \cap Z) + \varphi(Y_1 \cap Z) + \dots + \varphi(Y_n \cap Z) \subset$$
$$\subset \dot{Z}_S \cap \dot{Z} + \dot{Z}_1 \cap \dot{Z} + \dots + \dot{Z}_n \cap \dot{Z} \subset \dot{Z},$$

consequently $\dot{T}$ is strongly S_1-decomposable.

2.6.14. COROLLARY. Let $T \in B(X)$ be a strongly S-decomposable operator and Y a spectral maximal space of T such that $\sigma(T) \cap S = \varnothing$; then $\dot{T}$ is a strongly composable operator.

Proof. There follows by the preceding theorem, since $S_1 = \varnothing$.

2.7. THE PROPERTIES OF STRONGLY *S*-DECOMPOSABLE OPERATORS

There will be given some of the most important properties of the strongly *S*-decomposable operators: the demeanour at direct sums, at the Riesz-Dunfort functional calculus, at quasinilpotent equivalence.

2.7.1. PROPOSITION. *Let $T_\alpha \in B(X_\alpha)$ two strongly S-decomposable operators ($\alpha = 1,2$); then $T = T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ is a strongly S-decomposable operator, where $S = S_1 \cup S_2$.*

Proof. By proposition 2.6.2. and theorem 2.2.3. there follows that it will suffice to show that T satisfies strongly condition β_S (see definition 2.6.1.). Let Y be a spectral maximal space of T and $\mathrm{G} = \{G_{S'}\} \cup \{G_i\}_1^n$ an open S'-covering of $\sigma(T \mid Y)$, where $S' = S \cap \sigma(T \mid Y)$. Then, in accordance with proposition 2.1.7., $Y = Y_1 \oplus Y_2$, where Y_α is a spectral maximal space of T_α ($\alpha = 1,2$). If $y \in Y$, then $y = y^1 \oplus y^2$, with $y^\alpha \in Y_\alpha$ ($\alpha = 1,2$); since T_α ($\alpha = 1,2$) are strongly S-decomposable it follows that $T_\alpha \mid Y_\alpha$ verifies condition β_S, where $S'_\alpha = S_\alpha \cap \sigma(T_\alpha \mid Y_\alpha)$ ($\alpha = 1,2$) hence

$$y^\alpha = y^\alpha_{S'} + y^\alpha_1 + \ldots + y^\alpha_n \ (\alpha = 1,2)$$

and

$$\gamma_T(y^\alpha_{S'}) = \gamma_{T_\alpha \mid Y_\alpha}(y^\alpha_{S'}) \subset G_{S'} \ (\alpha = 1,2),$$
$$\gamma_{T_\alpha}(y^\alpha_i) = \gamma_{T_\alpha \mid Y_\alpha}(y^\alpha_i) \subset G_i \ (\alpha = 1,2; i = 1,2,\ldots,n).$$

Consequently

$$y = y^1 \oplus y^2 = (y^1_{S'_1} + y^1_1 + \ldots + y^1_n) + (y^2_{S'_2} + y^2_1 + \ldots + y^2_n) =$$
$$= (y^1_{S'_1} \oplus y^2_{S'_2}) + (y^1_1 \oplus y^2_1) + \ldots + (y^1_n \oplus y^2_n) = y_{S'} + y_1 + \ldots + y_n$$

and

$$\gamma_T(y_{S'}) = \gamma_{T \mid Y}(y_{S'}) = \gamma_{T_1 \mid Y_1}(y^1_{S'_1}) \cup \gamma_{T_2 \mid Y_2}(y^2_{S'_2}) \subset G_{S'},$$
$$\gamma_T(y_i) = \gamma_{T \mid Y}(y_i) = \gamma_{T_1 \mid Y_1}(y^1_i) \cup \gamma_{T_2 \mid Y_2}(y^2_i) \subset G_i \ (1 \le i \le n)$$

hence T satisfies strongly condition β_S.

2.7.2. DEFINITION. A S-decomposable operator $T \in B(X)$ is said to be *almost strongly S-decomposable* if for any spectral maximal space Y of T such that $\sigma(T \mid Y) \cap S = \varnothing$ or $\sigma(T \mid Y) \supset S$, we have that restriction $T \mid Y$ is a decomposable respectively S-decomposable operator.

2.7.3. *Remark.* The necessity of the definition above becomes established by the following: being given a S-decomposable (strongly S-decomposable) operator, we know about the existence of the spectral maximal spaces Y of T, that have the property that $\sigma(T \mid Y) \cap S = \varnothing$ or $\sigma(T \mid Y) \supset S$; these are the spaces which result form the relations $Y \oplus X_T(S) = X_T(\sigma(T \mid Y) \cup S)$ or $Y = X_T(\sigma(T \mid Y))$. However, we know nothing about the existence of the spectral maximal spaces Y of T that have the property that $\sigma(T \mid Y) \cap S = S'$ is a separated part of S (open and closed in S). Obviously strongly S-decomposable operators are almost strongly S-decomposable. It seems that strongly S-

decomposability (unlike the strongly decomposability) has not a such a favourable demeanour as the one of the S-decomposability (considering the properties from 2.2.1. and 2.2.17.).

2.7.4. PROPOSITION. *Let* $T = T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ *be a strongly S-decomposable operator; then* T_α $(\alpha = 1,2)$ *are almost strongly* S_α*-decomposable, where* $S_\alpha = S \cap \sigma(T_\alpha)$ $(\alpha = 1,2)$.

Proof. It will suffice to prove that if $F \subset \sigma(T_1)$ and $F \cap S_1 = \varnothing$ or $F \supset S_1$, then we also have $F \cap S = \varnothing$ or, respectively, $(F \cup S) \cap \sigma(T_1) \supset S_1$. If $F \cap S_1 = \varnothing$, we also have $F \cap S = (F \cap S) \cap \sigma(T_1) = F \cap (S \cap \sigma(T_1)) = F \cap \sigma(T_1) = \varnothing$, hence when $\sigma(T_1 \mid Y) \cap S_1 = \varnothing$ we also have $\sigma(T_1 \mid Y) \cap S = \varnothing$ (where Y is a spectral maximal space of T_1).

But it also follows that

$$X_{T_1 \oplus T_2}(\sigma(T_1 \mid Y_1) \cup S) = X_{T_1}(\sigma(T_1 \mid Y_1) \cup S) \oplus X_{T_2}(\sigma(T_1 \mid Y_1) \cup S) =$$
$$= [Y_1 + X_{T_1}(S)] \oplus [Y_2 + X_{T_2}(S)] = X_{T_1 \oplus T_2}(S) + Y$$

and one can easily verify that $Y = Y_1 \oplus Y_2$. $T \in T \mid Y_1 \oplus Y_2$ being decomposable, by proposition 2.2.6. there follows that $T_1 \mid Y_1$ is decomposable. Let now Y_1 be a maximal space of T_1 such that $\sigma(T_1 \mid Y_1) \supset S_1$. Then we have

$$X_{T_1 \oplus T_2}(\sigma(T_1 \mid Y_1) \cup S) = X_{T_1}(\sigma(T_1 \mid Y_1) \cup S) \oplus X_{T_2}(\sigma(T_1 \mid Y_1) \cup S) =$$
$$= X_{T_1}([\sigma(T_1 \mid Y_1) \cup S] \cap \sigma(T_1)) \oplus X_{T_2}(\sigma(T_1 \mid Y_1) \cup S) =$$
$$= Y_1 \oplus X_{T_2}(\sigma(T_1 \mid Y_1) \cup S)$$

whence it results $T_1 \mid Y_1$ is S_1-decomposable. Analogously, one verifies that T_2 is almost strongly S_2-decomposable.

2.7.5. THEOREM. *Let* $T = T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ *be a strongly decomposable operator. Then* T_1 *and* T_2 *are strongly decomposable.*

Proof. There follows by propositions 2.7.1. and 2.7.4.

2.7.6. PROPOSITION. *Let* $T \in B(X)$ *be a strongly S-decomposable operator and* $P \in B(X)$ *a projection commuting with T. Then* $T \mid PX$ *is almost strongly S-decomposable, where* $S_1 = \sigma(T \mid PX) \cap S$.

Proof. We have $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$, where $X_1 = PX$, $X_2 = (I - P)X$, $T_1 = T \mid X_1$, $T_2 = T \mid X_2$ and by proposition 2.7.4. we have that $T \mid PX$ is almost strongly S_1-decomposable.

2.7.7. COROLLARY. *Let* $T \in B(X)$ *be a strongly decomposable operator and* $P \in B(X)$ *a projection. Then* $T \mid PX$ *is strongly decomposable.*

Proof. There follows by the preceding proposition.

2.7.8. Proposition. *Let $T \in B(X)$ be a strongly S-decomposable operator and let σ be a separated part of $\sigma(T)$. Then $T \mid E(\sigma,T)X$ is strongly S_1-decomposable, where $S_1 = S \cap \sigma$ (for $E(\sigma,T)$ see corollary 2.2.8.)*

Proof. $X_1 = E(\sigma,T)X$ is a spectral maximal space of *T*. Let Y_1 be a spectral maximal space of $T \mid X_1$. Then by proposition 1.2. [2] Y_1 is also a spectral maximal space of T, hence $T \mid Y_1$ is S_1'-decomposable, where $S_1' = \sigma(T \mid Y_1) \cap S$. But $\sigma(T \mid Y_1) \cap S_1 =$ $= \sigma(T \mid Y_1) \cap (\sigma \cap S) = (\sigma(T \mid Y_1) \cap \sigma) \cap S = S_1'$, hence $(T \mid X_1) \mid Y_1$ is S_1'-decomposable, that is $T \mid E(\sigma,T)X$ is strongly S_1-decomposable.

2.7.9. Proposition. *Let $T \in B(X)$ be a strongly S-decomposable operator and let $f : G \to \mathbf{C}$ ($G \supset \sigma(T)$ open and connected) be an analytic function, injective on $\sigma(T)$. Then $f(T)$ is almost strongly S_1-decomposable.*

Proof. From the equalities $X_{f(T)}(F) = X_T(f^{-1}(F))$ (where $F \supset S_1 = f(S)$) and $X_{f(T)}(F \cup S_1) = X_T(f^{-1}(F) \cup S) = Y_F \oplus X_T(S) = Y_F \oplus X_{f(T)}(S_1)$ (where $F \cap S_1 = \varnothing$) and by proposition 2.2.9. there follows that the spectral maximal spaces *Y* of $f(T)$ that have the property $\sigma(f(T) \mid Y) \supset S_1$ or $\sigma(f(T) \mid Y) \cap S_1 = \varnothing$ are also spectral maximal spaces of *T*. One further performs the proof as for proposition 2.2.9., since a S_1-covering of $\sigma(f(T))$ is easily transformed trough f^{-1} into a *S*-covering of $\sigma(T)$.

2.8. A $(1,S)$-DECOMPOSABLE OPERATOR IS *S*-DECOMPOSABLE

During this paragraph we shall prove that a $(1,S)$-decomposable operator is *S*-decomposable. This result was inspired from the similar one concerning *2*-decomposable operators, which was recently obtained by M. Radjabalipour.

2.8.1. Proposition. *Let $T \in B(X)$, and let Y be an invariant, T-absorbing subspace of T (particularly, Y is a spectral maximal space of T) and let $\dot{T}$ be the operator induced by T in the quotient space $\dot{X} = X / Y$. Then we have the inclusion:*

$$S_{\dot{T}} \subset S_T \setminus \operatorname{Int} \sigma(T \mid Y).$$

Proof. By proposition 1.1.1. there follows that

$$S_{\dot{T}} \subset S_T \cup \sigma(T \mid Y).$$

But from the definition of the analytic residue S_T it results that $S_T = \overline{\operatorname{Int} S_T}$, hence we have the final inclusion

$$S_{\dot{T}} = S_T \cup \overline{\operatorname{Int} \sigma(\mathrm{T} \mid \mathrm{Y})}.$$

It will suffice to prove that $\operatorname{Int}\sigma(T \mid Y) \subset \mathbf{C} \setminus S_T = \Omega_T$. Let $G \subset \operatorname{Int}\sigma(T|Y)$ open and $f(\lambda)$ an analytic function on G taking values in $\dot{X}$ such that

$$(\lambda I - \dot{T})f(\lambda) = \dot{0} \ (\lambda \in \mathbf{C}).$$

Then there exists an open set $G_1 \subset G$ and an analytic function $f(\lambda)$ on G_1 such that $\overline{f(\lambda)} = \dot{f}(\lambda)$ and

$$(\lambda I - T)f(\lambda) = y(\lambda) \ (\lambda \in G_1)$$

with $y(\lambda) \in Y$ (see [18], lemma 2.1.). Since Y is T-absorbing and $\lambda \in G_1 \subset \sigma(T \mid Y_1)$ one obtains $f(\lambda) \in Y_1$, $\dot{f}(\lambda) \in \dot{Y}$, $\dot{f}(\lambda) = \dot{0}$ on G_1, hence $\dot{f}(\lambda) \equiv \dot{0}$; consequently $S_{\dot{T}} \subset S_T \setminus \operatorname{Int}\sigma(T|Y)$.

2.8.2. Corollary. *Having the preceding conditions, if moreover* $\sigma(T \mid Y) \cap S_T = \varnothing$, *we have*

$$S_{\dot{T}} = S_T.$$

Proof. By proposition 1.1.1. we have

$$S_T \subset S_{\dot{T}} \cup \sigma(T \mid Y),$$
$$S_{\dot{T}} \subset S_T \cup \sigma(T \mid Y)$$

hence, by the preceding proposition we have $S_T \subset S_{\dot{T}}$ and $S_{\dot{T}} \subset S_T$ hence $S_{\dot{T}} = S_T$.

2.8.3. *Remark.* By the preceding proposition and corollary there follows, as a particular case, the result obtained by Şt. Frunză in [53] namely that if T has the property of the single-valued extension property and Y is a spectral maximal space, then $\dot{T}$ also has the property of the single-valued extension; also, if $S_T = \varnothing$, Y is T-absorbing and $\sigma(T \mid Y) \supset S_T$, then $S_{\dot{T}} = \varnothing$.

2.8.4. Lemma. *Let* $T \in B(X)$, *Y an invariant subspace of T and* $\dot{T}$ *the operator induced by T in* $\dot{X} = X / Y$. *Then for* $F \supset S_{\dot{T}}$ *closed we have*

$$\dot{X}_{\dot{T}}(F) \subset \overline{X_T(F \cup \sigma(T \mid Y))}$$

Proof. Since $S_T \subset S_{\dot{T}} \cup \sigma(T \mid Y)$, the right member of the inclusion has sense, and from the relation

$$\sigma_T(x) \subset \sigma_{\dot{T}}(\dot{x}) \cup \sigma(T \mid Y)$$

(proposition 1.1.1.) it follows that if $\dot{x} \in \dot{X}_{\dot{T}}(F)$, then $\sigma_{\dot{T}}(\dot{x}) \subset F$, $\sigma_T(x) \subset \sigma_{\dot{T}}(\dot{x}) \cup \sigma(T \mid Y) \subset F \cup \sigma(T \mid Y)$, meaning $x \in X_T(F \cup \sigma(T \mid Y))$; consequently $\dot{x} \in \overline{X_T(F \cup \sigma(T \mid Y))}$.

2.8.5. Definition. A S-decomposable operator $T \in B(X)$ is said to have an almost S-found spectrum if for any spectral maximal space of T with $\sigma(T \mid Y) \cap S = \varnothing$ and any covering $\{G_j\}_{j=1}^m$ of $\sigma(T\mid Y)$ open with $\overline{G}_j \cap S = \varnothing$ $(j = 1,2,\dots,m)$ and $Y \subset Y_1 + Y_2 + \dots + Y_m$.

2.8.6. THEOREM. *Let $T \in B(X)$ a $(1,S)$-decomposable operator with $S_T = \varnothing$ (particularly* $\dim S \le 1$*). Then T is S-decomposable and its spectrum is almost S-found.*

Proof. Since T has the property of the single-valued extension and it is $(1,S)$-decomposable, we have that $X_T(F)$ is a spectral maximal space for any F having the property $F \cap S = \varnothing$ or $F \supset S$. One easily notices that it will suffice to prove that for any F closed with $F \cap S = \varnothing$ and any open covering of F $\{G_1, G_2\}$, where $(\overline{G}_1 \cup \overline{G}_2) \cap S = \varnothing$ we have $X_T(F) \subset X_T(\overline{G}_1) + X_T(\overline{G}_2)$ (one verifies that through induction: if $F \subset \bigcup_{j=1}^m G_j$ we take $G'_j \subset \overline{G}'_j \subset G_j$ such that $F \subset \bigcup_{i=1}^m G'_j$ and we obtain $X_T(F) \subset X_T(G'_1) + \dots + X_T(\overline{G}'_{m-1} \cup \overline{G}'_m) \subset X_T(\overline{G}_1) + \dots + X_T(\overline{G}_{m-2}) + X_T(\overline{G}_{m-1}) + X_T(\overline{G}_m)$.).
Let $H = \overline{G_1 \cap G_2}$ and let us set $Y = X_T(H)$; we shall also put $F_1 = F \setminus G_2$, $F_2 = F \setminus G_1$. Since $G_1 \cup G_2 \supset F$ it results $F_1 \cap F_2 = \varnothing$. By formula $\sigma(\dot{T}) \subset \sigma(T) \setminus (G_1 \cap G_2)$ (which follows by generalisation of a formula belonging to C. Apostol [2], see proposition 1.4.), where $\dot{T}$ is the operator induced by T in $\dot{X} = X/Y$, one obtains for $x \in X_T(F)$ that $\sigma_{\dot{T}}(\dot{x}) \subset F \setminus (G_1 \cap G_2) = F_2 \cup F_1$ (we also have $\sigma_{\dot{T}}(\dot{x}) \subset \sigma(T)(x) \subset F$ and $\sigma_{\dot{T}}(\dot{x}) \subset \sigma(\dot{T}) \setminus (G_1 \cap G_2)$). Consequently we have $(\lambda I - T)x(\lambda) = x$ for $\lambda \in \mathbf{C} \setminus F$ while $(\lambda I - \dot{T})\dot{x}(\lambda) = \dot{x}$ for $\lambda \in \mathbf{C} \setminus (F_1 \cup F_2)$.
We can choose two rectifiable Jordanian curves systems Γ_1, Γ_2 surrounding F_1, respectively F_2 and separating F_1 by F_2. We shall define now

$$\dot{\xi}_j = \frac{1}{2\pi i}\int_{\Gamma_j} \dot{x}(\lambda)d\lambda \quad (j = 1,2)$$

and

$$\dot{v}_j(z) = \frac{1}{2\pi i}\int (z-\lambda)^{-1}\dot{x}(\lambda)d\lambda \quad (j = 1,2)$$

for $z \notin \overline{D}_j$, where D_j is the domain bounded by Γ_j $(j = 1,2)$.
We obviously have $\dot{x} = \dot{\xi}_1 + \dot{\xi}_2$, and

$$(z-\dot{T})\dot{v}_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j} (z - \lambda + \lambda - \dot{T})(z-\lambda)^{-1}\dot{x}(\lambda)d\lambda =$$

$$= \dot{\xi}_j + \frac{1}{2\pi i}\int_{\Gamma_j} (z-\lambda)^{-1}\dot{x}\,d\lambda = \dot{\xi}_j \quad (j = 1,2)$$

since $\lambda \to (z-\lambda)^{-1}$ is analytic in $\overline{D}_j$. Consequently $\dot{\xi}_j \in \dot{X}_{\dot{T}}(\overline{D}_j)$ $(j = 1,2)$. We choose $x_j \in \dot{\xi}_j$ and we notice the fact that we have $\dot{x} = \dot{x}_1 + \dot{x}_2$, hence $x = x_1 + x_2 + y$, with. In accordance with lemma 2.8.4. (where $S_T = S_{\dot{T}} = \varnothing$) we obtain $x_1 + y \in X_T(\overline{D}_1 \cup H) + X_T(H) \subset X_T(\overline{G}_1)$ and $X_T(\overline{G}_2) \ni x_2$ which was to be proved.

2.8.7. THEOREM. *Let $T \in B(X)$ a $(1, S)$-decomposable operator with $S_T \neq \varnothing$. Then T is S-decomposable.*

Proof. Let Y a spectral maximal space of T such that $\sigma(T \mid Y) \cap S = \varnothing$ and let us note $F = \sigma(T \mid Y)$. We know that Y has the following form $Y \oplus X_T(S) = X_T(\sigma(T \mid Y) \cup S)$.
In order to prove the theorem, it will suffice to verify that for any open covering $\{G_1, G_2\}$ of F such that $(\overline{G}_1 \cup \overline{G}_2) \cap S = \varnothing$ and any $x \in Y$, there exists x_1, x_2, x_S such that $\gamma_T(x_1) \subset \overline{G}_1$, $\gamma_T(x_2) \subset \overline{G}_2$, $\gamma_T(x_S) \subset \overline{G}_S$ and $x = x_1 + x_2 + x_S$ ($G_S \supset G$). Thus it results that $(1, S)$-decomposability implies $(2, S)$-decomposability; through induction, one can prove (n, S)-decomposability for all n. Let $H = \overline{G_1 \cap G_2}$ and $Y_{\overline{H}}$ the spectral maximal space of T defined by the equality $Y_H \oplus X_T(S) = X_T(H \cup S)$. If we set by $\dot{T}$ the operator induced by T in the quotient space $\dot{X} = X / Y_H$, by formula in proposition 2.3.6. we have

$$\sigma(\dot{T}) \subset \sigma(T) \setminus (G_1 \cap G_2),$$

hence

$$\sigma(\dot{T}) \subset (\sigma(T) \setminus G_1) \cup (\sigma(T) \setminus G_2).$$

If $x \in Y$ then $\sigma_T(x) \subset F \cup S$ hence $\gamma_{\dot{T}}(\dot{x}) \subset \gamma_T(x) \subset F \cup S$. On the other hand $\gamma_{\dot{T}}(\dot{x}) \subset \sigma(\dot{T}) \subset (\sigma(T) \setminus G_1) \cup (\sigma(T) \setminus G_2)$.
If we set $F \setminus G_1 = F_2$, $F \setminus G_2 = F_1$, then by the relations above and by the fact that $S_{\dot{T}} = S_T$ (see proposition 2.8.2.) we obtain:

$$\sigma_{\dot{T}}(\dot{x}) = \gamma_{\dot{T}}(\dot{x}) \cup \dot{S}_{\dot{T}} \subset (F \cup S) \cap \sigma(\dot{T}) \subset [\sigma(T) \setminus (G_1 \cap G_2)] \cap (F \cup S) \subset$$
$$\subset (F_1 \cup F_2) \cup S$$

Consequently $\rho_{\dot{T}}(\dot{x}) \supset \mathrm{C}F_1 \cap \mathrm{C}F_2 \cap \mathrm{C}S = G$ and on set G we have the equality

$$(\lambda I - \dot{T})\dot{x}(\lambda) = \dot{x}.$$

One further leads the verification as in the preceding theorem. Let Γ_1, Γ_2, Γ_S three rectifiable Jordanian curves systems surrounding F_1, F_2, S and separating them. We shall define then

$$\dot{\xi}_j = \frac{1}{2\pi i} \int_{\Gamma_j} \dot{x}(\lambda) d\lambda \ (j = 1,2), \dot{\xi}_S = \frac{1}{2\pi i} \int_{\Gamma_S} \dot{x}(\lambda) d\lambda$$

and

$$\dot{v}_j(z) = \frac{1}{2\pi i} \int_{\Gamma_j} (z - \lambda)\dot{x}(\lambda) d\lambda \ (j = 1,2)$$

$$\dot{v}_S(z) = \frac{1}{2\pi i} \int_{\Gamma_S} (z - \lambda)\dot{x}(\lambda) d\lambda,$$

where $z \notin \overline{D}_j$ $(j=1,2)$, $z \notin \overline{D}_S, \overline{D}_j, \overline{D}_S$ being the domains bounded by Γ_j, Γ_S $(j=1,2)$.

We obviously have

$$\dot{x} = \dot{\xi}_1 + \dot{\xi}_2 + \dot{\xi}_S$$

and

$$(z-\dot{T})\dot{v}_j(z) = \frac{1}{2\pi i}\int_{\Gamma_j}(z-\lambda+\lambda-\dot{T})(z-\lambda)^{-1}\dot{x}(\lambda)d\lambda =$$

$$= \dot{\xi}_j + \frac{1}{2\pi i}\int_{\Gamma_j}(z-\lambda)^{-1}\dot{x}\,d\lambda = \dot{\xi}_j \quad (j=1,2)$$

$$(z-\dot{T})\dot{v}_S(z) = \dot{\xi}_S,$$

since $\lambda \to (z-\lambda)^{-1}$ is analytic in $\overline{D}_j$, $\overline{D}_S$.

Hence $\gamma_{\dot{T}}(\dot{\xi}_j) \subset \overline{D}_j$ $(j=1,2)$, $\gamma_{\dot{T}}(\dot{\xi}_S) \subset \overline{D}_S$ and we can choose $x_1' \in \dot{\xi}_1$, $x_2' \in \dot{\xi}_2$, $x_S' \in \dot{\xi}_S$, such that

$$x = x_1' + x_2' + x_S' + y \text{ (where } y \in Y_H\text{)}.$$

However we have $\gamma_T(x_j) \subset \gamma_{\dot{T}}(\dot{x}_j) \cup H \subset \overline{D}_j \cup H \subset G_j$ $(j=1,2)$ and $\gamma_T(x_S) \subset \subset \gamma_{\dot{T}}(\dot{x}_S) \cup H \subset \overline{D}_S \cup H$. Form the last inclusion it follows that $x_S' \in X_T(\overline{D}_S \cup H)$ hence $x_S' = x_S + y_1$, where $x_S \in X_T(\overline{D}_S)$ and $y_1 \in Y_H$. We can finally write

$$x = x_1 + x_2 + x_S,$$

where

$$x_1 = x_1' + y_1 + y,\ x_2 = x_2',\ \gamma_T(x_1) \subset \overline{G}_1,\ \gamma_T(x_2) \subset \overline{G}_2,$$

$\gamma_T(x_S) \subset \overline{G}_S$. By this the proof is over.

By proposition 2.4.1. and in accordance with the ones above we obtain the following result:

2.8.8. Theorem. *Let $T \in B(X)$, and let $S \subset \sigma(T)$ be a compact set such that $S_T \subset S$. Then the following conditions are equivalent:*

1°. T is S-decomposable;

2°. T is $(1,S)$-decomposable;

3°. $X_T(F)$ is closed for any closed $F \supset S$

and

$$\sigma(\dot{T}) = \overline{\sigma(T) \setminus \sigma(T \mid Y_{\overline{G}})},$$

where G is arbitrary open in $\sigma(T)$, $\overline{G} \cap S = \varnothing$, and $Y_{\overline{G}}$ is the spectral maximal space of T defined by the equality

$$X_T(\overline{G} \cup S) = X_T(S) \oplus Y_{\overline{G}}.$$

2.9 THE ADJUNCT OF A *S*-DECOMPOSABLE OPERATOR

This last paragraph contains several remarks concerning the adjunct of a S-decomposable operator when it has the property of the single-valued extension ($S_T = \varnothing$), particularly when $\dim S \le 1$. Denote by T^* the adjunct of an operator T.

2.9.1. PROPOSITION. *Let $T \in B(X)$ an S-decomposable operator with $S_T = \varnothing$. Then for any $F \subset \mathbf{C}$ closed such that $F \cap S = \varnothing$ or $F \supset S$, the space $X_T(\mathbf{C} \setminus F)^{\perp}$ is a spectral maximal space for T^* and $\sigma(T^* \mid X_T(\mathbf{C} \setminus F)^{\perp}) \subset F$.*

The proof is identical with the one given in [59] for decomposable (*2*-decomposable) operators, since the demeanour of the *S*-decomposable operator (more exactly of the $(1, S)$-decomposable operator) is in this case (and for sets having the property mentioned the text of the theorem) the same with the one of a *2*-decomposable operator.

2.9.2. THEOREM. If $T \in B(X)$ is a *S*-decomposable operator with $S_T = \varnothing$ (particularly $\dim S \le 1$), then T^* is also *S*-decomposable.

Proof. We shall prove (as in [59]) that T^* is $(1, S)$-decomposable. Let G, G_S two open sets covering the set $\sigma(T^*) = \sigma(T)$ and let D, D_S be another two sets covering $\sigma(T^*)$ such that $\overline{D} \subset G$, $\overline{D}_S \subset G_S$ and moreover $\overline{G} \cap S = \varnothing$, $D_S \supset S$. By setting $F = (\mathbf{C} \setminus D_S) \cap \sigma(T)$ and $F_S = (\mathbf{C} \setminus D) \cap \sigma(T)$ we obtain two closed, disjunct sets F, F_S hence $X_T(F \cup F_S) = X_T(F) \oplus X_T(F_S)$.
Let now u be an arbitrary element from X^* and let us define $\widetilde{u}_1 : X_T(F \cup F_S) \to \mathbf{C}$ by $\widetilde{u}_1(x_1 + x_2) = u(x_2)$, $x_1 + x_2 \in X_T(F) \oplus X_T(F_S)$. It is obvious that the functional $\widetilde{u}_1$ can be extended to a linear continuous functional $u_1 \in X^*$ (using theorem Hahn-Banach). But we also have $u_1 \in X_T(F)$ and by setting $u_2 = u - u_1$, it follows easily that $x_2 \in X_T(F_S)^{\perp}$. We notice the fact that $X_T(F) = X_T(\mathbf{C} \setminus D_S) \supset X_T(\mathbf{C} \setminus \overline{D}_S)$, $X_T(F_S) = X_T(\mathbf{C} \setminus D) \supset$ $\supset X_T(\mathbf{C} \setminus \overline{D})$, hence $X_T(F)^{\perp} = X_T(\mathbf{C} \setminus \overline{D}_S)^{\perp}$, $X_T(F_S)^{\perp} \subset X_T(\mathbf{C} \setminus \overline{D})^{\perp}$ and consequently

$$X^* = X_T(\mathbf{C} \setminus \overline{D}_S)^{\perp} + X_T(\mathbf{C} \setminus \overline{D})^{\perp}.$$

Applying the preceding proposition one obtains that T^* is *S*-decomposable, being $(1, S)$-decomposable.

2.9.3. COROLLARY. *If $T \in B(X)$ is S-decomposable with $S_T = \varnothing$, then $X^*_{T^*}(F) = X_T(\mathbf{C} \setminus F)^{\perp}$ for any closed $F \subset \mathbf{C}$ such that $F \cap S = \varnothing$ or $F \supset S$.*

Remark. By theorem 2.9.2., knowing that $S_T = \varnothing$ it doesn't follow that $S_{T^*} = \varnothing$.

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