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by

IOAN BACALU

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S - spectral decompositions III

By

IOAN BACALU

MULTIDIMENSIONAL SPECTRAL PROPERTIES

Through the third chapter we shall try to generalise some spectral theory results obtained for a single operator, namely decomposable operators (particularly spectral), for operators systems. Mainly, we shall try to extend for systems of operators some results obtained during the first two chapters for decomposable and *S*-decomposable operators. We shall first obtain several results concerning direct sums of systems, by proving that the direct sum of two systems verifies condition (L) [58] if and only if each system verifies condition (L); moreover we shall obtain the relations between the local spectra. We shall further study the direct sums of decomposable and spectral systems. In the second paragraph we prove the uniqueness of the spectral *S*-capacities for *S*-decomposable systems of operators, and we also emphasise the case when $\dim S = 0$. We shall further try the generalisation of the concepts of single residual extension, analytical residuum, spectral residual localisation etc. defined by F. H. Vasilescu in his degree paper, and by this we shall obtain a structure theorem for the spectral maximal spaces of the *S*-decomposable systems.

§3.1. DIRECT SUMS OF DECOMPOSABLE SYSTEMS

3.1.1. Lemma. *If $\Lambda^p[\sigma, X]$, $\Lambda^p[\sigma, Y]$ are the spaces of all exterior forms with p degree in s ($\sigma = (s_1, s_2, \dots, s_n)$) having coefficients in X respectively Y, then*

$$\Lambda^p[\sigma, X] \oplus \Lambda^p[\sigma, Y] = \Lambda^p[\sigma, X \oplus Y].$$

Proof. If $\varphi \in \Lambda^p[\sigma, X]$ and $\psi \in \Lambda^p[\sigma, Y]$ then

$$\varphi = \sum_{1 \le i_1 < \dots < i_p \le n} x_{i_1 i_2 \dots i_p} s_{i_1} \wedge s_{i_1} \wedge \dots \wedge s_{i_p},$$

$$\psi = \sum_{1 \le i_1 < \dots < i_p \le n} y_{i_1 i_2 \dots i_p} s_{i_1} \wedge s_{i_2} \wedge \dots s_{i_p},$$

(we put the same indexes on the expressions of both φ and ψ because when the monoms $s_{i_1} \wedge s_{i_2} \wedge \dots \wedge s_{i_p}$ do not appear, their coefficient is assumed to be 0).

For $\chi \in \Lambda^p[\sigma, X \oplus Y]$ we have

$$\chi = \sum_{1\le i_1<\ldots<i_p\le n}(x\oplus y)_{i_1i_2\ldots i_p} s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_p} =$$
$$= \sum_{1\le i_1<\ldots<i_p\le n} x_{i_1i_2\ldots i_p} s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_p} \oplus y_{i_1i_2\ldots i_p} s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_p} =$$
$$= \sum_{1\le i_1<\ldots<i_p\le n} x_{i_1i_2\ldots i_p} s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_p} \oplus \sum_{1\le i_1<\ldots<i_p\le n} y_{i_1i_2\ldots i_p} s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_p} =$$
$$= \varphi\oplus\psi, \text{ hence}$$

$\chi\in\Lambda^p[\sigma,X]\oplus\Lambda^p[\sigma,Y]$. From the equalities above it follows the inverse inclusion.

3.1.2. *Remark.* If in the preceding lemma we replace the system of undetermined σ with the system $\sigma\cup d\bar{z}=(s_1,s_2,\ldots,s_n,d\bar{z}_1,d\bar{z}_2,\ldots,d\bar{z}_n)$ and the spaces X and Y with $C^\infty(G,X)$ respectively $C^\infty(G,Y)$ ($G\subset C^n$ open), using moreover the obvious equality $C^\infty(G,X)\oplus C^\infty(G,Y)=C^\infty(G,X\oplus Y)$ we obtain

$$\Lambda^p[\sigma\cup d\bar{z},C^\infty(G,X)]\oplus\Lambda^p[\sigma\cup d\bar{z},C^\infty(G,Y)]=$$
$$=\Lambda^p[\sigma\cup d\bar{z},C^\infty(G,X\oplus Y)]$$

3.1.3 Lemma. *Let A, A', B, B' be modules over an algebra such that $A'\subset A$, $B'\subset B$, and h, k two arbitrary maps between arbitrary given sets. Then we have*

$$A/A'\oplus B/B'=A\oplus B/A'\oplus B',$$
$$\operatorname{Ker}h\oplus\operatorname{Ker}k=\operatorname{Ker}(h\oplus k),$$
$$\operatorname{Im}h\oplus\operatorname{Im}k=\operatorname{Im}(h\oplus k).$$

Proof. One easily proves by direct verification.

3.1.4. Proposition. *If $a=(a_1,a_2,\ldots,a_n)\subset B(X)$ and $b=(b_1,b_2,\ldots,b_n)\subset B(Y)$ are two systems of operators and H^p are the co-homology modules (see [58], [70]) then we have*

$$H^p(X,z-a)\oplus H^p(Y,z-b)=H^p(X\oplus Y,z-(a\oplus b)),$$
$$H^p(C^\infty(G,X),\alpha\oplus\bar\partial)\oplus H^p(C^\infty(G,Y),\beta\oplus\bar\partial)=$$
$$=H^p(C^\infty(G,X\oplus Y),(\alpha\oplus\beta)\oplus(\bar\partial\oplus\bar\partial))$$

for any $z\in\mathbf{C}^n$ and $G\subset\mathbf{C}^n$ open.

Proof. Recall that we denote by $\alpha\oplus\bar\partial$, $\beta\oplus\bar\partial$ and $(\alpha\oplus\beta)\oplus(\bar\partial\oplus\bar\partial)$ the cofrontier operators which act on external forms having undeterminates s and $d\bar{z}$ with coefficients in $C^\infty(G,X)$, $C^\infty(G,Y)$ and $C^\infty(G,X\oplus Y)$ as described in the relations:

$$[(\alpha\oplus\bar\partial)\varphi](z)=\left[(z_1-a_1)s_1+\ldots+(z_n-a_n)s_n+\frac{\partial}{\partial\bar{z}_1}d\bar{z}_1+\ldots+\frac{\partial}{\partial\bar{z}_n}d\bar{z}_n\right]\wedge\varphi(z),$$
$$[(\beta\oplus\bar\partial)\psi](z)=\left[(z_1-b_1)s_1+\ldots+(z_n-b_n)s_n+\frac{\partial}{\partial\bar{z}_1}d\bar{z}_1+\ldots+\frac{\partial}{\partial\bar{z}_n}d\bar{z}_n\right]\wedge\psi(z),$$

$$[((\alpha\oplus\beta)\oplus(\bar\partial\oplus\bar\partial))\chi](z)=[(z_1-(a_1\oplus b_1))s_1+\ldots+(z_n-(a_n\oplus b_n))s_n+$$
$$+\left(\frac{\partial}{\partial\bar z_1}\oplus\frac{\partial}{\partial\bar z_1}\right)\mathrm{d}\bar z_1+\ldots+\left(\frac{\partial}{\partial\bar z_n}\oplus\frac{\partial}{\partial\bar z_n}\right)\mathrm{d}\bar z_n\Bigg]\wedge\chi(z)$$

The assertions in the text follow both from the more general theorems concerning tensorial external forms and cohomology modules ([87], [93]) and from direct verifications using the preceding lemma. Indeed, we can write:

$$H^p(X,z-a)\oplus H^p(y,z-b) = [\mathrm{Ker}\,(z-a:\Lambda^p[\sigma,X]\to\Lambda^{p+1}[\sigma,X])/$$
$$/\,\mathrm{Im}\,(z-a:\Lambda^{p-1}[\sigma,X]\to\Lambda^p[\sigma,X])]\oplus$$
$$[\mathrm{Ker}\,(z-b:\Lambda^p[\sigma,Y]\to\Lambda^{p+1}[\sigma,Y])/\,\mathrm{Im}\,(z-b:\Lambda^{p-1}[\sigma,Y]\to\Lambda^{p-1}[\sigma,Y])]=$$
$$[\mathrm{Ker}\,(z-a:\Lambda^p[\sigma,X]\to\Lambda^{p+1}[\sigma,X])\oplus\mathrm{Ker}\,(z-b:\Lambda^p[\sigma,Y]\to\Lambda^{p+1}[\sigma,Y])]/$$
$$/[\mathrm{Im}\,(z-a:\Lambda^{p-1}[\sigma,X]\to\Lambda^p[\sigma,X])\oplus\mathrm{Im}\,(z-b:\Lambda^{p-1}[\sigma,Y]\to\Lambda^p[\sigma,Y])]=$$
$$=\mathrm{Ker}\,(z-(a\oplus b):\Lambda^p[\sigma,X\oplus Y]\to\Lambda^{p+1}[\sigma,X\oplus Y])/$$
$$/\,\mathrm{Im}\,(z-(a\oplus b):\Lambda^{p-1}[\sigma,X\oplus Y]\to\Lambda^p[\sigma,X\oplus Y])=$$
$$=H^p(X\oplus Y,z-(a\oplus b))$$

One easily and similarly verifies the second equality.

3.1.15. LEMMA. *Let* $a=(a_1,a_2,\ldots,a_n)\subset B(X)$ *and* $b=(b_1,b_2,\ldots,b_n)\subset B(Y)$ *be two commuting systems of operators. Then* $a\oplus b=(a_1\oplus b_1,a_2\oplus b_2,\ldots,a_n\oplus b_n)\subset B(X\oplus Y)$ *has the property that the corresponding Taylor spectra verify the equality*

$$\sigma(a\oplus b,X\oplus Y)=\sigma(a,X)\cup\sigma(b,Y).$$

Proof. We shall have to verify that $z-a=(z_1-a_1,z_2-a_2,\ldots,z_n-a_n)$ and $z-b=(z_1-b_1,z_2-b_2,\ldots,z_n-b_n)$ are simultaneously unsingular on X and Y if and only if $z-(a\oplus b)=(z_1-(a_1\oplus b_1),z_2-(a_2\oplus b_2),\ldots,z_n-(a_n\oplus b_n))$ is unsingular on $X\oplus Y$. This means that the complexes of cochains $F(X,z-a)$ and $F(Y,z-b)$ with the operators given by $\alpha=(z_1-a_1)s_1+\ldots+(z_n-a_n)s_n$ respectively $\beta=(z_1-b_1)s_1+\ldots+(z_n-b_n)s_n$ are simultaneously exact if and only if the complex of cochains $F(X\oplus Y,z-(a\oplus b))$ is exact; hence the cohomology modules

$$H^p(X,z-a)=0,\ H^p(Y,z-b)=0$$

if and only if

$$H^p(X\oplus Y,z-(a\oplus b))=0.$$

By proposition 3.1.4. it follows that

$$H^p(X\oplus Y,z-(a\oplus b))=H^p(X,z-a)\oplus H^p(Y,z-b)$$

hence the left member becomes 0 if and only if each term of the right member is null, q.e.d.

3.1.6. PROPOSITION. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *and* $b=(b_1,b_2,...,b_n)\subset B(Y)$. *Systems* a *and* b *verify condition* (L) *if and only if the system* $a\oplus b=(a_1\oplus b_1,...,a_n\oplus b_n)\subset B(X\oplus Y)$ *verifies condition* (L).

Proof. Suppose that a and b verify condition (L) hence

$$H^{n-1}(C^\infty(G,X),\alpha\oplus\bar\partial)=0,\ H^{n-1}(C^\infty(G,Y),\beta\oplus\bar\partial)=0$$

for any open $G\subset\mathbf{C}^n$. From proposition 3.1.4. it results

$$H^{n-1}(C^\infty(G,X\oplus Y),(\alpha\oplus\beta)\oplus(\bar\partial\oplus\bar\partial))=H^{n-1}(C^\infty(G,X),\alpha\oplus\bar\partial)\oplus$$
$$\oplus H^{n-1}(C^\infty(G,Y),\beta\oplus\bar\partial)=0$$

therefore $a\oplus b$ also verifies condition (L). Conversely, if $a\oplus b$ verifies condition (L) then

$$H^{n-1}(C^\infty(G,X\oplus Y),(\alpha\oplus\beta)\oplus(\bar\partial\oplus\bar\partial))=0$$

and from the equality above it follows that

$$H^{n-1}(C^\infty(G,X),\alpha\oplus\bar\partial)=H^{n-1}(C^\infty(G,X),\beta\oplus\bar\partial)=0$$

therefore a and b also verify condition (L).

$$H^{n-1}(C^\infty(G,X\oplus Y),(\alpha\oplus\beta)\oplus(\bar\partial\oplus\bar\partial))=0$$

Conversely, one performs the verification similarly.

3.1.7. PROPOSITION. *If* $a=(a_1,a_2,...,a_n)\subset B(X)$ *and* $b=(b_1,b_2,...,b_n)\subset B(Y)$, *are two systems of operators that verify condition* (L), *then we have the equalities:*

1° $$\sigma(a\oplus b,x\oplus y)=\sigma(a,x)\cup\sigma(b,y);$$

2° $$sp(a\oplus b,x\oplus y)=sp(a,x)\cup sp(a,y);$$

3° $$(X\oplus Y)_{[a\oplus b]}(F)=X_{[a]}(F)\oplus Y_{[b]}(F);$$

4° $$(X\oplus Y)_{a\oplus b}(F)=X_a(F)\oplus Y_b(F),$$

where $x\in X$, $y\in Y$ *and* $F\subset\mathbf{C}^n$.

Proof. Let $z\in\rho(a,x)\cap\rho(b,y)$; then there exists a proximity V of z and $2n$ analytic functions on V taking values in X respectively Y, $f_1,f_2,...,f_n$ and $g_1,g_2,...,g_n$ such that

$$x=(\zeta_1-a_1)f_1(\zeta)+...+(\zeta_n-a_n)f_n(\zeta)$$
$$y=(\zeta_1-b_1)g_1(\zeta)+...+(\zeta_n-b_n)g_n(\zeta)$$

for $\zeta=(\zeta_1,\zeta_2,...\zeta_n)\in V$. Hence, we have

$$(\zeta_1-(a_1\oplus b_1))(f_1(\zeta)\oplus g_1(\zeta))+\ldots+(\zeta_n-(a_n\oplus b_n))(f_n(\zeta)\oplus g_n(\zeta))=$$
$$=((\zeta_1-a_1)f_1(\zeta)+\ldots+(\zeta_n-a_n)f_n(\zeta))\oplus$$
$$\oplus((\zeta_1-b_1)g_1(\zeta)+\ldots+(\zeta_n-b_n)g_n(\zeta))=x\oplus y,$$

consequently $z\in\rho(a\oplus b,x\oplus y)$, meaning

$$\sigma(a\oplus b,x\oplus y)\subset\sigma(a,x)\cup\sigma(b,y).$$

Conversely, if $z\in\rho(a\oplus b,x\oplus y)$, then there exists a proximity V of z and n analytic functions on V taking values in $X\oplus Y$, $f_1\oplus g_1, f_2\oplus g_2,\ldots,f_n\oplus g_n$ so that

$$x\oplus y=(\zeta_1-(a_1\oplus b_1))(f_1(\zeta)\oplus g_1(\zeta))+\ldots+(\zeta_n-(a_n\oplus b_n))$$
$$(f_n(\zeta)\oplus g_n(\zeta))=((\zeta_1-a_1)f_1(\zeta)+\ldots+(\zeta_n-a_n)f_n(\zeta))\oplus$$
$$\oplus((\zeta_1-b_1)g_1(\zeta)+\ldots+(\zeta_n-b_n)g_n(\zeta))$$

hence

$$x=(\zeta_1-a_1)f_1(\zeta)+\ldots+(\zeta_n-a_n)f_n(\zeta),$$
$$y=(\zeta_1-b_1)g_1(\zeta)+\ldots+(\zeta_n-b_n)g_n(\zeta),$$

whence it results that $\rho(a\oplus b,x\oplus y)\subset\rho(x,a)\cup\rho(b,y)$, meaning

$$\sigma(a,x)\cup\sigma(a,y)\subset\sigma(a\oplus b,x\oplus y).$$

Let us verify the second equality. Let $z\in r(a,x)\cap r(b,y)$; there exists a proximity V of z and two forms

$$\varphi\in\Lambda^{n-1}\left[\sigma\cup d\bar{z},C^\infty(V,X)\right] \text{ and } \psi\in\Lambda^{n-1}\left[\sigma\cup d\bar{z},C^\infty(V,Y)\right]$$

such that

$$xs_1\wedge\ldots\wedge s_n=\left((\zeta_1-a_1)s_1+\ldots+(\zeta_n-a_n)s_n+\frac{\partial}{\partial\bar{z}_1}d\bar{z}_1+\ldots+\frac{\partial}{\partial\bar{z}_n}d\bar{z}_n\right)\wedge\varphi$$

$$ys_1\wedge\ldots\wedge s_n=\left((\zeta_1-b_1)s_1+\ldots+(\zeta_n-b_n)s_n+\frac{\partial}{\partial\bar{z}_1}d\bar{z}_1+\ldots+\frac{\partial}{\partial\bar{z}_n}d\bar{z}_n\right)\wedge\psi.$$

It follows that $\varphi\oplus\psi\in\Lambda^{n-1}\left[\sigma\cup d\bar{z},C^\infty(V,X\oplus Y)\right]$ and moreover

$$(\zeta_1-(a_1\oplus b_1))s_1+\ldots+(\zeta_n-(a_n\oplus b_n))s_n+\left(\left(\frac{\partial}{\partial\bar z_1}\oplus\frac{\partial}{\partial\bar z_1}\right)\mathrm{d}\bar z_1+\right.$$

$$\left.+\ldots+\left(\frac{\partial}{\partial\bar z_n}\oplus\frac{\partial}{\partial\bar z_n}\right)\mathrm{d}\bar z_n\right)\wedge(\varphi\oplus\psi)=(((\zeta_1-a_1)s_1+\ldots+$$

$$\left.+(\zeta_n-a_n)s_n+\frac{\partial}{\partial\bar z_1}\mathrm{d}\bar z_1+\ldots+\frac{\partial}{\partial\bar z_n}\mathrm{d}\bar z_n\right)\wedge\varphi\Big)\oplus(((\zeta_1-b_1)s_1+$$

$$\left.+\ldots+(\zeta_n-b_n)s_n+\frac{\partial}{\partial\bar z_1}\mathrm{d}\bar z_1+\ldots+\frac{\partial}{\partial\bar z_n}\mathrm{d}\bar z_n\right)\wedge\psi\Big)=$$

$$=(xs_1\wedge s_2\wedge\ldots\wedge s_n)\oplus(ys_1\wedge s_2\wedge\ldots\wedge s_n)=$$

$$=(x\oplus y)s_1\wedge s_2\wedge\ldots\wedge s_n=(x\oplus y)s,$$

hence $z\in r(a\oplus b,x\oplus y)$. Conversely, let now $z\in r(a\oplus b,x\oplus y)$; then there exists a proximity V of z and a form χ, $\chi\in\Lambda^{n-1}[\sigma\cup\mathrm{d}\bar z,C^\infty(V,X\oplus Y)]=\Lambda^{n-1}[\sigma\cup\mathrm{d}\bar z,C^\infty(V,X)]\oplus\Lambda^{n-1}[\sigma\cup\mathrm{d}\bar z,C^\infty(V,Y)]$ (meaning $\chi=\varphi\oplus\psi$, where φ, ψ belong to the first respectively the second term of the direct sum) such that

$$(x\oplus y)\wedge s_1\wedge s_2\wedge\ldots\wedge s_n=(\zeta_1-(a_1\oplus b_1))s_1+\ldots+$$

$$+(\zeta_n-(a_n\oplus b_n))s_n+\left(\frac{\partial}{\partial\bar z_1}\oplus\frac{\partial}{\partial\bar z_1}\right)\mathrm{d}\bar z_1+\ldots+\left(\frac{\partial}{\partial\bar z_n}\oplus\frac{\partial}{\partial\bar z_n}\right)\mathrm{d}\bar z_n\wedge(\varphi\oplus\psi)$$

From the equalities already written above there follows that

$$xs_1\wedge s_2\wedge\ldots\wedge s_n=((\zeta_1-a_1)s_1+\ldots+(\zeta_n-a_n)s_n+$$

$$\left.+\frac{\partial}{\partial\bar z_1}\mathrm{d}\bar z_1+\ldots+\frac{\partial}{\partial\bar z_n}\mathrm{d}\bar z_n\right)\wedge\varphi$$

$$ys_1\wedge s_2\wedge\ldots\wedge s_n=((\zeta_1-b_1)s_1+\ldots+(\zeta_n-b_n)s_n+$$

$$\left.+\frac{\partial}{\partial\bar z_1}\mathrm{d}\bar z_1+\ldots+\frac{\partial}{\partial\bar z_2}\mathrm{d}\bar z_n\right)\wedge\psi$$

where $\varphi\in\Lambda^{n-1}[\sigma\cup\mathrm{d}\bar z,C^\infty(V,X)]$, $\psi\in\Lambda^{n-1}[\sigma\cup\mathrm{d}\bar z,C^\infty(V,X)]$, hence $z\in r(a,x)\cup r(b,y)$. The first two equalities are completely verified. From equality 2° it follows that if $x\oplus y\in X\oplus Y_{[a\oplus b]}(F)$ then $sp(a\oplus b,x\oplus y)=sp(a,x)\cup sp(a,y)\subset F$, hence $x\in X_{[a]}(F)$ and $y\in Y_{[a]}(F)$, that is $x\oplus y\in X_{[a]}(F)\oplus Y_{[b]}(F)$. The inverse inclusion is verified similarly. For equality 4° we proceed as for equality 3°, using this time the first equality.

3.1.8. Proposition. *Let $a=(a_1,a_2,\ldots,a_n)\subset B(X)$ be a commuting sistem of operators verifying condition (L). Then $X_{[a]}(F)$ is a linear ultrainvariant variety for a, meaning it is invariant to all operators b commuting with each a_i $(i=1,2,\ldots,n)$. If $X_{[a]}(F)$ for*

$F \subset \mathbf{C}^n$ closed is open and $\sigma(a, X_{[a]}(F)) \subset F$ then $X_{[a]}(F)$ is a spectral maximal space for a, more exactly, for any subspace Y invariant to a with $\sigma(a, Y) \subset F$ we have $Y \subset X_{[a]}(F)$.

Proof. From the remark at definition 1.5.2. [58] it results that if $b \in B(X)$ is an operator commuting with all a_i $(i = 1,2,\dots,n)$ then $sp(a, bx) \subset sp(a, x)$, hence when $x \in X_{[a]}(F)$, we have $sp(a, bx) \subset sp(a, x) \subset F$, meaning $bx \in X_{[a]}(F)$.

In accordance with the same remark, if Y is a subspace (linear, closed) of X invariant to a then $sp(a, Y) \subset \sigma(a, Y)$ for any $y \in Y$, hence the inclusion $\sigma(a, Y) \subset F$ yields $Y \subset X_{[a]}(F)$.

3.1.9. Proposition. *A system $a = (a_1, a_2, \dots, a_n) \subset B(X)$ is decomposable if and only if there are verified the following conditions: (1) a satisfies condition (L), $X_{[a]}(F)$ is closed and $\sigma(a, X_{[a]}(F)) \subset F$ for any closed $F \subset \mathbf{C}^n$; (2) for any open finite covering $\{G_j\}_1^m$ of $\mathbf{C}^n$ we have $x = x_1 + x_2 + \dots + x_m$ with $sp(a, x_j) \subset G_j$ $(j = 1,2,\dots,m)$ for all $x \in X$.*

Proof. If a is decomposable then conditions (1), (2) are evidently met. We will also prove reciprocally. Let (1) and (2) satisfied. Then the application E defined by the equality

$$\mathsf{E}(F) = X_{[a]}(F)$$

for any closed $F \subset \mathbf{C}^n$ is a spectral capacity for a. Indeed, according to corollary 1.5.10 [58] we have $sp(a, x) = \varnothing$ implies $x = 0$, hence

$$\mathsf{E}(\varnothing) = X_{[a]}(\varnothing) = \{0\},\ \mathsf{E}(\mathbf{C}^n) = X_{[a]}(\mathbf{C}^n) = X$$

(using the fact that $sp(a, x) \subset \sigma(a, X) \subset \mathbf{C}^n$ for any $x \in X$). Let us verify the equality

$$\mathsf{E}\left(\bigcap_{i \in I} F_i\right) = \bigcap_{i \in I} \mathsf{E}(F_i)\ (F_i = \overline{F_i} \subset \mathbf{C}^n).$$

Let $x \in \mathsf{E}\left(\bigcap_{i \in I} F_i\right) = X_{[a]}\left(\bigcap_{i \in I} F_i\right)$, hence $sp(a, x) \subset F_i$ for all $i \in I$, meaning $x \in \bigcap_{i \in I} \mathsf{E}(F_i) = \bigcap_{i \in I} X_{[a]}(F_i)$. Conversely if $x \in \bigcap_{i \in I} X_{[a]}(F_i)$, $sp(a, x) \subset F_i$ for all $i \in I$ and hence $x \in \mathsf{E}\left(\bigcap_{i \in I} F_i\right)$. Let $\{G_j\}_{j=1}^m$ be an open and finite covering of $\mathbf{C}^n$ and $x \in X$ arbitrary. From the equality $x = x_1 + x_2 + \dots + x_m$ with $sp(a, x_j) \subset G_j$ $(j = 1,2,\dots,m)$ it follows

$$X = \sum_{j=1}^{m} X_{[a]}(\overline{G}_j) = \sum_{j=1}^{m} \mathsf{E}(\overline{G}_j).$$

In order to prove that E is attached to system a, hence that a is decomposable there further has to be verified the inclusions

$$a_i \mathsf{E}(F) \subset \mathsf{E}(F),\ \sigma(a, \mathsf{E}(F)) \subset F$$

for any closed $F \subset \mathbf{C}^n$. This follows by definition of $\mathsf{E}(F)$ and by the fact that $X_{[a]}(F)$ is invariant to each a_i.

3.1.10. Lemma. *Let* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$, $b = (b_1, b_2, \ldots, b_n) \subset B(Y)$ *be two systems of operators verifying condition* (L) *and* $F \subset \mathbf{C}^n$ *closed. Then subspaces* $X_{[a]}(F)$ *and* $Y_{[b]}(F)$ *are closed and* $\sigma(a, X_{[a]}(F)) \subset F$, $\sigma(b, Y_{[b]}(F)) \subset F$ *if and only if the subspace* $(X \oplus Y)_{[a \oplus b]}(F)$ *is closed and* $\sigma(a \oplus b, (X \oplus Y)_{[a \oplus b]}(F)) \subset F$.

Proof. Let $X_{[a]}(F)$, $Y_{[b]}(F)$ closed and

$$\sigma(a, X_{[a]}(F)) \subset F,\ \sigma(b, Y_{[b]}(F)) \subset F.$$

In accordance with proposition 3.1.6., $a \oplus b$ verifies condition (L) and by the equality

(1) $$X_{[a]}(F) \oplus X_{[b]}(F) = (X \oplus Y)_{[a \oplus b]}(F)$$

and by lemma 3.1.5. it follows that $(X \oplus Y)_{[a \oplus b]}(F)$ is closed and

$$\sigma(a \oplus b, (X \oplus Y)_{[a \oplus b]}(F)) = \sigma(a, X_{[a]}(F)) \cup \sigma(b, X_{[b]}(F)) \subset F.$$

Conversely, let $(X \oplus Y)_{[a \oplus b]}(F)$ be closed and $\sigma(a \oplus b, (X \oplus Y)_{[a \oplus b]}(F)) \subset F$. If we denote by P_X, P_Y the corresponding projections (meaning $P_X(X \oplus Y) = X$, $P_Y(X \oplus Y) = Y$) then according to equality (1) we evidently have $P_X((X \oplus Y)_{[a \oplus b]}(F)) = X_{[a]}(F)$, $P_Y((X \oplus Y)_{[a \oplus b]}(F)) = Y_{[b]}(F)$. Let us prove that $X_{[a]}(F)$, $Y_{[b]}(F)$ are closed. One easily verifies that P_X, P_Y commute with each $a_i \oplus b_i$ and since $(X \oplus Y)_{[a \oplus b]}(F)$ is ultrainvariant, there follows that it is invariant to the projections P_X, P_Y. Consequently P_X, P_Y are also projections in the Banach space $(X \oplus Y)_{[a \oplus b]}(F)$, hence the images X_1, Y_1 through P_X, P_Y of this Banach space are closed subspaces, $X_1 \oplus Y_1 = (X \oplus Y)_{[a \oplus b]}(F)$ and $\sigma(a, X_1) \subset F$, $\sigma(b, Y_1) \subset F$. There follows that $X_1 \subset X_{[a]}(F)$, $Y_1 \subset Y_{[b]}(F)$; but we also have $X_{[a]}(F) \oplus Y_{[b]}(F) = X_1 \oplus Y_1$, hence after all $X_1 = X_{[a]}(F)$, $Y_1 = Y_{[b]}(F)$ q.e.d.

3.1.11. Theorem. *Let* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$, $b = (b_1, b_2, \ldots, b_n) \subset B(Y)$ *be two systems of operators. The system* $a \oplus b = (a_1 \oplus b_1, a_2 \oplus b_2, \ldots, a_n \oplus b_n) \subset B(X \oplus Y)$ *is decomposable if and only if a and b are decomposable.*

Proof. In accordance with proposition 3.1.7. we have

$$(X \oplus Y)_{[a\oplus b]}[F] = X_{[a]}(F) \oplus Y_{[b]}(F).$$

By the preceding lemma there follows that a and b satisfy condition (1) from proposition 3.1.9. if and only if $a \oplus b$ satisfies this condition. From the equalities

$$(x_1 \oplus y_1) + (x_2 \oplus y_2) + \ldots + (x_m \oplus y_m) = (x_1 + x_2 + \ldots + x_m) \oplus$$
$$\oplus (y_1 + y_2 + \ldots + y_m)$$
$$sp(a \oplus b, x_j \oplus y_j) = sp(a, x_j) \cup sp(b, y_j) \; (j = 1, 2, \ldots, m)$$

it results that a and b satisfy condition (2) from proposition 3.1.9. if and only if $a \oplus b$ satisfies this condition. Consequently, in accordance with proposition 3.1.9. a and b are decomposable if and only if $a \oplus b$ is decomposable.

3.1.12. Corollary. *The systems* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ *and* $b = (b_1, b_2, \ldots, b_n) \subset B(Y)$ *are strongly decomposable if and only if* $a \oplus b = (a_1 \oplus b_1, a_2 \oplus b_2, \ldots, a_n \oplus b_n) \subset$ $\subset B(X \oplus Y)$ *is strongly decomposable.*

Proof. Let a and b strongly decomposable and Z a spectral maximal space of $a \oplus b$. By the preceding theorem there follows that

$$Z = (X \oplus Y)_{[a\oplus b]}(F) = X_{[a]}(F) \oplus Y_{[a]}(F)$$

where $F = \sigma(a \oplus b, Z)$, since $a \oplus b$ is decomposable. But a and b being strongly decomposable, the restrictions $a| \; X_{[a]}(F)$ and $b| \; X_{[b]}(F)$ are decomposable and hence $(a \oplus b)| \; Z$ is decomposable. Let now $a \oplus b$ be strongly decomposable and let $X_1 = X_{[a]}(\sigma(a, X_1))$ be a spectral maximal space of a, a and b being decomposable. Then $(a \oplus b)| \; (X \oplus Y)_{[a\oplus b]}(\sigma(a, X_1)) = a| \; X_{[a]}(\sigma(a_1, X_1)) \oplus b| \; Y_{[b]}(\sigma(a_1, X_1))$ is decomposable and according to the preceding theorem $a| \; X_{[a]}(\sigma(a_1, X_1))$ is decomposable hence a is strongly decomposable.

3.1.13. Proposition. *Let* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ *be a decomposable (strongly decomposable) system and* $p \in B(X)$ *a projection in* X *that commutes with each* a_i. *Then the restriction* $a| \; pX$ *is a decomposable (strongly decomposable) system.*

Proof. Since p commutes with all a_i $(i = 1, 2, \ldots, n)$, subspaces $X_1 = pX$ and $X_2 = (I - p)X$ are invariant to a and moreover we have

$$X = X_1 \oplus X_2, \; a = (a| \; X_1) \oplus (a| \; X_2).$$

By theorem 3.1.11. it follows that if a is a decomposable system then $a| pX$ is decomposable and by the preceding corollary it results that a strongly decomposable implies $a| pX$ also strongly decomposable.

3.1.14. Corollary. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *a decomposable (strongly decomposable) system and let* σ *be a separated part of* $\sigma(a, X)$. *If* $X = X_1 \oplus X_2$ *where* $\sigma(a, X_1)=\sigma$ *and* $\sigma(a, X_2)=\sigma(a, X)\setminus \sigma$ *is the decomposition corresponding to Taylor's theorem 4.9. [71] , then the restrictions* $a| X_1$, $a| X_2$ *are decomposable (respectively strongly decomposable) systems.*

Proof. There follows by the preceding proposition.

3.1.15. *Remark.* By corollary 3.1.12., proposition 3.1.13 and corollary 3.1.14. there results for the unidimensional case the following: 1° the operators $T_1 \in B(X_1)$, $T_2 \in B(X_2)$ are strongly decomposable if and only if the operator $T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ is strongly decomposable; 2° if the operator $T \in B(X)$ is strongly decomposable and $P \in B(X)$ is a projection in X then the operator $T| PX$ is strongly decomposable; 3° the strongly decomposable operator $T \in B(X)$ and σ a separated part of the spectrum $\sigma(T)$ implies $T| E(\sigma, T)X$ strongly decomposable where $E(\sigma, T)$ is the projector associated to σ (1.3.10. [37]).

3.1.16. Proposition. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a decomposable system and let* $p\in B(X)$ *be a projector commuting with a* ($p^2=p$, $pa_i=a_ip$, $i=1,2,...,n$). *By putting* $Y=pX$ *and* $b=a| Y$ *we have*

$$Y_{[b]}(\sigma)=pX_{[a]}(\sigma)$$

for any closed $\sigma\subset \mathbf{C}^n$.

Proof. We first verify that

$$Y\cap X_{[a]}(\sigma)=pX_{[a]}(\sigma).$$

Since $\sigma(a, x)=sp(a, x)$ for any $x\in X$ (corollary 2.2.4. [58]) and $sp(a, px)\subset sp(a, x)$ (remark to definition 1.5.2. [58]) there follows that for $x\in X_{[a]}(\sigma)$ we have $sp(a, px)=\sigma(a, px)\subset\sigma$ hence

$$pX_{[a]}(\sigma)\subset X_{[a]}(\sigma),$$

$$pX_{[a]}(\sigma)\subset Y\cap X_{[a]}(\sigma).$$

Let now $y\in Y\cap X_{[a]}(\sigma)$, that is $y=px$, $x\in X$ and $sp(a, y)\subset\sigma$. We shall have

$$pX_{[a]}(\sigma) = Y \cap X_{[a]}(\sigma).$$

Let now $y \in Y_{[b]}(\sigma)$, whence $\sigma(b, y) = sp(b, y) \subset \sigma$ (b is decomposable, according to proposition 3.1.13.); for $z \in \mathbf{C}^n \setminus \sigma$ there exists an open neighbourhood V of z and n analytic functions on V taking values in Y, $f_1, f_2, \ldots, f_n$ satisfying the identity

$$y \equiv (\zeta_1 - b_1)f_1(\zeta) + (\zeta_2 - b_2)f_2(\zeta) + \ldots + (\zeta_n - b_n)f_n(\zeta),$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in V$. Then we also have

$$x \equiv (\zeta_1 - a_1)f_1(\zeta) + (\zeta_2 - a_2)f_2(\zeta) + \ldots + (\zeta_n - a_n)f_n(\zeta)$$

hence $\sigma(a, y) \subset \sigma(b, y)$. This yields

$$Y_{[b]}(\sigma) \subset Y \cap X_{[a]}(\sigma).$$

Let $y \in Y \cap X_{[a]}(\sigma)$, hence $\sigma(a, y) \subset \sigma$ and $y = px$, $x \in X$. On the set $\mathbf{C}^n \setminus \sigma$ we have

$$x \equiv (\zeta_1 - a_1)f_1(\zeta) + (\zeta_2 - a_2)f_2(\zeta) + \ldots + (\zeta_n - a_n)f_n(\zeta)$$

with $f_i(\zeta) \in X$. By applying the projection p to the equality above we obtain

$$\begin{aligned} y = py = p^2x = p((\zeta_1 - a_1)f_1(\zeta) + \ldots + (\zeta_n - a_n)f_n(\zeta)) = \\ = (\zeta_1 - b_1)pf_1(\zeta) + \ldots + (\zeta_n - b_n)pf_n(\zeta) \in Y_{[b]}(\sigma) \end{aligned}$$

hence

$$Y_{[b]}(\sigma) = pX_{[a]}(\sigma).$$

3.1.17. Theorem. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$, $b = (b_1, b_2, \ldots, b_n) \subset B(Y)$. a and b are spectral if and only if the system $a \oplus b = (a_1 \oplus b_1, a_2 \oplus b_2, \ldots, a_n \oplus b_n) \subset B(X \oplus Y)$ is spectral.*

Proof. Suppose a and b are spectral and let E_a, E_b be their spectral measures. Let us verify that $E = E_a \oplus E_b$ defined by the relation

$$(E_a \oplus E_b)(B) = E_a(B) \oplus E_b(B)$$

($B \subset \mathbf{C}^n$ Borelian) is a spectral measure for $a \oplus b$. We evidently have

$$E(\emptyset) = E_a(\emptyset) \oplus E_b(\emptyset) = 0, \; E(\mathbf{C}^n) = E_a(\mathbf{C}^n) \oplus E_b(\mathbf{C}^n) = I$$

$$\begin{aligned} E\left(\bigcup_{k=1}^{\infty} B_k\right)(x \oplus y) = E_a\left(\bigcup_{k=1}^{\infty} B_k\right)x \oplus E_b\left(\bigcup_{k=1}^{\infty} B_k\right)y = \\ = \left(\sum_{k=1}^{\infty} E_a(B_k)x\right) \oplus \left(\sum_{k=1}^{\infty} E_b(B_k)y\right) = \\ = \sum_{k=1}^{\infty}(E_a(B_k)x \oplus E_b(B_k)y) = \sum_{k=1}^{\infty} E(B_k)(x \oplus y) \end{aligned}$$

for B_k Borelian, $B_i \cap B_j = \varnothing$, $i \neq j$. Also $(a_i \oplus b_i)E(B) = (a_i \oplus b_i)(E_a(B) \oplus E_b(B)) =$ $= a_i E_a(B) \oplus b_i E_b(B) = E_a(B)a_i \oplus E_b(B)b_i = E(B)(a_i \oplus b_i)$ and $\sigma(a \oplus b,\, E(B)(X \oplus Y)) =$ $= \sigma(a \oplus b,\, (E_a(B) \oplus E_b(B))(X \oplus Y)) = \sigma(a \oplus b,\, E_a(B)X \oplus E_b(B)Y) = \sigma(a,\, E_a(B)X) \cup$ $\cup\, \sigma(b,\, E_b(B)Y) \subset \overline{B} \cup B = \overline{B}$, whence it results that $a \oplus b$ is spectral. Conversely, let us suppose that $a \oplus b$ is spectral and let E be its spectral measure. Then E can be written like this: $E = E_a \oplus E_b$ where $E_a \in B(X)$, $E_b \in B(Y)$. One easily verifies that $E_a(\cdot)$ and $E_b(\cdot)$ are projectors and then the same as above that E_a and E_b are spectral measures of a respectively b. Indeed, we have $0 = E(\varnothing) = E_a(\varnothing) \oplus E_b(\varnothing)$, hence $E_a(\varnothing) = 0$, $E_b(\varnothing) = 0$ and $I = E(\mathbf{C}^n) = E_a(\mathbf{C}^n) \oplus E_b(\mathbf{C}^n)$, consequently $E_a(\mathbf{C}^n) = I_X$, $E_b(\mathbf{C}^n) = I_Y$. Then we can write

$$E\left(\bigcup_{k=1}^{\infty} E(B_k)\right)(x \oplus y) = \sum_{k=1}^{\infty} E(B_k)(x \oplus y) = \sum_{k=1}^{\infty} (E_a(B_k)x \oplus E_b(B_k)y) =$$
$$= \left(\sum_{k=1}^{\infty} (E_a(B_k)x)\right) \oplus \left(\sum_{k=1}^{\infty} (E_b(B_k)y)\right) = ,$$
$$= \left(E_a\left(\bigcup_{k=1}^{\infty} B_k\right)x\right) \oplus \left(E_b\left(\bigcup_{k=1}^{\infty} B_k\right)y\right),$$

whence it results that

$$E_a\left(\bigcup_{k=1}^{\infty} B_k\right)x = \sum_{k=1}^{\infty} E_a(B_k)x,\quad E_b\left(\bigcup_{k=1}^{\infty} B_k\right)y = \sum_{k=1}^{\infty} E_b(B_k)y$$

for $B_k \subset \mathbf{C}^n$ Borelian, $B_i \cap B_j = \varnothing$, $i \neq j$. From the inclusion

$$\sigma(a \oplus b,\, E(B)(X \oplus Y)) = \sigma(a \oplus b,\, E_a(B)X \oplus E_b(B)Y) =$$
$$= \sigma(a,\, E_a(B)X) \cup \sigma(E_b(B)Y) \subset \overline{B},$$

it follows that

$$\sigma(a,\, E_a(B)X) \subset \overline{B},\ \sigma(b,\, E_b(B)Y) \subset \overline{B}.$$

Finally we have $(a_i \oplus b_i)E(B) = (a_i \oplus b_i)(E_a(B) \oplus E_b(B)) = a_i E_a(B) \oplus b_i E_b(B) =$ $= E_a(B)a_i \oplus E_b(B)b_i = E(B)(a_i \oplus b_i)$ ($i = 1, 2, \ldots, n$) and $B \subset \mathbf{C}^n$ Borelian q.e.d.

3.1.18. Corollary. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ be a spectral system and $p \in B(X)$ a projection that commutes with each a_i. Then the restriction $a| \; pX$ is a spectral system.*

Proof. There follows by the preceding proposition ($a = (a| \; pX) \oplus (a| \; (i - p)X)$) or by proposition 3.1.16. and proposition 3.4.5.

§3.2. MULTIDIMENSIONAL SPECTRAL *S*-CAPACITIES

The aim of this paragraph is to show that the *S*-decomposable operators admit a single spectral *S*-capacity. The case $\dim S = 0$ is also studied. The definitions of the *S*-decomposable systems and of the spectral *S*-capacities is given in the preliminaries. If $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ is an operators system that commute and $\sigma(a, X)$ is the system's Taylor spectrum reported to *X*, we shall denote by $\mathsf{U}(\sigma(a, X))$ the algebra of the embryos of analytic functions defined in one neighbourhood of $\sigma(a, X)$. One knows that there exists a homomorphism from $\mathsf{U}(\sigma(a, X))$ to $B(X)$ so that $1 \to 1_X$ and $z_i \to a_i$ ($i = 1, 2, \ldots, n$) where 1 means the embryo associated to the function $z \to 1$ and z_i the embryo associated to the coordinate function [71]. We shall further make use of the following result, proved in [71].

3.2.1. Proposition. *Let Y, Z be two Banach spaces,* $\tau\colon Y \to Z$ *a continuous homomorphism and let* $b = (b_1, b_2, \ldots, b_n) \subset B(Y)$, $c = (c_1, c_2, \ldots, c_n) \subset B(Z)$ *be two systems of operators that commute such that* $\tau b_i = c_i \tau$ *for any* $i = 1, 2, \ldots, n$. *If* $f \in \mathsf{U}(\sigma(b, Y) \cup \sigma(c, Z))$ *then we also have* $\tau f(b) = f(c)\tau$.

3.2.2. Proposition. *Let* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ *and* $\sigma(a, X) = \sigma_1 \cup \sigma_2$ *with* $\sigma_1 \cap \sigma_2 = \varnothing$, σ_1, σ_2 *closed. If* $X = X_1 \oplus X_2$ *is the direct sum decomposition according to Taylor's theorem ([71], 4.9.), where* $\sigma(a, X_1) = \sigma_1$, $\sigma(a, X_2) = \sigma_2$, *then* X_1, X_2 *are spectral maximal spaces of a.*

Proof. Let *Y* be an invariant closed subspace of *X* to *a* such that $\sigma(a, y) \subset \sigma(a, X_1)$. We mark with p_2 the projection of *X* on X_2, with b_i the restriction of a_i at *Y*, $b_i = a_i| \; Y$, with c_i the restriction of a_i at X_2, $c_i = a_i| \; X_2$, and with τ the restriction of p_2 at *Y*, $\tau = p_2| \; Y$. Since p_2 commutes with a_i ($i = 1, 2, \ldots, n$) ([71], 4.9.) we have

$$\tau b_i = x_i \tau .$$

By setting $b = (b_1, b_2, \ldots, b_n)$, $c = (c_1, c_2, \ldots, c_n)$, we have

$$\sigma(b, Y) \cap \sigma(c, X_2) = \varnothing .$$

Let now *f* be the embryo of the analytic function equal with 1 in a neighbourhood of $\sigma(b, Y)$, and equal with 0 in a neighbourhood of $\sigma(c, X_2)$. In accordance with proposition 3.2.1. one obtains $p_2 l_Y = 0$ (since $f(b) = l_Y f(c) = 0$) meaning for $Y \subset X_1$; consequently X_1 is a spectral maximal space of a. Similarly for X_2.

3.2.3. THEOREM. *Let* $a = (a_1, a_2, ..., a_n) \subset B(X)$ *be a S-decomposable system and* E *a spectral S-capacity of a. Then* $\mathrm{E}(F)$ *is a spectral maximal space of a* ($F \subset \mathbf{C}^n$ *closed).*

Proof. Let Y be a invariant closed subspace of X to a with $\sigma(a, Y) \subset F$ for a certain closed set $F \subset \mathbf{C}^n$. To choose, let $F \supset S$. Then there exists an open S-covering of C^n $\{G_S, G\}$ such that $G_S \supset S$ and $\overline{G} \cap F = \varnothing$, and

$$X = \mathrm{E}(\overline{G}_S) + \mathrm{E}(\overline{G}).$$

According to an isomorphic theorem, the quotient space $X/\mathrm{E}(\overline{G}_S)$ is isomorphic with $\mathrm{E}(\overline{G})/\mathrm{E}(\overline{G}_S) \cap \mathrm{E}(\overline{G}) = \mathrm{E}(\overline{G})/\mathrm{E}(\overline{G}_S \cap \overline{G})$.

Taylor's theorem concerning the inclusion of the spectra ([71], lemma 1.2.) yields

$$\sigma(a, \mathrm{E}(\overline{G})/\mathrm{E}(\overline{G}_S \cap \overline{G}) \subset \sigma(a, \mathrm{E}(\overline{G}_S \cap \overline{G})) \cup \sigma(a, \mathrm{E}(\overline{G})) \subset (\overline{G}_S \cap \overline{G}) \cup \overline{G} = \overline{G},$$

meaning

$$\sigma(a, X/\mathrm{E}(\overline{G}_S)) \subset \overline{G}.$$

By denoting by φ the canonical map of X on $X/\mathrm{E}(\overline{G}_S)$, by b_i the restriction of a_i at Y, by c_i the operator induced by a_i in $Z = X/\mathrm{E}(\overline{G}_S)$ and by τ the restriction of φ at Y we shall put $b = (b_1, b_2, ..., b_n)$, $c = (c_1, c_2, ..., c_n)$. It follows

$$\sigma(b, Y) \cap \sigma(c, Z) \subset F \cap G = \varnothing.$$

If f is the embryo of the analytic function equal to 1 on $\sigma(b, Y)$ and by 0 on $\sigma(c, Z)$ then $f(b) = l_Y$ and $f(c) = 0$. By applying proposition 3.2.1. we obtain $\varphi \cdot l_Y = 0$ hence $Y \subset \mathrm{E}(\overline{G}_S)$. Since G_S is arbitrary with the property $G_S \supset F$ we infer that $Y \subset \cap \{\mathrm{E}(\overline{G}_S) \; G_S \supset F\} = \mathrm{E}(F)$. When $F \cap S = \varnothing$ one proceeds analogously.

3.2.4. COROLLARY. *Let* $a = (a_1, a_2, ..., a_n) \subset B(X)$ *be a S-decomposable system. Then a admits a single spectral capacity* E.

Proof. Let E and E_1 two spectral S-capacities of a. Then, in accordance with the preceding theorem $\mathrm{E}(F)$ and $\mathrm{E}_1(F)$ are spectral maximal spaces of a and from the inclusions

$$\sigma(a, \mathrm{E}(F)) \subset F, \; \sigma(a, \mathrm{E}_1(F)) \subset F$$

it follows

$$\mathrm{E}(F) \subset \mathrm{E}_1(F), \mathrm{E}_1(F) \subset \mathrm{E}(F),$$

hence the two spectral S-capacities coincide.

3.2.5. *Remark.* If E is the spectral S-capacity of the S-decomposable system $a=(a_1,a_2,...,a_n)\subset B(X)$ then $\mathsf{E}(F_1\cup F_2)=\mathsf{E}(F_1)\oplus\mathsf{E}(F_2)$ if F_1, F_2 are closed and disjunct $F_1,F_2\in\mathsf{F}_S$ meaning E is additive disjunct [11]. Indeed we have $\mathsf{E}(F_i)\subset\mathsf{E}(F_1\cup F_2)$ ($i=1,2$), hence $\mathsf{E}(F_1)\oplus\mathsf{E}(F_2)\subset\mathsf{E}(F_1\cup F_2)$; but $\mathsf{E}(F_1\cup F_2)=Y_{F_1}\oplus Y_{F_2}$ (see 4.9. [71]), where $\sigma(a,Y_{F_i})\subset F_i$ ($i=1,2$), hence $Y_{F_i}\subset\mathsf{E}(F_i)$ and $Y_{F_1}\oplus Y_{F_2}=\mathsf{E}(F_1)\oplus\mathsf{E}(F_2)$. This remark is also made in proposition 2.2.8. [58].

3.2.6. Proposition. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a S-decomposable system such that* $\dim S=0$. *Then a admits the following spectral decomposition: for any open covering* $\{G_j\}_1^m$ *of* $\mathbf{C}^n$ *there exists the spectral maximal spaces* $\{Y_j\}_1^m$ *of a such that* $X=\sum_{j=1}^m Y_j$ *and* $\sigma(a,Y_j)\subset G_j$ *(*$j=1,2,...,m$*).*

Proof. Let $\{G_j\}_1^m$ an open and finite covering of $\mathbf{C}^n$. By putting $G_j'=G_j\cap(\mathbf{C}^n\setminus S)$ and by observing that $\{G_j\}_1^m$ is also a covering of S, it will follow that there exists an open covering $\{G_j''\}_1^m$ of S such that $G_j''\subset G_j$, $G_i''\cap G_j''=\varnothing$ ($i\neq j,i,j=1,2,...,m$); indeed, this fact is a consequence of lemma 6.2. [13], because S is totally disconnected, that is $\dim S=0$. Then there will exist a covering

$$\{H_j\}_1^m\cup\{H_j'\}_1^m$$

of $\mathbf{C}^n$ such that $\overline{H}_j\subset G_j'$, $\overline{H}_j'\subset G_j''$ ($j=1,2,...,m$). Let us set $H_S=\bigcup_{j=1}^m H_j'$; then

$$\{H_S\}\cup\{H_j\}_1^m$$

is a S-covering of $\mathbf{C}^n$. There will exist the spectral maximal spaces

$$\{Y_S\}\cup\{Y_j'\}_1^m$$

of a such that

$$X=Y_S+\sum_{j=1}^m Y_j,\ \sigma(a,Y_S)\subset H_S,\ \sigma(a,Y_j)\subset H_j.$$

But $Y_S=Y_S^{(1)}\oplus Y_S^{(2)}\oplus....\oplus Y_S^{(m)}$ with $\sigma(a,Y_S^{(j)})\subset H_j'$ ($j=1,2,...,m$) according to theorem 4.9. [71]. It will be enough to show that there exists a spectral maximal space X_j of a such that $Y_S^{(j)}\subset X_j$, $Y_j\subset X_j$ and $\sigma(a,X_j)\subset G_j$ ($j=1,2,...,m$). By setting $F_1^{(j)}=\overline{H}_j\cup\overline{H}_j'$ and $F_1^{(j)}=S\cap(H_1'\cup H_2'\cup...\cup H_{j-1}'\cup H_{j+1}'\cup...\cup H_m')$ we notice that $F_1^{(j)}\cap F_2^{(j)}=\varnothing$ and $F_1^{(j)}\cup F_2^{(j)}\subset S$, hence

$$\mathsf{E}\left(F_1^{(j)} \cup F_2^{(j)}\right) = Y_1^{(j)} \oplus Y_2^{(j)}$$

(according to proposition 3.2.2. and using theorem 4.9. [71]) and the wanted spectral maximal space will be $X_j = Y_1^{(j)}$ ($j = 1,2,\ldots,m$). We will now give a lemma that will prove to be useful.

3.2.7. Lemma. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ a decomposable system. Then*

$$\sigma_{a_i}(x) = \pi_i \sigma(a, x)$$

for all $1 \le i \le n$ and any $x \in X$, where π_i is the projection of $\mathbf{C}^n$ on the plane $\mathbf{C}$ corresponding to the index i.

Proof. Let $z \in \sigma(z, x)$ and let us suppose that $\pi_i(z) = z_i \notin \sigma_{a_i}(x)$; then there exists an analytic function $f_i : V_{z_i} \to X$ such that

$$x \equiv (z_i - a_i) f_i(z_i) = (z_1 - a_1)0 + \ldots + (z_i - a_i) f_i(z) + \ldots + (z_n - a_n)0$$

hence $z \in \sigma(a, x)$, contradiction, meaning $\pi_i \sigma(a, x) \subset \sigma_{a_i}(x)$. Conversely, let $F = \sigma(a, x)$; from $x \in X_a(F) = X_{[a](F)}$ and $\sigma(a, X_{[a]}(F)) \subset F$ it results

$$\sigma_{a_i}(x) \subset \pi_i \sigma(a, X_{[a]}(F)) = \sigma(a_i \mid X_{[a](F)}) \subset \pi_i F = \pi_i \sigma(a, x)$$

whence follows the equality

$$\sigma_{a_i}(x) = \pi_i \sigma(a, x).$$

§3.3. RESTRICTIONS AND QUOTIENTS OF DECOMPOSABLE SYSTEMS

In this paragraph we shall generalise the result obtained in chapter I for decomposable operators namely we shall prove that the restriction and the quotient of a decomposable system related to one of its spectral maximal spaces are S-decomposable systems where S is the intersection of the spectra belonging to the restriction respectively the quotient.

3.3.1. Lemma. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ be an operators system meeting (L) condition and $\{Y_j\}_1^m$ a system of σ-stabile subspaces for a (meaning $a_i Y_j \subset Y_j$, $1 \le i \le n$, $1 \le j \le m$ and $\sigma(a, Y_j) \subset \sigma(a, X)$) such that $X = Y_1 + Y_2 + \ldots + Y_m$. Then we have the equality*

$$\sigma(a, X) = \bigcup_{j=1}^{m} \sigma(a, Y_j)$$

Proof. Obviously we have

$$\bigcup_{j=1}^{m}\sigma(a,Y_j)\subset\sigma(a,X).$$

If $y_j\in Y_j$, then $sp(a,ylj)\subset\sigma(a,Y_j)$ ($j=1,2,\ldots,m$) and $x=y_1+y_2+\ldots+y_m$ implies

$$sp(a,x)\subset\bigcup_{j=1}^{m}sp(a,y_j)\text{ ([58], 1.5.2.)}$$

hence

$$\sigma(a,X)=\bigcup_{x\in X}sp(a,x)\subset\bigcup_{x=y_1+\ldots+y_m}\left(\bigcup_{j=1}^{m}sp(a,y_j)\right)=$$

$$=\bigcup_{j=1}^{m}\left(\bigcup_{x=y_1+\ldots+y_m}sp(a,y_i)\right)\subset\bigcup_{j=1}^{m}\sigma(a,Y_j)$$

meaning what there was to prove.

3.3.2. *Lemma. Let $a=(a_1,a_2,\ldots,a_n)\subset B(X)$ be a decomposable system of operators and Y a spectral maximal space of a. Then the following equality takes place:*

$$\sigma(a,X/Y)=\overline{\sigma(a,X)\setminus\sigma(a,Y)}$$

where X/Y is the quotient space of X by Y.

This lemma represents the extent of a result belonging to Apostol. The proof is contained in [81] where it appears in the following way

$$\sigma(a,X/Y)\subset\sigma(a,X)\setminus\operatorname{Int}\sigma(a,Y).$$

We shall only verify the equivalence of the two expressions. We emphasise the fact that in the above inclusion the interior is considered in the topology of $\sigma(a,X)$. One knows that by denoting by 1 the total set, we have $\operatorname{Int}(X)=1-\overline{1-X}$ for $X\subset 1$ (see [67], paragraph 6). Accordingly to Taylor spectra inclusion theorem of [70] we have

$$\sigma(a,X)=\sigma(a,Y)\cup\sigma(a,X/Y)$$

hence

$$\sigma(a,X/Y)\subset\sigma(a,X)\setminus\operatorname{Int}\sigma(a,Y)=$$

$$=\sigma(a,X)\setminus\left(\sigma(a,X)\setminus\overline{\sigma(a,X)\setminus\sigma(a,Y)}\right)=\overline{\sigma(a,X)\setminus\sigma(a,Y)}\subset\sigma(a,X/Y)$$

3.3.3. Definition. Same as for an operator we shall define a *set-spectrum* for a system of operators $a=(a_1,a_2,\ldots,a_n)$ as being as compact set $\sigma\subset\sigma(a,X)$ such that there exists an invariant subspace Y to all a_i ($i=1,\ldots,n$) enjoying the property $\sigma(a,Y)=\sigma$.

3.3.4. Proposition. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a decomposable system and* $\sigma\subset\sigma(a,Y)$ *such that* $\sigma=\overline{\operatorname{Int}\sigma}$ *(in the topology of* $\sigma(a,X)$*). Then* σ *and* $\sigma'=\overline{\sigma(a,X)\setminus\sigma}$ *are sets-spectra for a and*

$$\sigma(a,X_{[a]}(\sigma))=\sigma,\ \sigma(a,X_{[a]}(\sigma'))=\sigma'.$$

Proof. Since $\sigma(a,X_{[a]}(s))\subset\sigma$ it will suffice to verify that

$$\sigma(a,X_{[a]}(\sigma))\subset\operatorname{Int}\sigma,$$

(the interior is considered in the topology of $\sigma(a,X)$). Let $\lambda_0\in\operatorname{Int}\sigma$; then there exists a polidisk $d=\{\lambda\in\mathbf{C}^n,|\lambda-\lambda_0|<r\}$ such that $d\cap\sigma(a,X)\subset\operatorname{Int}\sigma$. Let us put

$$d_1=\left\{\lambda;\lambda\in\sigma(a,X),|\lambda-\lambda_0|<\frac{r}{2}\right\},$$

$$G_0=\left\{\lambda;|\lambda-\lambda_0|<\frac{3}{4}r\right\},$$

$$G_1=\left\{\lambda;|\lambda-\lambda_0|>\frac{5}{8}\right\}.$$

It follows that $G_0\cup G_1=\mathbf{C}^n$ and $\overline{G}_1\cap d_1=\varnothing$, hence

$$X=\mathrm{E}(\overline{G}_0)+\mathrm{E}(\overline{G}_1)$$

and

$$\sigma(a,X)=\sigma(a,\mathrm{E}(\overline{G}_0))\cup\sigma(a,\mathrm{E}(\overline{G}_1))$$

(according to lemma 3.3.2.). Consequently $\sigma(a,\mathrm{E}(\overline{G}_1))\cap d_1=\varnothing$, whence it follows that $d_1\subset\sigma(a,\mathrm{E}(\overline{G}_0))\subset d\cap\sigma(a,X)\subset\operatorname{Int}\sigma\subset\sigma$, hence $X_{[a]}(\sigma(a,\mathrm{E}(\overline{G}_0)))=\mathrm{E}(\overline{G}_0)\subset\mathrm{E}(\sigma)$. Finally one obtains $\lambda_0\in d_1\subset\sigma(a,\mathrm{E}(\sigma))$ and $\sigma\subset\sigma(a,\mathrm{E}(\sigma))$. Since $\sigma'=\overline{\operatorname{Int}\sigma'}$ (in the topology of $\sigma(a,X)$) we also have $\sigma(a,X_{[a]}(\sigma'))=\sigma'$.

3.3.5. Corollary. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a decomposable system of operators and Y one of its spectral maximal spaces. Then there exists another spectral maximal space* Y_1 *of a such that* $\sigma(a,Y_1)=\sigma(a,X/Y)$.

Proof. From the equality $\sigma(a,X/Y)=\overline{\sigma(a,X)\setminus\sigma(a,Y)}$ and by the preceding proposition there follows that $\sigma=\sigma(a,X/Y)$ is a set-spectrum for a, hence $Y_1=\mathrm{E}(\sigma)$ and $\sigma=\sigma(a,\mathrm{E}(\sigma))$.

3.3.6. *Remarks.* (a) We notice that a system a is decomposable if and only if it admits a spectral capacity such as (σ,X) (see definition 2.1.1. [58]), where $\sigma=\sigma(a,X)$ and $\sigma(a,\mathrm{E}(\overline{G}))=\overline{G}$ for any open $G\subset\sigma(a,X)$ in the topology of $\sigma(a,X)$. There follows by the fact that $\overline{G}$ is a set-spectrum for a and the support of the spectral capacity E is precisely $\sigma(a,X)$. (b) If the system a is S-decomposable, then we can take $S\subset\sigma(a,X)$. Indeed if $S^*=S\cap\sigma(a,X)$ it will suffice to notice that the map $a=(a_1,a_2,...,a_n)\subset B(X)$ defined by the equality $\mathrm{E}^*(F)=\mathrm{E}(F\cup S)$ for $F\supset S^*$ and $\mathrm{E}^*(F)=\mathrm{E}(F\cap\sigma(a,X))$ if $F\cap S^*=\varnothing$ is a spectral S^*-capacity for a.

3.3.7. PROPOSITION. *Let $a=(a_1,a_2,...,a_n)\subset B(X)$ be a decomposable system of operators and Y a spectral maximal space of a. The restriction $b=(a_1\mid Y,a_2\mid Y,...,a_n\mid Y)$ of a is a S-decomposable system where $S=\sigma(a,Y)\cap\sigma(a,X/Y)$ and the spectral S-capacity E^* of b is given by the equality $\mathrm{E}^*(F)=\mathrm{E}(F)\cap Y$, for any $F\in\mathrm{F}_S$.*

Proof. We put $\sigma=\sigma(a,Y)$ and $\sigma_1=\sigma(a,X/Y)$; it will suffice to prove that b is σ_1-decomposable. We have $Y=\mathrm{E}(\sigma)$, $\mathrm{E}^*(\varnothing)=\mathrm{E}(\varnothing)\cap Y=\{0\}$ and $\mathrm{E}^*(\mathbf{C}^n)=\mathrm{E}(\mathbf{C}^n)\cap$ $\cap\mathrm{E}(\sigma)=Y$. Let $\{F_i\}_{i\in I}\subset\mathrm{F}_{\sigma_1}$; then

$$\mathrm{E}^*\left(\bigcap_{i\in I}F_i\right)=\mathrm{E}\left(\bigcap_{i\in I}F_i\right)\cap\mathrm{E}(\sigma)=\bigcap_{i\in I}(\mathrm{E}(F_i)\cap\mathrm{E}(\sigma))=\bigcap_{i\in I}\mathrm{E}^*(F_i).$$

Also it follows that $(a_i\mid Y)\mathrm{E}^*(F)=a_i\mathrm{E}(F\cap\sigma)\subset\mathrm{E}(F\cap\sigma)=\mathrm{E}^*(F)$ and $\sigma(a,\mathrm{E}^*(F))=$ $=\sigma(a,\mathrm{E}(F\cap\sigma))\subset F\cap\sigma\subset F$ for any $F\in\mathrm{F}_{\sigma_1}$. There is only left to be proved that for any σ_1-covering $\{G_{\sigma_1}\}\cup\{G_j\}_1^m$ of $\mathbf{C}^n$ we have

$$Y=\mathrm{E}^*(\overline{G}_{\sigma_1})+\sum_{j=1}^m\mathrm{E}^*(\overline{G}_j).$$

Indeed, we have

$$X=\mathrm{E}(\overline{G}_{\sigma_1})+\sum_{j=1}^m\mathrm{E}(\overline{G}_j)$$

and since $\sigma(a,X)=\sigma\cup\sigma_1$ and $\overline{G}_j\cap\sigma_1=\varnothing$ it follows that

$$\overline{G}_j\cap\sigma(a,X)=\overline{G}_j\cap\sigma\subset\sigma$$

hence $\mathrm{E}(\overline{G}_j)=\mathrm{E}(\overline{G}_j\cap\sigma)\subset\mathrm{E}(\sigma)=Y$ and $\mathrm{E}^*(\overline{G}_j)=\mathrm{E}(\overline{G}_j)$. If $y\in Y$ then

$$y=x_{\sigma_1}+x_1+...+x_m$$

where $x_{\sigma_1}\in\mathrm{E}(\overline{G}_{\sigma_1})$, $x_j\in\mathrm{E}(\overline{G}_j)\subset Y$, hence $x_{\sigma_1}=y-(x_1+...+x_m)\in Y\cap\mathrm{E}(\overline{G}_{\sigma_1})=$ $\mathrm{E}^*(\overline{G}_{\sigma_1})$, meaning what it was to be proved.

3.3.8. PROPOSITION. *Let $a=(a_1,a_2,...,a_n)\subset B(X)$ a decomposable system of operators and Y a spectral maximal space of a. Then the system $\dot{a}=(\dot{a}_1,\dot{a}_2,...,\dot{a}_n)$ induced by a on the quotient space $\dot{X}=X/Y$ is S-decomposable where $S=\sigma(a,Y)\cap\sigma(\dot{a},\dot{X})$.*

Proof. We put $\sigma=\sigma(a,Y)$ and we have $Y=X_{[a]}(\sigma(a,Y))$. It will suffice to prove that $\dot{a}$ is σ-decomposable (see remark 3.3.6. (b)) and it will result that $\dot{a}$ is S-decomposable. If E is the spectral capacity of a we will have $Y=\mathsf{E}(\sigma)$. The proof will consist in showing that the map induced by E in $\dot{X}$ defined by the equality

$$\mathsf{E}^{\cdot}(F)=\varphi(\mathsf{E}(F))$$

(where $\varphi: X\to X$ is the canonical map and $F\in\mathsf{F}_{\sigma}$) is a spectral σ-capacity for the system $\dot{a}$. Obviously $\mathsf{E}^{\cdot}(\varnothing)=\varphi(\mathsf{E}(\varnothing))=\varphi(\{0\})=\{\dot{0}\}$, $\mathsf{E}(\mathbf{C}^n)=\varphi(\mathsf{E}(\mathbf{C}^n))=\varphi(X)=\dot{X}$. Now let us verify the equality

$$\mathsf{E}^{\cdot}\left(\bigcap_{i\in I}F_i\right)=\bigcap_{i\in I}\mathsf{E}^{\cdot}(F_i).$$

Let $\{F_j\}_{j\in J}\subset\mathsf{F}_{\sigma}$ where $J\subset I$ with the property $F_i\supset\sigma$ for any $i\in J$. We will have to prove that

$$\varphi\left(\mathsf{E}\left(\bigcap_{i\in J}F_i\right)\right)=\bigcap_{i\in J}\varphi(\mathsf{E}(F_i)).$$

One knows that if a map f is surjective then $f(f^{-1}(A))=A$ for any $A\subset F$ ($f:E\to F$); this yields that if the images through the inverse map φ^{-1} of the canonical map φ are equal, then the two sets are equal (φ is surjective). We shall use this remark. Since $\sigma\subset\bigcap_{i\in J}F_i$, $Y\subset\mathsf{E}\left(\bigcap_{i\in J}F_i\right)$ we shall obtain

$$\varphi^{-1}\left(\varphi\left(\mathsf{E}\left(\bigcap_{i\in J}F_i\right)\right)\right)=\mathsf{E}\left(\bigcap_{i\in J}F_i\right)+Y=\mathsf{E}\left(\bigcap_{i\in J}F_i\right),$$

$$\varphi^{-1}\left(\bigcap_{i\in J}\varphi(\mathsf{E}(F_i))\right)=\bigcap_{i\in J}\varphi^{-1}(\varphi(\mathsf{E}(F_i)))=$$

$$=\bigcap_{i\in J}(\mathsf{E}(F_i)+Y)=\bigcap_{i\in J}\mathsf{E}(F_i)=\mathsf{E}\left(\bigcap_{i\in J}F_i\right).$$

We shall proceed as in the case when $\{F_i\}_{i\in J}\subset\mathsf{F}_{\sigma}$, $J\subset I$ with $F_i\cap\sigma=\varnothing$ for any $i\in J$. It follows

$$\varphi^{-1}\left(\varphi\left(\mathsf{E}\left(\bigcap_{i\in J}F_i\right)\right)\right)=\mathsf{E}\left(\bigcap_{i\in J}F_i\right)+Y,$$

$$\varphi^{-1}\left(\bigcap_{i\in J}\varphi(\mathsf{E}(F_i))\right)=\bigcap_{i\in J}\varphi^{-1}(\varphi(\mathsf{E}(F_i)))=\bigcap_{i\in J}(\mathsf{E}(F_i)+Y).$$

We shall use the equality

$$\mathsf{E}(F_1 \cup F_2) = \mathsf{E}(F_1) \oplus \mathsf{E}(F_2)$$

for $F_1 \cap F_2 = \varnothing$ and hence

$$\mathsf{E}\left(\bigcap_{i\in J} F_i\right) + \mathsf{E}(\sigma) = \mathsf{E}\left(\left(\bigcap_{i\in J} F_i\right) \cup \sigma\right) =$$

$$= \mathsf{E}\left(\bigcap_{i\in J} (F_i \cup \sigma)\right) = \bigcap_{i\in J} \mathsf{E}(F_i \cup \sigma) = \bigcap_{i\in J} (\mathsf{E}(F_i) + Y)$$

For the case when $H, F \in \mathsf{F}_\sigma$ with $H \supset \sigma$, $F \cap \sigma = \varnothing$ there follows

$$\varphi^{-1}(\varphi(\mathsf{E}(H \cap F))) = \mathsf{E}(H \cap F) + Y = \mathsf{E}((H \cap F) \cup \sigma) =$$
$$= (\mathsf{E}(H) + Y) \cap (\mathsf{E}(F) + y) = \mathsf{E}(H) \cap \mathsf{E}(F \cup \sigma),$$
$$\varphi^{-1}(\varphi(\mathsf{E}(H)) \cap \varphi(\mathsf{E}(F))) = (\mathsf{E}(H) + Y) \cap (\mathsf{E}(F) + Y) = \mathsf{E}(H) \cap \mathsf{E}(F \cup \sigma).$$

Finally let $\{F_i\}_{i\in I} \subset \mathsf{F}_\sigma$ arbitrary. We put $I = I_1 \cap I_2$ such that $F_i \supset \sigma$ if $i \in I_1$ and $F_i \cap \sigma = \varnothing$ if $i \in I_2$. In accordance with the ones above we can write

$$\mathsf{E}^{\cdot}\left(\bigcap_{i\in I} F_i\right) = \mathsf{E}^{\cdot}\left(\left(\bigcap_{i\in I_1} F_i\right) \cap \left(\bigcap_{i\in I_2} F_i\right)\right) = \mathsf{E}^{\cdot}\left(\bigcap_{i\in I_1} F_i\right) \cap \mathsf{E}^{\cdot}\left(\bigcap_{i\in I_2} F_i\right) = \bigcap_{i\in I} \mathsf{E}^{\cdot}(F_i).$$

One must also verify that $\mathsf{E}^{\cdot}(F)$ defined above is closed. Indeed, if $F \supset \sigma$, then $\varphi^{-1}(\mathsf{E}^{\cdot}(F)) = \mathsf{E}(F) + Y = \mathsf{E}(F) + \mathsf{E}(\sigma) = \mathsf{E}(F)$ is closed and hence $\mathsf{E}^{\cdot}(F)$ is also closed; when $F \cap \sigma = \varnothing$ we have $\varphi^{-1}(\mathsf{E}^{\cdot}(F)) = \mathsf{E}(F) + \mathsf{E}(\sigma) = \mathsf{E}(F \cup \sigma)$, consequently $\mathsf{E}^{\cdot}(F)$ is closed in this case also. The subspaces $\mathsf{E}^{\cdot}(F)$ are evidently invariant to all $\dot{a}_i$ induced by a_i on $\dot{X}$ $(i = 1,2,\dots,n)$. Let us prove that for any $F \in \mathsf{F}_\sigma$ we have $\sigma(\dot{a}, \mathsf{E}^{\cdot}(F)) \subset F$. If $F \supset \sigma$ we have $\sigma(\dot{a}, \mathsf{E}^{\cdot}(F)) \cup \sigma(a, \mathsf{E}(\sigma)) = \sigma(a, \mathsf{E}(F)) \subset F$ and when $F \cap \sigma = \varnothing$ it follows that $\mathsf{E}^{\cdot}(F)$ can be identified with $\mathsf{E}(F)$ (since $\mathsf{E}^{\cdot}(F) = \varphi(\mathsf{E}(F))$, $\varphi^{-1}(\mathsf{E}^{\cdot}(F)) = \mathsf{E}(F) \oplus Y$) hence we once again have $\sigma(\dot{a}, \mathsf{E}^{\cdot}(F)) \subset F$.

Let now $\{G_\sigma\} \cup \{G_j\}_1^m$ a σ-covering of $\mathbf{C}^n$. It follows that

$$X = \mathsf{E}(\overline{G}_\sigma) + \sum_{j=1}^m \mathsf{E}(\overline{G}_j)$$

hence

$$\dot{X} = \mathsf{E}^{\cdot}(\overline{G}_\sigma) + \sum_{j=1}^m \mathsf{E}^{\cdot}(\overline{G}_j)$$

meaning $\dot{a}$ is σ-decomposable.

3.3.9. THEOREM. *Let $a = (a_1, a_2, \dots, a_n) \subset B(X)$ a decomposable system and Y a spectral maximal space of a. Then both the restriction $b = a \mid Y$ of a to Y and the quotient*

$\dot{a}$ induced by a in the quotient space $\dot{X} = X / Y$ are S-decomposable operators, where $S = \sigma(a,Y) \cap \sigma(\dot{a},\dot{X})$.

Proof. It follows by proposition 3.3.7. and 3.3.8.

3.3.10. THEOREM. *Let $a = (a_1, a_2, ..., a_n) \subset B(X)$ a decomposable system of operators and Y a spectral maximal space of a such that $\dim S = 0$, where $S = \sigma(a,Y) \cap \sigma(\dot{a},\dot{X})$. Then the restriction $a \mid Y$ of a to Y is a decomposable system.*

Proof. Let E be the spectral capacity of a. Then the application E^* defined by the equality

$$E^*(F) = E(F \cap \sigma(a,Y)) = E(F) \cap Y$$

is a (σ, Y) type spectral capacity where $\sigma = \sigma(a,Y)$. But from proposition 3.2.6. it results that for any open covering $\{G_i\}_1^m$ of $\mathbf{C}^n$ there exists the spectral maximal spaces $\{Y_i\}_1^m$, $Y_i \subset Y$ of $a \mid Y$ such that

$$Y = \sum_{i=1}^{m} Y_i, \ \sigma(a, Y_i) \subset G_i, \ (i = 1,2,..., m).$$

Consequently

$$Y \subset \sum_{i=1}^{m} E(\overline{G_i}) \cap Y = \sum_{i=1}^{m} E^*(\overline{G_i}) \subset Y$$

hence $a \mid Y$ is decomposable.

3.3.11. COROLLARY. *With the same condition as in the theorem above the system $\dot{a}$ admits the following spectral decomposition: for any open covering $\{G_i\}_1^m$ of $\sigma(\dot{a},\dot{X})$ there exists the spectral maximal spaces $\{\dot{Y}_i\}_1^m$ of $\dot{a}$ such that*

$$\dot{X} = \sum_{i=1}^{m} Y_i, \ \sigma(\dot{a}, \dot{Y}_i) \subset G_i \ (i = 1,2,..., m).$$

Proof. It follows by theorem 3.3.9. and proposition 3.2.6.

3.3.12. DEFINITION. We shall denote by C the class of the compact sets $\sigma \in \mathbf{C}^n$ with $\dim \sigma \leq 1$ which enjoy moreover the property that for any subset $\sigma_1 \subset \sigma$, closed in σ, we have $\dim \partial\sigma_1 \leq 0$ ($\partial\sigma_1$ being the frontier of σ_1 in the topology of σ), meaning $\partial\sigma_1$ is totally disconnected. We remind that a decomposable system $a = (a_1, a_2, ..., a_n)$ is said to be strongly decomposable if the restriction $a \mid Y$ at any spectral maximal space Y of a is a decomposable system.

3.3.13. THEOREM. *If* $a=(a_1,a_2,...,a_n)\subset B(X)$ *is a decomposable system and* $\sigma(a,X)\in \mathbb{C}$ *, then a is strongly decomposable.*

Proof. From the formula $\sigma(\dot{a},\dot{X})=\overline{\sigma(a,X)\setminus\sigma(a,Y)}$ where Y is an arbitrary spectral maximal space of a and $\dot{a}=(\dot{a}_1,\dot{a}_2,...,\dot{a}_n)$ is the system induced by a in $\dot{X}=X/Y$, it results that $S=\sigma(a,Y)\cap\sigma(\dot{a},\dot{X})$ is a part of the frontier of $\sigma(a,Y)$ relative to $\sigma(a,X)$ and by theorem 3.3.10., a is a strongly decomposable system.

§3.4. RESTRICTIONS AND QUOTIENTS OF SPECTRAL SYSTEMS

During this paragraph the results obtained in [42], [43] for a single operator will be extended. There will be shown that a spectral system's restriction and quotient regarding an invariant subspace to the system are spectral systems if and only if that subspace is also invariant to the spectral measure of the system and hence that the restriction to an invariant subspace is a spectral system if and only if the quotient is a spectral system. We shall further study the case of the spectral systems having a spectrum of dimension 0 (totally disconnected).

3.4.1. DEFINITION. A $(\mathbb{C}^n,X)$ type *spectral measure* is a map: $\mathcal{B}(\mathbb{C}^n)\to B(X)$ ($\mathcal{B}(\mathbb{C}^n)$ being the family of all Borelian sets of $\mathbb{C}^n$) enjoying the following conditions: $E(\varnothing)=0$, $E(\mathbb{C}^n)=I$, $E(B_1\cap B_2)=E(B_1)E(B_2)$ for any $B_1,B_2\in\mathcal{B}(\mathbb{C}^n)$, $E\left(\bigcup_{k=1}^{\infty}B_k\right)x=\sum_{k=1}^{\infty}E(B_k)x$ for any sequences $(B_k)_{k\in\mathbb{N}}\subset\mathcal{B}(\mathbb{C}^n)$ of sets disjunct two by two. A commuting system $a=(a_1,a_2,...,a_n)$ is said to be a *spectral system* if there exists a $(\mathbb{C}^n,X)$ type spectral measure E such that $a_jE(B)=E(B)a_j$ and $\sigma(a,E(B)X)\subset\overline{B}$ for any $B\in\mathcal{B}(\mathbb{C}^n)$ and $1\le j\le n$.

3.4.2. LEMMA. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a spectral system and let E be its spectral measure. Then each operator* a_i *is spectral and its measure is given by the equality* $E_i(B)=E(\pi_i^{-1}B)$ *where* $B\in\mathcal{B}(\mathbb{C}^n)$ *and* π_i *is the corresponding projection.*

Proof. Let us notice that $\pi_i^{-1}B\in\mathcal{B}(\mathbb{C}^n)$ if $B\in\mathcal{B}(\mathbb{C}^n)$. Obviously, we have $E_i(\varnothing)=E(\pi_i^{-1}\varnothing)=E(\varnothing)=0$, $E_i(\mathbb{C})=E(\mathbb{C}^n)=I$ and $E(B_1\cap B_2)=E(\pi_i^{-1}(B_1\cap B_2))=$ $=E((\pi_i^{-1}B_1)\cap(\pi_i^{-1}B_2))=E_i(B_1)E_i(B_2)$ for $B_1,B_2\in\mathcal{B}(\mathbb{C}^n)$. If $(B_k)_{k\in\mathbb{N}}$ is a sequence of disjunct sets $B_k\in\mathcal{B}(\mathbb{C})$, then $(\pi_i^{-1}B_k)_{k\in\mathbb{N}}\subset\mathcal{B}(\mathbb{C}^n)$ is a sequence of sets disjunct two by two hence $E_i\left(\bigcup_{k=1}^{\infty} B_k\right)x = E\left(\pi_i^{-1}\left(\bigcup_{i=1}^{\infty} B_k\right)\right)x = E\left(\bigcup_{k=1}^{\infty}\left(\pi_i^{-1}(B_k)\right)\right)x = \sum_{k=1}^{\infty} E\left(\pi_i^{-1} B_k\right)x$. There further follows that $a_i E_i(B) = a_i E\left(\pi_i^{-1} B\right) = E_i(B) a_i$ and $\sigma(a_i \mid E_i(B)X) = \pi_i \sigma(a, E_i(B)X) = \pi_i \sigma\left(a, E\left(\pi_i^{-1}(B)\right)X\right) = \pi_i\left(\overline{\pi_i^{-1}(B)}\right) = \pi_i\left(\pi_i^{-1}\left(\overline{B}\right)\right) = \overline{B}$ for any $B \in \mathrm{B}\left(\mathbf{C}^n\right)$ (for the inclusion $\overline{\pi_i^{-1}(B)} \subset \pi_i^{-1}\left(\overline{B}\right)$ see [68], page 85) hence a_i is a spectral operator.

3.4.3. Proposition. *Let* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ *be a spectral system and* Y *be an invariant subspace to a. If the restriction* $b = a \mid Y = (a_1 \mid Y, a_2 \mid Y, \ldots, a_n \mid Y)$ *is a spectral system and E is the spectral measure of a, then Y is invariant to E and* $E \mid Y$ *is the spectral measure of b.*

Proof. Let E_Y the spectral measure of b, $x \in Y$ and $F \subset \mathrm{F} = \left\{F \subset \mathbf{C}^n, F = \overline{F}\right\}$; then according to proposition 3.1.3 [58] we have $E_Y(E)x \in E_Y Y = Y_{[b]}(F)$, hence $\sigma(a, E_Y(F)x) = sp(a, E_y(F)x) \subset sp(b, E_y(F)x) \subset F$ whence $E_Y(F)x \in X_{[a]}(F) = E(F)X$ such that

$$E_Y(F)x = E(F)z$$

with $z \in X$ and hence $E(F)E_Y(F)x = E^2(F)z = E(F)z = E_Y(F)x$. Let now $F_1 = \overline{F_1} \subset \mathbf{C}^n \setminus F$. In accordance with the above we will obtain

$$0 = E(F)E(F_1)E_Y(F_1)x = E(F)E_Y(F_1)x.$$

Since $\mathbf{C}^n$ is a metric space and $\mathbf{C}^n \setminus F$ being open it is of F_σ type, hence $\mathbf{C}^n \setminus F = \bigcup_{n \in \mathrm{N}} F_n$ with $F_n = \overline{F}_n$, $F_n \subset F_{n+1}$. We shall have

$$E(F)E_Y\left(\mathbf{C}^n \setminus F\right)x = \lim_{n \to \infty} E(F)E_Y(F_n)x = 0,$$

whence

$$E(F)x = E(F)E_Y\left(\mathbf{C}^n\right)x = E(F)\left(E_Y(F) + E_Y\left(\mathbf{C}^n \setminus F\right)\right)x = \\ = E(F)E_Y(F)x = E_Y(F)x.$$

By using now the regularity of measures $\left\langle E(\cdot)x, x^*\right\rangle$ ($x^* \in X^*$) and $\left\langle E_Y(\cdot), x^*\right\rangle$ and the fact that for $G \subset \mathbf{C}^n$ open and $F_n \subset F_{n+1}$, F_n closed we obtain

$$E_Y(G) = E_Y\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n \to \infty} E_Y(F_n) = \lim_{n \to \infty} E(F_n) = E(G)$$

hence

$$\left\langle E(B)x, x^*\right\rangle = \inf_{G \supset B}\left\langle E(G)x, x^*\right\rangle = \inf_{G \supset B}\left\langle E_Y(G)x, x^*\right\rangle = \left\langle E_Y(B)x, x^*\right\rangle$$

for any Borelian set $B \subset \mathbf{C}^n$, hence $E \mid Y = E_Y$.

3.4.4. Lemma. *Let* $a = (a_1, a_2, ..., a_n) \subset B(X)$ *be a spectral system and let* $A \subset \mathbf{C}^n$ *be Borelian. Then the restriction* $b = a \mid E(A)X$ *is a spectral system with the spectral measure* E_A *given by the relation* $E_A(B) = E_A(A \cap B)$ *for any* $B \subset \mathbf{C}^n$ *Borelian.*

Proof. One easily verifies that E_A is a spectral measure; the fact that E_A is a spectral measure for b follows by the equality

$$b_j E_A(B) = b_j E(A \cap B) = E(A \cap B) b_j = E_A(B) b_j$$

($b_j = a_j \mid Y$, $Y = E(A)X$) and from the relations

$$\sigma(b, E_A(B)Y) = \sigma(b, E(A \cap B)X) = \sigma(a, E(A \cap B)X) \subset \overline{B}.$$

3.4.5. Proposition. *Let* $a = (a_1, a_2, ..., a_n) \subset B(X)$ *be a spectral system, let E be its spectral measure and Y an invariant subspace to a and E. Then the restriction* $b = a \mid Y$ *is a spectral system and* $\sigma(a, Y) \subset \sigma(a, X)$.

Proof. Since the restriction of the measure E to Y, $E \mid Y = E_Y$ is a spectral measure and $b_j E_Y(B) = E_Y(B) b_j$ ($b_j = a_j \mid Y$), it is only left for us to prove that

$$\sigma(b, E_Y(B)) \subset \overline{B}$$

for any B Borelian. But for spectral systems, the Taylor spectrum is equal with the spectrum in bi-commute, from the formula of Cauchy-Weil it easily follows that $\sigma(a, Y) \subset \sigma(a, X)$. By replacing a with $a \mid E(B)X$ and Y with $E(B)Y$ and also using the preceding lemma one obtains

$$\sigma(b, E_Y(B)Y) = \sigma(a \mid Y, E(B)Y) \subset \sigma(a, E(B)X) \subset \overline{B}$$

($B \subset \mathbf{C}^n$ Borelian), hence b is a spectral system.

3.4.6. Theorem. *Let* $a = (a_1, a_2, ..., a_n) \subset B(X)$ *be a spectral system and let Y be an invariant subspace to a. Then the restriction* $b = a \mid Y$ *is a spectral system if and only if Y is invariant to the spectral measure E of a.*

Proof. It follows from proposition 3.4.3. and 3.4.6.

3.4.7. Proposition. *Let* $a = (a_1, a_2, ..., a_n) \subset B(X)$ *be a spectral system and let Y be a subspace of X invariant to a and the spectral measure E of a. Then the system* $\dot{a} = (\dot{a}_1, \dot{a}_2, ..., \dot{a}_n)$ *induced by a on the quotient space* $\dot{X} = X / Y$ *is spectral.*

Proof. Let $\dot{E}$ the map defined by $\dot{E}(B)\dot{x} = \overset{\bullet}{\overline{E(B)x}}$. The definition is coherent because $E(B)Y \subset Y$ and the left member of the equality does not depend on the choice of the representative of class $\dot{x}$. $\dot{E}$ is a spectral measure of $\dot{a}$. Indeed, $\dot{E}(\varnothing)\dot{x} = \overset{\bullet}{\overline{E(\varnothing)x}} = \dot{0}$,

$$\dot{E}(\mathbf{C}^n)\dot{x} = \overline{E(\mathbf{C}^n)x} = \dot{x}, \quad \dot{E}\left(\bigcup_{k=1}^{\infty} B_k\right)\dot{x} = \overline{E\left(\bigcup_{k=1}^{\infty} B_k\right)x} = \sum_{k=1}^{\infty} \dot{E}(B_k)\dot{x}, \quad \dot{a}_j\dot{E}(B)\dot{x} = \dot{a}_j\overline{E(B)x} =$$

$$= \overline{a_j E(B)x} = \overline{E(B)a_j x} = \dot{E}(B)\dot{a}_j\dot{x} \quad (B, B_k \text{ Borelian, } B_i \cap B = \varnothing,\ i \neq j,\ 1 \leq i, j, k \leq \infty)$$

for any $\dot{x} \in \dot{X}$, hence

$$\dot{E}(\varnothing) = \dot{0},\ \dot{E}(\mathbf{C}^n) = I,\ \dot{E}(B)\dot{a}_j = \dot{a}_j\dot{E}(B).$$

Obviously we also have

$$\dot{E}(B_1 \cap B_2) = \dot{E}(B_1)\dot{E}(B_2).$$

The only item left to be verified is the inclusion

$$\sigma\left(\dot{a}, \dot{E}(B)\dot{X}\right) \subset \overline{B},$$

for any $B \subset \mathbf{C}^n$ Borelian. We also have

$$\dot{E}(B)\dot{X} = \overline{E(B)X} = \overline{E(B)X + Y},$$

and by a known theorem related to isomorphism we obtain

$$\dot{E}(B)\dot{X} = E(B)X + Y / Y = E(B)X / E(B)X \cap Y.$$

But $E(B)X$ and $E(B)X \cap Y$ are invariant to a and E, hence $a \setminus E(B)X$ is spectral and

$$\sigma(a, E(B)X \cap Y) \subset \sigma(a, E(B)X)$$

(according to proposition 3.4.5.); whence, by using Taylor's theorem of spectra inclusion [70] it follows that

$$\sigma\left(\dot{a}, \dot{E}(B)\dot{X}\right) \subset \sigma\left(\dot{a}, E(B)X / E(B)X \cap Y\right) \subset$$
$$\subset \sigma(a, E(B)X) \cup \sigma(a, E(B)X \cap Y) \subset \overline{B}$$

for any $B \subset \mathbf{C}^n$ Borelian, hence $\dot{a}$ is spectral.

3.4.8. Proposition. *Let $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ be a spectral system having the measure E, let Y be a subspace invariant to a and $\dot{a} = (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_n)$ the system induced by a on the quotient space $\dot{X} = X / Y$. If $\dot{a}$ is a spectral system with the spectral measure $\dot{E}$ then Y is also invariant to the spectral measure of a and $\dot{E}$ is equal with the spectral measure induced by E on $\dot{X}$.*

Proof. Since a and $\dot{a}$ are spectral systems, they are decomposable; therefore $\sigma(a, x) = sp(a, x)$ ([58] 1.2.4.). From the equality

$$x = (\zeta_1 - a_1)f_1(\zeta) + \ldots + (\zeta_n - a_n)f_n(\zeta)$$

it results

$$\dot{x} = (\zeta_1 - a_1)\dot{f}_1(\zeta) + \ldots + (\zeta_n - a_n)\dot{f}_n(\zeta)$$

where $f_i(\cdot)$ are analytic and $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)$ hence $sp(\dot{a}, \dot{x}) \subset sp(a, x)$. But $E(F)X = X_{[a]}(F)$, $\dot{E}(F)\dot{X} = \dot{X}_{[\dot{a}]}(F)$ for $F \subset \mathbf{C}^n$ closed (see proposition 3.1.3. [58]), consequently from the inclusions

$$sp(\dot{a}, \dot{x}) \subset sp(a, x) \subset F$$

it follows that

$$\overline{\dot{X_{[a]}(F)}} \subset \dot{X}_{[\dot{a}]}(F) = \dot{E}(F)\dot{X}$$

and consequently the equality $\dot{E}(F)\dot{x} = \dot{x}$ (if $sp(\dot{a}, \dot{x}) \subset F$ then $\dot{x} \in \dot{E}(F)\dot{X}$ and conversely) implies

$$\dot{E}(F)\overline{\dot{E(F)x}} = \overline{\dot{E(F)x}}$$

hence

$$\dot{E}(F)\overline{\dot{E(F_0)x}} = \dot{E}(F)\dot{E}(F_0)\overline{\dot{E(F_0)x}} = \{\dot{0}\}$$

for any $x \in X$. Since $F' = \mathbf{C}^n \setminus F$ is open, there exists a growing sequence $\{F_n\}_{n\in\mathbf{N}}$ of closed sets from $\mathbf{C}^n$ such that $F' = \bigcup_{n=1}^{\infty} F_n$, $F_n \cap F = \varnothing$ hence

$$\dot{E}(F)\overline{\dot{E(F_n)x}} = \{\dot{0}\};$$

from the continuity of the measures $E(\cdot)x$, the limitation of $\dot{E}(F)$ and by the relation $\|\dot{x}\| \le \|x\|$ one obtains

$$\dot{E}(F)\overline{\dot{E(F')x}} = \dot{E}(F)\overline{\dot{\lim_{n\to\infty} E(F_n)x}} = \dot{E}(F)\lim_{n\to\infty}\overline{\dot{E(F_n)x}} =$$
$$= \lim_{n\to\infty}\dot{E}(F)\overline{\dot{E(F_n)x}} = \{\dot{0}\}$$

and hence

$$\dot{E}(F)\dot{x} = \dot{E}(F)\overline{\dot{E(F)x + E(F')x}} = \overline{\dot{E(F)x}} \quad (x \in X).$$

If G is an open set from $\mathbf{C}^n$, there exists a growing sequence of closed sets $H_m \subset \mathbf{C}^n$ such that $G = \bigcup_{m=1}^{\infty} H_m$ therefore similarly as above it follows that

$$\dot{E}(G)\dot{x} = \overline{\dot{E(G)x}}.$$

Finally, if $B \subset \mathbf{C}^n$ is a Borelian set, from the regularity of the measures $\langle \dot{E}(\cdot)\dot{x}, \dot{x}^* \rangle$ ($\dot{x}^* \in \dot{X}^*$), trough customary methods, one proves that

$$\dot{E}(B)\dot{x} = \overline{\dot{E(B)x}}$$

for any $x \in X$. By this last equality it follows that Y is also invariant to the spectral measure E of a and $\dot{E}$ is precisely the spectral measure induced by E on $\dot{X}$.

3.4.9. THEOREM. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a spectral system and let Y be a subspace of X invariant to a. Then the system* $\dot{a}=(\dot{a}_1,\dot{a}_2,...,\dot{a}_n)$ *induced by a on the quotient space* $\dot{X}=X/Y$ *is spectral if and only if Y is also invariant to the spectral measure E of a.*

Proof. There follows by propositions 3.4.7. and 3.4.8.

3.4.10. THEOREM. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a spectral system and let Y be a subspace invariant to a. Then the following three conditions are equivalent:*

1° Y is also invariant to the spectral measure E of a;

2° the restriction $a\,|\,Y=(a_1\,|\,Y,a_2\,|\,Y,...,a_n\,|\,Y)$ *is a spectral system;*

3° the quotient system $\dot{a}=(\dot{a}_1,\dot{a}_2,...,\dot{a}_n)$ *induced by a on* $\dot{X}=X/Y$ *is spectral.*

Proof. There follows by theorems 3.4.6. and 3.4.9.

3.4.11. DEFINITION. Let $a=(a_1,a_2,...,a_n)\subset B(X)$ be a spectral system, let Y be a subspace invariant to a. By marking with Y_m the intersection of all closed subspaces of X that contain Y and moreover are invariant both to a and to the spectral measure E of a, the restriction $a\,|\,Y_m$ will be a spectral system (according to the preceding theorem) which we shall say to be the *minimal spectral extension* of the restriction $a\,|\,Y$.

3.4.12. PROPOSITION. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a spectral system, let Y be a subspace of X invariant to both a and* $a\,|\,Y_m$, *the minimal spectral extension of Y. Then*

$$\sigma(a,Y_m)\subset\sigma(a,Y).$$

Proof. In accordance with proposition 3.1.3. [58] we have

$$Y\subset E(\sigma(a,Y))X=X_{[a]}(\sigma(a,Y))=\mathrm{E}\,(\sigma(a,Y))$$

and since $E(\sigma(a,Y))X$ is a closed subspace invariant to both a and E, we have $Y_m\subset E\sigma(a,Y)X$. By applying proposition 3.4.5. to the system $a\,|\,E(\sigma(a,Y))$ and to its restriction $a\,|\,Y_m$ one obtains

$$\sigma(a,Y_m)\subset\sigma(a,E(\sigma,Y))\subset\sigma(a,Y).$$

3.4.13. PROPOSITION. *Let* $a=(a_1,a_2,...,a_n)\subset B(X)$ *be a system of operators with the spectrum* $\sigma(a,X)$ *totally disconnected* ($\dim\sigma(a,X)=0$) *and Y a subspace of X invariant to a. Then*

$$\sigma(a,Y)\subset\sigma(a,X).$$

Proof. We have $\dim\sigma(a_i, X)=0$ $(i=1,2,\ldots,n)$ and hence by proposition 4.11., [41] we obtain

$$\sigma(a_i, Y)\subset\sigma(a_i, X).$$

This yields that $\dim\sigma(a_i, Y)=0$, hence

$$\dim(\sigma(a_1, Y)\times\sigma(a_2, Y)\times\ldots\times\sigma(a_n, Y))=0.$$

But from the inclusion

$$\sigma(a, Y)\subset\sigma(a_1, Y)\times\sigma(a_2, Y)\times\ldots\times\sigma(a_n, Y)$$

it also results that $\dim\sigma(a, Y)=0$. Let now $z\in\sigma(a, Y)$ be a point such that $z\notin\sigma(a, X)$. Since $\dim\sigma(a, Y)=0$ there exists a decomposition of $\sigma(a, Y)$ in separated parts,

$$\sigma(a, Y)=\sigma_1\cap\sigma_2$$

such that $z\in\sigma_1$ and $\sigma_1\cap\sigma(a, X)=\varnothing$. By remark 3.3. [81] we obtain that $Y_1=\{0\}$ where $Y=Y_1+Y_2$, $\sigma(a, Y_i)\subset\sigma_i$ $(i=1,2)$ hence $\sigma(a, Y)\subset\sigma(a, X)$.

Remark. The proof of the preceding theorem belongs to F.-H. Vasilescu.

3.4.14. THEOREM. *Let* $a=(a_1, a_2, \ldots, a_n)\subset B(X)$ *be a spectral system with* $\dim\sigma(a, X)=0$. *Then for any closed subspace* $Y\subset X$ *invariant to a, the restriction* $a\,|\,Y$ *is a spectral system.*

Proof. According to theorem 3.4.6. it will be enough to show that *Y* is also invariant to the spectral measure *E* of *a*. By the preceding proposition it follows that

$$\sigma(a, Y)\subset\sigma(a, X) \text{ and } \dim\sigma(a, Y)=0.$$

Let σ be a separated part of $\sigma(a, X)$ and $F(\sigma)$ the projector associated to σ that results from Taylor's theorem 4.9. [75]. Let us verify that $F(\sigma)=E(\sigma)$. We have

$$\sigma(a, F(\sigma))=\sigma,\ \sigma(a, F(\sigma(a, X)\setminus\sigma)X)=\sigma(a, X)\setminus\sigma$$

and

$$F(\sigma)+F(\sigma(a, X)\setminus\sigma)=I.$$

By other means

$$F(\sigma)X\subset X_{[a]}(\sigma)=E(\sigma)X$$

$$F(\sigma(a, X)\setminus\sigma)X\subset E(\sigma(a, X)\setminus\sigma)X$$

see [58], proposition 3.1.3., whence according to lemma 1.12. [41] one obtains

$$E(\sigma)F(\sigma)=F(\sigma),\ E(\sigma)F(\sigma(a, X)\setminus\sigma)=0$$

hence $F(\sigma)=E(\sigma)$. But $\sigma(a,Y)=\sigma'\cup\sigma''$ with $\sigma'\subset\sigma$, $\sigma''\subset\sigma(a,X)\setminus\sigma$ and hence $\sigma'\cap\sigma''=\varnothing$, σ',σ'' being separated parts of $\sigma(a,Y)$. Therefore we shall be allowed to write

$$Y=Y'\oplus Y''$$

with $Y',Y''\subset Y$ being invariant to both a and $\sigma(a,Y')=\sigma'$, $\sigma(a,Y'')=\sigma''$. But $E(\sigma)X$ and $E(\sigma(a,X)\setminus\sigma)X$ are spectral maximal spaces of a ([58], 3.1.3.), consequently

$$Y'\subset E(\sigma)X\,,\; Y''\subset E(\sigma(a,x)\setminus\sigma)X\,.$$

Hence if $y\in Y$, then $y=y'+y''$, $y',y''\in Y$ and $y'=E(\sigma)z'$, $y''=E(\sigma(a,X)\setminus\sigma)z''$ whence

$$\begin{aligned}E(\sigma)y&=E(\sigma)y'+E(\sigma)y''=\\&=E^2(\sigma)z'+E(\sigma)E(\sigma(a,X)\setminus\sigma)z''=E(\sigma)z'=y'\in Y\end{aligned}$$

meaning $E(\sigma)Y\subset Y$. According to theorem 1, paragraph 21, [67] a compact set $\sigma\subset\mathbf{C}^n$ having dimension 0 can be written as a countable reunion of closed-open sets in the relative topology (there exists a countable base $(\sigma_n)_{n\in\mathbf{N}}$ formed out of closed-open sets in σ). Let now $G\subset\mathbf{C}^n$ be open and let $(\sigma_i)_{i\in\mathbf{N}}$ such that

$$G\cap\sigma(a,X)=\bigcup_{i\in\mathbf{N}}\alpha_{n_i}\,.$$

By the ones above there follows that for any $y\in Y$ we have

$$E(G\cap\sigma(a,X))x=E\left(\bigcup_{i\in\mathbf{N}}\sigma_{n_i}\right)x=\sum_{i\in\mathbf{N}}E(\sigma_{n_i})x\in\overline{Y}=Y\,,$$

hence $E(G)Y\subset Y$, whence $E(B)Y\subset Y$ for any $B\subset\mathbf{C}^n$ Borelian. Consequently $a\,|\,Y$ is a spectral system.

3.4.15. Corollary. *Let $a=(a_1,a_2,\ldots,a_n)\subset B(X)$ be a spectral system and Y a closed invariant subspace to a such that $\sigma(a,Y)=0$. Then $a\,|\,Y$ is a spectral system.*

Proof. Let $a\,|\,Y_m$ be the minimal spectral extension of $a\,|\,Y$; then, according to proposition 3.4.12. we have

$$\sigma(a,Y_m)\subset\sigma(a,Y),$$

hence

$$\dim\sigma(a,Y_m)=0\,.$$

Since the system $a\,|\,Y_m$ is spectral, by the preceding theorem there follows that $a\,|\,Y$ is a spectral system.

3.4.16. *Remark.* (1) If $a=(a_1,a_2,...,a_n)\subset B(X)$ is a spectral system and Y is an invariant subspace to a such that $\dim\sigma(a,Y)=0$ (particularly $\sigma(a,Y)$ is discreet) then the quotient system $\dot{a}=(\dot{a}_1,\dot{a}_2,...,\dot{a}_n)$ induced by a on the space $\dot{X}=X/Y$ is spectral. There follows by the preceding corollary and by theorem 3.4.10. (2) If $a=(a_1,a_2,...,a_n)\subset B(X)$ is a spectral system and Y is a spectral maximal space of a, then the restriction $a|Y$ and the quotient $\dot{a}$ are spectral systems, because $Y=X_{[a]}(\sigma(a,Y))$ is also invariant to the spectral measure E of a.

§3.5. RESIDUAL SPECTRAL PROPERTIES FOR OPERATORS SYSTEMS

Across this paragraph we shall try to generalise for operators systems some of the results obtained by F.-H. Vasilescu for a single operator: residual single valued extension, analytic residuum, the problem of local spectra etc.

Most of the proofs are adaptations of the ones from [58] with minor changes. We shall regularly use the equality $\mathsf{B}(U,X)=\mathsf{C}^\infty(U,X)$ [81].

3.5.1. Definition. Let $a=(a_1,a_2,...,a_n)\subset B(X)$ be a commuting operators system and $S_a\subset \mathbf{C}^n$ a compact minimal set having the property that $H^{n-1}(\mathsf{C}^\infty(G,X),\alpha\oplus\bar\partial)=0$ for any open $G\subset\mathbf{C}^n$ with $G\cap S_a=\varnothing$ (minimal means that S_a is the intersection of all compact sets having the specified property). We shall denote by $d(a,x)$ the reunion of all open sets $V\subset\mathbf{C}^n$ with the property that there exists a form $\psi\in\Lambda^{n-1}[\sigma\cup d\bar z,\mathsf{C}^\infty(V,X)]$ satisfying the equality $sx=(\alpha\oplus\bar\partial)\psi$ meaning

$$xs_1\wedge s_2\wedge\ldots\wedge s_n=\left((z_1-a_1)s_1+\ldots+(z_n-a_n)s_n+\frac{\partial}{\partial\bar z_1}d\bar z_1+\ldots\right)\wedge\psi(z)$$

(we remind that there exist sets V with this property, for example the solving set $r(a,X)$). We shall also denote by

$$g(a,x)=\mathbf{C}^n\setminus d(a,x),$$
$$r(a,x)=d(a,x)\cap(C^n\setminus S_a)$$
$$sp(a,x)=\mathbf{C}^n\setminus r(a,x)=g(a,x)\cup S_a.$$

The set $r(a,x)$ will be said to be the *solvent set* of x related to a, $sp(a,x)$ will be said to be the *spectrum* of x related to a and S_a will be called the *spectral residuum* of a.

We shall call *analytic solvent set* of x related to a and we shall denote by $\rho(a,x)$ the set

$$\rho(a,x) = \delta(a,x) \cap (\mathbf{C}^n \setminus S_a)$$

where $\delta(a,x)$ is the set of $z \in \mathbf{C}^n$ for which there exists an open neighbourhood V of z and n analytic function on V taking values in X, $f_1, f_2, \ldots, f_n$ satisfying the identity

$$x = (\zeta_1 - a_1) f_1(\zeta) + \ldots + (\zeta_n - a_n) f_n(\zeta),\ \zeta \in V.$$

We shall understand trough the *analytic spectrum* of x related to a the set

$$\sigma(a,x) = \mathbf{C}^n \setminus \rho(a,x) = \gamma(a,x) \cup S_a$$

where

$$\gamma(a,x) = \mathbf{C}^n \setminus \delta(a,x).$$

We shall prove that for an operators system that admit a spectral S-capacity we have $g(a,x) = \gamma(a,x)$, $d(a,x) = \delta(a,x)$, $\rho(a,x) = r(a,x)$, $p(a,x) = \sigma(a,x)$.

3.5.2. Proposition. *For a commuting operators system* $a = (a_1, a_2, \ldots, a_n) \subset B(X)$ *we have:*

1° $x = 0$ *implies* $g(a,x) = \varnothing$, $sp(a,x) = S_a$*;*

2° $g(a, x+y) \subset g(a,x) \cup g(a,y)$, $sp(a,x+y) \subset sp(a,x) \cup sp(a,y)$, $(\forall) x, y \in X$*;*

3° $g(a, by) \subset g(a,x)$, $sp(a,by) \subset sp(a,x)$ *if* $ba_i = a_i b$, $b \in B(X)$, $x \in X$*;*

4° $g(a,y) \subset sp(a,y) \subset \sigma(a,Y)$

where Y *is a (linear, closed) subspace of* X *invariant to all* a_i *and* $y \in Y$*.*

Proof. 1° follows from the fact that for $x = 0$ and any neighbourhood $V \subset \mathbf{C}^n$, the form $\psi = 0 \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^\infty(V,X)]$ verifies the relation $sx = (\alpha \oplus \bar{\partial})\psi$ meaning

$$xs_1 \wedge s_2 \wedge \ldots \wedge s_n = \Big((z_1 - a_1)s_1 + \ldots + (z_n - a_n)s_n + \\ + \frac{\partial}{\partial \bar{z}_1} d\bar{z}_1 + \ldots + \frac{\partial}{\partial \bar{z}_2} d\bar{z}_2 \Big) \wedge \psi(z)$$

Let $z \in d(a,x) \cap d(a,y)$ and $z \in V_i$ such that there exist the forms $\psi_i \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^\infty(V_i, X)]$ ($i = 1,2$) verifying the equalities

$$sx = (\alpha \oplus \bar{\partial})\psi_1,\ sy = (\alpha \oplus \bar{\partial})\psi_2.$$

Then the form $\psi_1 + \psi_2 \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^\infty(V_1 \cup V_2, X)]$ verifies the equality

$$s(x+y) = (\alpha \oplus \bar{\partial})(\psi_1 + \psi_2),$$

hence 2° is verified. The inclusions from 3° result from the fact that by considering the form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^\infty(V,X)]$ such that

$$sx = (\alpha \oplus \bar{\partial})\psi$$

and by applying operator b to the coefficients of ψ, its commuting with each a_i ($i = 1,2,\dots,n$) implies

$$(\alpha \oplus \bar{\partial})b\psi = b(\alpha \oplus \bar{\partial})\psi = bxs$$

(admitting the equality on components). The last inclusion, 4°, follows by the remark that on the solvent set $r(a,Y)$ it is satisfied the equality $xs = (\alpha \oplus \bar{\partial})\psi$.

Remark. If $n = 1$, the spectral residuum S_a defined above coincides with S_T defined by F.-H. Vasilescu in [76] ($a = T$). Indeed, in this case

$$H^0(C^\infty(G,X)\alpha \oplus \partial) = 0$$

for any open $G \subset \mathbf{C}$ such that $G \cap S_a = \varnothing$; if for $f \in C^\infty(G,X)$ we have

$$[(\alpha \oplus \bar{\partial})f](z) = (z-a)f(z)s + \frac{\partial f(z)}{\partial \bar{z}} \mathrm{d}\bar{z}$$

then $(\alpha \oplus \bar{\partial})f = 0$ means $(z-a)f(z) = 0$ and $\frac{\partial f(z)}{\partial \bar{z}} = 0$. The operator $\alpha \oplus \bar{\partial}$ has a null nucleus on $C^\infty(G,X)$ if and only if the only analytic function f on G verifying $(z-a)f(z) = 0$ is the identical null function.

We shall further use a lemma that we proved in [58].

3.5.3. LEMMA. *Let V_1, V_2 two open sets in $\mathbf{C}^n$ such that $V_1 \cap V_2 \neq \varnothing$. Then for any $f \in C^\infty(V_1 \cap V_2, X)$ there exist $f_i \in C^\infty(V_i, X)$ ($i = 1,2$) such that $f = f_1 - f_2$ on $V_1 \cap V_2$.*

3.5.4. LEMMA. *Let $a = (a_1, a_2, \dots, a_n) \subset B(X)$ be an operators system with the spectral residuum S_a and V_i ($i = 1,2$) two open sets in $\mathbf{C}^n \setminus S_a$ such that there exists the forms $\psi_i \in \Lambda^{n-1}[\sigma \cup \mathrm{d}\bar{z}, C^\infty(V_i, X)]$ with the property that $sx = (\alpha \oplus \bar{\partial})\psi_i$ on V_i. Then there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup \mathrm{d}\bar{z}, C^\infty(V_1 \cup V_2, X)]$ having the following property $sx = (\alpha \oplus \bar{\partial})\psi$ on $V_1 \cup V_2$.*

Proof. When $V_1 \cap V_2 = \varnothing$ we can consider $\psi(z) = \psi_i(z)$ for $z \in V_i$ ($i = 1,2$) and we have $sx = (\alpha \oplus \bar{\partial})\psi$ on $V_1 \cup V_2$. If $V_1 \cap V_2 \neq \varnothing$ we have

$$(\alpha \oplus \bar{\partial})(\psi_2 - \psi_1) = 0 \text{ on } V_1 \cap V_2.$$

Since $V_1 \cap V_2 = G \subset \mathbf{C}^n \setminus S_a$, it results that there exists a form $\varphi \in \Lambda^{n-2}[\sigma \cup \mathrm{d}\bar{z}, C^\infty(V_1 \cap V_2, X)]$ such that

$$\psi_2 - \psi_1 = (\alpha \oplus \bar{\partial})\varphi.$$

Indeed, the nucleus of the co-frontier operator $\alpha \oplus \bar{\partial}$,

$$\operatorname{Ker}\left(\alpha \oplus \bar{\partial} : \Lambda^{n-1}\left[\sigma \cup d\bar{z}, C^{\infty}(V_1 \cap V_2, X)\right] \to \Lambda^{n}\left[\sigma \cup d\bar{z}, C^{\infty}(V_1 \cap V_2, X)\right]\right) =$$
$$= \operatorname{Im}\left(\alpha \oplus \bar{\partial} : \Lambda^{n-2}\left[\sigma \cup d\bar{z}, C^{\infty}(V_1 \cap V_2, X)\right] \to \Lambda^{n-1}\left[\sigma \cup d\bar{z}, C^{\infty}(V_1 \cap V_2, X)\right]\right)$$

By applying the preceding lemma to the coefficients of φ there follows

$$\varphi = \varphi_1 - \varphi_2 \text{ where } \varphi_i \in \Lambda^{n-2}\left[\sigma \cup d\bar{z}, C^{\infty}(V_i, X)\right] \ (i = 1,2).$$

Consequently

$$(\alpha \oplus \bar{\partial})(\varphi_1 - \varphi_2) = (\alpha \oplus \bar{\partial})\varphi = \varphi_2 - \varphi_1$$

whence $\psi_1 + (\alpha \oplus \bar{\partial})\psi_1 = \psi_2 + (\alpha \oplus \bar{\partial})\psi_2$ on $V_1 \cap V_2$. By putting $\psi_i' = \psi_i + (\alpha \oplus \bar{\partial})\varphi_i$ ($i = 1,2$) we shall obtain $sx = (\alpha \oplus \bar{\partial})\psi_i'$ on V_i ($i = 1,2$) and $\psi_1' = \psi_2'$ on $V_1 \cap V_2$. Hence by defining $\psi(z) = \psi_i'(z)$ for $z \in V_i$ ($i = 1,2$) one obtains a form as the one required in the text of the lemma. The lemma is proved.

3.5.5. Corollary. *Let $\{V_i\}_{i=1}^{m}$ be a finite family of open sets from $\mathbf{C}^n \setminus S_a$ such that the equation $sx = (\alpha \oplus \bar{\partial})\psi$ has a solution ψ on each of them. If $K \subset r(a,x)$ is a compact set, there exists an open neighbourhood V of K ($V \subset r(a,x)$) on which the equation $sx = (\alpha \oplus \bar{\partial})\psi$ has a solution.*

Proof. Let $\{K_\nu\}_{\nu=1}^{\infty}$ be a growing sequence of compact sets such that $r(a,x) = \bigcup_{\nu=1}^{\infty} K_\nu$. We shall prove that there exists a corresponding sequence of forms $\psi_\nu \in \Lambda^{n-1}\left[\sigma \cup d\bar{z}, C^{\infty}(r(a,x), X)\right]$ that verify the equality $sx = (\alpha \oplus \bar{\partial})\psi_\nu$ on a neighbourhood of K_ν. Then $\psi = \lim_{\nu \to \infty} \psi_\nu$ exists and it is a global solution. We shall start with K_1. By corollary 3.5.5. there exists a form ψ_1 defined in an open neighbourhood of K_1 and satisfying the equality $sx = (\alpha \oplus \bar{\partial})\psi_1^*$ on this neighbourhood. Since the space $C^{\infty}(r(a,x), X)$ is invariant to multiplication with scalar functions of a C^{∞} class ([71], 2.16.1.) we can assume, without limiting the generality, that ψ_1^* is defined on $r(a,x)$; indeed, by multiplying ψ_1^* with a suitable scalar function, the new form can be extended to $r(a,x)$ and we will obtain a form ψ_1 on $r(a,x)$ verifying the equality $sxs = (\alpha \oplus \bar{\partial})\psi_1$ on a neighbourhood of K_1. We will now suppose that the forms $\psi_1, \psi_2, \ldots, \psi_i$ from the desired sequences were already determined and let us determine ψ_{i+1}. According to the preceding corollary there exists a neighbourhood V_{i+1} of the set K_{i+1} and a form ψ_{i+1} defined on this neighbourhood satisfying the equality $sx = (\alpha \oplus \bar{\partial})\psi_{i+1}^*$, and we are allowed to suppose moreover that ψ_{i+1}^* is defined on the whole $r(a,x)$. But

$sx = (\alpha \oplus \bar{\partial})\psi_i$ on a vicinity V_i of K_i, hence by subtraction we obtain $(\alpha \oplus \bar{\partial})(\psi^*_{i+1} - \psi_i) = 0$ on $V_i \cap V_{i+1}$; since $V_i \cap V_{i+1} \subset \mathbf{C}^n \setminus S_a$, it will result that there exists a form φ' such that $\psi^*_{i+1} - \psi_i = (\alpha \oplus \bar{\partial})\varphi'$ on $V_i \cap V_{i+1}$, and we may suppose that φ' is defined on $r(a,x)$. We will put $\psi_{i+1} = \psi^*_{i+1} - (\alpha \oplus \bar{\partial})\varphi'$ and obtain a form defined on $r(a,x)$ equal with ψ_i on $V_i \cap V_{i+1}$ and satisfying the equality $sx = (\alpha \oplus \bar{\partial})\varphi_{i+1}$ on the neighbourhood V_{i+1} of K_{i+1}. By this the demonstration ends.

3.5.7. *Remark.* A local version of the Cauchy-Weil formula (1.2.4., [58]) can be establish on the same way as in [58] formula 1.5.1. Let $a = (a_1, a_2, ..., a_n) \subset B(X)$ be a commuting operators system with the spectral residuum S_a and U an open neighbourhood of $sp(a,x)$; obviously $U \supset S_a$. We shall prove that there exists a form $\chi \in \Lambda^n[\sigma \cup d\bar{z}, C_0^\infty(\mathbf{C}^n, X)]$ in the same co-homology class related to $\alpha \oplus \bar{\partial}$ as sx and such that $\text{suport}(\chi) \subset U$. According to theorem 3.5.6. there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^\infty(r(a,x), X)]$ such that $sx = (\alpha \oplus \bar{\partial})\psi$. Let U_1 and U_2 two open neighbourhoods relatively compact of $sp(a,x)$, such that

$$sp(a,x) \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U} \subset U$$

and let us consider scalar C^∞-function h on $\mathbf{C}^\infty$, $h = 1$ outside U_2 and $h = 0$ on U_1. By using h let us define the form $\tilde{\psi}$ by $\tilde{\psi} = h\psi$ on $r(a,x)$ and $\tilde{\psi} = 0$ on U_1. This form has the coefficients in $C^\infty(\mathbf{C}^n, X)$ and satisfies the condition $sx = (\alpha \oplus \bar{\partial})\tilde{\psi}$ outside the relatively compact set U_2. Hence by setting $\chi = sx - (\alpha \oplus \bar{\partial})\tilde{\psi}$ wi will obtain a form defined on $\mathbf{C}^n$ with $\text{suport}(\chi) \subset \overline{U}_2 \subset U$, that is precisely the form having the specified properties. Considering formula 1.2.4. [58] and using form χ above we can write

$$x = \frac{1}{(2\pi i)^n} \int_U (-1)^n \pi_\chi \wedge dz_1 \wedge \ldots \wedge dz_n$$

which will yield the local version of Cauchy-Weil formula.

3.5.8. Proposition. *Let $a = (a_1, a_2, ..., a_n) \subset B(X)$ be a S-decomposable system, let D be an open polidisk with $D \cap S = \varnothing$, let p be an integer, $0 \le p \le n-1$ and let $\psi \in \Lambda^p[\sigma, U(D, X)]$ such that $\sigma\psi = 0$ where α is defined by*

$$(\alpha\psi)(z) = ((z_1 - a_1)s_1 + (z_2 - a_2)s_2 + \ldots + (z_n - a_n)s_n) \wedge \psi(z).$$

Then for any polidisk $D' \subset D$ with $\overline{D}' \subset D$ there exists a form $\psi \in \Lambda^{p-1}[\sigma, U(D', X)]$ such that $\psi = \alpha\varphi$ on D'.

The proof of proposition 2.1.3. presented in [58] is also true in this case, with a single comment, that D isn't any more any polidisk of $\mathbf{C}^n$, but a polidisk that doesn't cross S.

3.5.9. THEOREM. *If* $a=(a_1,a_2,...,a_n)\subset B(X)$ *is S-decomposable then* $S\supset S_a$.

Proof. With minor differences, the proof is identical with the one for the decomposable systems ($S=\varnothing$) ([58], proposition 2.1.4.) where $S_a=\varnothing$ is called property (L). It will have to show that for any polidisk $U\subset\mathbf{C}^n$ such that $U\cap S=\varnothing$ we have $H^i(\mathrm{U}(V,X),\alpha)=0$ $(0\le i\le n-1)$. We note that $H^i(\mathrm{U}(U,X),\alpha)=0$ implies $H^i(C^\infty(G,X),\alpha\oplus\bar\partial)=0$ $(0\le i\le n-1)$ where U is any open polidisk from $\mathbf{C}^n$, G is any open set $G\subset\mathbf{C}^n$ such that $U\cap S=\varnothing$, $G\cap S=\varnothing$; the proof is given in [58], theorem 1.5.16. for any U, $G\subset\mathbf{C}^n$.

One motivates this through induction on I, beginning with $i=0$. Let $f\in\mathrm{U}(U,X)$ such that $\alpha f=0$; according to the preceding proposition we shall have $f=0$ on any polidisk D' with $D'\subset U$ and $f=0$ on U. Suppose that for any open polidisk $D\subset\mathbf{C}^n$ with $D\cap S=\varnothing$ we have $H^{i-1}(\mathrm{U}(D,X),\alpha)=0$ with i fixed, $0\le i\le n-1$ and let us prove that $H^i(\mathrm{U}(U,X),\alpha)=0$.

Let $\{D_\nu\}$ be a sequence of polidisks, $D_\nu\cap S=\varnothing$, such that $D_\nu\subset\overline{D}_{\nu+1}$ for any ν with $\bigcup_{\nu=1}^{\infty}D_\nu=U$ and $\psi\in\Lambda^i[\sigma,\mathrm{U}(U,X)]$ such that $\alpha\psi=0$. By applying the preceding proposition for D_2, we infer that there exists a form $\varphi_1\in\Lambda^{i-1}[\sigma,\mathrm{U}(D_2,X)]$ such that $\psi=\alpha\varphi_1$ on D_2; analogously we can find a form φ_2' on D_3 with $\psi=\alpha\varphi_2'$ on D_3. One obtains $\alpha(\varphi_1-\varphi_2')=0$ on D_2 whence, by applying the inductive hypothesis, we infer that there exists a form $\chi\in\Lambda^{i-2}[\sigma,\mathrm{U}(D_2,X)]$, such that $\varphi_1-\varphi_2'=\alpha\psi$. We shall retain from the Taylor's decomposition of χ on D_2 a sufficient number of terms, such that χ' (the retained part) verifies $\|\alpha\chi-\alpha\chi'\|<\frac{1}{2}$ on $\overline{D}_1$. Thinking analogously, we can define a sequence of forms φ_ν, $\varphi_\nu\in\Lambda^{i-1}[\sigma,\mathrm{U}(D_{\nu+1},X)]$ enjoying the properties: $\psi=\alpha\varphi_\nu$ on $D_{\nu+1}$ and $\|\varphi_{\nu+1}-\varphi_\nu\|<\frac{1}{2^{\nu+1}}$ on D. The sequence φ_ν obviously converges to a form having the analytic coefficients on U and satisfying $\psi=\alpha\varphi$ on U, q.e.d.

In 3.2. we proved the uniqueness of the spectral S-capacities for S-decomposable operators systems. We shall now prove this on other ways, emphasising the connection between the spectral *S*-capacity related to an operator and certain linear subspaces, described using the local spectrum, which is most useful.

Let a be a commuting system of operators on the space *X*, $a \subset B(X)$, with the spectral residuum S_a. If H is an arbitrary set from $\mathbf{C}^n$ such that $H \supset S_a$, we shall put $X_{[a]}(H)=\{x, x \in X, sp(a,x) \subset H\}$ and $X_a(H)=\{x, x \in X, \sigma(a,x) \subset H\}$; $X_{[a]}(H)$ and $X_a(H)$ are linear subspaces of *X* and $X_a(H) \subset X_{[a]}(H)$.

3.5.10. Theorem. *If* $a=(a_1,a_2,\ldots,a_n)$ *is S-decomposable then* $\mathrm{E}(F)=X_{[a]}(F)$ *for any closed set* $F \supset S$.

Proof. According to theorem 3.5.9., $S \supset S_a$, hence $F \supset S_a$ and $X_{[a]}(F)$ has sense. The inclusion $\mathrm{E}(F) \subset X_{[a]}(F)$ follows by the fact that $sp(a,x) \subset \sigma(a,\mathrm{E}(F))$ (proposition 3.5.2.). One proves the inverse inclusion exactly as in [58] theorem 2.2.1. with the only remark that *F* is no longer arbitrary, but $F \supset S$.

3.5.11. Corollary. *Let a be a S-decomposable system. Then for any closed* $F \supset S$, *the subspace* $X_{[a]}(F)$ *is spectral maximal space of a; more precisely, for any subspace Z invariant to a such that* $\sigma(a,Z) \subset F$, *we have* $Z \subset X_{[a]}(F)$; *moreover* $\sigma(a, X_{[a]}(F)) \subset F$.

Proof. The inclusion $\sigma(a, X_{[a]}(F)) \subset F$ follows by the preceding theorem, since $\sigma(a,\mathrm{E}(F)) \subset F$. If Z is invariant to a with $\sigma(a,Z) \subset F$ then any $z \in \mathbf{Z}$, $sp(a,z) \subset \sigma(a,Z) \subset F$ hence $z \in X_{[a]}(F)$, meaning $Z \subset X_{[a]}(F)$.

3.5.12. Proposition. *If a is S-decomposable then for any* $x \in X$, *we have* $sp(a,x)=\sigma(a,x)$.

Proof. Let us prove first that $sp(a,x) \subset \sigma(a,x)$ or equivalent with this $\sigma(a,x) \subset r(a,x)$. Let $z \in \delta(a,x)$ and according to definition 1 let us consider an open neighbourhood V of z and n analytic functions defined on V taking values in *X*, $f_1, f_2, \ldots, f_n$ that verify the equality $x \equiv (\zeta_1 - a_1)f_1(\zeta)+\ldots+(\zeta_n - a_n)f_n(\zeta)$, $\zeta \in V$. We consider the $n-1$ degree form defined on *V*, $\psi(\zeta)=\sum_{i=1}^{n}(-1)^{i-1} f_i(\zeta)s_1 \wedge s_2 \wedge \ldots \wedge s_i \wedge \ldots \ldots \wedge s_n$. This form can be considered as an element of $\Lambda^{n-1}[\sigma \cup d\bar{z}, C^{\infty}(V,X)]$ and it easily verifies the equality $sx=(\alpha \oplus \bar{\partial})\psi$ on V taking into account the analyticity of the functions f_i ($\delta f_i = 0$); hence it results that $V \subset d(a,x)$, that is $\delta(a,x) \subset d(a,x)$ or $g(a,x) \subset \gamma(a,x)$ whence $g(a,x) \cup S_a = sp(a,x) \subset \gamma(a,x) \cup S_a = \sigma(a,x)$. For the inverse inclusion $\sigma(a,x) \subset sp(a,x)$, let $z \in r(a,x)$ and let D be an open polidisk with its centre in

z such that $D \subset r(a,x)$. Since $x \in X_{[a]}(sp(a,x) = \mathcal{E}(sp(a,x)))$ hence by theorem 1.1.3. [58] there exist the analytic functions $f_1, f_2, ..., f_n$ defined on D and taking values in X, such that $x = \sum_{i=1}^{n} (\zeta_i - a_i) f_i(\zeta)$, $\zeta \in D$. That means that $z \in \rho(a,x)$, hence $r(a,x) \subset \rho(a,x)$ whence $\sigma(a,x) \subset sp(a,x)$.

3.5.13. COROLLARY. *If a is a S-decomposable system then for any* $H \subset \mathbf{C}^n$ *with* $H \supset S$ *we have* $X_{[a]}(H) = X_a(H)$.

Proof. It easily follows by the preceding proposition.

3.5.14. PROPOSITION. *If a is an arbitrary system of operators, then* $\sigma(a,X) = \bigcup_{x \in X} sp(a,x)$.

Proof. The inclusion $\bigcup_{x \in X} sp(a,x) \subset \sigma(a,X)$ results from proposition 3.5.2., $sp(a,x) \subset \sigma(a,X)$. Conversely, if $z \in \bigcap_{x \in X} r(a,x)$, then $H^* \times (\mathcal{U}(z,X), \alpha) = 0$; since $z \in S_a$, there exists an open polidisk D, with $D \cap S_a = \varnothing$ and for which, according to theorem 3.5.6., we have $H^i(\mathcal{U}(D,X), \alpha) = 0$ $(i = 0,1,...,n-1)$. Then by corollary 1.4.3. [58] there follows that $z \in r(a,x)$, hence $\bigcup_{x \in X} sp(a,x) \supset \sigma(a,X)$.

3.5.15. DEFINITION. The support of the spectral S-capacity is the set-support $\mathcal{E} = \bigcap \{F;\ F \text{ closed}, \mathcal{E}(F) = X\}$.

3.5.16. PROPOSITION. *If a is a S-decomposable system and* $\mathcal{E}$ *is its spectral S-capacity then* suport$\mathcal{E} = \sigma(a,X)$.

Proof. The inclusion $\sigma(a,X) \subset$ suport$\mathcal{E}$ results from the fact that for any closed F such that $E(F) = X$, we have $\sigma(a,x) = \sigma(a, \mathcal{E}(F)) \subset F$, whence $\sigma(a,x) \subset \subset \bigcap \{F, F \text{ closed}, \mathcal{E}(\mathrm{F}) = X\} =$ suport$\mathcal{E}$. For the inverse inclusion let $z_0 \in r(a,X)$ and let us prove that $z_0 \in$ suport$\mathcal{E}$. Let V be an open neighbourhood of z_0 such that $\overline{V} \subset r(a,X)$ and let F be a closed set such that $z_0 \notin F$, $F \supset S$ and $X = \mathcal{E}(F) + \mathcal{E}(\overline{V})$; this is possible because $z_0 \in S$ $(S \subset \sigma(a,X))$. Let $x \in \mathcal{E}(\overline{V})$; since $V \subset r(a,X)$ it results that $sx = (\alpha \oplus \bar{\partial})\psi$ in a neighbourhood $(\mathbf{C}^n \setminus V)$ of the spectrum $sp(a,X)$, hence by applying formula 1.2.4. [58] we infer $x = 0$, hence $\mathcal{E}(\overline{V}) = \{0\}$, that is $\mathcal{E}(F) = X$; hence from $z_0 \in F$ it follows $z_0 \in$ suport$\mathcal{E}$, just what there was to prove.

3.5.17. COROLLARY. *If a is a S-decomposable system then for any closed set F we have* $\sigma(a, \mathcal{E}(F)) \subset \sigma(a,X)$.

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