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VIA ABSTRACT COGALOIS THEORY

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TOMA ALBU

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TOMA ALBU[1]

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1 Koç University, Department of Mathematics, Rumeli Feneri Yolu, 34450 Sariyer – Istanbul, Turkey and Institute of Mathematics "Simion Stoilow" of the Romanian Academy P.O. Box 1-764 RO – 014700 Bucharest 1, Romania
e-mail: talbu@ku.edu.tr , Toma.Albu@imar.ro

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TOMA ALBU

Koç University, Department of Mathematics

Rumeli Feneri Yolu, 34450 Sariyer - Istanbul, TURKEY

and

Institute of Mathematics "Simion Stoilow" of the Romanian Academy

P.O. Box 1 - 764
RO - 014700 Bucharest 1, ROMANIA

e-mail: talbu@ku.edu.tr, Toma.Albu@imar.ro

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Abstract

An Abstract Cogalois Theory for arbitrary profinite groups has been initiated by T. Albu and Ş.A. Basarab (*J. Pure. Appl. Algebra*, 2005, to appear). The aim of this paper is two-fold: firstly, to present the abstract group theoretic versions of various types of Kummer field extensions, and secondly, to show how some basic results of the field theoretic Cogalois Theory, like the Kneser Criterion, the General Purity Criterion, etc., can be very easily deduced form their abstract versions.

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Introduction

The *Class Field Theory* of Abelian field extensions of global and local fields can be also performed for arbitrary profinite groups, and therefore, an *Abstract Galois Theory* for such profinite groups was developed within the *Abstract Class Field Theory* (see e.g., [8]).

A dual theory to the *Abstract Galois Theory* that also applies to arbitrary profinite groups, called *Abstract Cogalois Theory*, was initiated in [4]. Roughly speaking, *Cogalois Theory* (see [3]) investigates field extensions, finite or not, which possess a Cogalois correspondence. This theory is somewhat dual to the very classical *Galois Theory* dealing with field extensions possessing a Galois correspondence.

The basic concepts of Cogalois Theory, namely that of *G-Kneser* and *G-Cogalois field extension*, as well as their main properties have been generalized in [4] to arbitrary profinite groups. More precisely, let Γ be an arbitrary profinite group, and let A be any subgroup of the Abelian group $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on the discrete group A. The concepts of *Kneser* subgroup and *Cogalois* subgroup of the group $Z^1(\Gamma, A)$ of all continuous 1-cocycles of Γ with coefficients in A have been defined and their main properties have been established in [4]. Their proofs, involving cohomological as well as topological tools, are completely different from that of their field theoretic correspondents.

The aim of this paper is two-fold: firstly, to present the abstract versions for arbitrary profinite groups of various types of *Kummer field extensions*, and secondly to show how some basic results of the field theoretic Cogalois Theory, like the *Kneser Criterion*, the *General Purity Criterion*, the uniqueness of the Kneser group of a G-Cogalois field extension, etc., can be very easily deduced form their abstract versions.

0 Notation and Preliminaries

Throughout this and the next two sections of the paper Γ will always denote a fixed profinite group, and A will be a fixed subgroup of the Abelian group $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on A endowed with the discrete topology, i.e., A is a discrete Γ-module.

For any topological group T we denote by $\mathbb{L}(T)$ the lattice of all subgroups of T, and by $\overline{\mathbb{L}}(T)$ the lattice of all closed subgroups of T. The notation $U \leqslant T$ means that U is a subgroup of T. For any $U \leqslant T$ we denote by $\mathbb{L}(T\,|\,U)$ (resp. $\overline{\mathbb{L}}(T\,|\,U)$) the lattice of all subgroups (resp. closed subgroups) of T lying over U. If $X \subseteq T$, then $\langle X \rangle$ will denote the subgroup generated by X.

An additive Abelian group D is said to be a *group of bounded order* if there exists a positive integer m such that $mD = \{0\}$, and the least such m is called the *exponent* of D and is denoted by $\exp(D)$.

As usually, we denote by $Z^1(\Gamma, A)$ the torsion Abelian group of all *continuous 1-cocycles* of Γ with coefficients in A and by $B^1(\Gamma, A)$ its subgroup consisted of all 1-*coboundaries*.

The *evaluation map*

$$\langle -,- \rangle : \Gamma \times Z^1(\Gamma, A) \longrightarrow A, \ \ \langle \sigma, h \rangle = h(\sigma),$$

defines for any $\Delta \leqslant \Gamma$, $G \leqslant Z^1(\Gamma, A)$, and $g \in Z^1(\Gamma, A)$ the following sets

$$\Delta^{\perp} := \{\, h \in Z^1(\Gamma, A) \,|\, \langle \sigma, h \rangle = 0, \ \forall\, \sigma \in \Delta \,\},$$

$$G^{\perp} := \{\, \sigma \in \Gamma \,|\, \langle \sigma, h \rangle = 0,\ \forall\, h \in G \,\},$$

$$g^{\perp} := \{\, \sigma \in \Gamma \,|\, \langle \sigma, g \rangle = 0 \,\}.$$

Then $\Delta^{\perp} \leqslant Z^1(\Gamma, A)$, $g^{\perp} = \langle\, g \,\rangle^{\perp}$, $G^{\perp} \in \overline{\mathbb{L}}(\Gamma)$, and the maps

$$\mathbb{L}(Z^1(\Gamma, A)) \longrightarrow \overline{\mathbb{L}}(\Gamma),\ G \mapsto G^{\perp},$$

$$\overline{\mathbb{L}}(\Gamma) \longrightarrow \mathbb{L}(Z^1(\Gamma, A)),\ \Delta \mapsto \Delta^{\perp},$$

establish a *Galois connection* between the lattices $\mathbb{L}(Z^1(\Gamma, A))$ and $\overline{\mathbb{L}}(\Gamma)$, i.e., they are order-reversing maps and $X \leqslant X^{\perp\perp}$ for any element X of $\mathbb{L}(Z^1(\Gamma, A))$ or $\overline{\mathbb{L}}(\Gamma)$ (see [4, Proposition 0.1 (1)]).

The following notation from [4], with D an additive Abelian torsion group, will be used throughout this paper.

- $\mathbb{N}$ denotes the set $\{1,\, 2,\, \dots\}$ of all positive integers;
- $D[n] := \{\, x \in D \,|\, nx = 0 \,\}$ for any $n \in \mathbb{N}$;
- $\mathcal{O}_D := \{\, m \in \mathbb{N} \,|\, \exists\, x \in D$ of order $m \,\}$;
- $\mathbb{P}$ denotes the set of all positive prime numbers;
- $\mathcal{P} := (\,\mathbb{P} \setminus \{2\}) \cup \{4\}$;
- $\mathcal{P}_n := \{\, p \in \mathcal{P} \,|\, p \,|\, n \,\}$ for any $n \in \mathbb{N}$;
- $\mathcal{P}_D := \mathcal{O}_D \cap \mathcal{P}$;
- $\widehat{r}$ denotes for any $r \in \mathbb{Q}$ its coset in the quotient group $\mathbb{Q}/\mathbb{Z}$;
- $\mathcal{P}(\Gamma, A) := \{\, p \in \mathcal{P} \,|\, \widehat{1/p} \in A \setminus A^{\Gamma} \,\}$;
- $\varepsilon_n \in B^1(\Gamma, A)$ denotes for any $n \in \mathbb{N}$ with $\widehat{1/n} \in A$ the coboundary

  $$\varepsilon_n(\sigma) := \sigma\, \widehat{1/n} - \widehat{1/n},\ \sigma \in \Gamma,$$

  associated with $\widehat{1/n}$;
- If $\widehat{1/4} \in A \setminus A^{\Gamma}$, then the map $\varepsilon_4' \in Z^1(\Gamma, A)$ is defined by

  $$\varepsilon_4'(\sigma) := \begin{cases} \widehat{1/4} & \text{if } \ \sigma\, \widehat{1/4} = -\widehat{1/4}, \\ \widehat{0} & \text{if } \ \sigma\, \widehat{1/4} = \widehat{1/4}. \end{cases}$$

1 Some basic facts of Abstract Cogalois Theory

An Abstract Cogalois Theory for arbitrary profinite groups has been developed in [4]. We present below some of its basic concepts and results needed in the sequel.

As above, Γ denotes a fixed profinite group and A is a fixed discrete subgroup of the Abelian group $\mathbb{Q}/\mathbb{Z}$ such that Γ acts continuously on A.

The next two definitions from [4] are the abstract versions of the concepts of *Kneser* and *Cogalois field extensions.*

Definition 1.1. *A finite subgroup K of $Z^1(\Gamma, A)$ is said to a* Kneser group of $Z^1(\Gamma, A)$ *if $(\Gamma : K^{\perp}) = |K|$. An arbitrary subgroup of $Z^1(\Gamma, A)$ is said to a* Kneser group of $Z^1(\Gamma, A)$ *if any of its finite subgroups is a Kneser group of $Z^1(\Gamma, A)$.* □

Definition 1.2. *A subgroup G of $Z^1(\Gamma, A)$ is said to be a* Cogalois group of $Z^1(\Gamma, A)$ *if it is a Kneser group of $Z^1(\Gamma, A)$ and the maps*

$$(-)^{\perp} : \mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^{\perp}) \text{ and } G \cap (-)^{\perp} : \overline{\mathbb{L}}(\Gamma|G^{\perp}) \longrightarrow \mathbb{L}(G)$$

are lattice anti-isomorphisms, inverse to one another. □

The next result is the abstract version of the field theoretic *Kneser Criterion* [7].

Theorem 1.3. (THE ABSTRACT KNESER CRITERION [4, Theorem 1.20]). *The following assertions are equivalent for $G \leqslant Z^1(\Gamma, A)$.*

(1) *G is a Kneser group of $Z^1(\Gamma, A)$.*

(2) *$\varepsilon_p \notin G$ whenever $4 \neq p \in \mathcal{P}(\Gamma, A)$ and $\varepsilon'_4 \notin G$ whenever $4 \in \mathcal{P}(\Gamma, A)$.* □

As in [4], a subgroup D of an Abelian group C is said to be *quasi n-pure*, where $n \in \mathbb{N}$ is a given positive integer, if $C[n] \subseteq D$, or equivalently, if $C[n] = D[n]$. For $M \subseteq \mathbb{N}$, C is *quasi M-pure* if C is quasi n-pure for all $n \in M$.

The next result is the abstract version of the field theoretic *General Purity Criterion* [1, Theorem 2.3].

Theorem 1.4. (THE QUASI-PURITY CRITERION [4, Theorem 2.5]). *The following statements are equivalent for a subgroup G of $Z^1(\Gamma, A)$.*

(1) *G is Cogalois.*

(2) *The subgroup A^{Γ} of $A^{G^{\perp}}$ is quasi $\mathcal{P}_G$-pure.*

(3) *$G^{\perp} \nsubseteq \varepsilon_p^{\perp}$ for all $p \in \mathcal{P}_G \cap \mathcal{P}(\Gamma, A)$.* □

Definitions 1.1 and 1.2 above give the abstract correspondents for subgroups of $Z^1(\Gamma, A)$ of Kneser and Cogalois field extension. If we move now via the maps $(-)^{\perp}$ from subgroups of $Z^1(\Gamma, A)$ to subgroups of the given profinite group Γ, then one can define as follows the abstract versions for the later ones of the concepts of radical, simple radical, Kneser, and Cogalois field extension.

Definition 1.5. *A subgroup* Δ *of* Γ *is said to be* G-radical *if* $\Delta = G^{\perp}$ *for some* $G \leqslant Z^1(\Gamma, A)$*. A* radical subgroup *of* Γ *is a subgroup which is* G*-radical for some* $G \leqslant Z^1(\Gamma, A)$*. A subgroup* Δ *of* Γ *is called* simple radical *if there exists* $g \in Z^1(\Gamma, A)$ *such that* $\Delta = g^{\perp}$*.* □

Observe that any radical subgroup of Γ is necessarily closed, and any simple radical subgroups of Γ is open.

Definition 1.6. ([6]). *A subgroup* Δ *of* Γ *is said to be* G-Kneser *if* Δ *is* G*-radical and* G *is a Kneser group of* $Z^1(\Gamma, A)$*.* Δ *is said to be a* Kneser *group if* Δ *is* G*-Kneser for some* $G \leqslant Z^1(\Gamma, A)$*.* □

Definition 1.7. *A subgroup* Δ *of* Γ *is said to be* Cogalois *if there exists a Cogalois group* G *of* $Z^1(\Gamma, A)$ *such that* $\Delta = G^{\perp}$*.* Δ *is said to be* strongly Cogalois *if* $\Delta = \Delta^{\perp\perp}$ *and* $\Delta^{\perp}$ *is a Cogalois group of* $Z^1(\Gamma, A)$*.* □

If Δ is Cogalois, then the Cogalois group G of $Z^1(\Gamma, A)$ for which $\Delta = G^{\perp}$ is uniquely determined by [4, Corollary 2.12], and we say in this case that Δ is *G-Cogalois*.

Lemma 1.8. *The following statements are equivalent for a radical subgroup* Δ *of* Γ*.*

(1) Δ *is strongly Cogalois.*

(2) $\Delta^{\perp}$ *is a Kneser group of* $Z^1(\Gamma, A)$*.*

Proof. (2) $\Longrightarrow$ (1): Since Δ is a radical subgroup of Γ, there exists $G \leqslant Z^1(\Gamma, A)$ such that $\Delta = G^{\perp}$, and so $\Delta^{\perp\perp} = (G^{\perp})^{\perp\perp} = G^{\perp} = \Delta$. If $G := \Delta^{\perp}$ is not a Cogalois group of $Z^1(\Gamma, A)$, then we are going to show that G is not Kneser. As G is not Cogalois, it follows by the Quasi-Purity Criterion (Theorem 1.4) that there exists $p \in \mathcal{P}(\Gamma, A) \cap \mathcal{P}_G$ such that $\Delta = G^{\perp} \leqslant \varepsilon_p^{\perp}$, and hence $\varepsilon_p^{\perp\perp} \leqslant \Delta^{\perp} = G$. Consequently, $\varepsilon_p \in \varepsilon_p^{\perp\perp} \leqslant G$ if $p \neq 4$, and $\varepsilon_4' \in \varepsilon_4^{\perp\perp} \leqslant G$ if $p = 4$. By the Abstract Kneser Criterion (Theorem 1.3) we deduce that G is not Kneser.

(1) $\Longrightarrow$ (2): This is trivial since any Cogalois group of $Z^1(\Gamma, A)$ is also Kneser. □

2 Kummer groups of cocycles

In this section we introduce four types *Kummer groups of cocycles* which are the abstract group theoretic correspondents of the various types of Kummer field extensions studied in Galois Theory and Cogalois Theory (see [3]) and prove that any of them is a Cogalois group of cocycles.

Definitions 2.1. *Let* $G \leqslant Z^1(\Gamma, A)$*, and let* $n \in \mathbb{N}$*.*

(1) G *is said to be a* classical n-Kummer group *if* $nG = \{0\}$ *and* $A[n] \subseteq A^{\Gamma}$*.*

(2) *G is said to be a* generalized n-Kummer group *if* $nG = \{0\}$ *and* $A^{G^\perp}[n] \subseteq A^\Gamma$.

(3) *G is said to be an* n-Kummer group with few cocycles *if* $nG = \{0\}$ *and* $A^{G^\perp}[n] \subseteq A[2]$.

(4) *G is said to be an* n-quasi-Kummer group *if* $nG = \{0\}$ *and* $A[p] \subseteq A^\Gamma$ *for every* $p \in \mathcal{P}_n$.

We say that G is a classical Kummer group *(resp. a* generalized Kummer group, Kummer group with few cocycles, quasi-Kummer group*) if G is a classical m-Kummer group (resp. a generalized m-Kummer group, m-Kummer group with few cocycles, m-quasi-Kummer group) for some $m \in \mathbb{N}$.* □

Observe that $A[2] \subseteq \{\widehat{0}, \widehat{1/2}\} \subseteq A^\Gamma$, and so, any n-Kummer group with few cocycles is a generalized n-Kummer group. Clearly, any classical n-Kummer group is both a generalized n-Kummer group and an n-quasi-Kummer group.

Proposition 2.2. *Any generalized Kummer group and any quasi-Kummer group is Cogalois. In particular, any classical Kummer group and any Kummer group with few cocycles is Cogalois.*

Proof. Let $G \leqslant Z^1(\Gamma, A)$. If G is a generalized Kummer group, then there exists $n \in \mathbb{N}$ such that $nG = \{0\}$ and $A^{G^\perp}[n] \subseteq A^\Gamma$. If $p \in \mathcal{P}_G$, then clearly $p \,|\, n$, and hence $A^{G^\perp}[p] \subseteq A^{G^\perp}[n] \subseteq A^\Gamma$, which shows that the subgroup A^Γ of $A^{G^\perp}$ is quasi $\mathcal{P}_G$-pure. By Theorem 1.4 we deduce that G is a Cogalois group of $Z^1(\Gamma, A)$.

If G is a quasi-Kummer group, then there exists $n \in \mathbb{N}$ such that $nG = \{0\}$ and $A[p] \subseteq A^\Gamma$ for every $p \in \mathcal{P}_n$. Observe that $\mathcal{P}_G \subseteq \mathcal{P}_n$; hence $A^{G^\perp}[p] \subseteq A[p] \subseteq A^\Gamma$ for every $p \in \mathcal{P}_G$, which shows that the subgroup A^Γ of $A^{G^\perp}$ is quasi $\mathcal{P}_G$-pure. Again by Theorem 1.4 we deduce that G is a Cogalois group of $Z^1(\Gamma, A)$. □

Corollary 2.3. *Let $G \leqslant Z^1(\Gamma, A)$ be one of any of the four types of Kummer groups of cocycles introduced in Definition 2.1. Then the maps*

$$(-)^\perp : \mathbb{L}(G) \longrightarrow \overline{\mathbb{L}}(\Gamma|G^\perp) \quad \text{and} \quad G \cap (-)^\perp : \overline{\mathbb{L}}(\Gamma|G^\perp) \longrightarrow \mathbb{L}(G)$$

are lattice anti-isomorphisms, inverse to one another. □

Proof. By Proposition 2.2, G is a Cogalois group of $Z^1(\Gamma, A)$, and so, according to Definition 1.2, we are done. □

Proposition 2.4. *Let $G \leqslant Z^1(\Gamma, A)$ be a Cogalois group of bounded order such that $A[\exp(G)] \subseteq A^{G^\perp}$. Then G is a quasi-Kummer group.*

Proof. Set $n = \exp(G)$, and let $p \in \mathcal{P}_n$. Then $A[p] \subseteq A[n] \subseteq A^{G^\perp}$, and hence $A[p] \subseteq A^{G^\perp}[p]$. Since G is a Cogalois group of $Z^1(\Gamma, A)$, A^Γ is a quasi $\mathcal{P}_G$-pure subgroup of $A^{G^\perp}$ by Theorem 1.4; hence $A^{G^\perp}[p] \subseteq A^\Gamma$ for all $p \in \mathcal{P}_G$. Since $n = \exp(G)$, we have $\mathcal{P}_G := \mathcal{O}_G \cap \mathcal{P} = \mathcal{P}_n$. Consequently, $A[p] \subseteq A^\Gamma$ for every $p \in \mathcal{P}_n$, which shows that G is an n-quasi-Kummer group. □

Proposition 2.5. *Any generalized Kummer group $G \leqslant Z^1(\Gamma, A)$ with $A[\exp(G)] \subseteq A^{G^\perp}$ is a classical Kummer group.*

Proof. Let $n = \exp(G)$. Since $n \mid m$, we have $A[n] \subseteq A^{G^\perp}[n] \subseteq A^{G^\perp}[m] \subseteq A^\Gamma$ by hypotheses. This shows that G is a classical n-Kummer group. □

Remarks 2.6. (1) The result in Proposition 2.4 is the abstract version of the following field theoretic result: *Any Galois n-bounded G-Cogalois extension E/F is an n-quasi-Kummer extension*, cf. [3, Thm. 13.4.3]. The condition $A[n] \subseteq A^{G^\perp}$ in Proposition 2.4 corresponds to the fact that the primitive n-th of unity ζ_n belongs to E, which in turn, is a consequence of the fact that E/F is a Galois extension. It is not clear whether or not we can replace it by another condition, e.g., by the following one: *G is a Γ-submodule of* $Z^1(\Gamma, A)$ (see also Corollary 4.6).

(2) The same question holds for Proposition 2.5, which is the abstract version of [3, Thm. 13.4.4]: *Any Galois generalized Kummer extension is a classical Kummer extension*. □

3 A field theoretic ↔ abstract Cogalois Theory dictionary

In this section we establish a dictionary relating the basic notions of the field theoretic Cogalois Theory to their correspondents in the group theoretic Abstract Cogalois Theory; this will allows us to recover in the next section some main results of the former theory from the later one.

Throughout this section Ω/F denotes a fixed Galois extension with the (profinite) Galois group $\Gamma := \mathrm{Gal}(\Omega/F)$. In particular, we can take as Ω an algebraic separable closure $\widetilde{F}^{\mathrm{sep}}$ of the base field F, in which case Γ is the absolute Galois group of F.

For any nonempty subset $S \subseteq \Omega$ we denote by $\mu(S)$ the set of all roots of unity contained in S, and for $n \in \mathbb{N}$, $\mu_n(S)$ will denote the set of all n-th roots of unity contained in S. If $\Omega = \widetilde{F}^{\mathrm{sep}}$ and the characteristic $\mathrm{Char}(F)$ of F is p, then the multiplicative torsion group $\mu(\Omega)$ is isomorphic in a non-canonical way to the additive group $\mathbb{Q}/\mathbb{Z}$ if $p = 0$, respectively to its subgroup $\bigoplus_{q \in \mathbb{P} \setminus \{p\}} (\mathbb{Q}/\mathbb{Z})(q)$ for $p \neq 0$, where $(\mathbb{Q}/\mathbb{Z})(q)$ denotes the q-primary component of the torsion Abelian group $\mathbb{Q}/\mathbb{Z}$. Thus, in general, the group $A := \mu(\Omega)$ is isomorphic to a uniquely determined subgroup of $\mathbb{Q}/\mathbb{Z}$, and the canonical action of Γ on Ω induces a continuous action of the profinite group Γ on the discrete Abelian torsion group A.

Assigning to the exact sequence of topologically discrete Γ-modules

$$\{1\} \longrightarrow A \longrightarrow \Omega^* \longrightarrow \Omega^*/A \longrightarrow \{1\}$$

the corresponding exact sequence of cohomology groups in low dimensions

$$\{1\} \longrightarrow A^\Gamma \longrightarrow \Omega^{*\Gamma} \longrightarrow (\Omega^*/A)^\Gamma \longrightarrow H^1(\Gamma, A) \longrightarrow H^1(\Gamma, \Omega^*),$$

where $H^1(\Gamma, \Omega^*) = \{1\}$ by the *Hilbert's Theorem 90* (see e.g., [9]), we obtain the canonical epimorphism of Abelian torsion groups

$$\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A),\ x \mapsto (\sigma \in \Gamma \mapsto (\sigma x)\, x^{-1} \in A),$$

whose kernel is F^*, where

$$T(\Omega/F) = \{\, x \in \Omega^* \mid (\sigma x)\, x^{-1} \in A,\ \forall\, \sigma \in \Gamma\,\} = \{\, x \in \Omega^* \mid \exists\, n \in \mathbb{N},\ x^n \in F\,\}.$$

The quotient $T(\Omega/F)/F^*$, which is exactly the torsion subgroup of the quotient group Ω^*/F^*, is called in Cogalois Theory (see [3]) the *Cogalois group* of the field extension Ω/F and is denoted by $\mathrm{Cog}(\Omega/F)$. Thus, the epimorphism ψ induces a canonical isomorphism

$$\varphi : \mathrm{Cog}(\Omega/F) \xrightarrow{\sim} Z^1(\Gamma, A)$$

(see also [2, Corollary 1.2] or [3, Theorem 15.1.2]), which identifies in a canonical way the subgroups $G \leqslant Z^1(\Gamma, A)$ investigated in the frame of Abstract Cogalois Theory with the subgroups $\mathbb{G}/F^* := \varphi^{-1}(G) \leqslant \mathrm{Cog}(\Omega/F)$ investigated in the frame of field theoretic Cogalois Theory. In particular, for every intermediate field E of Ω/F, the restriction of ψ to $T(E/F) = T(\Omega/F) \cap E$ induces an isomorphism from the torsion group $\mathrm{Cog}(E/F) := T(E/F)/F^*$ of E^*/F^* onto the subgroup $\Gamma_E^{\perp}$ of $Z^1(\Gamma, A)$, where $\Gamma_E := \mathrm{Gal}(\Omega/E)$.

The lattice $\mathbb{I}(\Omega/F)$ of all intermediate fields of the extension Ω/F, the lattice $\mathbb{L}(T(\Omega/F)|F^*)$ of all subgroups of $T(\Omega/F)$ lying over F^*, the lattice $\overline{\mathbb{L}}(\Gamma)$ of all closed subgroups of Γ, and the lattice $\mathbb{L}(Z^1(\Gamma, A))$ of all subgroups of $Z^1(\Gamma, A)$ are related as shown in the commutative diagram below

$$\begin{array}{ccc} \mathbb{L}(T(\Omega/F)|F^*) & \rightleftarrows & \mathbb{I}(\Omega/F) \\ \downarrow & & \downarrow \\ \mathbb{L}(Z^1(\Gamma, A)) & \rightleftarrows & \overline{\mathbb{L}}(\Gamma) \end{array}$$

where the left vertical arrow is the lattice isomorphism induced by ψ, the right vertical arrow is the canonical lattice anti-isomorphism $E \mapsto \Gamma_E$ with inverse $\Delta \mapsto E^{\Delta}$ given by the Fundamental Theorem of Infinite Galois Theory, the horizontal top arrows are the sup-semilattice morphism $\mathbb{G} \mapsto F(\mathbb{G})$ and the inf-semilattice morphism $E \mapsto T(E/F)$, while the horizontal bottom arrows are the sup-semilattice anti-morphism $G \mapsto G^{\perp}$ and the inf-semilattice anti-morphism $\Delta \mapsto \Delta^{\perp}$ defined in Section 0. Note that the commutativity of the diagram above follows at once from [2, Theorem 1.8] or [3, Theorem 15.1.7].

The next result is essentially a reformulation of the corresponding results from [2] or [3] involving the lattices and the maps above.

Proposition 3.1. *Let E be an intermediate field of the given Galois extension Ω/F, let $\Gamma_E = \mathrm{Gal}(\Omega/E)$, let $A = \mu(\Omega)$, let $\mathbb{G} \in \mathbb{L}(T(\Omega/F)|F^*)$, and let $G = \psi(\mathbb{G})$, where ψ is the canonical group epimorphism*

$$\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A),\ x \mapsto (\sigma \in \Gamma \mapsto (\sigma x)\, x^{-1} \in A),$$

defined above. Then, the following statements hold.

(1) *The extension E/F is $\mathbb{G}$-radical if and only if the subgroup Γ_E of Γ is G-radical. In particular, E/F is a radical extension (resp. a simple radical extension) if and only if Γ_E is a radical subgroup (resp. a simple radical subgroup) of Γ.*

(2) *The extension E/F is $\mathbb{G}$-Kneser if and only if the subgroup Γ_E of Γ is G-Kneser. In particular, E/F is a Kneser extension if and only if Γ_E is a Kneser subgroup of Γ.*

(3) *The extension $F(\mathbb{G})/F$ is $\mathbb{G}$-Kneser if and only if G is a Kneser group of $Z^1(\Gamma, A)$.*

(4) *The extension E/F is $\mathbb{G}$-Cogalois if and only if the subgroup Γ_E of Γ is Cogalois. In this case, G is the unique Cogalois group of $Z^1(\Gamma, A)$ for which $\Gamma_E = G^{\perp}$.*

(5) *The extension $F(\mathbb{G})/F$ is $\mathbb{G}$-Cogalois if and only if G is a Cogalois group of $Z^1(\Gamma. A)$.*

(6) *The extension E/F is Cogalois if and only if the subgroup Γ_E of Γ is strongly Cogalois.*

Proof. (1) is a reformulation of [2, Theorem 1.8] or [3, Theorem 15.1.7].

(2) is a reformulation of [2, Corollary 1.10 (1)] or [3, Corollary 15.1.8 (1)].

(4) is a reformulation of [2, Corollary 1.10 (2)] or [3, Corollary 15.1.8 (2)]. The uniqueness of G is assured by [4, Corollary 2.12].

(6) Denote $\mathbb{H} = T(E/F)$ and $H = \psi(\mathbb{H}) = \Gamma_E^{\perp}$. By (2), the extension E/F is $\mathbb{H}$-Kneser if and only if $\Gamma_E = H^{\perp} = \Gamma_E^{\perp\perp}$ and $\Gamma_E^{\perp}$ is a Kneser group of $Z^1(\Gamma, A)$. By Lemma 1.8, this means precisely that Γ_E is strongly-Cogalois. □

The connection between various types of Kummer field extensions and their abstract correspondents is given by the next result.

Proposition 3.2. *Let E/F be an arbitrary separable algebraic extension, let $\Omega := \widetilde{F}^{\mathrm{sep}}$, $\Gamma = \mathrm{Gal}(\Omega/F)$, $A = \mu(\Omega)$, and let $n \in \mathbb{N}$ be relatively prime with the characteristic exponent of F.*

Then, the extension E/F is a classical n-Kummer extension (resp. a generalized n-Kummer extension, an n-Kummer extension with few roots of unity, an n-quasi-Kummer extension) if and only if there exists a unique classical n-Kummer group (resp. a generalized n-Kummer group, an n-Kummer group with few cocycles, an n-quasi-Kummer group) G, $G \leqslant Z^1(\Gamma, A)$, such that $\Gamma_E := \mathrm{Gal}\,(\Omega/E) = G^{\perp}$.

Proof. We may assume that $E \subseteq \Omega$. Assume that E/F is a classical n-Kummer extension. Then there exists a group $\mathbb{G} \in \mathbb{L}(T(E/F)|F^*)$ such that $E = F(\mathbb{G})$, $\mathbb{G}^n \subseteq F^*$, and $A[n] = \mu_n(\Omega) \subseteq \mu_n(F) \subseteq A^\Gamma$. Let $G = \psi(\mathbb{G})$. Then $nG = \{0\}$, and so G is a classical n-Kummer group. By Proposition 3.1 (1) we have $\Gamma_E := \mathrm{Gal}(\Omega/E) = G^\perp$. Now observe that G is Cogalois by Proposition 2.2, so the uniqueness of G follows from Proposition 3.1 (4).

Conversely, assume that there exists a classical n-Kummer group G of $Z^1(\Gamma, A)$ such that $\mathrm{Gal}\,(\Omega/E) = G^\perp$. If we denote $\mathbb{G} = \psi^{-1}(G)$, then $E = F(\mathbb{G})$ by Proposition 3.1 (1). Since clearly $\mathbb{G}^n = 1$ and $\mu_n(\Omega) = A[n] \subseteq A^\Gamma \subseteq F$, we deduce that E/F is a classical n-Kummer extension.

The cases of generalized n-Kummer extensions, n-Kummer extensions with few roots of unity, and n-quasi-Kummer extensions follow in the same manner as above from the following simple facts: $A[n] = \mu_n(\Omega)$ (in particular $A[2] = \{-1, 1\}$) and $A^{\mathrm{Gal}(\Omega/L)} = \{\, x \in \mu(\Omega) \mid \sigma x = x,\ \forall \sigma \in \mathrm{Gal}(\Omega/L)\,\} = \mu(L)$ for any intermediate field L of the given Galois extension Ω/F. □

Remark 3.3. According to Proposition 3.2, all the types of Kummer groups of cocycles defined in Section 2 are abstract versions of corresponding field extensions from the field theoretic Kummer Theory. So, the counterexamples from the field theoretic Kummer Theory provided in [3], converted into Kummer groups of cocycles via Proposition 3.2, show that. except the obvious inclusions indicated just after Definition 2.1, no other inclusions between these four types of Kummer groups of cocycles do exist.

4 Field theoretic via Abstract Cogalois Theory

The results of the previous section permit us to retrieve easily most of the results of field theoretic Cogalois Theory from the basic results of Abstract Cogalois Theory mentioned in Section 1. We will illustrate this by presenting only three of them.

Theorem 4.1. (The infinite Kneser Criterion [5, Theorem 2.1] or [3, Theorem 11.1.5]). *Let E/F be an arbitrary separable $\mathbb{G}$-radical extension. For any positive integer n, let $\zeta_n \in \Omega := \widetilde{F^{\mathrm{sep}}}$ denote a primitive n-th root of unity. Then, the following assertions are equivalent.*

(1) *E/F is a $\mathbb{G}$-Kneser extension.*

(2) *$\zeta_p \in \mathbb{G} \Longrightarrow \zeta_p \in F$ for every odd prime p, and $1 \pm \zeta_4 \in \mathbb{G} \Longrightarrow \zeta_4 \in F$.*

Proof. We may assume that $E \subseteq \Omega$. Set $\Gamma := \mathrm{Gal}(\Omega/F)$ and $A := \mu(\Omega)$, and let

$$\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A),\ x \mapsto (\sigma \in \Gamma \mapsto (\sigma x)\, x^{-1} \in A),$$

be the canonical group-epimorphism defined at the beginning of Section 3. Then

$$A^\Gamma = \mu(F) \text{ and } \mathcal{P}(\Gamma, A) = \{\, p \mid p \text{ odd prime or 4 such that } \zeta_p \notin F\,\}.$$

By assumption, $E = F(\mathbb{G})$, with $F^* \leqslant \mathbb{G} \leqslant T(\Omega/F)$. Setting $G := \psi(\mathbb{G}) \leqslant Z^1(\Gamma, A)$ we have $\Gamma_E = G^\perp$ by Proposition 3.1 (1). Consequently, by Proposition 3.1 (3), the extension E/F is $\mathbb{G}$-Kneser if and only if G is a Kneser subgroup of $Z^1(\Gamma, A)$.

For every odd prime $p \neq \mathrm{Char}(F)$, $\varepsilon_p := \psi(\zeta_p) \in Z^1(\Gamma, A)$ is the coboundary assigning to any $\sigma \in \Gamma$ the p-th root of unity $(\sigma\zeta_p)\zeta_p^{-1} \in A[p]$. Obviously, $\varepsilon_p \in G$ if and only if $\zeta_p \in \mathbb{G}$. Observe that if $p = \mathrm{Char}(F) > 2$, then $\zeta_p \in A[p] = \{1\} \subseteq A^\Gamma$.

Assume that $\mathrm{Char}(F) \neq 2$. Since $1 - \zeta_4 \in T(\Omega/F)$, we can consider the continuous cocycle $\psi(1 - \zeta_4) \in Z^1(\Gamma, A)$, which by definition works as follows:

$$\psi(1-\zeta_4)(\sigma) = \sigma(1-\zeta_4)\cdot(1-\zeta_4)^{-1} = (1-\sigma\zeta_4)\cdot(1-\zeta_4)^{-1},\ \forall\, \sigma \in \Gamma.$$

Since for any $\sigma \in \Gamma$, we have either $\sigma\zeta_4 = \zeta_4$ or $\sigma\zeta_4 = -\zeta_4$, we deduce that

$$\psi(1-\zeta_4)(\sigma) = \begin{cases} \zeta_4 & \text{if } \ \sigma\zeta_4 = -\zeta_4, \\ 1 & \text{if } \ \sigma\zeta_4 = \zeta_4. \end{cases}$$

Thus, $\psi(1-\zeta_4)$ is nothing else than the multiplicative version of the cocycle ε_4', defined in Section 0 and appearing in the statement of the Abstract Kneser Criterion (Theorem 1.3). A simple calculation shows that $\psi(1+\zeta_4) = (\psi(1-\zeta_4))^{-1}$ in the multiplicative group $Z^1(\Gamma, A)$, so $\varepsilon_4' \in G \iff 1 \pm \zeta_4 \in \mathbb{G}$. Observe that if $\mathrm{Char}(F) = 2$, then $\zeta_4 \in A[4] = \{1\} \subseteq A^\Gamma$.

To finish the proof it remains to apply Proposition 3.1 (3) and the Abstract Kneser Criterion (Theorem 1.3). □

Corollary 4.2. *Let E/F be a separable $\mathbb{G}$-radical extension (i.e., $E = F(\mathbb{G})$ for some $F^* \leqslant \mathbb{G} \leqslant T(E/F)$), which is not $\mathbb{G}$-Kneser. Assume that the extension E/F is minimal with respect to the property not being $\mathbb{G}$-Kneser, that is, for any proper subgroup $\mathbb{G}'$ of $\mathbb{G}$, the extension $F(\mathbb{G}')/F$ is $\mathbb{G}'$-Kneser Then, the extension E/F is cyclic having either the form $E = F(\zeta_p)$ with $p \neq \mathrm{Char}(F)$ an odd prime number and $\zeta_p \notin F$, or the form $F(\zeta_4)$ with $\mathrm{Char}(F) \neq 2$ and $\zeta_4 \notin F$.*

Proof. With $\Omega = \widetilde{F}^{\mathrm{sep}}$, Γ, and A as above, let E/F be a subextension of Ω/F satisfying the minimality condition from the statement. Using the canonical group epimorphism

$$\psi : T(\Omega/F) \longrightarrow Z^1(\Gamma, A),\ x \mapsto (\sigma \in \Gamma \mapsto (\sigma x)x^{-1} \in A),$$

as well as Proposition 3.1, we deduce that $G = \psi(\mathbb{G})$ is a minimal non-Kneser group of $Z^1(\Gamma, A)$. According to [4, Lemma 1.18], it follows that either $G = \langle \varepsilon_p \rangle \cong \mathbb{Z}/p\mathbb{Z}$ for some odd prime number $p \neq \mathrm{Char}(F)$ such that $\zeta_p \notin F$, or $G = \langle \varepsilon_4' \rangle \cong \mathbb{Z}/4\mathbb{Z}$, with $\mathrm{Char}(F) \neq 2$ and $\zeta_4 \notin F$. Consequently, $\mathbb{G} = F^*\langle \zeta_p \rangle$ in the former case and $\mathbb{G} = F^*\langle 1 + \zeta_4 \rangle$ in the latter one. The result now follows easily. □

Remark 4.3. The inverse implication in Corollary 3.5 may fail. Indeed, $F(\zeta_4)/F$ is $F^*\langle \zeta_4 \rangle$-Cogalois, in particular Kneser, whenever $\mathrm{Char}(F) \neq 2$ and $\zeta_4 \notin F$. Also, for every odd prime p, if the characteristic exponent of F is relatively prime with $p(p-1)$, $\zeta_p \notin F$, and $\zeta_{p-1} \in F$, then there exists $\theta \in E := F(\zeta_p)$ such that $E = F(\theta)$ and $\theta^{p-1} \in F$, therefore E/F is an $F^*\langle \theta \rangle$-Cogalois extension, in particular Kneser. □

Theorem 4.4. (The General Purity Criterion [1, Theorem 2.3] or [3, Theorem 12.1.4]). *The following assertions are equivalent for an arbitrary separable $\mathbb{G}$-radical extension E/F.*

(i) *E/F is $\mathbb{G}$-Cogalois.*

(ii) *E/F is $\mathcal{P}_{\mathbb{G}}$-pure, i.e., $\zeta_p \in E \Longrightarrow \zeta_p \in E$ for every $p \in \mathcal{P}_{\mathbb{G}} := \mathcal{P} \cap \mathcal{O}_{\mathbb{G}/F^*}$.*

Proof. We may assume that $E \subseteq \Omega := \widetilde{F}^{\text{sep}}$. Set $\Gamma := \text{Gal}(\Omega/F)$ and $A := \mu(\Omega)$. Since E/F is a $\mathbb{G}$-radical extension, we have $E = F(\mathbb{G})$ with $F^* \leqslant \mathbb{G} \leqslant T(\Omega/F)$. If $G := \psi(\mathbb{G}) \leqslant Z^1(\Gamma, A)$, then $\Gamma_E := \text{Gal}(\Omega/E) = G^{\perp}$ by Proposition 3.1 (1), so E/F is $\mathbb{G}$-Cogalois if and only if G is a Cogalois subgroup of $Z^1(\Gamma, A)$ by Proposition 3.1 (5). Since for any $p \in \mathcal{P}_{\mathbb{G}}$ we have $A^{\Gamma}[p] = \mu_p(F)$ and $A^{G^{\perp}}[p] = \mu_p(E)$, we deduce that the $\mathcal{P}_{\mathbb{G}}$-purity of the extension E/F is equivalent to the quasi $\mathcal{P}_G$-purity of the embedding $A^{\Gamma} \leqslant A^{G^{\perp}}$. The result follows now at once by applying Theorem 1.4. □

Theorem 4.5. ([5, Theorem 3.12] or [3, Theorem 12.1.10]). *If E/F is an algebraic separable extension which is simultaneously $\mathbb{G}$-Cogalois and $\mathbb{H}$-Cogalois, then $\mathbb{G} = \mathbb{H}$.* □

Proof. Apply [4, Corollary 2.12] and Proposition 3.1 (4). □

Corollary 4.6. *Let Ω/F be a Galois extension, $\Gamma := \text{Gal}(\Omega/F)$, $A := \mu(\Omega)$, and let $F^* \leqslant \mathbb{G} \leqslant T(\Omega/F)$ be such that $E := F(\mathbb{G})$ is a $\mathbb{G}$-Cogalois extension of F. If $G := \psi(\mathbb{G}) \leqslant Z^1(\Gamma, A)$, then, the following assertions are equivalent.*

(1) *G is a Γ-submodule of $Z^1(\Gamma, A)$, i.e., it is stable under the action of Γ.*

(2) *E/F is a Galois extension.*

(3) *$\sigma x \in E$ for all $\sigma \in \Gamma$ and $x \in \mathbb{G}$.*

Proof. First, observe that $\Gamma_E := \text{Gal}(\Omega/E) = G^{\perp}$ by Proposition 3.1 (4). Now, by [4, Corollary 2.14], G is a Γ-submodule of $Z^1(\Gamma, A)$ if and only if $G^{\perp}$ is a normal subgroup of Γ if and only if E/F is a Galois extension. □

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