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Smoothness and differentials in positive characteristic

Cristodor Ionescu

Abstract

We give several cases in which the formal projectivity of the differential module implies the formal smoothness of the algebra, in the case of rings containing a field of positive characteristic. We also obtain some smoothness criteria for algebras with differential basis, in the same case of rings of characteristic $p > 0$.

1 Introduction

All the rings considered will be commutative and with unit. The terminology will be that of [M]. Throughout the paper p will be a positive prime number. All the rings considered will be supposed to contain a field of characterisitc p. If $u : A \longrightarrow B$ is a ring morphism and E is a B−module, $H_i(A, B, E), i \geq 1$ will denote the André-Quillen homology(see [BR]). If p is a prime number, by a ring of characyeristic p we mean a ring containing a field of characterisitc p. If $u : A \longrightarrow B$ is a morphism of noetherian local rings, formally smooth will mean smooth in the topology of the maximal ideal of B.
Let $u : A \longrightarrow B$ be a morphism of noetherian rings and suppose that B is endowed with the topology given by the powers of an ideal I. It is well-known that there is a strong connection between the I-smoothness of B over A and the projectivity of the module of differentials of B over A, denoted by $\Omega_{B/A}$. Namely, one has:

Theorem 1.1 ([BR], 3.1) *If u is I-smooth, then $\Omega_{B/A}$ is formally projective with respect to the I-adic topology.*

Theorem 1.1 is true in general, for any noetherian rings. The converse is not always true, as many examples show(see for instance [BR]). There are, however, several results dealing with the converse obtained by Kunz, Radu, Suzuki. Our aim is to add some cases in which the converse is true, for rings of positive prime characteristic.

2 Local rings of positive prime characteristic

In this section we shall give some results concerning local rings of characteristic p, whose module of differentials are formally projective in the topology of the maximal ideal. We start with the case of algebras over a field k. The case of the perfect field k is settled already(see [BR], 15.4), so we consider the case when $k \neq k^p$.

Theorem 2.1 *Let k be a field of characteristic $p > 0$ and let (B, m, K) be a noetherian local k-algebra. Suppose that:*
a) $\mathrm{rk}_K H_1(k, K, K) < \infty$;
b) $k \subseteq K^p$;
c) $\mathrm{Reg}(\widehat{B}) \neq \emptyset$;
d) $\Omega_{B/k}$ *is a formally projective B-module.*
Then B is a formally smooth k-algebra.

Proof: It is sufficient to show that the assertion is true when B is complete. Indeed, $(\Omega_{\widehat{B}/k})^\frown \cong (\Omega_{B|k} \otimes_B \widehat{B})^\frown$ is formally projective as a $\widehat{B}$-module, so that $\Omega_{\widehat{B}/k}$ is a formally projective $\widehat{B}$-module. On the other hand, if $\widehat{B}$ is formally smooth over k, then B is also formally smooth over k. By a) it follows that there exists a noetherian local complete, formally smooth k-algebra A and a surjective morphism $v : A \longrightarrow B$(cf. [FR]). Let $I := \ker(v)$. By ([BR],15.3) we can suppose that the canonical morphism

$$\widehat{v_0} : (\Omega_{A/k \otimes_A B})^\frown \longrightarrow (\Omega_{B/k})^\frown$$

is an isomorphism. Then I is a k-differential ideal of A and it is sufficient to show that $I = 0$. Let $Q \in \mathrm{Reg}(B)$. Consider $I = Q_1 \cap \ldots \cap Q_r$ a reduced primary decomposition of Q. Then there is $i \in \{1, \ldots, n\}$, such that $Q_i \in \mathrm{Min}(I)$ and such that $Q_i \subseteq Q$. But then Q_i is also a k-differential ideal and from ([BR], 14.11) it follows that $Q_i = 0$. Then $I = 0$ and consequently B is formally smooth over k.

Corollary 2.2 *Let k be a field of positive characteristic $p > 0$ and let (B, m, K) be a noetherian local k-algebra. Suppose that:*
a) $\mathrm{rk}_K H_1(k, K, K) < \infty$;
b) $k \subseteq K^p$;
c) B is a Nagata ring;
d) $\mathrm{Reg}(B) \neq \emptyset$;
e) $\Omega_{B/k}$ *is a formally projective B-module.*
Then B is a formally smooth k-algebra.

Proof: Let $P \in \mathrm{Reg}(B), Q \in \mathrm{Min}(P\widehat{B})$. Then $(\widehat{B})_Q$ is a formally smooth B_P-algebra and consequently $(\widehat{B})_Q$ is regular.

Remark 2.3 2.1 and 2.2 are generalizations of ([BR],16.12, 16.13).

For the next results we need the following considerations: let A be a ring of prime characteristic $p > 0$. Denote by $A^{(p)}$ the A-algebra A given by the Frobenius endomorphism of A. For a ring morphism $u : A \longrightarrow B$ we have the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{u} & B & & \\ \downarrow F_A & & \downarrow F_A \otimes 1_B & & \\ A^{(p)} & \longrightarrow & A^{(p)} \otimes_A B & \xrightarrow{\omega_{B/A}} & B^{(p)} \end{array}$$

where F_A is the Frobenius morphism of A and

$$\omega_{B/A}(a \otimes b) = u(a) \cdot b^p, \ \forall a \in A, b \in B.$$

If A is as before, we have an inductive system

$$A^{(p)} \xrightarrow{F_A} A^{(p^2)} \xrightarrow{F_A} A^{(p^3)} \xrightarrow{F_A} \ldots \xrightarrow{F_A} A^{(p^k)} \xrightarrow{F_A} \ldots$$

We denote by $A^{(p^\infty)}$ the inductive limit of this system and by

$$\omega^\infty_{B/A} : A^{(p^\infty)} \otimes_A B \longrightarrow B^{(p^\infty)}, \ \omega^\infty_{B/A} := \varinjlim \omega^r_{B/A}.$$

Theorem 2.4 *Let k be a field of characteristic $p > 0$ and let (B, m, K) be a noetherian local k-algebra. Suppose that:*
a) $\mathrm{rk}_K H_1(k, K, K) < \infty$;
b) $(B \otimes_k k^{(p)})\widehat{}$ is a reduced ring;
c) $\Omega_{B/k}$ is a formally projective B-module.
Then B is a formally smooth k-algebra.

Proof: Since

$$(B \otimes_k k^{(p)})\widehat{} \cong (\widehat{B} \otimes_k k^{(p)})\widehat{}$$

we can suppose as above that B is complete. From a) and ([R], cor. 2), it follows that $B \otimes_A k^{(p)}$ is noetherian and then also $(B \otimes_A k^{(p)})\widehat{}$ is noetherian.

There exists a local formally smooth k-algebra A and a surjective morphism $v : A \longrightarrow B$ (cf. [FR]). By ([BR], 15.3) we can also suppose that

$$(\Omega_{A/k} \otimes_A B)\widehat{\ } \cong (\Omega_{B/k})\widehat{\ }.$$

Let $I = \ker(v)$. Then I is a k-differential ideal and it is enough to show that $I = (0)$. Let

$$A'' := A \otimes_k k^{(p)},\ B'' := B \otimes_k k^{(p)};$$
$$A' := (A \otimes_k k^{(p)})\widehat{\ },\ B' := (B \otimes_k k^{(p)})\widehat{\ }.$$

Clearly we have that A' and B' are noetherian complete local rings. We have an isomorphism

$$\widehat{v}_0 : (\Omega_{A'/k^{(p)}} \otimes_{A'} B') \longrightarrow (\Omega_{B'/k^{(p)}})\widehat{\ }.$$

Indeed, we have successively

$$(\Omega_{B'/k^{(p)}})\widehat{\ } \cong (\Omega_{B''|k^{(p)}} \otimes_{B''} B')\widehat{\ } \cong (\Omega_{B|k^{(p)}} \otimes_B B'' \otimes_{B''} B')\widehat{\ } \cong$$
$$\cong ((\Omega_{B|k})\widehat{\ } \otimes_B B')\widehat{\ } \cong ((\Omega_{A|k} \otimes_A B)\widehat{\ } \otimes_B B')\widehat{\ } \cong (\Omega_{A|k} \otimes_A B \otimes_B B')\widehat{\ } \cong$$
$$\cong (\Omega_{A|k} \otimes_A A'' \otimes_{A''} B'')\widehat{\ } \cong (\Omega_{A''|k^{(p)}} \otimes_{A''} B')\widehat{\ } \cong (\Omega_{A''|k^{(p)}} \otimes_{A''} A' \otimes_{A'} B')\widehat{\ } \cong$$
$$\cong ((\Omega_{A''|k^{(p)}} \otimes_{A''} A')\widehat{\ } \otimes_{A'} B)\widehat{\ } \cong ((\Omega_{A'|k^{(p)}})\widehat{\ } \otimes_{A'} B')\widehat{\ }$$

It follows that IA' is a k-differential ideal. But

$$IA' = P_1 \cap \ldots \cap P_r,\ P_i \in \operatorname{Spec}(A'),\ i = 1, ..., r.$$

Then P_i are k-differential ideals, so that $IA' = 0$. Since $A'' = A \otimes_k k^{(p)}$ is a noetherian local ring, A' is a faithfully flat A''-algebra and consequently A' is faithfully flat over A. It follows that $I = (0)$.

Corollary 2.5 *Let k be a field of characteristic $p > 0$ and (B, m, K) a noetherian local k-algebra. Suppose that:*
a) $\operatorname{rk}_K H_1(k, K, K) < \infty$;
b) $(B \otimes_k k^{(p)})\widehat{\ }$ *is a reduced ring;*
c) $\Omega_{B/k}$ *is a formally projective B-module.*
Then $(B \otimes_k k^{(p)})\widehat{\ }$ *is a noetherian regular ring.*

Proof: Since B is formally smooth over k, it follows that $B \otimes_k k^{(p)}$ is a noetherian regular ring. Consequently $(B \otimes_k k^{(p)})\widehat{\ }$ is also a noetherian regular ring.

Remark 2.6 2.2 and 2.4 are generalizations of results of N. Radu, where $k^{(p)}$ was replaced by the algebraic closure of k.

When k is not necessarily a field, we can generalize 2.4 as follows:

Theorem 2.7 *Let $u : (A, m, k) \longrightarrow (B, n, K)$ be a morphism of noetherian local rings of characteristic p. Suppose that:*
a) A has geometrically regular formal fibers;
b) $\mathrm{rk}_K H_1(k, K, K) < \infty$;
c) $(B \otimes_A A^{(p)})\widehat{}$ is a reduced ring;
d) For any prime ideal $\wp$ of A, $[k(\wp) : (k(\wp)^p] < \infty$;
e) $\Omega_{B/A}$ is a formally projective B-module.
Then u is formally smooth.

Proof: The proof is similar to the proof of 2.4, so that we only point out the main steps. We can suppose that A and B are complete because $(\widehat{B} \otimes_{\widehat{A}} \widehat{A}^{(p)}) \cong (B \otimes_A A^{(p)})\widehat{}$. By a), b) and ([R], Th. 7) it follows that $B \otimes_A A^{(p)}$ is noetherian. By [FR] there exists a local complete formally smooth A-algebra C and a surjective morphism $v : C \longrightarrow B$. Let

$$I = \ker(v), B'' := B^{(p)}, C'' := A^{(P)} \otimes_A C,$$

$$B' := \widehat{B''}, C' := \widehat{C''}.$$

We can also suppose that $(\Omega_{C|A} \otimes_C B)\widehat{} \cong (\Omega_{B|A})\widehat{}$ and then I is an A-differential ideal. As in the proof of 2.4 we have an isomorphism

$$\widehat{v}_0 : (\Omega_{C'/A^{(p)}} \otimes_{C'} B')\widehat{} \longrightarrow (\Omega_{B'/A^{(p)}})\widehat{}$$

Write $IA' = P_1 \cap \ldots \cap P_r$, where $P_i, i = 1, \ldots, r$ are prime ideals of C and then IC' is a C'-differential ideal. Let $Q_i := P_i \cap A'$. Since A has geometrically regular formal fibers and C is formally smooth over A, by the theorem on the localisation of formal smoothness(see [BR], 11.3) the morphism $A_{Q_i} \longrightarrow C_{P_i}$ is also formally smooth. Thus, taking account on d), we can apply ([BR], 14.12) and we obtain that $P_i = 0$ and consequently $IA' = 0$. Now as above it follows that $I = 0$.

Corollary 2.8 *In the same conditions as in Theorem 2.7, u is a regular morphism. Consequently, $\Omega_{B|A}$ is even a flat B-module.*

Proof: Since A is quasi-excellent, by the localization of formal smoothness([BR], 11.3) it follows that u is regular. Then $\Omega_{B|A}$ is clearly flat over B.

We can also obtain a criterion similar to 2.5 using the ring $A^{(p^\infty)}$ instead of $A^{(p)}$.

Theorem 2.9 *Let k be a field of characteristic $p > 0$ and (B, m, K) a noetherian local k-algebra. Suppose that:*
a) K is an extension of finite type of k;
b) $(k^{(p^\infty)} \otimes_k B)\hat{}$ is a reduced ring;
c) $\Omega_{B/k}$ is formally projective B-module.
Then B is formally smooth k-algebra.

Proof: As above we can suppose that B is complete. By [D] and a) it follows that $(k^{(p^\infty)} \otimes_k B)$ is a noetherian ring. As in the preceeding proofs, we can find a noetherian complete local k-algebra C which is formally smooth and a surjective morphism $C \longrightarrow B$. Now we continue as in theorem 2.4.

3 Algebras with differential basis

In the last section we shall obtain criteria for the smoothness of algebras having differential basis. We need the following important result of Tyç([T]).

Theorem 3.1 *Let A be a noetherian ring and B a noetherian A-algebra. Then any differential basis of B over A is a p-basis of B over A.*

Proof: See [A] for a complete proof.

Proposition 3.2 *Let $u : A \longrightarrow B$ be a morphism of noetherian rings of characteristic p. Suppose that:*
a) B has a differential basis over A;
b) u is a reduced morphism.
Then u is smooth.

Proof: Since u is reduced, by ([D], Theorem 3) it follows that the morphism $\omega_{B/A} : A^{(p)} \otimes_A B \longrightarrow B^{(p)}$ is injective. Let $\{x_i\}_{i\in I}$ a differential basis of B over A. Then from 3.1 it follows that $\{x_i\}_{i\in I}$ is also a p-basis of B over A, so that $\{x_i\}_{i\in I}$ is a basis of $B^{(p)}$ over $A^{(p)} \otimes_A B$. Then $B^{(p)}$ is an $A^{(p)} \otimes_A B$-free module via $\omega_{B/A}$. By ([D],Theorem 2) it folows that u is regular. But $\Omega_{B/A}$ is free and then u is smooth(see for example [BR], 17.10, 17.11).

The last result is a generalization of a theorem obtained by A. Tyç([T],Theorem 2).

Theorem 3.3 *Let $u : A \longrightarrow B$ be a morphism of noetherian rings of characteristic p. Suppose that:*
a) B has a differential basis over A;
b) A has a p-basis;
c) $A^{(p)} \otimes_A B$ is reduced.
Then u is smooth.

Proof: By a) and 3.1, B has also a p-basis over A and $\Omega_{B/A}$ is a free B-module, so it is enough to show that u is regular. By ([D], Theorem 2) we have to show that $\omega_{B/A}$ is flat.
Let $\{x_i\}_{i\in I}$ be a p-basis of B over A. We shall show that $\{x_i\}_{i\in I}$ is a p-basis of $B^{(p)}$ over $A^{(p)} \otimes_A B$. Since $\mathrm{Im}(\omega_{B/A}) = A[B^p], \{x_i\}_{i\in I}$ is a system of p-generators of $B^{(p)}$ over $A^{(p)} \otimes_A B$ and it is enough to show that $\{x\}_{i\in I}$ is p-free over $A^{(p)} \otimes_A B$, i.e. $\{1, x_i, ..., x_i^{p-1}\}_{i\in I}$ is free over $A^{(p)} \otimes_A B$. Suppose that we have a relation

$$\sum_{i\in I}\sum_{s=0}^{p-1}(a_{is} \otimes b_{is}) \cdot x_i^s = 0, a_{is} \in A^{(p)}, b_{is} \in B.$$

Then we have

$$\sum_{i\in I}\sum_{s=0}^{p-1} u(a_{is})b_{is}^p x_i^s = 0$$

and it follows that

$$u(a_{is})b_{is}^p = 0, \forall i \in I, \forall s = 0, \dots, p-1.$$

Let $\{y_j\}_{j\in J}$ be a p-basis of A. Then

$$a_{is} = \sum_{j\in J}\sum_{k=0}^{p-1} \alpha_{jis}^p y_j^k, \ \alpha_{jis} \in A.$$

It follows that

$$a_{is} \otimes b_{is} = (\sum_{j,k} \alpha_{jis}^p y_j^k) \otimes b_{is} = \sum_{j,k} 1 \otimes (\alpha_{jis}^p y_j^k b_{is})$$

and then

$$(a_{is} \otimes b_{is})^p = \sum_{j,k} 1 \otimes (\alpha_{jis}^{p^2} y^{pk} b_{is}^p) = 1 \otimes (\sum_{j,k} \alpha_{jis}^p y^k)^p b_{is}^{p^2} = 1 \otimes (u(a_{is})b_{is}^p)^p = 0$$

so that $(a_{is} \otimes b_{is})^p = 0$. But by c) it follows that $a_{is} \otimes b_{is} = 0$.

Remark 3.4 The original result of Tyç was settled in the case when A was a field of characteristic $p > 0$.

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