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On the differentiability of Sobolev functions on metric measure spaces

MARCELINA MOCANU

ABSTRACT. We give applications to the Stepanov differentiability theorem of [BRZ] in doubling metric measure spaces supporting a Poincaré inequality. For $1 \leq p < \infty$ we prove that the differential of a Sobolev mapping from $N^{1,p}(X)$ is an average L^p- integral pointwise differential, at almost every point of X. A differentiability result for monotone Sobolev functions is established. We study the regularity of quasiminimizers of the Dirichlet energy integral, by using a Cacciopoli type estimate, Gehring's Lemma in doubling metric measure spaces and a Calderón type theorem.

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1.INTRODUCTION

Geometric function theory has been an important source of inspiration for the recent developments of analysis on metric measure spaces. The study of Sobolev spaces and $p-$harmonic functions on metric measure spaces led to the definition of a concept of differentiability in this general setting.

In his seminal paper [C] Cheeger proved that every metric space with a doubling measure supporting a Poincaré inequality admits a strong measurable differentiable structure, with which Lipschitz functions can be differentiated almost everywhere. Note that quasiconformal theory and nonlinear potential theory are currently studied on metric measure spaces with the properties mentioned above.

In their recent paper [BRZ] Balogh, Rogovin and Zürcher used Cheeger's extension of Rademacher differentiability theorem to prove a generalization of this result, the following extension of Stepanov's differentiability theorem.

Theorem 1.1.[BRZ] *Let (X, d, μ) be a doubling metric measure space. Assume that there exists a strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}$ for (X, d, μ) with respect to $LIP(X)$, such that the sets X_α are mutually disjoint.*

Then each function $f : X \to \mathbb{R}$ is μ−a.e. differentiable in $S(f) := \{x \in X : Lip\, f(x) < \infty\}$ with respect to the structure $\{(X_\alpha, \varphi_\alpha)\}$.

The above theorem has far-reaching consequences. It turns out that the class of Cheeger differentiable functions is very rich. The applications of Stepanov's differentiability theorem given in [BRZ] include a Calderon-type differentiability theorem and a theorem on the differentiability almost everywhere of the post-composition with Lipschitz functions of quasiconformal mappings between Ahlfors regular spaces.

The aim of this paper is to give another applications to Theorem 1.1, which extend to metric measure spaces classical results on the differentiability of some Sobolev functions on Euclidean domains.

The paper is organized as follows.

In the second section we state the needed definitions and preliminary results In Section 3 we give alternative proofs for some results of [BRZ], which extend the Calderon differentiability theorem to metric measure spaces. We establish a Calderon-Zygmund type theorem, which improves a theorem of Keith [3] and extends a theorem of Reshetnyak [4] on differentiability of $W^{1,p}$ functions in the sense of $W^{1,p}$. In Section 4 we prove a result on the differentiability of monotone functions on doubling metric measure spaces. In Section 5 we investigate the regularity of some quasiminimizers of the Dirichlet energy integral. Using a Caccioppoli type inequality of [KSh] we prove that the minimal weak upper gradients of quasiminimizers satisfy a weak reverse Hölder inequality, and consequently they have a higher integrability property, by a Gehring- type lemma of [Z-G]. In particular, in the borderline case we obtain the differentiability of quasiminimizers , with respect to any strong measurable differentiable structure for the metric measure space.

2. Preliminaries

In what follows we assume that (X, d, μ) is a metric measure space, where the measure μ is Borel regular, positive and finite on balls. An open ball centered at $x \in X$,of radius $r > 0$ will be denoted by $B(x, r)$. If $B := B(x, r)$ and $\sigma > 0$ then σB stands for $B(x, \sigma r)$.

A function between metric spaces $f : (X, d) \to (Y, \rho)$ is called L−Lipschitz if $\rho(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$ and is said to be Lipschitz if it is L−Lipschitz for some $L > 0$, respectively locally Lipschitz if it is Lipschitz on each ball $B \subset X$. If f is Lipschitz , let $LIPf$ be the infimum of all $L > 0$ for which f is L−Lipschitz. We will denote by $LIP(X)$ the set of all real Lipschitz functions on (X, d) and by $LIP_{loc}(X)$ the set of functions $u : X \to \mathbb{R}$ such that u is Lipschitz on every ball in X. In order to describe the scaled oscillations of a function $u : X \to \mathbb{R}$ which is not necessarily Lipschitz, we use the upper Lipschitz constant defined by

$$Lip\, u(x) = \limsup_{r\to 0} \frac{1}{r} \sup_{y\in B(x,r)} |u(x)-u(y)|.$$

Note that $Lip\, u(x) = \limsup_{y\to x} \frac{|u(x)-u(y)|}{d(x,y)}$ if x is a limit point of X and $Lip\, u(x) = 0$ if x is isolated (see [C], [Ke], [BRZ]).

Let $1 \le p < \infty$. The $p-$modulus of a family of paths Γ in X, denoted by $Mod_p(\Gamma)$, is the number $\inf_\rho \int_X \rho^p d\mu$, where the infimum is taken over all non-negative Borel measurable functions ρ such that for all rectifiable paths γ which belong to Γ we have $\int_\gamma \rho ds \ge 1$.

The basic concept of the "first-order calculus" on metric measure spaces is that of upper gradient, which generalizes the norm of the gradient of a real-valued C^1-function on a Euclidean domain. Let $u : X \to \mathbb{R}$. A non-negative Borel measurable function g is said to be an *upper gradient* of u if for all rectifiable paths $\gamma : [a,b] \to X$ the following inequality holds

$$|u(\gamma(a)) - u(\gamma(b))| \le \int_\gamma g ds. \tag{2.1}$$

We say that g is a $p-$*weak upper gradient* of u if (2.1) holds for Mod_p-almost every compact rectifiable path γ .

The Newtonian spaces introduced by Shanmugalingam [Sh1]are the Sobolev type spaces based on the notion of weak upper gradient.Let $\widetilde{N}^{1,p}(X)$ be the collection of all real-valued $p-$integrable functions u on X that possess a $p-$integrable $p-$weak upper gradient. This space can be endowed with the seminorm $\|u\|_{\widetilde{N}^{1,p}} := \|u\|_p + \inf \|g\|_p$, where the infimum is taken over all $p-$integrable $p-$weak upper gradients g of u. If u and v are functions in $\widetilde{N}^{1,p}$, we set $u \sim v$ if $\|u-v\|_{\widetilde{N}^{1,p}} = 0$. Then $\sim$ is an equivalence relation. The quotient space $N^{1,p}(X) := \widetilde{N}^{1,p}(X)/\sim$ equipped with the norm $\|u\|_{N^{1,p}} := \|u\|_{\widetilde{N}^{1,p}}$, is the *Newtonian space* corresponding to the index p.

If $1 < p < \infty$ then each function $u \in N^{1,p}(X)$ has a minimal $p-$integrable $p-$weak upper gradient in X, denoted by g_u, in the sense that if g is another $p-$weak upper gradient of u, then $g_u \le g$ $\mu-$a.e.in X ([C], Theorem 2.18).

Let Ω be an open subset of X. The Newtonian space $N^{1,p}(\Omega)$ is defined in an obvious way. We say that a function $u : \Omega \to \mathbb{R}$ belongs to the *local Newtonian space* $N^{1,p}_{loc}(\Omega)$ if $u \in N^{1,p}(E)$ for every measurable set $E \subset\subset \Omega$ ([KM]). If $u \in N^{1,p}_{loc}(\Omega)$ with $1 < p < \infty$, then u has a minimal $p-$weak upper gradient g_u in Ω, in the following sense: whenever $D \subset\subset \Omega$ is an open set and $g_{u,D}$ is a minimal $p-$weak upper gradient of u in D we have $g_u \le g_{u,D}$ $\mu-$a.e.in D .

The *p−capacity* of a set $E \subset X$ is defined by $C_p(E) = \inf_u \|u\|^p_{N^{1,p}}$, where the infimum is taken over all functions $u \in N^{1,p}(X)$ with $u = 1$ on E. The *Newtonian space with zero boundary values* $N^{1,p}_0(E)$ is the set of functions $u : E \to \mathbb{R}$ for which there exists a function $\widetilde{u} \in N^{1,p}(X)$ such that $\widetilde{u} = u$, μ−almost everywhere in E and $C_p(\{x \in X \setminus E : \widetilde{u}(x) \neq 0\}) = 0$.

The theory of quasiconformal mappings between two metric spaces, the non-linear potential theory on metric measure spaces and the theory of Cheeger differentiability are relevant if the metric measure spaces we use are doubling and support Poincaré inequalities.

The metric measure space (X, d, μ) is said to be *doubling* if there is a constant $C_d \geq 1$ so that

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)) \tag{2.2}$$

for every ball $B(x, r)$ in X. By the doubling condition (2.2) there exist some constants $C_b > 0$ and Q such that

$$\frac{\mu(B(x, r))}{\mu(B(x_0, r_0))} \geq C_b \left(\frac{r}{r_0}\right)^Q \tag{2.3}$$

whenever $x \in B(x_0, r_0)$ and $0 < r \leq r_0$. Every such Q will be called a *homogeneous dimension* of the given metric measure space. For instance, (2.3) holds with $Q = \log_2 C_d$ and some C_b depending only on C_d.

An important advantage of doubling metric measure spaces is the valability of Lebesgue differentiation theorem. For $f \in L^p_{loc}(X)$ we call x_0 a *Lebesgue point* of f if $\lim_{r \to 0} \frac{1}{\mu(B(x_0,r))} \int_{B(x_0,r)} |f(x) - f(x_0)|^p \, d\mu = 0$. When μ is doubling, if $f \in L^p_{loc}(X)$ then μ−a.e. $x_0 \in X$ is a Lebesgue point of f (see [He], Theorem 1.8) . In particular, $\lim_{r \to 0} \frac{1}{\mu(B(x_0,r))} \int_{B(x_0,r)} |f(x)|^p \, d\mu = |f(x_0)| < \infty$ for μ−a.e. $x_0 \in X$. If $A \subset X$ and $x_0 \in A$, the latter equation implies (taking f as the characteristic function of $A \cap B(x_0, R)$ for some $R > 0$) $\lim_{r \to 0} \frac{\mu(A \cap B(x_0,r))}{\mu(B(x_0,r))} = 1$, i.e. x_0 is a *point of density* of A, for μ−a.e. $x_0 \in A$.

Remark 2.1. Let $\rho \in L^1_{loc}(X)$ and $\lambda > 0$. If x is a limit point of X then

$$\limsup_{y \to x} \frac{1}{\mu(B(x, \lambda d(x,y)))} \int_{B(x,\lambda d(x,y))} \rho \, d\mu \leq \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \rho \, d\mu \tag{2.4}$$

Let $u \in L^1_{loc}(X)$ and g be a measurable non-negative function on X. Let $p > 0$. The pair (u, g) is said to *satisfy a weak* $(1,p)$−*Poincaré inequality* if there exist some constants $C_P > 0$ and $\tau \geq 1$ such that

$$\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)}|u-u_{B(x,r)}|\,d\mu \le C_P r\left(\frac{1}{\mu(B(x,\tau r))}\int\limits_{B(x,\tau r)} g^p d\mu\right)^{1/p} \tag{2.5}$$

for every ball $B(x,r)$. Here $u_{B(x,r)} = \frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)} ud\mu$ (see [HaKo], page 9).

We say that the metric measure space (X,d,μ) *supports a weak* $(1,p)-$*Poincaré inequality* (for locally integrable functions) if there exist some constants $C_P > 0$ and $\tau \ge 1$ such that (u,g) satisfy (2.5) whenever $u \in L^1_{loc}(X)$ and g is an upper gradient of u.

Note that,when (X,d,μ) supports a weak $(1,p)-$Poincaré inequality, if $u \in L^1_{loc}(X)$ has a $p-$integrable $p-$weak upper gradient g then (2.5) holds, since g is the limit in $L^p(X)$ of a sequence of upper gradients (g_n) of u and each pair (u,g_n) satisfies (2.5) . If (X,d,μ) supports a weak $(1,p)-$Poincaré inequality for some $p>0$ then (X,d,μ) supports a weak $(1,q)-$Poincaré inequality for every $q>p$, by Hölder inequality.

One of the self-improving features of Poincaré inequalities, that we will use as a key tool, consists in the fact that a weak $(1,p)-$Poincaré inequality implies a weak $(t,p)-$Poincaré inequality, for some $t>1$. This follows from a result of Hajłasz and Koskela ([HaKo], Theorem 5.1; see also [KSh],page 406).

Lemma 2.2. *Let (X,d,μ) be a doubling metric measure space with a homogeneous dimension Q. Let $p>0$. Assume that the pair (u,g) satisfies the weak $(1,p)-$Poincaré inequality (2.5) .*

Then for every $0<t<\frac{Qp}{Q-p}$ if $p<Q$ and for every $t>0$ if $p\ge Q$ the pair (u,g) satisfies the weak $(t,p)-$Poincaré inequality

$$\left(\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)}|u-u_{B(x,r)}|^t\,d\mu\right)^{1/t} \le Cr\left(\frac{1}{\mu(B(x,5\tau r))}\int\limits_{B(x,5\tau r)} g^p d\mu\right)^{1/p}, \tag{2.6}$$

for every ball $B(x,r)$ in X.

Moreover, if $p>Q$ then u has a locally Hölder continuous representative satisfying

$$|u(x)-u(y)| \le Cr^{Q/p}d(x,y)^{1-Q/p}\left(\frac{1}{\mu(B(a,5\tau r))}\int\limits_{B(a,5\tau r)} g^p d\mu\right)^{1/p} \tag{2.7}$$

for all $x,y\in B(a,r)$, where $B(a,r)$ is an arbitrary ball and $C>0$ is a constant.

The essence of Cheeger's study on the infinitesimal behavior of Lipschitz functions on doubling metric measure spaces has been synthetized by Keith in the following definition.

Definition 2.3. ([Ke],[BRZ]) Let (X, d, μ) be a metric measure space, let $\mathcal{C} \subset LIP(X)$ be a vector space of functions and let $\{(X_\alpha, \varphi_\alpha)\}$ be a finite or countable collection such that each set $X_\alpha \subset X$ is measurable with positive measure, and such that each $\varphi_\alpha = (\varphi_\alpha^1, ..., \varphi_\alpha^{N(\alpha)}) : X \to \mathbb{R}^{N(\alpha)}$ is a function with $\varphi_\alpha^i \in \mathcal{C}$ for every $1 \leq i \leq N(\alpha)$, where $N(\alpha)$ is a non-negative integer . (Here φ_α will be viewed to be the empty function if $N(\alpha) = 0$). Then $\{(X_\alpha, \varphi_\alpha)\}$is said to be a *strong measurable differentiable structure* for (X, d, μ) with respect to $\mathcal{C}$ if the following is true:

(i) $\mu(X \setminus \bigcup\limits_\alpha X_\alpha) = 0.$

(ii) There exists a non-negative integer N such that $N(\alpha) \leq N$ for every *coordinate patch* $(X_\alpha, \varphi_\alpha)$ and

(iii) For every $f \in \mathcal{C}$ and each coordinate patch $(X_\alpha, \varphi_\alpha)$, there exists a unique (up to a set of measure zero) measurable function $d^\alpha f : X_\alpha \to \mathbb{R}^{N(\alpha)}$ such that for almost every $x \in X_\alpha$

$$\lim_{y \to x} \frac{|f(y) - f(x) - < d^\alpha f(x), \varphi_\alpha(y) - \varphi_\alpha(x)|}{d(y, x)} = 0, \tag{2.8}$$

where $< \cdot, \cdot >$ is the usual inner product on $\mathbb{R}^{N(\alpha)}$.

If in the above definition $\mathcal{C} = LIP(X)$ we will simply say that $\{(X_\alpha, \varphi_\alpha)\}$is a strong measurable differentiable structure for (X, d, μ).

Remark 2.4. In [BRZ] the definition of a strong measurable differentiable structure contains the additional assumption

(iv) The sets X_α are mutually disjoint.

Note that to every $\{(X_\alpha, \varphi_\alpha)\}$ strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}$ we can associate a strong measurable differentiable structure for which the domains of the coordinate patches are mutually disjoint.

We may assume that $\alpha \in \mathbb{N} \setminus \{0\}$. Let $Y_1 := X_1$ and $Y_\alpha := X_\alpha \setminus \bigcup\limits_{i=1}^{\alpha-1} X_i$ for $\alpha \geq 2$, and let ψ_α be the restriction of φ_α to Y_α. Then $\{(Y_\alpha, \psi_\alpha) : \alpha \in \mathbb{N} \setminus \{0\}, \mu(Y_\alpha) > 0\}$ is a strong measurable differentiable structure such that the sets Y_α are mutually disjoint.

In order to keep the presentation as simple as possible, we will adopt the point of view from [BRZ] by using only strong measurable differentiable structures which satisfy condition (iv), without saying this every time.

A function $f : X \to \mathbb{R}$ (not necessarily in $LIP(X)$) is said to be *differentiable* at $x \in X_\alpha$ if there exists $d^\alpha f(x) \in \mathbb{R}^{N(\alpha)}$ such that (2.8) holds ([BRZ]). Assume that a function $f : X \to \mathbb{R}$ is differentiable $\mu-$a.e. on X. Then for each coordinate patch $(X_\alpha, \varphi_\alpha)$ there exists a function (which is unique up to a set of zero measure) $d^\alpha f : X_\alpha \to \mathbb{R}^{N(\alpha)}$ so that (2.8) holds at $\mu-$a.e.point $x \in X_\alpha$. Setting

$df(x) := d^\alpha f(x)$ if $x \in X_\alpha$, for each α, we get a function $df : X \to \bigcup_\alpha \mathbb{R}^{N(\alpha)}$, called the Cheeger differential of f (see [C] page 32 for a more general approach, using sections of an L^∞ vector bundle).

Remark 2.5. The existence of a strong measurable differentiable structure for X with respect to $LIP(X)$ implies the differentiability $\mu-$a.e. on X of every locally Lipschitz function.

Cheeger proved that every doubling metric measure space (X, d, μ), that supports a weak $(1,p)-$Poincaré inequality for some $1 \leq p < \infty$,admits a strong measurable differentiable structure with respect to $LIP(X)$. In particular, Rademacher differentiability theorem can be extended to such metric measure spaces.

A strong measurable differentiable structure for a doubling metric measure space (X, d, μ) supporting a weak $(1,p)-$Poincaré inequality gives rise to a non-trivial $D-$structure on X (in the sense from [T],[Sh2]) and consequently to a Sobolev type space $H^{1,p}(X)$, to which the differential operator can be uniquely extended.

Recall that a *measurable vector bundle* of (real) Banach spaces on X is a collection $F = \{F_x\}$ such that to $\mu-$a.e. $x \in X$ there corresponds a Banach space $(F_x, \|\cdot\|_x)$. A *section* of the measurable vector bundle F is a real function ω on X such that for $\mu-$a.e. $x \in X$ we have $\omega(x) \in F_x$. The set of all sections of F will be denoted by $\Gamma(F)$. To every $\omega \in \Gamma(F)$ we associate the non-negative function $|\omega|$ defined by $|\omega|(x) = \|\omega(x)\|_x$. Define $L^p(X;F) := \{\omega \in \Gamma(F) : |\omega| \in L^p(X)\}$.We say that $\omega_n \to 0$ in $L^p(X;F)$ if $|\omega_n| \to 0$ in $L^p(X)$.

Set $\mathcal{L} := LIP_{loc}(X)$. Let $F = \{F_x\}$ a measurable vector bundle of Banach spaces on X. Assume that a mapping $D : \mathcal{L} \to \Gamma(F)$ has the following properties: D is linear, $|D|$ is a measurable function, $|Du|(x) \leq Lipu(x)$ for all $u \in \mathcal{L}$ and $\mu-$a.e. $x \in X$ and $Du(x) = 0$ for $\mu-$a.e. $x \in \{u = c\}$whenever $c \in \mathbb{R}$. The triple (X, F, D) is called a *weak $D-$structure* on X ([T], [Sh2]). A $D-$structure on X is a weak $D-$structure (X, F, D) such that D is a derivation, i.e. $D(uv) = v\ Du + u\ Dv$ for all $u, v \in \mathcal{L}$. A weak $D-$structure (X, F, D) is said to be non-trivial if for each $u \in \mathcal{L}$ there is an upper gradient ρ for u which equals $|Du|$a.e.

Example 2.6. Let $\{(X_\alpha, \varphi_\alpha)\}$ be a strong differentiable structure on (X, d, μ), with respect to $LIP(X)$, such that the sets X_α are mutually disjoint. For each $x \in \bigcup_\alpha X_\alpha$ (hence, for a.e. $x \in X$) there is an unique $\alpha = \alpha(x)$ such that $x \in X_{\alpha(x)}$. Let $F_x := \mathbb{R}^{N(\alpha(x))}$ be endowed with the norm $\|\lambda\|_x := Lip < \lambda, \varphi_{\alpha(x)} > (x)$ ([Ke, Lemma 6.9]). Then $F = \{F_x\}$ is a measurable vector bundle of Banach spaces, the cotangent bundle T^*X.

For $u \in \mathcal{L}$, define $du(x) = d^{\alpha(x)}(x)$ whenever u is differentiable at $x \in \bigcup_\alpha X_\alpha$. Consider the differentiation operator $d : \mathcal{L} \to \Gamma(T^*X)$. From (2.8) it follows that the differentiation operator is linear and is a derivation and that for every $x \in X_\alpha$ such that u is differentiable at x we have $Lipu(x) = \|d^\alpha u(x)\|_x$. This shows that the function $x \longmapsto \|du(x)\|_x$ is measurable on X , since $Lipu$ is measurable. Moreover, if the metric measure space is doubling it follows that $du(x) = Lipu(x) = 0$ for μ–a.e. $x \in \{u = c\}$whenever $c \in \mathbb{R}$ ([BRZ, Proposition 2.9]). If (X, d, μ) is doubling and supports a weak $(1,p)$–Poincaré inequality for some $1 \leq p < \infty$, then there exists a minimal upper gradient g_u of u such that $g_u(x) = \|du(x)\|_x$ for a.e. $x \in X$ ([C], [T, Theorem 2.1.1 (ii)]).

It follows that (X, T^*X, d) is a non-trivial D–structure whenever (X, d, μ) is doubling and supports a weak $(1,p)$–Poincaré inequality for some $1 \leq p < \infty$.

Let (X, F, D) be a weak D–structure on X and let $1 \leq p < \infty$. We define the space $H^{1,p}(X, F, D)$ as the closure of the collection $\mathcal{L}_{1,p} := \{u \in \mathcal{L} : \|u\|_{L^p(X)} + \| \, |Du| \, \|_{L^p(X)} < \infty\}$ under the norm $\|u\|_{1,p} := \|u\|_{L^p(X)} + \| \, |Du| \, \|_{L^p(X)}$. We have $f \in H^{1,p}(X, F, D)$ if and only if there exists a Cauchy sequence (u_n) in $(\mathcal{L}_{1,p}, \|\cdot\|_{1,p})$ such that $u_n \to f$ in $L^p(X)$.

The following theorem is a generalized form of a result of [FHK], which extends to metric measure spaces a theorem of Semmes [Se].

Theorem 2.7.([T, Theorem 1.4.2]. *Let (X, d, μ) be doubling and supporting a weak $(1,p)$–Poincaré inequality for some $1 \leq p < \infty$. Let (X, F, D) be a non-trivial weak D–structure on X. Then for every sequence (u_n) in $\mathcal{L}_{1,p}$ such that $u_n \to 0$ in $L^p(X)$ and $Du_n \to s$ in $L^p(X; F)$ it follows that $s = 0$.*

Corollary 2.8. *Let (X, d, μ) and (X, D, F) be as in Theorem 2.7.*

Then the following hold true:

(i) For every $f \in H^{1,p}(X, F, D)$ there exists $\omega_f \in L^p(X; F)$ such that for every Cauchy sequence (u_n) in $(\mathcal{L}_{1,p}, \|\cdot\|_{1,p})$ with $u_n \to f$ in $L^p(X)$ we have $Du_n \to \omega_f$ in $L^p(X; F)$.

(ii) Defining $Df := \omega_f$ for $f \in H^{1,p}(X, F, D)$ we obtain an extension of D from $\mathcal{L}_{1,p}$ to $H^{1,p}(X, F, D)$ and $\|f\|_{1,p} := \|f\|_{L^p(X)} + \| \, |Df| \, \|_{L^p(X)}$ is a norm on $H^{1,p}(X, F, D)$.

Proof. Fix $f \in H^{1,p}(X, F, D)$. Let (u_n) be a Cauchy sequence in $(\mathcal{L}_{1,p}, \|\cdot\|_{1,p})$ such that $u_n \to f$ in $L^p(X)$. Since $\lim\limits_{m,n \to \infty} \int\limits_X |du_m - du_n|^p \, d\mu = 0$, by standard arguments (used to prove that $L^p(X)$ is a Banach space) it follows that there is a section ω of F such that $Du_n \to \omega_f$ in $L^p(X; F)$. The fact that ω does not depend on the choice of the sequence (u_n) with the above properties follows from the preceding theorem. Now (ii) follows easily from (i).

Remark 2.9. Let (X, d, μ) be doubling and supporting a weak $(1,p)$–Poincaré inequality for some $1 \leq p < \infty$. Let (X, F, D) be the non-trivial D–structure considered in Example 2.6. The Sobolev-type space $H^{1,p}(X) := H^{1,p}(X, F, D)$ with the norm $\|\cdot\|_{1,p}$ is isometrically isomorphic to the Newtonian space $N^{1,p}(X)$ with the norm $\|\cdot\|_{N^{1,p}}$, by [T, Remark 2.2.3] and [Shan, Theorem 4.10].

In [BRZ] is explained in a constructive manner how the differentiation operator can be extended from $LIP_{loc}(X) \cap N^{1,p}(X)$ to $N^{1,p}(X)$.

3. Calderon differentiability theorem and a Calderon-Zygmund theorem

Calderón differentiability theorem shows that every function in the Sobolev space $W^{1,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain and $p > n$, is differentiable a.e. and its pointwise gradient and distributional gradient agree a.e.This fundamental result has been extended to in [BRZ], Theorems 4.1 and 4.4. In what follows, we will give shorter proofs to the results mentioned above and we will prove an extension of a Calderón-Zygmund theorem.

Theorem 3.1. ([BRZ], Theorem 4.1) *Let (X, d, μ) be a doubling metric measure space, with a homogeneous dimension Q. Assume that $u : X \to \mathbb{R}$ is a measurable function and $g \in L^p_{loc}(X)$, with $p \geq 1$ and $p > Q$. If the pair (u, g) satisfies a weak $(1,p)-$Poincaré inequality then:*

(i) u has a locally $(1 - Q/p)-$Hölder continuous representative;

(ii) $Lip\, u(x) < \infty$ at every Lebesgue point of g^p;

(iii) u is $\mu-$a.e. differentiable with respect to any strong measurable differentiable structure for (X, d, μ), if such a structure exists.

Proof. By Lemma 2.2 there exists a representative of u satisfying (2.7), in particular (i) holds. Setting $a := x$ and $r := 2d(x, y)$ in (2.7) we get

$$\frac{|u(x) - u(y)|}{d(x,y)} \leq C \left(\frac{1}{\mu(B(x, \lambda d(x,y)))} \int\limits_{B(x,\lambda d(x,y))} g^p d\mu \right)^{1/p}, \tag{3.1}$$

for every $y \neq x$, where $C > 0$ is a constant and $\lambda = 10\tau$.

If $x \in X$ is not isolated, by (3.1), (2.4) and $Lip\, u(x) = \limsup\limits_{y \to x} \frac{|u(x)-u(y)|}{d(x,y)}$ it follows

$$Lip\, u(x) \leq C \limsup_{r \to 0} \left(\frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} g^p d\mu \right)^{1/p},$$

hence, if x is a Lebesgue point of g^p we have $Lip\, u(x) \leq Cg(x)$. Since $Lip\, u(x) = 0$ whenever x is an isolated point, (ii) is proven.

Now (iii) follows from (ii) and Stepanov differentiability theorem, Theorem 1.1.

Remark 3.2. The proof of [BRZ] has the advantage that it still works if we assume that u and g are given only on a domain $\Omega \subset X$.

The following lemma is contained in the proofs of Proposition 7.32 from [Ke] and Theorem 4.4 from [BRZ]. We give the proof for the sake of completness.

Lemma 3.3. *Let (X, d, μ) , p, $\{(X_\alpha, \varphi_\alpha)\}$ and D be as in Theorem 3.4. Assume that* $u \in H^{1,p}(X)$ and $\omega \in LIP_{loc}(X)$. *Define* $g(x) := \|Du(x) - d\omega(x)\|_x$

for a.e. $x \in X$. Then $g \in L^p_{loc}(X)$ and the pair $(u - \omega, g)$ satisfies a weak $(1,p)$–Poincaré inequality. Moreover, if ω is a constant function then $g \in L^p(X)$.

Proof. Let (u_n) be a sequence of functions in $LIP_{loc}(X) \cap H^{1,p}(X)$ such that (u_n) converges to u in $L^p(X)$ and $du_n \to Du$ in $L^p(X; T^*X)$:

$$\lim_{n\to\infty} \int_X \|du_n(x) - Du(x)\|_x^p \, d\mu = 0$$

In what follows, $B = B(x_0, r)$ is an arbitrary ball and $\tau B := B(x_0, \tau r)$.

(i) Set $g_n(x) := \|du_n(x)\|_x$ for a.e. $x \in X$, $n \geq 1$. We have $\rho_n \in L^p(X)$ and $\|\rho_n - \rho\|_{L^p(X)} \to 0$ as $n \to \infty$, hence $\rho \in L^p(X)$ and $\|\rho_n\|_{L^p(X)} \to \|\rho\|_{L^p(X)}$ as $n \to \infty$.

Since ρ_n is a p–weak upper gradient of u_n,

$$\frac{1}{\mu(B)} \int_B |u_n - (u_n)_B| \, d\mu \leq Cr \left(\frac{1}{\mu(\tau B)} \int_{\tau B} \rho_n^p d\mu \right)^{1/p},$$

for each $n \geq 1$. Letting $n \to \infty$ we obtain the corresponding weak $(1,p)$–Poincaré inequality for the pair (u, ρ).

(ii) Let $v_n := u_n - \omega$ for $n \geq 1$ and $v := u - \omega$. We have $v_n \in LIP_{loc}(X)$ and $dv_n(x) = du_n(x) - d\omega(x)$ for a.e. $x \in X$. For a.e. $x \in X$, set $\zeta(x) := Du(x) - d\omega(x)$, $g_n(x) = \|du_n(x) - d\omega(x)\|_x$ and $g(x) = \|Du(x) - d\omega(x)\|_x$. For every ball B, we have:

a) $v_n \to v$ in $L^p(B)$;

b) $g_n \in L^p(B)$ and $\|g_n - g\|_{L^p(X)} \to 0$ as $n \to \infty$, hence $g \in L^p(B)$ and $\|\rho_n\|_{L^p(B)} \to \|\rho\|_{L^p(B)}$ as $n \to \infty$. If ω is a constant, we may take X instead of B in the preceding argument.

In particular, a) implies $(v_n)_B \to v_B$ as $n \to \infty$.

Since g_n is a p–weak upper gradient of v_n,

$$\frac{1}{\mu(B)} \int_B |v_n - (v_n)_B| \, d\mu \leq Cr \left(\frac{1}{\mu(\tau B)} \int_{\tau B} g_n^p d\mu \right)^{1/p},$$

for each $n \geq 1$. Letting $n \to \infty$ we obtain the corresponding weak $(1,p)$–Poincaré inequality for the pair $(u - \omega, g)$.

Theorem 3.4. *Assume that (X, d, μ) is a doubling metric measure space, with a homogeneous dimension Q and supports a weak $(1,p)$–Poincaré inequality for some $p \geq 1$. Let $\{(X_\alpha, \varphi_\alpha)\}$ be a strong measurable differentiable structure for (X, d, μ), such that X_α are mutually disjoint. Let D be the unique extension of the differential operator d from $Lip_{loc}(X)$ to $H^{1,p}(X)$. Let $s > 0$ such that $p > \frac{Qs}{Q+s}$.*

Then for every $u \in H^{1,p}(X)$, for each α and $\mu-$a.e. $x_0 \in X_\alpha$ we have

$$\lim_{r\to 0} \frac{1}{r}\left(\frac{1}{\mu(B(x_0,r)}\int_{B(x_0,r)} |u(x)-u(x_0)-<Du(x_0),\varphi_\alpha(x)-\varphi_\alpha(x_0)>|^s\, d\mu\right)^{1/s}.$$

Moreover, if $p > Q$ then u is differentiable $\mu-$a.e. with respect to the given strong measurable differentiable structure and $du(x_0) = Du(x_0)$ for $\mu-$a.e. $x_0 \in X$.

Proof. Using, if necessary, Hölder inequality we may assume that $s \geq 1$.

Fix a coordinate chart $(X_\alpha, \varphi_\alpha)$ and let $x_0 \in X_\alpha$. Denote $Du(x_0) = \lambda = (\lambda_1, \ldots \lambda_{N(\alpha)}) \in \mathbb{R}^{N(\alpha)}$. Consider the function $v : X \to \mathbb{R}$ defined by $v(x) := u(x) - u(x_0) - < Du(x_0), \varphi_\alpha(x) - \varphi_\alpha(x_0) >$. Notice that $v(x_0) = 0$.

Define $\omega(x) := u(x_0) + < Du(x_0), \varphi_\alpha(x) - \varphi_\alpha(x_0) >$ for $x \in X$. Then $\omega \in LIP(X)$ and $d\omega(x) = \sum_{i=1}^{N(\alpha)} \lambda_i d\varphi_\alpha^i(x)$ for a.e. $x \in X$.

Set $\zeta(x) := Du(x) - d\omega(x)$ and $g(x) := \|\zeta(x)\|_x$, for a.e. $x \in X$.

By Lemma 3.3, we have $v, g \in L^p_{loc}(X)$ and the pair (v, g) satisfies a weak $(1,p)-$Poincaré inequality . According to Lemma 2.2, the pair (v, g) satisfies a weak $(t,p)-$Poincaré inequality, where $1 \leq t < \frac{Qp}{Q-p}$ if $p < Q$ and $t \geq 1$ if $p \geq Q$. In particular, for every $r > 0$

$$\left(\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)} \left|v - v_{B(x_0,r)}\right|^t d\mu\right)^{1/t} \leq Cr\left(\frac{1}{\mu(B(x_0,5\tau r))}\int_{B(x_0,5\tau r)} g^p d\mu\right)^{1/p},$$

The above inequality and Minkowski's inequality imply

$$\left(\frac{1}{\mu(B(x_0,r))}\int_{B(x_0,r)} |v|^t\, d\mu\right)^{1/t} \leq Cr\left(\frac{1}{\mu(B(x_0,5\tau r))}\int_{B(x_0,5\tau r)} g^p d\mu\right)^{1/p} + \left|v_{B(x_0,r)}\right| \tag{3.2}$$

for every $r > 0$.

Using the weak $(1,p)-$Poincaré inequality satisfied by (v, g) together with the doubling property of μ and assuming that x_0 is a Lebesgue point for v it is proven in [Ke], Lemma 7.38 that

$$\left|v_{B(x_0,r)}\right| \leq C\, r \sup_{h\leq \tau r}\left(\frac{1}{\mu(B(x_0,h))}\int_{B(x_0,h)} g^p d\mu\right)^{1/p}. \tag{3.3}$$

In the proof of Theorem 4.4 [BRZ] the authors have shown that

$$\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)} g^p d\mu = 0, \tag{3.4}$$

provided that $x \in X_\alpha$ satisfies the following conditions:

a) x is a Lebesgue point of u ;

b) x is a density point of X_α and

c) $\lim_{r\to 0}\frac{1}{\mu(B(x,r))}\int\limits_{B(x,r)\setminus X_\alpha} \|Du(y)\|_y^p \, d\mu = 0.$

Since the function $y \longmapsto \|Du(y)\|_y$ is in $L^p(X)$ and the measure μ is doubling, condition c) is satisfied for a.e. $x \in X_\alpha$. Let S_α be the set of all $x \in X_\alpha$ that satisfy conditions a),b), c) and, in addition, are Lebesgue points of u. Then $\mu(X_\alpha \setminus S_\alpha) = 0$.

Assuming $x_0 \in S_\alpha$, applying (3.2), (3.3) and (3.4) we get

$$\lim_{r\to 0}\frac{1}{r}\left(\frac{1}{\mu(B(x_0,r))}\int\limits_{B(x_0,r)} |v|^t \, d\mu\right)^{1/t} = 0.$$

By our assumptions on s we can take $t = s$ in the equation above and the first claim follows.

If $p > Q$, applying the second part of Lemma 2.2 and an argument from the proof of Theorem 3.1, it follows that v has a locally $(1 - Q/p)-$Hölder continuous representative and $\frac{|v(x)-v(x_0)|}{d(x,x_0)} \leq C\left(\frac{1}{\mu(B(x_0,\lambda d(x,x_0)))}\int\limits_{B(x_0,\lambda d(x,x_0))} g^p d\mu\right)^{1/p}$ whenever $x \neq x_0$. If $x_0 \in S_\alpha$ (actually, if $x_0 \in X_\alpha$ satisfies conditions a), b) and c) then applying (3.4) and (2.4) and taking into account that $v(x_0) = 0$, it follows that

$$\limsup_{x\to x_0}\frac{|v(x)|}{d(x,x_0)} = 0.$$

We proved that u is differentiable at x_0 with respect to the strong measurable differentiable structure $\{(X_\alpha, \varphi_\alpha)\}$ and, by the uniqueness of the differential, $du(x_0) = Du(x_0)$.

Corollary 3.5. *Let $u \in W^{1,p}_{loc}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain. If $s > 0$ and $p > \frac{ns}{n+s}$ then for almost all points $a \in \Omega$ we have*

$$\lim_{r\to 0}\frac{1}{r^{n+s}}\int\limits_{B(a,r)} |u(x) - u(a) - < \nabla u(a), x - a >|^s \, dx = 0.$$

Proof. Let $X = \mathbb{R}^n$ with the Euclidean distance and the Lebesgue measure, endowed with the canonical strong measurable differentiable structure $\{(X_1, \varphi_1)\}$

where $X_1 = X$ and φ_1 is the identity of X. We may take $Q = n$. We have $H^{1,p}(X) = W^{1,p}(\mathbb{R}^n)$.

Since the claim is local, it suffices to prove it for the restriction of u to an arbitrary ball $B \subset \Omega$. Since every Euclidean ball is a smooth domain, extending $u|_B$ with zero outside B we get a Sobolev function in $W^{1,p}(\mathbb{R}^n)$, which we denote by $\widetilde{u}$. Applying Theorem 3.4 to $\widetilde{u}$ the claim follows.

Remark 3.6.

1. For $s = 1$ the above theorem gives a result of Björn [B], proved under more general assumptions by Keith ([Ke], Proposition 7.32). Assuming in addition that the metric space is complete, Keith proved that $Du(x)$ is an approximate diferential of u for a.e. $x \in X$.

2. Corollary 3.5 implies the Calderón-Zygmund theorem for mappings $f \in W^{1,p}_{loc}(\Omega, \mathbb{R}^n)$ (see [IM], Theorem 4.4.2).

3. Taking $s = p$ in Corollary 3.5 we give an alternative proof to the following theorem of Reshetnyak (see [Re], Theorem 4.3, page 334): If $\Omega \subset \mathbb{R}^n$ is a domain, $p \geq 1$ and $u \in W^{1,p}_{loc}(\Omega)$ is a function, then for almost every point $a \in \Omega$ the distributional gradient $\nabla u(a)$ is the differential of u at a in the sense of the convergence in $W^{1,p}$. This means that $\lim_{h\to 0} \|R_{h,a}\|_{W^{1,p}(B)} = 0$, where B is the unit ball in $\mathbb{R}^n$ and $R_{h,a}(Y) = \frac{1}{h}(u(a+hY) - u(a) - < \nabla u(a), hY >)$. To see this , we notice that by Corollary 3.5, for almost every $a \in \Omega$ we have $\lim_{h\to 0} \|R_{h,a}\|^p_{L^p(B)} = \lim_{h\to 0} \frac{1}{|h|^{n+p}} \int_{B(a,|h|)} |u(x) - u(a) - < \nabla u(a), x - a >|^p \, dx = 0$, while $\lim_{h\to 0} \|\nabla R_{h,a}\|^p_{L^p(B)} = \Omega_n \lim_{h\to 0} \frac{1}{\mu(B(a,|h|))} \int_{B(a,|h|)} |\nabla u(x) - \nabla u(a)|^p \, dx = 0$ for every Lebesgue point a of ∇u. (Here Ω_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$).

4. Differentiability of monotone Sobolev functions

Let $\Omega \subset \mathbb{R}^n$ be a domain. A function $u : \Omega \to \mathbb{R}$ is said to be monotone if u is continuous and $osc(u, D) = osc(u, \partial D)$ for every domain $D \subset\subset \Omega$. Here $osc(u, A) = \sup\{|u(x) - u(y)| : x, y \in A\}$.

Using a method of Väisälä which is an $n-$dimensional version of a technique used by Gehring and Lehto, Rickman [Ri] showed that every monotone Sobolev function $u \in W^{1,p}(\Omega)$ with $p > n - 1$ is differentiable almost everywhere . An important consequence of this theorem is a proof of the differentiability a.e. of quasiregular mappings. Recently, Onninen [O] proved a sharp integrability condition on the partial derivatives of a weakly monotone Sobolev functions, that guarantees the differentiability a.e. of the function. Unfortunately, the methods from the proofs of Rickman and Onninen are unlikely to be extended to the setting of metric measure spaces.

Recall that in a metric measure space supporting a Poincaré inequality every ball whose complement is non-empty has a non-empty boundary, hence small spheres are non-empty. Denote $S(x_0, r) = \{x \in X : d(x, x_0) = r\}$, $x_0 \in X$, $r > 0$.

We will extend Rickman 's lemma to doubling metric measure spaces supporting a Poincaré inequality. The main tool we use is a Sobolev embedding theorem on spheres proven by Hajłasz and Koskela ([HaKo], Theorem 7.1):

Lemma 4.1. ([HaKo]) *Let (X,d,μ) be a doubling metric measure space, with a homogeneous dimension Q. Assume that the pair (u,g) satisfies a weak $(1,p)-$Poincaré inequality for some $p > Q-1$, $p > 0$. Let $x_0 \in X$ and $r_0 > 0$. Then:*

(i) The restriction of u to $S(x_0,r)$ is uniformly $(1-(Q-1)/p)-$Hölder continuous for almost every $0 < r < r_0$;

(ii) There exists a constant $C_1 > 0$, depending only on p, Q, C_P, C_b, C_d and a radius $r_0/2 < r < r_0$ such that:

$$|u(x)-u(y)| \leq C_1 d(x,y)^{1-(Q-1)/p} r_0^{(Q-1)/p} \left(\frac{1}{\mu(B(x_0,5\tau r_0))} \int_{B(x_0,5\tau r_0)} g^p d\mu \right)^{1/p} \tag{4.1}$$

for every $x,y \in S(x_0,r)$.

In what follows, we say that a function $u : X \to \mathbb{R}$ is monotone if

$$osc(u, B(x_0,r)) \leq osc(S(x_0,r)),$$

for every $x_0 \in X$ and every $r > 0$ such that $S(x_0,r)$ is non-empty (see [HaKo], page 36). Note that this definition of monotone functions is less restrictive than the definition used in the Euclidean case.

Theorem 4.2. *Let (X,d,μ) be a doubling metric measure space, with a homogeneous dimension Q. Assume that for every point $x_0 \in X$ there exists $R(x_0) > 0$ such that $S(x_0,r)$ is non-empty for every $0 < r < R(x_0)$. Let $u : X \to \mathbb{R}$ be a monotone function. Assume that the pair (u,g) satisfies a weak $(1,p)-$Poincaré inequality with $p > Q-1$, $p \geq 1$* and $g \in L^p_{loc}(X)$.

Then $Lip\, u(x) < \infty$ for $\mu-$ a.e. $x \in X$. If X admits a strong measurable differentiable structure then u is differentiable $\mu-$ a.e. with respect to this structure.

Proof. Let $x_0 \in X$ be fixed. Set $R := R(x_0)$. For every $z \in B(x_0,R/2)$ there exists a positive integer $k = k(z)$ such that

$$2^{-k-1}R \leq d(z,x_0) < 2^{-k}R. \tag{4.2}$$

Apply to u Lemma 4.1 with $r_0 = 2^{-k+1}R$. There exists a radius $2^{-k}R < r < 2^{-k+1}R$ such that (4.1) holds for every $x,y \in S(x_0,r)$. It follows that

$$osc(u, S(x_0,r)) \leq C_2 r \left(\frac{1}{\mu(B(x_0,5\tau r_0))} \int_{B(x_0,5\tau r_0)} g^p d\mu \right)^{1/p}, \tag{4.3}$$

where $C_2 = 2^{(Q-1)/p}C_1$. Since u is a monotone function, $|u(z) - u(x_0)| \leq osc(u, B(x_0, r)) \leq osc(u, S(x_0, r))$. Using this inequality, (4.2) , (4.3) and the doubling property of μ we get

$$\frac{|u(z) - u(x_0)|}{d(z, x_0)} \leq C_3 \left(\frac{1}{\mu(B(x_0, \lambda d(z, x_0)))} \int\limits_{B(x_0, \lambda d(z, x_0))} g^p d\mu \right)^{1/p},$$

where $C_3 = 4C_d C_2$ and $\lambda = 20\tau$.

Assuming that x_0 is a Lebesgue point of g^p, taking $\limsup_{z \to x_0}$ in the latter inequality we obtain $Lip\, u(x_0) \leq C_3 g(x_0) < \infty$.

The second claim follows by Stepanov differentiability theorem, Theorem 1.1.

Corollary 4.3. *Assume that (X, d, μ) is a doubling metric measure space, with a homogeneous dimension Q and supports a weak $(1,p)-$Poincaré inequality for some $p \geq 1$. If $p > Q - 1$ then every monotone function $u \in N^{1,p}(X)$ is differentiable $\mu-a.e.$ with respect to any strong measurable differentiable structure for (X, d, μ).*

5. Regularity of quasiminimizers

In what follows $1 < p < \infty$ and $\Omega \subset X$ is an open set .

A function $u \in N^{1,p}_{loc}(\Omega)$ is said to be a *quasiminimizer* (for the Dirichlet $p-$energy integral) on Ω if there exists a constant $K \geq 1$ such that for all bounded open sets Ω' for which the closure is in Ω and $v \in N^{1,p}(\Omega')$ with $u - v \in N^{1,p}_0(\Omega')$ we have

$$\int\limits_{\Omega' \cap \{u \neq v\}} g_u^p d\mu \leq \int\limits_{\Omega' \cap \{u \neq v\}} g_v^p d\mu,$$

where g_u and g_v are the minimal $p-$weak upper gradients of u and v respectively.

Regularity properties of quasiminimizers have been thoroughly studied by Kinnunen and Shanmugalingam [KSh] by using De Giorgi method. They proved that quasiminimizers satisfy Harnack inequality, a strong maximum principle and are locally Hölder continuous, if the metric measure space is doubling and supports a $(1, q)-$Poincaré inequality for some $1 < q < p$. The potential theory of quasiminimizers has been studied in [KM].

A very important tool in the study of quasiminimizers is the following Caccioppoli type estimate on distribution sets.

Definition 5.1. ([KSh])We say that a function $u \in N^{1,p}_{loc}(\Omega)$ belongs to the *De Giorgi class* $DG_p(\Omega)$ if there exists a constant $c > 0$ such that for all $k \in \mathbb{R}$, $z \in \Omega$ and $0 < r < R < diam(X)/3$ so that $B(z, R) \subset \Omega$, we have

$$\int_{A_z(k,r)} g_u^p d\mu \leq \frac{c}{(R-r)^p} \int_{A_z(k,R)} (u-k)^p d\mu, \tag{5.1}$$

where $A_z(k,\rho) := \{x \in B(z,\rho) : u(x) > k\}$.

Kinnunen and Shanmugalingam proved that for every quasiminimizer $u \in N^{1,p}_{loc}(\Omega)$ in Ω the functions $\pm u$ belong to the De Giorgi class $DG_p(\Omega)$ ([KSh], Proposition 3.3). We derive further consequences of this property.

Let $u \in N^{1,p}_{loc}(\Omega)$ be a quasiminimizer. For all $k \in \mathbb{R}$ the function $u-k$ is a quasiminimizer, with a minimal $p-$weak upper gradient $g_{u-k} = g_u$. Applying (5.1) to $u-k$ and $(-u+k)$ and adding the two inequalities we get

$$\int_{B(z,r)} g_u^p d\mu \leq \frac{c}{(R-r)^p} \int_{B(z,R)} |u-k|^p \, d\mu.$$

For $k := u_{B(z,R)}$ this implies

$$\left(\frac{1}{\mu(B(z,r))} \int_{B(z,r)} g_u^p d\mu \right)^{1/p} \leq \frac{c}{(R-r)} \rho_p(r,R) \left(\frac{1}{\mu(B(z,R))} \int_{B(z,R)} \left|u - u_{B(z,R)}\right|^p d\mu \right)^{1/p}, \tag{5.2}$$

where we denoted $\rho_p(r,R) = \left(\frac{\mu(B(z,R))}{\mu(B(z,r))}\right)^{1/p}$.

From now on we assume that the metric measure space (X,d,μ) is doubling and supports a $(1,q)-$Poincaré inequality for some $1 < q < p$. We show that the minimal $p-$ weak upper gradient of a quasiminimizer satisfies a reverse Hölder inequality.

By Lemma 2.2, for every $u \in N^{1,p}_{loc}(X)$ the following $(p,q)-$Poincaré inequality holds, for all balls $B(z,R)$:

$$\left(\frac{1}{\mu(B(z,R))} \int_{B(z,R)} \left|u - u_{B(z,R)}\right|^p d\mu \right)^{1/p} \leq CR \left(\frac{1}{\mu(B(z,5\tau R))} \int_{B(z,5\tau R)} g_u^q d\mu \right)^{1/q}, \tag{5.3}$$

where $q < Q$ and $q < p < \frac{Qq}{Q-q}$ or $p > q \geq Q$.

Combining (5.2) and (5.3), then taking $R = 2r$ we obtain the following weak reverse Hölder inequality:

$$\left(\frac{1}{\mu(B(z,r))} \int_{B(z,r)} g_u^p d\mu \right)^{1/p} \leq b \left(\frac{1}{\mu(B(z,\sigma r))} \int_{B(z,\sigma r)} g_u^q d\mu \right)^{1/q}, \tag{5.4}$$

where $\sigma = 10\tau$ and b is a constant depending only on the data of X, $z \in X$ is an arbitrary point and the radius $0 < r < diam(X)/6$.

We proved the following

Proposition 5.2. *Let (X, d, μ) be a doubling metric measure space, with a homogeneous dimension Q, supporting a $(1, q)-$Poincaré inequality for some $1 < q < \infty$. Let $1 < q < p$ such that $p < \frac{Qq}{Q-q}$ if $q < Q$. Then there exist some constants $\sigma > 1$ and $C > 0$ depending only on the data of X such that, for every quasiminimizer $u \in N^{1,p}_{loc}(X)$, the minimal $p-$weak upper gradient g_u of u satisfies the weak reverse Hölder inequality (5.4).*

In the Euclidean case Gehring's lemma is a crucial tool which enables us to derive from reverse Hölder inequalities satisfied by a function (in applications, by the norm of the gradient of a solution to a certain PDE) a degree of integrability higher than the a priori one (see [IM]). Gehring's Lemma has been extended to doubling metric measure spaces by Zatorska-Goldstein, in a generalized and weak form.

Lemma 5.3.[Z-G] *Let (X, d, μ) be a doubling metric measure space, with a homogeneous dimension $Q > 1/2$.* Let $q_0 > 1$, $s \in [q_0, 2Q]$ and $\sigma > 1$. *Assume the functions f, g to be nonnegative and such that $g \in L^q_{loc}(X)$, $f \in L^{r_0}_{loc}(X)$ for some $r_0 > s$. Assume that there exist constants $b > 1$ and θ such that for every ball B the following inequality holds :*

$$\begin{aligned}\frac{1}{\mu(B)}\int_B g^s d\mu \;&\leq\; b\left[\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g d\mu\right)^s + \frac{1}{\mu(\sigma B)}\int_{\sigma B} f^{\,s} d\mu\right] + \qquad (5.5)\\ &\theta\frac{1}{\mu(\sigma B)}\int_{\sigma B} g^s d\mu\end{aligned}$$

Then there exist positive constants θ_0, ε_0 and C such that if $0 \leq \theta < \theta_0$ then for every $s \leq t < s + \varepsilon_0$ we have $g \in L^t_{loc}(X)$ and

$$\left(\frac{1}{\mu(B)}\int_B g^t d\mu\right)^{1/t} \leq C\left[\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g^s d\mu\right)^{1/s} + \left(\frac{1}{\mu(\sigma B)}\int_{\sigma B} f^{\,t} d\mu\right)^{1/t}\right] \qquad (5.6)$$

Here $\theta_0 = \theta_0(q_0, Q, C_d, \sigma)$, $\varepsilon_0 = \varepsilon_0(b, q_0, Q, C_d, \sigma)$ and $C = C(b, q_0, Q, C_d, \sigma)$.

Theorem 5.4. *Let (X, d, μ) be a doubling metric measure space, with a homogeneous dimension $Q > 1/2$, supporting a weak $(1, q)-$Poincaré inequality for some $1 < q < \infty$. Let p such that $q < p \leq 2Qq$ and, in addition, $p < \frac{Qq}{Q-q}$ if $q < Q$.*

Then there exist positive constants ε_0 and C such that for every quasiminimizer $u \in N^{1,p}_{loc}(X)$ and for each $0 \le \varepsilon < q\varepsilon_0$ we have $g_u \in L^{p+\varepsilon}_{loc}(X)$ and, moreover

$$\left(\frac{1}{\mu(B)}\int_B g_u^{p+\varepsilon} d\mu\right)^{1/(p+\varepsilon)} \le C\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g_u^p d\mu\right)^{1/p}.$$

Here ε_0 and C depend only on Q, C_d, τ and on q, p.

Proof. Let $u \in N^{1,p}_{loc}(X)$ be a quasiminimizer. According to (5.4), inequality (5.5) is satisfied for $g := g_u^q$, $f = 0$, $s = p/q$ and $\theta = 0$,whenever B is a ball in X. We fix an arbitrary number $1 < q_0 \le s \le 2Q$. Applying Lemma 5.3 it follows that there exist positive constants ε_0 and C, depending only on q, p and the data of X, such that for every $s \le t < s + \varepsilon_0$ we have $g_u^q \in L^t_{loc}(X)$ and

$$\left(\frac{1}{\mu(B)}\int_B g_u^{qt} d\mu\right)^{1/qt} \le C\left(\frac{1}{\mu(\sigma B)}\int_{\sigma B} g_u^{qs} d\mu\right)^{1/qs}$$

for every ball B. Setting $\varepsilon := qt - p$ the proof is completed.

Corollary 5.5. *Let (X, d, μ) be a doubling metric measure space , with a homogeneous dimension $Q > 1$, supporting a weak $(1, q)-$Poincaré inequality for some $1 < q < Q$. If $u \in N^{1,Q}_{loc}(X)$ is a quasiminimizer then:*

(i) u has a representative which is locally $\alpha-$Hölder continuous for some $0 < \alpha < 1$ not depending on u;

(ii) $Lip\, u(x) < \infty$ for $\mu-a.e.$ $x \in X$ and

(iii) u is differentiable $\mu-a.e.$ with respect to any strong measurable differentiable structure for (X, d, μ) .

Proof. We may assume that $Q/2 < q < Q$, since a $(1, q)-$Poincaré inequality implies a $(1, q')-$Poincaré inequality for every $q \le q' < \infty$. Then $p := Q$ satisfies the conditions $q < p \le 2Qq$ and $p < \frac{Qq}{Q-q}$.

Let $u \in N^{1,Q}_{loc}(X)$ be a quasiminimizer. According to Theorem 5.4, there exists $\varepsilon > 0$ (depending only on q and on the data of X) such that $g_u \in L^{Q+\varepsilon}_{loc}(X)$. Notice that the pair (u, g_u) satisfies a weak $(1, Q+\varepsilon)-$Poincaré inequality. Then we may apply Theorem 3.1 (the basic Calderón type result from [BRZ]), whence all the claims follow.

Remark 4.1.

1. We cannot weaken the assumptions of Corollary 5.5 so that $u \in N^{1,p}_{loc}(X)$ for some $q < p < Q$; we get $g_u \in L^{p+\varepsilon}_{loc}(X)$ for some positive ε and we have no guarantee that $p + \varepsilon > Q$.

2. It would be interesting to extend the results from this paragraph to the case when the quasiminimizer is defined only on a domain $\Omega \subset X$, not on the entire space X.

3. By a deep result of Keith and Zhong [KeZ] , if the metric measure space (X, d, μ) is *complete*,doubling and supports a weak $(1,p)-$Poincaré inequality for some $1 < p < \infty$ then there exists $\varepsilon > 0$ such that (X, d, μ) supports a weak $(1,q)-$Poincaré inequality for every $q > p - \varepsilon$. It turns out that in the last corollary it suffices to assume that (X, d, μ) supports a weak $(1, Q)-$Poincaré if the metric space is complete.

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