Viscous flow of the molten powder in meniscus zone
with elastic boundary

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Abstract. This paper is concerned with the hydrodynamic behaviour of the molten powder in meniscus zone, when the solidifying shell is considered an elastic medium. The physical problem is modelled by the coupled system: Stokes and Sophie-Germain equations. A variational approach of the problem allows us to obtain existence, uniqueness and regularity results. The variation of the pressure of the molten powder with respect to its viscosity is studied. We propose an algorithm to determine the maximum value of the viscosity which corresponds to a desired configuration of the molten powder flow.

1. Introduction

In this paper we generalize the problem studied in [1]. In [1] we proposed a mathematical model to describe the hydrodynamic behaviour of the molten powder in meniscus zone. This model corresponds to the simplified case when the surfaces to be lubricated are supposed to be as those of rigid bodies and it fails to explain the relation between the deformation of the solidifying shell and the pressure of the molten powder film.

The purpose of the present paper is to propose a model which represents a better description of the considered physical problem. We study the case when the solidifying shell is an elastic surface. In [2] the authors discuss some numerical results concerning this case.

The molten powder occupies the mold-strand space which represents a thin gap between two surfaces: the mold, $S_1$, and the strand, $S_2$. Usually, these are two nonparallel surfaces since the meniscus has a certain radius of curvature. To simplify the geometry of the flow domain, we shall suppose that this radius is such that the flow of the molten powder between two parallel surfaces represents a good approximation of the problem.

In section 2 we introduce and discuss the mathematical model for the considered fluid-structure interaction problem. The slow flow of the powder film is simulated by the Stokes equations and the solidifying shell behaviour is described by the Sophie-Germain equation for the transversal displacement.

Section 3 deals with the variational formulation of the problem. Existence, uniqueness and regularity results are obtained.

In the last section we study the variation of the pressure of the molten powder with respect to its viscosity. We propose a method to determine the maximum value of the viscosity for which the flow of the molten powder has a desired configuration.

2. The model for hydrodynamic behaviour of the molten powder in the mold-strand gap

We study the non steady motion of the powder film between two parallel plane surfaces: the mold, $S_1$, which is a rigid surface and the solidifying shell of the strand, $S_2$, which is elastic. Let $Ox_1x_3$ be the plane of the surface $S_1$ and $Ox_2$ the axis normal to the mold. We denote by $\varepsilon$ the ratio of the thickness of the powder film to its length, with $0 < \varepsilon << 1$. If we suppose that the motion does
We define \( v = u - u_0 \) and we obtain for \((v, p, d)\) the following problem:

\[
\begin{align*}
\rho f \frac{\partial v}{\partial t} - \mu \Delta v + \nabla p &= \bar{f} \text{ in } D_x \times (0, T), \\
\text{div } v &= 0 \text{ in } D_x \times (0, T), \\
v &= 0 \text{ on } (\partial D_x \setminus \Gamma_e) \times (0, T), \\
v_1 &= 0 \text{ and } v_2 = \frac{\partial d}{\partial t} \text{ on } \Gamma_e \times (0, T), \\
v(x, 0) &= 0 \text{ in } D_x,
\end{align*}
\]

(2.5)

\[
\begin{align*}
\rho h^2 \frac{\partial^2 d}{\partial t^2} + \frac{h^3 E}{12} \frac{\partial^4 d}{\partial x_1^4} + \nu \frac{\partial^4 d}{\partial x_1^4} \left( \frac{\partial d}{\partial t} \right) &= g(x_1, t) + (T_f n)_2 \text{ on } \Gamma_e \times (0, T), \\
d(0, t) = d(1, t) &= \frac{\partial d}{\partial x_1}(0, t) = \frac{\partial d}{\partial x_1}(1, t) = 0 \text{ in } (0, T), \\
d(x_1, 0) &= \frac{\partial d}{\partial t}(x_1, 0) = 0 \text{ in } (0, 1),
\end{align*}
\]

where \( \bar{f} = f - \rho f \frac{\partial u_0}{\partial t} + \mu \Delta u_0 \). Boundary condition (2.5) for \( v_2 \) uses the Eulerian coordinates for the velocity \( v \) and the Lagrangian coordinates for the transversal displacement of the wall \( d \), and so this condition is a linear approximation for a more precise non-linear interface condition (see [5], [6]) which takes into account the passage from one coordinate system to another one. However, in the case of small strains, the error of this linearization is of the same order that the error of the linearization of the strain tensor in the linear elasticity theory.

In the sequel the velocity will be the new function \( v \); it will give the properties of \( u \).

The unknowns of this system are: the velocity of the fluid, \( v \), the pressure of the fluid, \( p \), and the displacement of the elastic wall, \( d \). The fluid flow is described by the non-stationary Stokes equations. A "viscous" type term, \( \nu \frac{\partial^2 d}{\partial x_1^2} \) was added to the usual fourth-order equation for the normal displacement. This additional term will ensure more regularity for the unknowns.

The action of the viscous fluid on the elastic wall is represented by the stress tensor \( T_f = T_f(u, p) \) which is defined by

\[
T_f(u, p) = pI - \mu (\nabla u + (\nabla u)^T).
\]

On the boundary \( \Gamma_e \) \( n = (0, 1) \); hence

\[
(T_f n)_2 = p - 2\mu \frac{\partial u_2}{\partial x_2} \text{ on } \Gamma_e \times (0, T).
\]

If we formally consider \( \text{div } u = 0 \) on \( \Gamma_e \times (0, T) \), from (2.3) it follows that:

\[
\frac{\partial u_2}{\partial x_2} = 0.
\]

Hence, the surface force exerted by the fluid on the elastic boundary can be defined by:

(2.6)

\[
(T_f n)_2 = p.
\]

The compatibility condition for the coupled system which describes the physical problem is

\[
0 = \int_{\partial D_x} u(x, t) \cdot n \, ds_x = \int_0^1 u_2(x_1, \varepsilon, t) \, d\varepsilon = \frac{d}{dt} \left( \int_0^1 d(x_1, t) \, dx_1 \right).
\]

It follows that \( \int_0^1 d(x_1, t) \, dx_1 = \text{constant} \) for all \( t \in (0, T) \). Using next the initial condition for \( d \), we obtain the constant equal to zero.
We take \( b = 0 \) in (3.3). Then:
\[
\langle \rho_f \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} - \mathbf{f}, \varphi \rangle = 0 \quad \forall \varphi \in \{ w \in (H^3_0(D)) : \text{div} \, w = 0 \text{ in } D_e \}.
\]
From De Rham lemma, it follows that there exists a distribution \( q \), unique up to an additive distribution of \( t \), such that
\[
\rho_f \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} - \mathbf{f} = -\nabla q.
\]
Multiplying the previous relation by \( \varphi \in V^c \) with \( \varphi_2 = b \) on \( \Gamma_e \) and using (3.3), we get:
\[
\rho f \int_0^1 \frac{\partial^2 d}{\partial t^2} b + \frac{h^3 E}{12} \int_0^1 \frac{\partial^2 d}{\partial x_1^2} \frac{\partial^2 b}{\partial x_1^2} + \nu \int_0^1 \frac{\partial^2 d}{\partial x_1^2} \frac{\partial^2 b}{\partial t^2} + \int_0^1 g b + \int_0^1 q b
\]
for all \( b \in B_0 \). It follows that
\[
\rho f \frac{\partial^2 d}{\partial t^2} + \frac{h^3 E}{12} \frac{\partial^2 d}{\partial x_1^2} + \nu \frac{\partial^2 d}{\partial x_1^2 \partial t^2} - g - q / x_3 = e = h(t).
\]
The system (2.5) is satisfied with \( p = q + h \).

In the sequel we shall establish existence, uniqueness and regularity results for (3.3).

**Theorem 3.1** The variational problem (3.3) has a unique solution \((\mathbf{v}, d)\) with \( \mathbf{v} \in L^\infty(0,T; (H^1(D_e))^2), \frac{\partial \mathbf{v}}{\partial t} \in L^2(0,T; L^2(D_e))^2 \), \( d \in L^2(0,T; B_0), \frac{\partial d}{\partial t} \in L^\infty(0,T; L^2(0,1)) \), \( \frac{\partial^2 d}{\partial t^2} \in L^2((0,1) \times (0,T)) \). Moreover, if we introduce the pressure as in the proof of Proposition 3.1, we obtain that \( p \in L^2(0,T; H^1(D_e)) \) and that it is unique.

**Proof.**
Let us start with the proof of the uniqueness of the solution of (3.3). Consider \((\mathbf{v}_1, d_1)\) and \((\mathbf{v}_2, d_2)\) two weak solutions. We define \((\mathbf{v}, d) = (\mathbf{v}_1 - \mathbf{v}_2, d_1 - d_2)\). Then, \((\mathbf{v}, d)\) satisfies the equation:
\[
\rho f \int_{D_e} \frac{\partial v}{\partial t} \varphi + \mu \int_{D_e} \nabla \mathbf{v} \cdot \nabla \varphi + \rho f \int_0^1 \frac{\partial^2 d}{\partial t^2} b + \frac{h^3 E}{12} \int_0^1 \frac{\partial^2 d}{\partial x_1^2} \frac{\partial^2 b}{\partial x_1^2} + \nu \int_0^1 \frac{\partial^2 d}{\partial x_1^2 \partial t^2} \frac{\partial^2 b}{\partial x_1^2} = 0
\]
for all \( \varphi \in V^c \) and \( b \in B_0 \) with \( \varphi_2 = b \) on \( \Gamma_e \).

We take \( \varphi = \mathbf{v}(t) \) and \( b = \frac{\partial d}{\partial t}(t) \) as test functions. This choice is possible since \( d \) satisfies (2.7) and hence \( \frac{\partial d}{\partial t} \in L^2(0,T; B_0) \); moreover \( \nu_2 = \frac{\partial d}{\partial t} \) on \( \Gamma_e \). It follows that
\[
\frac{\rho f}{2} \frac{d}{dt} \int_{D_e} \mathbf{v}^2 + \mu \int_{D_e} (\nabla \mathbf{v})^2 + \rho f \frac{d}{dt} \int_0^1 (\frac{\partial d}{\partial t})^2 + \frac{h^3 E}{24} \frac{d}{dt} \int_0^1 (\frac{\partial^2 d}{\partial x_1^2})^2 + \nu \int_0^1 (\frac{\partial^2 d}{\partial x_1^2 \partial t})^2 = 0.
\]
Integrating from 0 to \( t \) this equality and taking into account the initial conditions, we obtain a.e. in \((0,T)\):
\[
\frac{\rho f}{2} \int_{D_e} \mathbf{v}^2(t) + \mu \int_0^t \int_{D_e} (\nabla \mathbf{v})^2(s) \, ds + \frac{\rho f}{2} \int_0^t (\frac{\partial d}{\partial t}(t))^2 + \frac{h^3 E}{24} \int_0^t (\frac{\partial^2 d}{\partial x_1^2})^2(t) + \nu \int_0^t (\frac{\partial^2 d}{\partial x_1^2 \partial t})^2(s) \, ds = 0.
\]
So, \( \mathbf{v} = 0 \) a.e. in \( D_e \times (0,T) \). Moreover, \( \frac{\partial d}{\partial t} = 0 \) and hence \( d = d(x_1) \). We also have \( \frac{\partial^2 d}{\partial x_1^2} = 0 \) which yields \( d(x_1) = \alpha x_1 + \beta \). Using the boundary conditions \( d(0,t) = d(1,t) = 0 \), we obtain \( \alpha = \beta = 0 \). Consequently, \( d = 0 \) a.e. in \((0,1) \times (0,T)\).

Hence, the problem (3.3) has a unique solution.

For proving the existence and the regularity of the functions \( v \) and \( d \) we shall use the Galerkin method.
we get that (3.6)-(3.7) can be written as a second order system of $m+n$ linear differential equations with the unknowns $a_i(t)$ and $b_j(t)$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$:

$$\begin{aligned}
&\left\{ \begin{array}{l}
\rho_f \dot{a}_i(t) + \mu \sum_{k=1}^{m} q_{ki} a_k(t) + \rho_f \sum_{k=1}^{n} p_{ki} \dot{b}_k(t) + \mu \sum_{k=1}^{n} r_{ki} b_k(t) = \int_{D_e} \hat{f} \cdot \psi_i \\
\rho_f \sum_{k=1}^{m} p_{jk} \dot{a}_k(t) + \mu \sum_{k=1}^{m} r_{jk} a_k(t) + \rho_f \sum_{k=1}^{n} s_{kj} \dot{b}_k(t) + \mu \sum_{k=1}^{n} t_{kj} b_k(t) \\
+ \rho \delta b_j(t) + \frac{h^2 \nu}{12} \sum_{k=1}^{n} v_{kj} \dot{b}_k(t) + \nu \sum_{k=1}^{n} v_{kj} \dot{b}_k(t) = \int_{D_e} \hat{f} \cdot \varphi_j + \int_{0}^{1} g \beta_j
\end{array} \right. \\
a_k(0) = b_k(0) = \dot{b}_k(0) = 0.
\end{aligned} \tag{3.8}$$

In the sequel, the basis $\{ \psi_i \}$ will be chosen by considering the eigenfunctions of the following Stokes problem:

$$\begin{aligned}
&\left\{ \begin{array}{l}
-\mu \Delta \psi_i + \nabla \varphi_i = \lambda_i \psi_i \text{ in } D_e, \\
\div \psi_i = 0 \text{ in } D_e, \\
\psi_i = 0 \text{ on } \partial D_e,
\end{array} \right.
\end{aligned}$$

with $\lambda_i > 0$, $\forall i \in \mathbb{N}^*$.  

Multiplying the Stokes equation by $\psi_j$ and integrating over $D_e$ we get $\mu \varphi_{ij} = \lambda_i \delta_{ij}$.  

Multiplying now (3.4) by $\psi_i$ and integrating over $D_e$ we obtain $r_{ij} = \delta_{ij} = 0$.

The previous second order system can be written as the following first order differential system:

$$\begin{aligned}
\text{Find} \\
X(t) = \begin{bmatrix} a_1(t) \\ \vdots \\ a_m(t) \end{bmatrix}, \\
Y(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \\
Z(t) = \begin{bmatrix} \dot{b}_1(t) \\ \vdots \\ \dot{b}_n(t) \end{bmatrix}
\end{aligned}$$

solution to

$$\begin{aligned}
\dot{Y} &= Z, \\
\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & \lambda_m \end{bmatrix} X + \rho_f I_m \dot{X} + \begin{bmatrix} p_{11} & \cdots & p_{1n} \\
\vdots & \ddots & \vdots \\
p_{m1} & \cdots & p_{mn} \end{bmatrix} Y + \begin{bmatrix} \mu \begin{bmatrix} t_{11} & \cdots & t_{1n} \end{bmatrix} + \nu \begin{bmatrix} v_{11} & \cdots & v_{1n} \end{bmatrix} \\
\vdots & \ddots & \vdots \\
\mu \begin{bmatrix} t_{m1} & \cdots & t_{mn} \end{bmatrix} + \nu \begin{bmatrix} v_{m1} & \cdots & v_{mn} \end{bmatrix} \end{bmatrix} Z \\
\end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\rho_f}{12} \begin{bmatrix} \rho \delta & \cdots & \rho \delta \\
p_{11} & \cdots & \cdots & \cdots \\
p_{m1} & \cdots & \cdots & \cdots \end{bmatrix} \dot{X} + \frac{h^2 \nu}{12} \begin{bmatrix} v_{11} & \cdots & v_{1n} \\
\vdots & \ddots & \vdots \\
v_{m1} & \cdots & v_{mn} \end{bmatrix} Y + \begin{bmatrix} \mu \begin{bmatrix} t_{11} & \cdots & t_{1n} \end{bmatrix} + \nu \begin{bmatrix} v_{11} & \cdots & v_{1n} \end{bmatrix} \\
\vdots & \ddots & \vdots \\
\mu \begin{bmatrix} t_{m1} & \cdots & t_{mn} \end{bmatrix} + \nu \begin{bmatrix} v_{m1} & \cdots & v_{mn} \end{bmatrix} \end{bmatrix} Z \\
+ \rho_f \begin{bmatrix} s_{11} & \cdots & s_{1n} \\
\vdots & \ddots & \vdots \\
s_{m1} & \cdots & s_{mn} \end{bmatrix} + \rho \delta I_n \dot{Z} = \begin{bmatrix} \int_{D_e} \hat{f} \cdot \varphi_1 + \int_{0}^{1} g \beta_1 \\
\vdots \\
\int_{D_e} \hat{f} \cdot \varphi_n + \int_{0}^{1} g \beta_n \end{bmatrix}
\end{aligned}$$

The initial conditions become:

$$X(0) = Y(0) = Z(0) = 0.$$
with \( C_2 = \frac{h^3E}{k} \), we obtain the following estimates:

\[
\begin{align*}
\left\| \frac{\partial \mathbf{v}}{\partial t} \right\|_{L^2(0,T;L^2(D_e)^3)} & \leq C \left( \| \mathbf{f} \|_{L^2(0,T;L^2(D_e)^3)} + \| g \|_{L^2((0,1) \times (0,T))} \right), \\
\left\| \nabla \mathbf{v} \right\|_{L^\infty(0,T;L^2(D_e)^3)} & \leq C \left( \| \mathbf{f} \|_{L^2(0,T;L^2(D_e)^3)} + \| g \|_{L^2((0,1) \times (0,T))} \right), \\
\left\| \frac{\partial^2 \mathbf{u}}{\partial t^2} \right\|_{L^2((0,1) \times (0,T))} & \leq C \left( \| \mathbf{f} \|_{L^2(0,T;L^2(D_e)^3)} + \| g \|_{L^2((0,1) \times (0,T))} \right), \\
\left\| \frac{\partial^3 \mathbf{u}}{\partial x_1^2 \partial t} \right\|_{L^\infty(0,T;L^2(D_e)^3)} & \leq C \left( \| \mathbf{f} \|_{L^2(0,T;L^2(D_e)^3)} + \| g \|_{L^2((0,1) \times (0,T))} \right),
\end{align*}
\]

(3.10)

with \( C \) a constant depending on the data.

The estimates (3.9) and (3.10) allow us to pass to the limit in (3.6) and (3.7) with \( m, n \to \infty \) which yields:

\[
\rho_f \int_{D_e} \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \mu \int_{D_e} \nabla \mathbf{v} \cdot \nabla \mathbf{v} = \int_{D_e} \tilde{f} \cdot \mathbf{v}
\]

for all \( i \in \{1, \ldots, m\} \) and

\[
\rho_f \int_{D_e} \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \mu \int_{D_e} \nabla \mathbf{v} \cdot \nabla \mathbf{v} + \rho h \int_0^1 \frac{\partial^2 \mathbf{u}}{\partial t^2} \mathbf{u} + \frac{h^3E}{12} \int_0^1 \frac{\partial^2 \mathbf{u}}{\partial x_1^2} \mathbf{u} + \nu \int_0^1 \frac{\partial^2 \mathbf{u}}{\partial x_1^2} \mathbf{u} + \int_0^1 \frac{\partial^2 \mathbf{u}}{\partial x_1^2} \mathbf{u} = \int_{D_e} \tilde{f} \cdot \mathbf{v} + \int_0^1 g \beta_j
\]

(3.12)

for all \( j \in \{1, \ldots, n\} \).

We shall obtain the variational formulation (3.3) by using the previous two equalities. Indeed, \( \{\psi_i\} \) being a basis for \( V_0 \) it follows that for any \( \varphi \in (D(D_e))^2 \) with \( \text{div} \varphi = 0 \) in \( D_e \), we have:

\[
\langle \rho_f \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v}, \varphi \rangle = \langle \tilde{f}, \varphi \rangle.
\]

Using De Rham lemma, as before, and the regularity of \( \mathbf{v} \), we obtain a function \( p \in L^2(0,T;H^1(D_e)) \), unique up to an additive function of \( t \), such that

\[
\rho_f \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} + \nabla p = \mathbf{f}.
\]

Since \( \text{div} \mathbf{v} = 0 \), for any \( \varphi \in V^c \) we obtain:

\[
\rho_f \int_{D_e} \frac{\partial \mathbf{v}}{\partial t} \cdot \varphi + \mu \int_{D_e} \nabla \varphi \cdot \nabla \varphi + \int_{V^c} p \varphi = \int_{D_e} \tilde{f} \cdot \varphi
\]

which is (3.1).

Hence, for any \( \varphi_j \), solution of (3.4) it follows:

\[
\rho_f \int_{D_e} \frac{\partial \mathbf{v}}{\partial t} \cdot \varphi_j + \mu \int_{D_e} \nabla \varphi \cdot \nabla \varphi_j = \int_{D_e} \tilde{f} \cdot \varphi_j - \int_{V^c} (\varphi_j) \partial \mathbf{v}.
\]

Introducing this equality in (3.12) and using (3.11), we obtain:

\[
\rho h \int_0^1 \frac{\partial^2 \beta_j}{\partial t^2} + \frac{h^3E}{12} \int_0^1 \frac{\partial^2 \mathbf{v}}{\partial x_1^2} \beta_j + \nu \int_0^1 \frac{\partial^2 \beta_j}{\partial x_1^2} + \int_0^1 g \beta_j = \int_0^1 \mathbf{f} \cdot \beta_j
\]

for all \( j \).

As \( \{\beta_j\} \) is a basis of the space \( B_0 \) we get, for any \( b \in B_0 \):

\[
\rho h \int_0^1 \frac{\partial^2 b}{\partial t^2} + \frac{h^3E}{12} \int_0^1 \frac{\partial^2 \mathbf{v}}{\partial x_1^2} \beta_j + \nu \int_0^1 \frac{\partial^2 b}{\partial x_1^2} + \int_0^1 g b = \int_0^1 \mathbf{f} \cdot \beta_j
\]
Majorating the right hand side of the previous inequality we obtain:

\[
\begin{aligned}
\rho_f \int_{D_T} (v_n - v)^2(t) + \frac{\mu}{2} \int_0^t \int_{D_T} |\nabla(v_n - v)|^2 + \rho \int_0^t \left( \frac{\partial (d_n - d)}{\partial t} \right)^2 \\
+ \frac{h^3 E}{12} \int_0^t \left( \frac{\partial^2 (d_n - d)}{\partial x_1^2} \right)^2 + 2\nu \int_0^t \int_0^t \left( \frac{\partial^3 (d_n - d)}{\partial x_1^2 \partial t} \right)^2 \\
\leq c_1 (\mu_n - \mu)^2 \int_0^t \int_{D_T} |\nabla v_n|^2 + c_2 \int_0^t \int_{D_T} (\bar{f}_n - \bar{f})^2.
\end{aligned}
\]  
(4.2)

We take now as test function in (3.3), corresponding to \((\varphi, b) = (v_n, \frac{\partial d_n}{\partial t})\) and we get the boundedness of \(\{\nabla v_n\}_n\) in \(L^2(0, T; (L^2(D_e))^4)\).

This property together with (4.2) yield the following convergences when \(n \to \infty\):

\[
\begin{aligned}
\{v_n \to v\} \text{ in } L^\infty(0, T; (L^2(D_e))^2), \\
\nabla v_n \to \nabla v \text{ in } L^2(0, T; (L^2(D_e))^4), \\
\frac{\partial d_n}{\partial t} \to \frac{\partial d}{\partial t} \text{ in } L^\infty(0, T; L^2(0, 1)), \\
\frac{\partial^2 d_n}{\partial x_1^2} \to \frac{\partial^2 d}{\partial x_1^2} \text{ in } L^\infty(0, T; L^2(0, 1)), \\
\frac{\partial^3 d_n}{\partial x_1^2 \partial t} \to \frac{\partial^3 d}{\partial x_1^2 \partial t} \text{ in } L^2((0, 1) \times (0, T)).
\end{aligned}
\]  
(4.3)

To obtain the continuity of the function \(J\) we need a convergence for the sequence \(\{p_n\}_n\). For this purpose we repeat the previous computations for the test function \((\varphi, b) = \left(\frac{\partial (v_n - v)}{\partial t}, \right.\frac{\partial^2 (d_n - d)}{\partial t^2}\right)\). It follows that

\[
\begin{aligned}
\rho_f \int_{D_T} \left( \frac{\partial (v_n - v)}{\partial t} \right)^2 + (\mu_n - \mu) \int_0^t \int_{D_T} \nabla v \cdot \nabla \frac{\partial (v_n - v)}{\partial t} + \frac{\mu_n}{2} \int_0^t \int_{D_T} |\nabla (v_n - v)|^2 \\
+ \rho \int_0^t \left( \frac{\partial^2 (d_n - d)}{\partial t^2} \right)^2 + h^3 E \frac{\partial^2 (d_n - d)}{\partial x_1^2} \int_0^t \left( \frac{\partial^3 (d_n - d)}{\partial x_1^2 \partial t} \right)^2 \\
= \int_{D_T} (\bar{f}_n - \bar{f}) \cdot \frac{\partial (v_n - v)}{\partial t}.
\end{aligned}
\]

We compute the second term of the left hand side of the previous equality:

\[
\int_{D_T} \nabla v \cdot \nabla \frac{\partial (v_n - v)}{\partial t} = \frac{d}{d t} \int_{D_T} \nabla v \cdot (v_n - v) - \int_{D_T} \nabla \frac{\partial v}{\partial t} \cdot \nabla (v_n - v).
\]

Hence, integrating on \((0, t)\), we get:

\[
\begin{aligned}
\rho_f \int_0^t \int_{D_T} \left( \frac{\partial (v_n - v)}{\partial t} \right)^2 + (\mu_n - \mu) \int_0^t \int_{D_T} |\nabla (v_n - v)|^2 + 2\rho \int_0^t \left( \frac{\partial^2 (d_n - d)}{\partial t^2} \right)^2 \\
+ \frac{h^3 E}{6} \int_0^t \int_0^t \left( \frac{\partial^3 (d_n - d)}{\partial x_1^2 \partial t} \right)^2 + \frac{1}{\rho_f} \int_0^t \int_{D_T} (\bar{f}_n - \bar{f})^2 - 2(\mu_n - \mu) \int_{D_T} \nabla v \cdot \nabla (v_n - v) \\
+ 2(\mu_n - \mu) \int_0^t \int_{D_T} \nabla \frac{\partial v}{\partial t} \cdot \nabla (v_n - v).
\end{aligned}
\]  
(4.4)