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COMPLETELY IRREDUCIBLE MEET DECOMPOSITIONS IN
LATTICES, WITH APPLICATIONS TO GROTHENDIECK
CATEGORIES AND TORSION THEORIES

by

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Preprint nr.10/2008

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COMPLETELY IRREDUCIBLE MEET DECOMPOSITIONS IN LATTICES, WITH APPLICATIONS TO GROTHENDIECK CATEGORIES AND TORSION THEORIES

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- *Dedicated to the memory of Mark L. Teply (1942 - 2006)* -

Abstract

The aim of this paper is to investigate decompositions of elements in upper continuous modular lattices as intersections of (completely) irreducible elements. Thus, we extend from modules to such lattices the results of J. Fort [*Math. Z.* **103** (1967), 363-388] and consider a similar setting where irreducible submodules are replaced by completely irreducible submodules. We also extend from ideals to lattices some results of L. Fuchs, W. Heinzer, and B. Olberding [*Trans. Amer. Math. Soc.* **358** (2006), 3113-3131] concerning decompositions of ideals in arbitrary commutative rings as (irredundant) intersections of completely irreducible ideals. Applications of these results are given to Grothendieck categories and module categories equipped with a hereditary torsion theory.

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Introduction

The aim of this paper is two-fold: first, to extend from modules to lattices the results of Fort [8] concerning modules rich in coirreducibles and to consider a similar setting where irreducible submodules are replaced by completely irreducible submodules, obtaining so the concept of lattice rich in completely coirreducibles, and second to extend from ideals to lattices some results of Fuchs, Heinzer, and Olberding [10] concerning decompositions of ideals in arbitrary commutative rings as (irredundant) intersections of completely irreducible ideals.

In doing so, we were inspired by the ideal or module theoretical situation, where the concept of essential ideal or submodule plays a key role. Though (meet) decompositions of elements in lattices have been extensively studied in the literature (see, e.g., Crawley and Dilworth [6], Ern  [7], Walendziak [21], etc.) it is surprising that the concept of essential element was not yet involved.

The paper consists of 3 sections. Section 0 contains the basic definitions and facts, needed in the sequel, on subdirectly irreducible posets and completely irreducible elements in lattices.

In Section 1 we first extend from modules to upper continuous modular lattices the main result of Fort [8] concerning the characterization of modules M_R rich in coirreducibles by means of irredundant irreducible decompositions of 0 in any submodule of M . Then, we consider a similar problem by replacing coirreducible submodules with completely coirreducible elements. It turns out that the lattices having this property, we called lattices rich in completely coirreducibles, are exactly the atomic lattices. Then, we extend from ideals to lattices some results of Fuchs, Heinzer, and B. Olberding [10] concerning decompositions of ideals in arbitrary commutative rings as (irredundant) intersections of completely irreducible ideals.

Section 2 contains applications of the obtained latticial results to Grothendieck categories and module categories equipped with a hereditary torsion theory.

0 Subdirectly irreducible posets and completely irreducible elements in lattices

In this section we present the basic terminology and results, needed in the sequel, on subdirectly irreducible posets and completely irreducible elements in lattices.

All posets considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1. For a poset P and elements $a \leq b$ in P we write

$$b/a := [a, b] = \{ x \in P \mid a \leq x \leq b \},$$

$$[a, b[:= \{ x \in P \mid a \leq x < b \},$$

$$]a, b] := \{ x \in P \mid a < x \leq b \}.$$

An *initial subinterval* (resp. a *quotient interval*) of b/a is any interval c/a (resp. b/c) for some $c \in b/a$. If $x < y$ are elements of a poset P and there is no $z \in P$ such that $x < z < y$, then we say that x is *covered* by y (or y *covers* x), and we write $x \prec y$ (or $y \succ x$). We say that the interval b/a is *simple* if $a \prec b$. An element $a \in P$ is said to be an *atom* of P if the interval $a/0$ is simple, or equivalently, if $0 \prec a$. We denote by $\mathcal{A}(P)$ the set, possibly empty, of all atoms of P . A *coatom* of P is an element $b \in P$ such that $b \prec 1$, i.e., a maximal element of $P \setminus \{1\}$. We denote by $\mathcal{A}^0(P)$ the set, possibly empty, of all coatoms of P .

We denote by \mathcal{L} (resp. \mathcal{M} , \mathcal{C} , \mathcal{U}) the class of all lattices with 0 and 1 (resp. modular lattices with 0 and 1, complete lattices, upper continuous lattices). Throughout this paper a lattice will always mean a member of \mathcal{L} , and $(L, \leq, \wedge, \vee, 0, 1)$, or more simply, just L , will always denote such a lattice. If $L \in \mathcal{C}$, then for every subset S of L we denote $\bigwedge S = \bigwedge_{x \in S} x$ and $\bigvee S = \bigvee_{x \in S} x$. An element e of a lattice L is said to be *essential* in L if $e \wedge x \neq 0$ for each $0 \neq x \in L$. If $L \in \mathcal{C}$, then the *socle* $\text{Soc}(L)$ of L is the join of all atoms of L . As in Năstăsescu and Van Oystaeyen [18], a lattice L is said to be *semi-Artinian* if for any $1 \neq x \in L$, the lattice $1/x$ has at least an atom. As in Crawley and Dilworth [6], a poset P is said to be *atomic* (resp. *strongly atomic*, resp. *weakly atomic*) if for every $0 \neq x \in P$ there exists an atom $a \in P$ such that $a \leq x$ (resp. for every $x < y$ in P the interval y/x contains an atom, resp. for every $x < y$ there exist $a, b \in P$ such that $x \leq a \prec b \leq y$). A lattice $L \in \mathcal{U} \cap \mathcal{M}$ is strongly atomic if and only if it is semi-Artinian (see, e.g., Năstăsescu and Van Oystaeyen [18, Proposition 1.9.3]). As in Năstăsescu and Van Oystaeyen [18], for a lattice $L \in \mathcal{C}$ we define the *radical* $r_L = \bigwedge_{b \in \mathcal{A}^0(L)} b$ of L as the meet of all coatoms of L , putting $r_L = 1$ if L has no coatoms.

For all undefined notation and terminology on lattices, the reader is referred to Birkhoff [5], Crawley and Dilworth [6], Grätzer [13], and/or Stenström [19].

Throughout this paper R will denote an associative ring with nonzero identity element, and $\text{Mod-}R$ the category of all unital right R -modules. The notation M_R will be used to designate a unital right R -module M , and the lattice of all submodules of M_R will be denoted by $\mathcal{L}(M)$. The notation $N \leq M$ (resp. $N < M$) means that N is a submodule (resp. proper submodule) of M . We denote by \mathbb{N} the set $\{0, 1, 2, \dots\}$ of all natural numbers, by \mathbb{N}^* the set $\mathbb{N} \setminus \{0\}$ of strictly positive natural numbers, by \mathbb{Z} the ring of rational integers, and by \mathbb{R} the field of real numbers; when considered as posets, all of them are assumed to have the usual order.

Clearly, a module M_R is *cyclic* if and only if it satisfies the following condition:

$$\exists x_0 \in M, \forall N \in \text{Mod-}R, \forall f \in \text{Hom}_R(N, M) \text{ with } x_0 \in \text{Im}(f) \implies f \text{ is an epimorphism.}$$

Dually, a module M_R is said to be *cocyclic* if it satisfies the following condition:

$$\exists x_0 \in M, \forall N \in \text{Mod-}R, \forall g \in \text{Hom}_R(M, N) \text{ with } x_0 \notin \text{Ker}(g) \implies g \text{ is a monomorphism.}$$

To the best of our knowledge, the notion of cocyclic module appears for the first time in the literature in Fuchs [9, Section 3].

The next result provides various characterizations of cocyclic modules, that will naturally lead below (see Definition 0.2) to the most general concept of subdirectly irreducible poset. See Proposition 0.5 for an extension of these characterizations to complete lattices.

Proposition 0.1. *The following statements are equivalent for a nonzero module M_R .*

- (1) M is cocyclic.
- (2) $\bigcap_{0 \neq X \leq M} X \neq 0$.
- (3) The poset $\mathcal{L}(M) \setminus \{0\}$, ordered by inclusion, has a least element.
- (4) M has a simple essential socle.
- (5) M is subdirectly irreducible.

Proof. See Wisbauer [22, 14.8]. □

Recall that a module M_R is called *subdirectly irreducible* if any representation of M as a *subdirect product* of other modules is trivial, i.e., for every family $(M_i)_{i \in I}$ of right R -modules and for every monomorphism $\varepsilon : M \hookrightarrow \prod_{i \in I} M_i$ such that $\pi_j \circ \varepsilon$ is an epimorphism $\forall j \in I$, $\exists i \in I$ such that $\pi_i \circ \varepsilon$ is an isomorphism, where $\pi_j : \prod_{i \in I} M_i \rightarrow M_j$, $j \in I$, are the canonical projections.

Obviously, for any module M_R we have

$$M = \sum_{C \in \mathcal{C}(M)} C$$

where $\mathcal{C}(M) := \{C \leq M \mid C \text{ is cyclic}\}$.

Dually, for any module M_R we have the following less obvious fact (see, e.g., Wisbauer [22, 14.9] or Remarks 0.15 (2))

$$0 = \bigcap_{X \in \mathcal{S}(M)} X$$

where $\mathcal{S}(M) := \{X \leq M \mid M/X \text{ is cocyclic}\}$.

In the sequel we present some definitions and results on subdirectly posets and lattices from Albu, Iosif, and Teply [2] along with some new ones.

Definitions 0.2. (Albu, Iosif, and Teply [2]). (a) A poset P is said to be subdirectly irreducible (or colocal), abbreviated SI, if $P \neq \{0\}$ and the set $P \setminus \{0\}$ has a least element; i.e., there exists an element $0 \neq x_0 \in P$ such that $x_0 \leq x$ for every $0 \neq x \in P$.

(b) An element $s \in P$ is said to be a subdirectly irreducible element of P if the interval $1/s$ is a subdirectly irreducible poset; in this case, the interval $]s, 1]$ has a (unique) least element, denoted by s^* and called the cover of s .

(c) Dually, a poset P is said to be local (or cosubdirectly irreducible) if $P \neq \{1\}$ and the set $P \setminus \{1\}$ has a last element; i.e., there exists an element $1 \neq x_1 \in P$ such that $x \leq x_1$ for every $1 \neq x \in P$. \square

By Proposition 0.1, a module M_R is subdirectly irreducible (resp. local) if and only if the lattice $\mathcal{L}(M_R)$ of all submodules of M_R is subdirectly irreducible (resp. local).

For any poset P we denote by $\mathcal{S}(P)$ the set of all subdirectly irreducible elements of P ; i.e.,

$$\mathcal{S}(P) := \{x \in P \mid 1/x \text{ is SI}\}.$$

For any module M_R we set $\mathcal{S}(M_R) := \mathcal{S}(\mathcal{L}(M_R))$. Notice that this set has been denoted in Albu and Rizvi [4] by $\mathcal{C}^0(M_R)$.

Definitions 0.3. (a) A lattice L is said to be coirreducible (or uniform) if $L \neq \{0\}$ and $x \wedge y \neq 0$ for any nonzero elements $x, y \in L$.

(b) A lattice L is said to be completely coirreducible (or completely uniform) if $L \neq \{0\}$ and $\bigwedge_{i \in I} x_i \neq 0$ for any nonempty family $(x_i)_{i \in I}$ of nonzero elements $x_i \in L$, or shortly, if $\bigwedge S \neq 0$ for every $\emptyset \neq S \subseteq L \setminus \{0\}$.

(c) An element x of a lattice L is said to be meet irreducible, or just irreducible, if $x \neq 1$ and whenever $x = a \wedge b$ for $a, b \in L$, then $x = a$ or $x = b$.

(d) An element x of a lattice L is said to be completely meet irreducible, or just completely irreducible, abbreviated CI, if $x \neq 1$ and whenever $x = \bigwedge_{i \in I} a_i$ for a nonempty family $(a_i)_{i \in I}$ of elements of L , then $x = a_j$ for some $j \in I$, or shortly, if $x \neq 1$ and whenever $x = \bigwedge S$ for some $\emptyset \neq S \subseteq L$, then necessarily $x \in S$. \square

Notice that sometimes in the literature the last element 1 of a lattice is considered to be (completely) irreducible. However, in this paper, any (completely) irreducible element of any lattice is always assumed to be $\neq 1$. Clearly, an element $x \in L$ is irreducible (resp. completely irreducible) if and only if the lattice $1/x$ is coirreducible (resp. completely coirreducible). For any lattice L we denote by $\mathcal{I}(L)$ the set of all irreducible elements of L , and by $\mathcal{I}^c(L)$ the set of all completely irreducible elements of L . For any module M_R we set $\mathcal{I}(M_R) := \mathcal{I}(\mathcal{L}(M_R))$ and $\mathcal{I}^c(M_R) := \mathcal{I}^c(\mathcal{L}(M_R))$.

Remarks 0.4. (1) The set $\mathcal{I}^c(L)$ may be empty; for instance, take as L the interval $[0, 1]$ of real numbers. However, for any nonzero module M_R we have $\mathcal{I}^c(M_R) \neq \emptyset$ (see, e.g., Albu and Rizvi [4, Lemma 0.2]).

(2) Clearly, for any lattice L we have $\mathcal{I}^c(L) \subseteq \mathcal{I}(L)$. In general, the inclusion $\mathcal{I}^c(L) \subseteq \mathcal{I}(L)$ may be strict. Indeed, if L is the interval $[0, 1]$ of the set \mathbb{R} of real numbers, then $\mathcal{I}^c(L) = \emptyset$ and $\mathcal{I}(L) = L$.

(3) The set $\mathcal{I}(L)$ may be also empty (see Albu, Iosif, and Teply [2, Remarks 0.3 (3)]). However, for any nonzero module M_R we have $\mathcal{I}(M_R) \neq \emptyset$ by (1) and (2).

(4) If $L \in \mathcal{C}$, then clearly $s \in L$ is a *subdirectly element* of L if and only if s is *completely irreducible*, so $\mathcal{S}(L) = \mathcal{I}^c(L)$. In the sequel, for the term of subdirectly irreducible element of any lattice, we will always use the term of completely irreducible element. \square

Proposition 0.5. *The following statements are equivalent for a lattice $L \in \mathcal{C}$, $L \neq \{0\}$.*

- (1) L is subdirectly irreducible.
- (2) $\bigwedge_{x \in L \setminus \{0\}} x \neq 0$.
- (3) L is completely coirreducible.
- (4) L has an atom a such that $a \leq x, \forall x \in L \setminus \{0\}$.
- (5) L has an atom a that is essential in L .
- (6) L is coirreducible and $\text{Soc}(L) \neq 0$.

Proof. (1) \implies (2): If x_0 is the least element of $L \setminus \{0\}$, then $0 \neq x_0 \leq \bigwedge_{x \in L \setminus \{0\}} x$.

(2) \iff (3) is clear.

(3) \implies (4): If $a := \bigwedge_{x \in L \setminus \{0\}} x$, then a is an atom, and $a \leq x, \forall x \in L \setminus \{0\}$.

(4) \implies (5): For any $x \in L \setminus \{0\}$, we have $a \leq x$, so $x \wedge a = a \neq 0$, i.e., a is essential in L .

(5) \implies (6): Let $x, y \in L \setminus \{0\}$. Then $a \wedge x \neq 0$ and $a \wedge y \neq 0$ since a is essential in L , so $a \wedge x = a$ and $a \wedge y = a$ since a is an atom of L . It follows that $a \leq x$ and $a \leq y$, which implies that $0 \neq a \leq x \wedge y$, i.e., L is coirreducible. Clearly $a \leq \text{Soc}(L)$, so $\text{Soc}(L) \neq 0$.

(6) \implies (1): Since $\text{Soc}(L) \neq 0$, L has at least an atom, say a . We claim that a is the single atom of L ; indeed, if a' is another atom of L , then $a \wedge a' \neq 0$ since L is coirreducible, so $a \wedge a' = a = a'$. It follows that $\text{Soc}(L) = \{a\}$. For every $x \in L \setminus \{0\}$ we have $0 \neq x \wedge a \leq a$, so $x \wedge a = a$, and then $a \leq x$. This shows that a is the least element of $L \setminus \{0\}$; hence L is subdirectly irreducible. \square

Corollary 0.6. *If $L \in \mathcal{C}$ is a semi-Artinian lattice, in particular a strongly atomic lattice, then $\mathcal{I}(L) = \mathcal{I}^c(L)$.*

Proof. If $x \in \mathcal{I}(L)$, then the lattice $1/x$ is coirreducible, and $\text{Soc}(1/x) \neq 0$ since L is semi-Artinian lattice, so $1/x$ is a subdirectly irreducible lattice by Proposition 0.5, i.e., $x \in \mathcal{I}^c(L)$, as desired. \square

Proposition 0.7. *The following statements are equivalent for $L \in \mathcal{C}$ and $1 \neq c \in L$.*

- (1) *c is completely irreducible in L .*
- (2) *There exists $0 \neq x_0 \in L$ such that c is maximal with respect to $x_0 \not\leq c$.*

Proof. (1) \implies (2): Since the lattice L was assumed to be complete, we can consider its element $y_0 := \bigwedge_{x > c} x$. Then $y_0 > c$ since c is CI, so $y_0 \not\leq c$. Now, c is maximal with respect to $y_0 \not\leq c$, since for any $d > c$ we have $y_0 \leq d$ by the definition of y_0 . Thus (2) holds with x_0 as y_0 .

(2) \implies (1): If we consider again the element $y_0 := \bigwedge_{x > c} x$ of L , then clearly $y_0 \geq c$. In order to prove that c is CI, i.e., $1/c$ is a SI lattice, by Proposition 0.5, we have to show that $y_0 \neq c$. We have $x_0 \leq x$ for every $x > c$ by the maximality condition of c in (2). This implies that $x_0 \leq y_0$. We cannot have $y_0 = c$ since then $x_0 \leq c$, which contradicts (2). \square

Observe that if $x_0 \neq 0$ is any compact element of $L \in \mathcal{C}$, then the set $L_0 = \{x \in L \mid x_0 \not\leq x\}$ is inductive, so the Zorn's Lemma can be applied to find a maximal element c of L_0 , which is completely irreducible by Proposition 0.7.

The concept of a *lattice rich in subdirectly irreducible*, indispensable in the evaluation of the dual Krull dimension of lattices, has been introduced by Albu, Iosif, and Teply [2]. The definitions below are slight variations of it.

Definition 0.8. (a) *A poset P is said to be rich in subdirectly irreducibles, abbreviated RSI, if for every $a < b$ in P , the interval b/a has a subdirectly irreducible quotient interval b/c , with $a \leq c < b$.*

(b) *A poset P is said to be weakly rich in subdirectly irreducibles, abbreviated WRSI, if for every $a \neq 1$ in P , the interval $1/a$ has a subdirectly irreducible quotient interval $1/c$ with $a \leq c$.*

(c) *A lattice L is called rich in completely irreducibles, abbreviated RCI (resp. weakly rich in completely irreducibles, abbreviated WRCI) if the poset L is RSI (resp. WRSI).* \square

Of course, any RSI poset is also WRSI but not conversely: indeed, the subset $[0, 1/2] \cup \{1\}$ of \mathbb{R} is WRSI but not RSI. Observe that a poset P is RSI if and only if the poset $b/0$ is WRSI for any $0 \neq b \in P$.

For the concepts of Krull dimension, dual Krull dimension, and Gabriel dimension of a lattice the reader is referred to Năstăsescu and Van Oystaeyen [18].

Proposition 0.9. (Albu, Iosif, and Teply [2, Proposition 1.2]). *The following assertions are equivalent for $L \in \mathcal{U} \cap \mathcal{M}$.*

- (1) L is RCI.
- (2) For each $a < b$ in L there exist $x < y$ in b/a such that y/x is simple, in other words, L is weakly atomic.
- (3) For each $a < b$ in L there exist $x < y$ in b/a such that y/x is compact.
- (4) For each $a < b$ in L there exist $x < y$ in b/a such that y/x is compactly generated.
- (3) For each $a < b$ in L there exist $x < y$ in b/a such that y/x has (dual) Krull dimension.
- (6) For each $a < b$ in L there exist $x < y$ in b/a such that y/x has Gabriel dimension. \square

Corollary 0.10. (Albu, Iosif, and Teply [2, Corollary 1.3]). *Let $L \in \mathcal{U} \cap \mathcal{M}$. If L has Gabriel dimension, then L is RCI. In particular, if L is Artinian, semi-Artinian, Noetherian, or has (dual) Krull dimension, then L is RCI.* \square

Corollary 0.11. (Albu, Iosif, and Teply [2, Corollary 1.4]). *Any compactly generated lattice $L \in \mathcal{U} \cap \mathcal{M}$ is RCI.* \square

Remarks 0.12. (1) The poset reduced to 0 is by definition RSI.

(2) Clearly, we can express equivalently the property of a poset P being RSI as follows: $\mathcal{S}(b/a) \neq \emptyset$ for every $a < b$ in P . Thus, if $P \neq \{0\}$ is SI, then $\mathcal{S}(P) \neq \emptyset$, but not conversely; if $P = [0, 1/2] \cup \{1\} \subseteq \mathbb{R}$, then $\mathcal{S}(P) = \{1/2\} \neq \emptyset$, but P is not RSI since $\mathcal{S}([0, 1/2]) = \emptyset$.

(3) For any module M_R , the lattice $\mathcal{L}(M)$ is RCI by Albu and Rizvi [4, Lemma 0.2] or by Corollary 0.11.

(4) Clearly, any strongly atomic lattice is RCI, but not conversely. Indeed, if M_R is a module that is not semi-Artinian, then the lattice $\mathcal{L}(M)$ is RCI by (3) but is not semi-Artinian, or equivalently, not strongly atomic.

(5) Any Noetherian poset P is RSI since for any $a < b$ in P , the interval $[a, b[$ contains a maximal element c , and so, the interval b/c is simple, in particular SI.

(6) Clearly, any RSI poset is also weakly atomic, but not conversely. Indeed, let

$$L = \{x_{n,k_n} \mid n \in \mathbb{N}^*, 1 \leq k_n \leq 2^n\} \cup \{0\} \cup \{1\}$$

be the lattice whose Hasse diagram is indicated below:

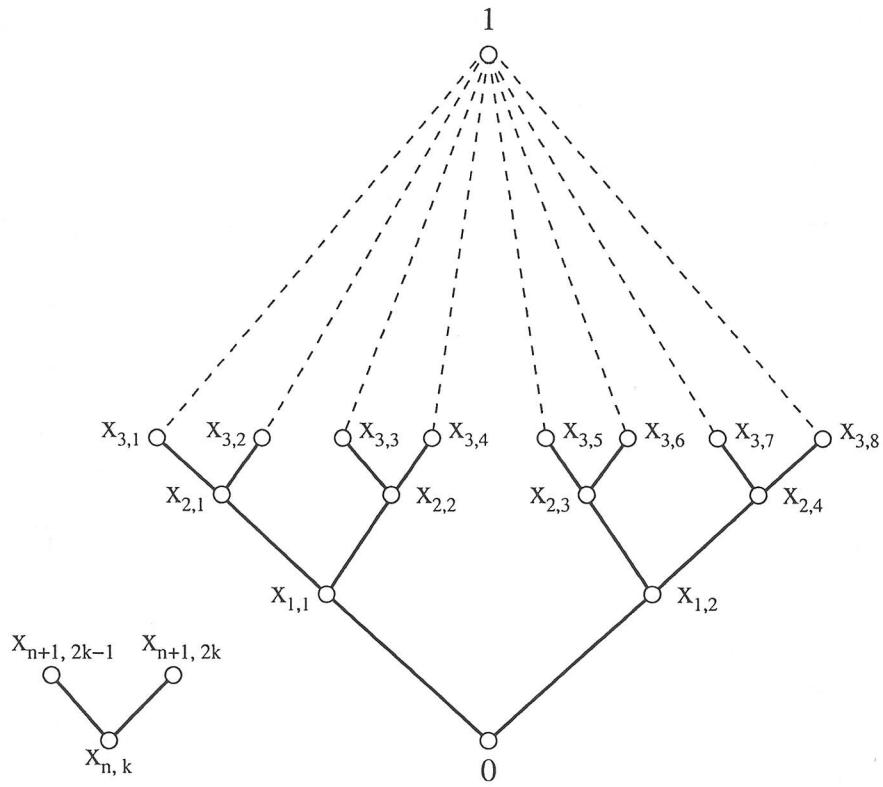


Figure 1:

Then L is an Artinian lattice, so also weakly atomic, which has no irreducible element, so no CI element too; thus, L is not WRCI, and, a fortiori, not RCI. Observe that L is neither upper continuous nor semi-modular.

(7) Examples involving torsion theories of lattices that are RCI are provided in Albu, Iosif, and Teply [2, Section 2].

(8) An example of an RCI upper continuous modular lattice that is not compactly generated is the following one given in Crawley and Dilworth [6, p. 16]: let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the chain of

all natural numbers with a largest element ∞ adjoined, and let L be the set of all functions $f : \mathbb{N} \rightarrow \overline{\mathbb{N}}$ such that $f(n) = \infty$ for all but finitely many $n \in \mathbb{N}$. With the partial order in L defined by $f \leq g \iff f(n) \leq g(n), \forall n \in \mathbb{N}$, L becomes an upper continuous modular lattice having the following two properties: the interval $1/f$ of L is an atomic lattice for all $f \neq o$, and o is the only compact element of L , where o is the zero function, $o(n) = 0$ for all $n \in \mathbb{N}$. So, L is RCI but not compactly generated. It would be interesting to find such an example based on lattices of type $\text{Sat}_\tau(M)$ for a certain hereditary torsion theory τ on the category $\text{Mod-}R$ and a module M_R . A possible candidate for that could be the example in Albu, Iosif, and Teply [2, Remarks 2.6 (3)].

(9) Examples from general topology of weakly or strongly atomic lattices that are upper continuous and distributive but not compactly generated are provided in Ern  [7]. \square

Proposition 0.13. (Albu, Iosif, and Teply [2, Lemma 1.6]). *The following statements are equivalent for a nonzero lattice $L \in \mathcal{U} \cap \mathcal{M}$.*

- (1) *The lattice L is RCI.*
- (2) *For every $a < b$ in L , one has $a = \bigwedge_{x \in \mathcal{I}^c(b/a)} x$.*

In particular, if L is RCI, then $0 = \bigwedge_{x \in \mathcal{I}^c(L)} x$. \square

Definition 0.14. *A lattice L is said to be with completely irreducible decomposition, abbreviated CID, if every $1 \neq a \in L$ can be written as a meet of a family of CI elements of L , or equivalently $a = \bigwedge_{x \in \mathcal{I}^c(1/a)} x$.* \square

Remarks 0.15. (1) Consider the subset $L = \{0\} \cup [1/2, 1]$ of \mathbb{R} . Then 0 is the only CI element of the lattice L , but for every $0 < a < b$ in L , the interval $[a, b]$ has no CI elements; in particular, the lattice L is not RCI. This example shows that a lattice $L \in \mathcal{U} \cap \mathcal{M}$ may satisfy the property $0 = \bigwedge_{x \in \mathcal{I}^c(L)} x$ of Proposition 0.13 without being necessarily RCI.

(2) By Remarks 0.12 (3) and Proposition 0.13, any proper submodule of any module M_R is an intersection of CI submodules of M .

(3) Proposition 0.13 can be expressed by saying that a lattice L is RCI if and only if, for every $0 \neq b \in L$, the lattice $b/0$ is with CID. In particular, any RCI lattice is a lattice with CID. The converse may be not true. Indeed, consider the following example due to Ern  [7]: let $L := \{(x, y) \mid x, y \in [0, 1], x + y \leq 1\} \cup \{(1, 1)\}$, where $[0, 1]$ is the unit interval in the set \mathbb{R} of all real numbers. Then L , ordered componentwise by the usual \leq relation, is a complete semimodular lattice. Since the only covering pairs in L are $(x, 1 - x) \prec (1, 1)$, $x \in [0, 1]$, it follows that L is not weakly atomic; so it is not RCI too. However, every element of L can be written as an irredundant intersection of at most two coatoms, so, L is a lattice with CID.

(4) Observe that if L is a lattice with CID, and $a, b \in L$, then $b \not\leq a$ if and only if there exists a CI element $c \in L$ such that $a \leq c$ and $b \not\leq c$. Indeed, assume that for any CI element $c \in L$ such that $a \leq c$ we also have $b \leq c$. Since L is with CID, we can write $a = \bigwedge_{i \in I} c_i$ with CI irreducible elements $c_i, i \in I$. Then $b \leq c_i, \forall i \in I$, so $b \leq \bigwedge_{i \in I} c_i = a$. This shows the nontrivial implication.

(5) In the next section we will discuss when a decomposition of an element of a lattice $L \in \mathcal{U} \cap \mathcal{M}$ with CID as a meet of irreducible/completely irreducible elements is irredundant or unique. \square

As in Walendziak [21], a lattice L is said to satisfy the condition (P) (resp. (M)) if for any two elements $a \prec b$ in L , the set $P(a, b) = \{x \in L \mid a = b \wedge x\}$ has a maximal element (resp. if for every $c \in \mathcal{I}^c(L)$ and for every $a \in L$ with $a \not\leq c$ one has $c \wedge a \in \mathcal{I}^c(a/(a \wedge c))$). Of course, the condition (M) can be expressed equivalently as follows: for every $c \in L$ with $1/c$ SI and for every $a \in L$ with $a \not\leq c$, the interval $a/(a \wedge c)$ is SI. Clearly, any upper continuous lattice satisfies the condition (P), and any modular lattice satisfies the condition (M).

The existence of completely irreducible decompositions in complete lattices that are more general than upper continuous modular lattices is given by the following result.

Proposition 0.16. (Walendziak [21, Theorem 1, Theorem 2, Corollary 2]). *If $L \in \mathcal{C}$ satisfies the conditions (P) and (M), then $1 \neq a \in L$ has a decomposition into CI elements if and only if for every $x > a$ in L there are $u, v \in x/a$ with $u \prec v$. In particular, a complete lattice satisfying the conditions (P) and (M) is a lattice with CID if and only if it is weakly atomic.* \square

We end this section with an extension of Proposition 0.13 to complete lattices satisfying the condition (P).

Proposition 0.17. *The following statements are equivalent for a nonzero lattice $L \in \mathcal{C}$ satisfying the condition (P).*

(1) *The lattice L is RCI.*

(2) *For every $a < b$ in L , one has $a = \bigwedge_{x \in \mathcal{I}^c(b/a)} x$.*

Proof. (1) \implies (2): Observe first that if a lattice L is RCI, then so is any of its intervals $[x, y]$, and $\mathcal{I}^c(L) \neq \emptyset \iff L \neq \{0\}$. Therefore, it is sufficient to prove only that $0 = \bigwedge_{x \in \mathcal{I}^c(L)} x$ if the lattice L is RCI. Set $y = \bigwedge_{x \in \mathcal{I}^c(L)} x$, and assume that $y \neq 0$. Since L is RCI, there exist $a \prec b$ in $y/0$, and since L satisfies the condition (P), there exists an element $m \in P(a, b) := \{x \in L \mid a = b \wedge x\}$ which is maximal in $P(a, b)$, and then it is necessarily CI by Ern  [7, Lemma]. So, $m \geq y \geq b$, and hence $a = m \wedge b = b$, which is a contradiction. Consequently, $y = 0$, and we are done.

(2) \implies (1): Since $\bigwedge \emptyset = 1$, we have $\mathcal{I}^c(b/a) \neq \emptyset$ for any $a < b$ in L , which says exactly that L is RCI. \square

Observe that condition (2) in Proposition 0.17 expresses the fact that any element $a \in L$ has a decomposition in CI elements in any interval $[a, b]$ of L , $a < b$, and this does not require the condition (M).

1 Lattices rich in coirreducibles/completely coirreducibles, and irredundant meet decompositions

The aim of this section is two-fold. First we extend from modules to upper continuous modular lattices the main result of Fort [8] concerning the characterization of modules M_R rich in coirreducibles by means of irredundant irreducible decompositions of 0 in any submodule of M . Then, we consider a similar problem by replacing coirreducible submodules with subdirectly irreducible submodules. It turns out that the lattices having this property, we called lattices rich in completely coirreducibles, are exactly the atomic lattices. Note that the existence of irredundant meet decompositions has been explored in the literature, also for lattices that are not necessarily modular or upper continuous (see, e.g., Crawley and Dilworth [6], Stern [20], Walendziak [21], etc.), but no connection with essential elements has been so far considered.

As in Fort [8], a module M_R is said to be *rich in coirreducibles*, abbreviated RC (or *rich in uniforms*, abbreviated RU), if $M \neq 0$ and every of its nonzero submodules contains a coirreducible (or uniform) submodule. The next result characterizes RC modules.

Theorem 1.1. (Fort [8, Théorème 3]). *The following statements are equivalent for a nonzero module M_R .*

- (1) M is RC.
- (2) M is an essential extension of a direct sum of coirreducible submodules of M .
- (3) The injective hull $E_R(M)$ of M is an essential extension of a direct sum of indecomposable injective modules.
- (4) 0 has an irredundant irreducible decomposition in every nonzero submodule of M . \square

Our first aim is to extend the characterization above from modules to upper continuous modular lattices, and then, to consider a similar problem, where irredundant irreducible decompositions are replaced by irredundant completely irreducible decompositions.

For brevity, we say that an element c of a lattice L is *coirreducible* (or *uniform*) if the interval $c/0$ is coirreducible, that is, if $c \neq 0$ and $x \wedge y \neq 0$ for all $x, y \in]0, c]$. We denote by $\mathcal{C}(L)$ the set, possibly empty, of all coirreducible elements of L . Similarly, we say that an element $s \in L$ is *completely coirreducible* (or *cocompletely irreducible*, or *cosubdirectly irreducible*) if the interval $s/0$ is subdirectly irreducible, that is, if $s \neq 0$ and $\bigwedge_{i \in I} x_i \neq 0$ for any nonempty family $(x_i)_{i \in I}$ of elements with $x_i \in]0, s]$, $i \in I$. We denote by $\mathcal{C}^c(L)$ the set, possibly empty, of all completely coirreducible elements of L . Clearly $\mathcal{A}(L) \subseteq \mathcal{C}^c(L) \subseteq \mathcal{C}(L)$. Observe that if $0 < x \leq y$ are elements in L and y is coirreducible (resp. completely coirreducible), then so is also x . There exist nonzero modules M_R having no coirreducible submodules (see, e.g., Fort [8, Théorème 2]), and for such M we have $\mathcal{C}(\mathcal{L}(M)) = \emptyset$.

Definition 1.2. A lattice L is said to be *rich in coirreducibles* or *rich in uniforms*, abbreviated RC or RU (resp. *rich in completely coirreducibles* or *rich in completely uniforms*, abbreviated RCC or RCU), if $L \neq \{0\}$ and for any $0 \neq x \in L$ there exists $c \in \mathcal{C}(L)$ (resp. $c \in \mathcal{C}^c(L)$) such that $c \leq x$. \square

Examples 1.3. (1) Since any atom of a lattice is a coirreducible element, it follows that any atomic lattice is RCC. Conversely, since any subdirectly irreducible interval $x/0$ of any lattice L contains a (unique) atom, it follows that any RCC lattice is atomic. Consequently, a lattice is RCC if and only if it is atomic. In particular, any Artinian or semi-Artinian lattice is RCC.

(2) Any Noetherian lattice $L \in \mathcal{M}$ is RC. To show that, observe first that any element $1 \neq x \in L$ can be written as a finite intersection of irreducible elements of L (see, e.g., Năstăsescu and Van Oystaeyen [18, Proposition 1.4.4]). It follows that any element $1 \neq z \in L$ can be written as a finite irredundant intersection of irreducible elements of L . Now, let $0 \neq x \in L$. Then $x/0$ is a Noetherian lattice, so we can write 0 as a finite irredundant intersection $0 = \bigwedge_{1 \leq i \leq n} x_i$ of irreducible elements of $x/0$. If $n = 1$, then $0 = x_1$ is irreducible in $x/0$, so x is a coirreducible element. If $n \geq 2$, set $y_1 := \bigwedge_{2 \leq i \leq n} x_i$. Since the decomposition $0 = \bigwedge_{1 \leq i \leq n} x_i$ is irredundant, we have $y_1 \neq 0$. But $0 = x_1 \wedge y_1$, so $y_1/0 = y_1/(x_1 \wedge y_1) \simeq (x_1 \vee y_1)/x_1 \subseteq x/x_1$, and since the interval x/x_1 is coirreducible, it follows that so is also $y_1/0$, in other words, x contains the coirreducible element y_1 , as desired.

(3) Let $L \in \mathcal{U} \cap \mathcal{M}$. If L has Gabriel dimension $g(L)$, in particular, if L is semi-Artinian or has (dual) Krull dimension, then L is RC. Indeed, let $0 \neq x \in L$. Then, the interval $x/0$ of L has Gabriel dimension $g(x/0) > 0$, hence it contains a Gabriel γ -simple interval $c/0$, $0 < c \leq x$, for some ordinal $\beta \leq g(x/0)$ (see Năstăsescu and Van Oystaeyen [18, 3.4]). We claim that c is a coirreducible element of L , i.e., the interval $c/0$ is coirreducible. If not, then there would exist $a, b \in]0, c]$ such that $0 = a \wedge b$. Since $c/0$ is Gabriel γ -simple, we have $g(c/a) < \gamma$ and $g(c/b) < \gamma$. By modularity, we have $a/0 = a/(a \wedge b) \simeq (a \vee b)/b \subseteq c/b$, which implies

that $g(a/0) \leq g(c/b) < \gamma$, hence $g(c/0) = \max\{g(c/a), g(a/0)\} < \gamma$, which is a contradiction. Consequently, for each $0 \neq x \in L$, there exists a $c \in L$, $0 < c \leq x$, such that c is coirreducible, which means exactly that L is RC, as desired.

(4) Clearly any RCC lattice is RC, but not conversely. Indeed, the lattice $\mathcal{L}(\mathbb{Z})$ of all subgroups of the Abelian group \mathbb{Z} is RC by (2), but not RCC by (1) since the \mathbb{Z} -module \mathbb{Z} does not contain any simple submodule. \square

In order to extend Theorem 1.1 from RC modules to RC lattices and also to consider the case of RCC lattices, we need the next lemmas, which are the latticial versions of the corresponding results used in its proof in Fort [8]. Note that their statements and proofs are given in a parallel manner, by replacing the word “irreducible” (resp. “coirreducible”) with “completely irreducible” (resp. “completely coirreducible”).

Recall that a subset A of a lattice $L \in \mathcal{C}$ is said to be *join independent*, or just *independent*, if $0 \notin A$ and $a \wedge \bigvee(A \setminus \{a\}) = 0$ for all $a \in A$. If $L \in \mathcal{U}$, then $A \subseteq L$ is independent if and only if every finite subset of A is independent. Alternatively, we say that a family $(x_i)_{i \in I}$ of elements of a lattice $L \in \mathcal{C}$ is *independent* if $x_i \neq 0$ and $x_i \wedge (\bigvee_{j \in I \setminus \{i\}} x_j) = 0$ for every $i \in I$, and in that case, necessarily $x_p \neq x_q$ for each $p \neq q$ in I ; so, the two definitions of independence, using subsets or families of elements of L , are essentially the same.

Lemma 1.4. *Let $L \in \mathcal{U} \cap \mathcal{M}$ and let $(x_i)_{i \in I}$ be an independent family of elements of L . If $y \in L$ is such that $y \wedge (\bigvee_{i \in I} x_i) \neq 0$, then there exist $i \in I$, $0 \neq x'_i \leq x_i$, and $0 \neq y' \leq y$ such that $x'_i/0 \simeq y'/0$.*

Proof. Observe that the proof of Fort [8, Proposition 2] works not only for the lattice $\mathcal{L}(M_R)$ of all submodules of a module M_R , but also for any upper continuous modular lattice L . \square

It is known that for any upper continuous lattice L and for any $\emptyset \neq B \subseteq L$, any independent subset A of B is contained in a maximal independent subset S of B (see e.g., Crawley and Dilworth [6, p.46]), and $\bigvee_{x \in S} x$ is called a *maximal direct join* of elements of B and denoted by $\bigvee_{x \in S} x$. In particular, if $A = \emptyset$ and $B = \mathcal{C}(L)$ (resp. $B = \mathcal{C}^c(L)$, resp. $B = \mathcal{A}(L)$) then one obtains a *maximal direct join of coirreducibles* (resp. a *maximal direct join of completely coirreducibles*, resp. a *maximal direct join of atoms*) of L .

Corollary 1.5. *Let $L \in \mathcal{U} \cap \mathcal{M}$, let $x \in L$, and let m be a maximal direct join of coirreducibles (resp. a maximal direct join of completely coirreducibles) of L . Then $m \wedge x \neq 0$ if and only if there exists $y \in \mathcal{C}(L)$ (resp. $y \in \mathcal{C}^c(L)$) such that $y \leq x$.* \square

Lemma 1.6. *Let $L \in \mathcal{U} \cap \mathcal{M}$. Then there exists $t \in L$ which is maximal with respect to the property that there exists no $y \in \mathcal{C}(L)$ (resp. $y \in \mathcal{C}^c(L)$) such that $y \leq t$. If m is a*

maximal direct join of coirreducibles (resp. a maximal direct join of completely coirreducibles), then $t \wedge m = 0$ and $t \vee m$ is an essential element of L .

Proof. Let $T := \{u \in L \mid c \not\leq u, \forall c \in \mathcal{C}(L)\}$ (resp. $T := \{u \in L \mid c \not\leq u, \forall c \in \mathcal{C}^c(L)\}$). If $\emptyset \neq D \subseteq T$ is a chain, then $u_0 := \bigvee_{d \in D} d \in T$, for otherwise $u_0 \geq c$ for some $c \in \mathcal{C}(L)$ (resp. $c \in \mathcal{C}^c(L)$), hence, by upper continuity, $c = u_0 \wedge c = \bigvee_{d \in D} (d \wedge c) = 0$; indeed, if $d \wedge c \neq 0$ for some $d \in D$, then $d \geq d \wedge c \in \mathcal{C}(L)$ (resp. $d \geq d \wedge c \in \mathcal{C}^c(L)$), which is a contradiction. This shows that T is an inductive set, so, by Zorn's Lemma, it has a maximal element, say t , and $t \wedge m = 0$ by Corollary 1.5.

We are now going to show that $t \vee m$ is an essential element of L , i.e., if $z \in L$ is such that $(t \vee m) \wedge z = 0$, then $z = 0$. We claim that $(z \vee t) \wedge m = 0$. Indeed, by modularity, we have

$$(z \vee t) \wedge m \leq (z \vee t) \wedge (t \vee m) = ((z \wedge (t \vee m)) \vee t) \wedge m = 0 \vee t = t.$$

On the other hand, $(z \vee t) \wedge m \leq m$, so $(z \vee t) \wedge m \leq t \wedge m = 0$, which proves our claim.

By Corollary 1.5, $z \vee t$ does not contain any coirreducible (resp. completely coirreducible) element of L , so $z \vee t \in T$. Since t is maximal in T it follows that $t = z \vee t$. Then $z \leq t$, and so $0 = (t \vee m) \wedge z = z$, as desired. \square

Lemma 1.7. *Let $L \in \mathcal{U} \cap \mathcal{M}$, and let $x \leq y$. Then x is irreducible (resp. completely irreducible) in $y/0$ if and only if there exists an irreducible (resp. completely irreducible) element z in L such that $x = z \wedge y$.*

Proof. “ \implies ”: Consider the set $S := \{u \in L \mid u \wedge y = x\}$. Since the lattice L is upper continuous, the set S is inductive, so, by Zorn's Lemma, it has a maximal element, say z . Assume that x is irreducible (resp. completely irreducible) in $y/0$, let I be a set of two elements (resp. arbitrary set), and let $(z_i)_{i \in I}$ be a family of elements of L such that $z = \bigwedge_{i \in I} z_i$. Then $x = z \wedge y = \bigwedge_{i \in I} (z_i \wedge y)$. By hypothesis, there exists $j \in I$ such that $x = z_j \wedge y$. Since z is maximal in S , we deduce that $z = z_j$, which proves that z is an irreducible (resp. completely irreducible) element in L .

“ \impliedby ”: Assume that z is an irreducible (resp. completely irreducible) element in L . By modularity, we have $y/x = y/(z \wedge y) \simeq (z \vee y)/z \subseteq 1/z$. Since the interval $1/z$ is coirreducible (resp. SI), so is also y/x , in other words, x is irreducible (resp. completely irreducible) in $y/0$, as desired. \square

Definition 1.8. *Let $L \in \mathcal{C}$, and let $x \in L$. An irreducible (meet) decomposition, abbreviated ID (resp. completely irreducible (meet) decomposition, abbreviated CID) of x in L is a family $(x_i)_{i \in I}$ of irreducible (resp. completely irreducible) elements of L such that*

$$x = \bigwedge_{i \in I} x_i.$$

We also say shortly that $x = \bigwedge_{i \in I} x_i$ is an ID (resp. CID) of x . The decomposition $x = \bigwedge_{i \in I} x_i$ is said to be an irredundant irreducible decomposition, abbreviated IID (resp. irredundant completely irreducible decomposition, abbreviated ICID) if $\bigwedge_{j \in I \setminus \{i\}} x_j \not\leq x_i$ for every $i \in I$, in other words, none of the x_i 's can be omitted without changing the intersection.

Alternatively, using the notation $\bigwedge C$ for $\bigwedge_{c \in C} c$, we say that a representation $x = \bigwedge A$ with $\emptyset \neq A \subseteq \mathcal{I}(L)$ (resp. $\emptyset \neq A \subseteq \mathcal{I}^c(L)$) is an irreducible (meet) decomposition (resp. completely irreducible (meet) decomposition) of x in L , which is called irredundant in case $\bigwedge(A \setminus \{a\}) \neq x$ for every $a \in A$. \square

Lemma 1.9. *Let x be an essential element of a lattice $L \in \mathcal{U} \cap \mathcal{M}$. Then $0 = \bigwedge_{i \in I} x_i$ is an IID (resp. ICID) of 0 in $x/0$ if and only if there exists a family $(y_i)_{i \in I}$ of elements of L such that $x_i = x \wedge y_i$, $\forall i \in I$, and $0 = \bigwedge_{i \in I} y_i$ is an IID (resp. ICID) of 0 in L .*

Proof. “ \implies ”: Assume that $0 = \bigwedge_{i \in I} x_i$ is an IID (resp. ICID) of 0 in $x/0$. Then, by Lemma 1.7, there exists a family $(y_i)_{i \in I}$ of irreducible (resp. completely irreducible) elements of L such that $x_i = x \wedge y_i$, $\forall i \in I$. We claim that $0 = \bigwedge_{i \in I} y_i$ is an IID (resp. ICID) of 0 in L . First, $\bigwedge_{i \in I} y_i = 0$ since $(\bigwedge_{i \in I} y_i) \wedge x = \bigwedge_{i \in I} x_i = 0$ and x is an essential element of L . Second, the decomposition $\bigwedge_{i \in I} y_i = 0$ is irredundant, for otherwise, $y_i \geq \bigwedge_{j \in I \setminus \{i\}} y_j$ for some $i \in I$, hence $x_i \geq \bigwedge_{j \in I \setminus \{i\}} x_j$, which is a contradiction.

“ \impliedby ” can be proved using similar arguments. \square

Lemma 1.10. *The following statements are equivalent for a nonzero lattice $L \in \mathcal{U} \cap \mathcal{M}$.*

- (1) L is atomic.
- (2) L is RCC.
- (3) $\text{Soc}(L)$ is essential in L .

Proof. (1) \iff (2) has been shown in Examples 1.3 (1).

(1) \implies (3): First, we are going to show that $\text{Soc}(L)$ coincides with any maximal direct join s of atoms of L . Since, by definition, $\text{Soc}(L)$ is the join of all atoms of L , we clearly have $s \leq \text{Soc}(L)$. Now, let $a \in \mathcal{A}(L)$. Then $a \wedge s$ is either 0 or a . Since s is a maximal direct join of atoms, $a \wedge s \neq 0$, so $a \wedge s = a$, i.e., $a \leq s$ for each $a \in \mathcal{A}$, and then, $\text{Soc}(L) = \bigvee_{a \in \mathcal{A}(L)} a \leq s$, as desired. To show that $\text{Soc}(L)$ is essential in L , let $0 \neq x \in L$, and assume that $x \wedge \text{Soc}(L) = 0$. Let $a \in \mathcal{A}(L)$ with $a \leq x$. Then $a \vee \text{Soc}(L)$ is a direct join of atoms, which contradicts the fact that $\text{Soc}(L)$ is a maximal direct join of atoms of L .

(3) \implies (1): Let $0 \neq x \in L$. Then $x \wedge \text{Soc}(L) \neq 0$, and since $\text{Soc}(L)$ is a direct union $\bigvee_{i \in I} a_i$ of atoms, by Lemma 1.4, there exist $i \in I$, $0 \neq a'_i \leq a_i$ and $0 \neq x' \leq x$ such that $a'_i/0 \simeq x'/0$. But $a'_i = a_i$, so x' is an atom contained in x , as desired. \square

Remarks 1.11. (1) For the implication (1) \implies (3) in Lemma 1.10 we needed the condition that L is upper continuous, while for the opposite implication, we required that $L \in \mathcal{U} \cap \mathcal{M}$. Note that the equivalence (1) \iff (2) is valid for any poset.

(2) An atomic lattice is not necessarily strongly atomic. Indeed, if F is any field, I is any infinite set, and R is the product ring F^I of I copies of R , then $\text{Soc}(R)$ is essential in R , but R is not a semi-Artinian ring, so the lattice $\mathcal{L}(R)$ of all ideals of R is atomic but not strongly atomic. \square

Lemma 1.12. (Grzeszczuk and Puczyłowski [15, Lemma 1.4]). *Let $L \in \mathcal{U} \cap \mathcal{M}$, and let X be an independent subset of L . Then, $(\bigvee A) \wedge (\bigvee B) = 0$ for any disjoint subsets A, B of X . \square*

Lemma 1.13. *Let $L \in \mathcal{U} \cap \mathcal{M}$, and let X be an independent subset of L . Then, for any $n \in \mathbb{N}$, $n \geq 2$, and for any finite family $(A_i)_{1 \leq i \leq n}$ of subsets of X , we have*

$$\bigwedge_{1 \leq i \leq n} (\bigvee A_i) = \bigvee (\bigcap_{1 \leq i \leq n} A_i).$$

In particular, if $\bigcap_{1 \leq i \leq n} A_i = \emptyset$, then $\bigwedge_{1 \leq i \leq n} (\bigvee A_i) = 0$.

Proof. We proceed by induction on n . Let $n = 2$. If $A_1 \cap A_2 = \emptyset$, then the result is exactly Lemma 1.12. So, we may assume that $A_1 \cap A_2 \neq \emptyset$, and then, we can partition A_1 as $A_1 = (A_1 \setminus A_2) \cup (A_1 \cap A_2)$. By modularity, we have

$$\begin{aligned} (\bigvee A_1) \wedge (\bigvee A_2) &= (\bigvee A_2) \wedge ((\bigvee (A_1 \cap A_2)) \vee (\bigvee (A_1 \setminus A_2))) = \\ &= (\bigvee (A_1 \cap A_2)) \vee ((\bigvee A_2) \wedge (\bigvee (A_1 \setminus A_2))) = \bigvee (A_1 \cap A_2) \end{aligned}$$

since $(\bigvee A_2) \wedge (\bigvee (A_1 \setminus A_2)) = 0$ by Lemma 1.12. If $n > 2$ and the result is true for $n - 1$, then by the inductive hypothesis and step $n = 2$, we have

$$\bigwedge_{1 \leq i \leq n} (\bigvee A_i) = (\bigvee A_1) \bigwedge (\bigwedge_{2 \leq i \leq n} (\bigvee A_i)) = (\bigvee A_1) \bigwedge (\bigvee (\bigcap_{2 \leq i \leq n} A_i)) = \bigvee (\bigcap_{1 \leq i \leq n} A_i),$$

which finishes the proof. \square

The next result shows that Lemma 1.13 also holds for infinite families of subsets of an independent set.

Proposition 1.14. *Let $L \in \mathcal{U} \cap \mathcal{M}$, and let X be an independent subset of L . Then, for any family $(A_i)_{i \in I}$ of subsets of X , we have*

$$\bigwedge_{i \in I} (\bigvee A_i) = \bigvee (\bigcap_{i \in I} A_i).$$

In particular, if $\bigcap_{i \in I} A_i = \emptyset$, then $\bigwedge_{i \in I} (\bigvee A_i) = 0$.

Proof. Set $A := \bigcup_{i \in I} A_i$, $a_i := \bigvee A_i$, $a := \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} (\bigvee A_i)$, $B_i := A \setminus A_i$, $b_i := \bigvee B_i$, and $b := \bigvee (\bigcap_{i \in I} A_i)$. Since $\bigcap_{i \in I} A_i \subseteq A_j$ for each $j \in I$, we have $\bigvee (\bigcap_{i \in I} A_i) \leq \bigvee A_j$, and so $\bigvee (\bigcap_{i \in I} A_i) \leq \bigwedge_{i \in I} (\bigvee A_i)$, i.e., $b \leq a$. Thus, it remains to prove only that $a \leq b$.

We have

$$\bigvee_{i \in I} b_i = \bigvee_{i \in I} (\bigvee B_i) = \bigvee_{i \in I} (\bigvee (A \setminus A_i)) = \bigvee_{i \in I} (\bigcup (A \setminus A_i)) = \bigvee (A \setminus \bigcap_{i \in I} A_i),$$

so

$$\bigvee A = (\bigvee (A \setminus \bigcap_{i \in I} A_i)) \vee (\bigvee (\bigcap_{i \in I} A_i)) = (\bigvee b_i) \vee b.$$

Denote by $\mathcal{F}(I)$ the poset, ordered by inclusion, of all finite nonempty subsets of I , and for every $F \in \mathcal{F}(I)$ set $b_F := \bigvee_{i \in F} b_i$. Then $\bigvee_{i \in I} b_i = \bigvee_{F \in \mathcal{F}(I)} b_F$, and so

$$a = a \wedge (\bigvee A) = a \wedge ((\bigvee b_i) \vee b) = a \wedge (\bigvee_{F \in \mathcal{F}(I)} (b_F \vee b)) = \bigvee_{F \in \mathcal{F}(I)} (a \wedge (b_F \vee b)),$$

since L is upper continuous and $\mathcal{F}(I)$ is a directed set.

Let $F = \{i_1, \dots, i_k\}$ be an arbitrary but fixed nonempty finite subset of I . Then

$$a = \bigwedge_{i \in I} a_i \leq a_{i_1} \wedge \dots \wedge a_{i_k} = (\bigvee A_{i_1}) \wedge \dots \wedge (\bigvee A_{i_k}) = \bigvee (A_{i_1} \cap \dots \cap A_{i_k})$$

by Lemma 1.13. On the other hand, we have

$$b_F \vee b = ((\bigvee B_{i_1}) \vee \dots \vee (\bigvee B_{i_k})) \vee b = (\bigvee (B_{i_1} \cup \dots \cup B_{i_k})) \vee b = (\bigvee (A \setminus (A_{i_1} \cap \dots \cap A_{i_k}))) \vee b.$$

We deduce that

$$a \wedge (b_F \vee b) \leq \bigvee (A_{i_1} \cap \dots \cap A_{i_k}) \wedge ((\bigvee (A \setminus (A_{i_1} \cap \dots \cap A_{i_k}))) \vee b).$$

For simplicity, denote $A_F := A_{i_1} \cap \dots \cap A_{i_k}$, and then the inequality above becomes

$$a \wedge (b_F \vee b) \leq (\bigvee A_F) \wedge ((\bigvee (A \setminus A_F)) \vee b).$$

We claim that

$$(\bigvee A_F) \wedge ((\bigvee (A \setminus A_F)) \vee b) = b.$$

Indeed $b = \bigvee (\bigcap_{i \in I} A_i) \leq \bigvee A_F$, so, by modularity and by Lemma 1.12, we obtain

$$(\bigvee A_F) \wedge ((\bigvee (A \setminus A_F)) \vee b) = ((\bigvee A_F) \wedge (\bigvee (A \setminus A_F))) \vee b = 0 \vee b = b.$$

This shows that $a \wedge (b_F \vee b) \leq b$ for all $F \in \mathcal{F}(I)$, which implies that

$$a = \bigvee_{F \in \mathcal{F}(I)} (a \wedge (b_F \vee b)) \leq b,$$

as desired. \square

If $(M_i)_{i \in I}$ is an independent family of submodules of a module M_R , then clearly $\bigcap_{i \in I} M'_i = 0$, where $M'_i := \sum_{i \in I \setminus \{j\}} M_i$ for all $i \in I$. The next result, needed in the proof of Theorem 1.16, is a very particular case of Proposition 1.14, and shows that this simple fact on modules can be extended, with some effort, to any upper continuous modular lattice. We are very indebted to Patrick F. Smith for very helpful discussions about how this latticial extension can be proved.

Corollary 1.15. *Let $L \in \mathcal{U} \cap \mathcal{M}$, let I be an arbitrary nonempty set, let $(x_i)_{i \in I}$ be an independent family in L , and for each $i \in I$, set $x'_i := \bigvee_{i \in I \setminus \{j\}} x_i$. Then $0 = \bigwedge_{i \in I} x'_i$. \square*

As in Grzeszczuk and Puczyłowski [14, Proposition 2] a *basis* of an arbitrary lattice L is a maximal independent subset of L consisting only of coirreducible elements of L .

Theorem 1.16. *The following statements are equivalent for a nonzero lattice $L \in \mathcal{U} \cap \mathcal{M}$.*

- (1) L is RC (resp. RCC).
- (2) There exists a join of an independent family of coirreducible (resp. completely coirreducible) elements of L that is essential in L .
- (3) L has a basis (resp. a basis consisting only of completely coirreducible elements).
- (4) For every $0 \neq x \in L$ there exists a nonempty set I_x such that 0 can be written as an irredundant intersection

$$0 = \bigwedge_{i \in I_x} x_i$$

of irreducible elements (resp. completely irreducible elements) x_i in $x/0$, $i \in I_x$.

Moreover, the equivalent conditions (1) – (4) for an RCC lattice can be reformulated as follows:

- (1)' L is an atomic lattice.
- (2)' $\text{Soc}(L)$ is essential in L .
- (3)' L has a basis consisting only of atoms of L .
- (4)' For every $0 \neq x \in L$ there exists a nonempty set I_x such that 0 can be written as an irredundant intersection

$$0 = \bigwedge_{i \in I_x} x_i$$

of coatoms x_i in $x/0$, $i \in I_x$, in other words, the radical $r_{x/0}$ of $x/0$ is zero and an irredundant intersection of coatoms.

Proof. (1) \iff (2) follows immediately from Lemma 1.6.

(1) \implies (3): Let B be a maximal independent subset of $\mathcal{C}(L)$ (resp. $\mathcal{C}^c(L)$), which exists by Zorn's Lemma. We claim that B is also a maximal independent subset of L . Indeed, if not, there exists an element $0 \neq y \in L \setminus B$ such that $B \cup \{y\}$ is independent, so $y \wedge (\bigvee B) = 0$. Since L is RC (resp. RCC), there exists a coirreducible (resp. completely coirreducible) element $c \in L$ with $c \leq y$. Then $c \wedge (\bigvee B) \leq y \wedge (\bigvee B) = 0$, so $c \wedge (\bigvee B) = 0$, and then, necessarily $c \notin B$. This implies that $B \cup \{c\}$ is an independent subset of $\mathcal{C}(L)$ (resp. $\mathcal{C}^c(L)$) strictly including B , which contradicts the maximality of B .

(3) \implies (1): Let B be a basis of L (resp. a basis of L consisting only of completely coirreducible elements). Then $\bigvee B$ is a maximal direct join of coirreducibles (resp. completely coirreducibles) of L . If L would not be RC (resp. RCC), then, by Lemma 1.6, there would exist a nonzero element $t \in L$ such that $t \wedge (\bigvee B) = 0$, so $B \cup \{t\}$ would be an independent subset of L strictly containing B , contradicting the maximality of B .

(1) \implies (4): Let $0 \neq x \in L$. Since L is RC (resp. RCC), so is also the interval $x/0$. By the equivalence (1) \iff (2) applied for the lattice $x/0$, there exists a direct join $m = \bigvee_{i \in I} c_i$ of coirreducible (resp. completely coirreducible) elements $c_i, i \in I$, in $x/0$ that is essential in $x/0$.

If $I = \{i\}$ is a singleton then $m = c_i$ is an essential coirreducible (resp. completely coirreducible) element in $x/0$, that is, $0 = 0$ is an IID (resp. ICID) of 0 in $m/0$. By Lemma 1.9, this is also an IID (resp. ICID) of 0 in $x/0$.

If I has at least two elements, then for every $j \in I$ we set $c'_j := \bigvee_{i \in I \setminus \{j\}} c_i$. Since $m/c'_j \simeq c_j/0$, it follows that c'_j is an irreducible (resp. completely irreducible) element of $m/0$ for every $j \in I$.

By Corollary 1.15, we have $0 = \bigwedge_{i \in I} c'_i$. We claim that $\bigwedge_{i \in I \setminus \{j\}} c'_i \not\leq c'_j$ for all $j \in I$. Indeed, if not, then, there exists $j \in I$ such that $\bigwedge_{i \in I \setminus \{j\}} c'_i \leq c'_j$; hence $c_j \leq \bigwedge_{i \in I \setminus \{j\}} c'_i \leq c'_j$, and so $0 = c_j \wedge c'_j = c_j$, which is a contradiction.

This shows that $0 = \bigwedge_{i \in I} c'_i$ is an IID (resp. ICID) of 0 in $m/0$. Apply now Lemma 1.9 to see that this decomposition can be extended to an IID (resp. ICID) of 0 in $x/0$, as desired.

(4) \implies (1): Let $0 \neq x \in L$, and let $0 = \bigwedge_{i \in I} x_i$ be an IID (resp. ICID) of 0 in $x/0$. Assume that the set I has at least two elements. Then, for every $i \in I$, set $\bar{x}_i := \bigwedge_{j \in I \setminus \{i\}} x_j$. By modularity, we have $\bar{x}_i/0 = \bar{x}_i/(x_i \wedge \bar{x}_i) \simeq (x_i \vee \bar{x}_i)/x_i \subseteq x/x_i$. Since x_i is irreducible (resp. completely irreducible) in $x/0$, it follows that x/x_i is coirreducible (resp. completely coirreducible), and hence, so is also its initial subinterval $(x_i \vee \bar{x}_i)/x_i$. This implies that the interval $\bar{x}_i/0$ is coirreducible (resp. completely coirreducible), i.e., \bar{x}_i is coirreducible (resp. completely coirreducible) and $\bar{x}_i \leq x$, as desired. Observe that if $I = \{i\}$ is a singleton, then

$x_i = 0$, so 0 is irreducible (resp. completely irreducible) in $x/0$, in other words, x itself is coirreducible (resp. completely coirreducible).

(1)' \iff (2)' is exactly (1) \iff (3) in Lemma 1.10.

(2)' \iff (3)' \iff (4)': Observe that instead of completely coirreducible elements in (2) and (3) we may take atoms; indeed, if $(c_i)_{i \in I}$ is an independent set of completely coirreducible elements of L such that $\bigvee_{i \in I} c_i$ is essential in L , then $a := \bigvee_{i \in I} a_i$ is also an essential element of L , where, for every $i \in I$, a_i is the unique essential atom in the subdirectly irreducible interval $c_i/0$. Then, the corresponding elements $a'_j := \bigvee_{i \in I \setminus \{j\}} a_i$ are coatoms in $a/0$. \square

Remarks 1.17. (1) The equivalence (1) \iff (3) in Theorem 1.16, but only for coirreducibles, has been also established in Grzeszczuk and Puczyłowski [14, Proposition 2] for modular lattices L which are not necessarily upper continuous, using a completely different approach, namely the embedding of L into the modular upper continuous lattice $Id(L)$ of all ideals of L . According to Grzeszczuk and Puczyłowski [14, Theorem 1], any two bases of an RC modular lattice L have the same cardinality, called the *Goldie dimension* of L .

(2) One may ask whether condition (3) in Theorem 1.16 can be replaced by the following weaker one:

The element 0 has an irredundant decomposition $0 = \bigwedge_{i \in I} x_i$ in irreducible elements (resp. completely irreducible elements) x_i in L .

The answer is *no* for the case of completely irreducibles, as the following example, due Fuchs, Heinzer, and Olberding [10, Example 2.4], shows. Let $\{p_0, p_1, \dots\}$ be the set of all positive prime numbers in \mathbb{Z} , and consider the direct product ring $T = \prod_{i \in \mathbb{N}} \mathbb{Z}/p_i^2 \mathbb{Z}$. Let R be the subring of T generated by $(1, 1, \dots)$, (p_0, p_1, p_2, \dots) , and $(0, \dots, p_n, 0, \dots)$ for each $n \in \mathbb{N}$. The ideal 0 of R has an ICID in R and $\text{Soc}(R) = \sum_{n \in \mathbb{N}} Rr_n$ is not an essential submodule of R_R . So, by Theorem 1.16, the lattice $\mathcal{L}(R)$ of all ideals of R is not RCC, but 0 has an ICID in $\mathcal{L}(R)$.

For decompositions in irreducible elements we are looking for a module M_R that is not rich in coirreducibles such that 0 has an IID in M . This is also an open problem mentioned in Fort [8, Problème, p. 383]. According to Fort [8, Théorème 1, Proposition 5] (see also Lemmas 1.6 and 1.9), the module M should be a direct sum of a module without any coirreducibles with one that is a maximal direct sum of coirreducibles. \square

The next results, extending some results of Fuchs, Heinzer, and Olberding [10] from ideals to lattices, investigate when a CI element in an ICID of a given element is relevant, i.e., cannot be omitted.

Definition 1.18. Let L be a lattice, and let $a \leq c$ be elements of L . We say that c is a relevant completely irreducible divisor, abbreviated an RCID, of a if a has a decomposition as an intersection of completely irreducibles elements of L in which c appears and is relevant, i.e., cannot be omitted. \square

Proposition 1.19. Let $a \leq c$ be elements of a lattice $L \in \mathcal{M}$ with CID such that $c \in \mathcal{I}^c(L)$. Then c is an RCID of a if and only if c is not an essential element in the interval $1/a$.

Proof. “ \implies ”: If c is an RCID of a , then we can write $a = c \wedge b$ where b is a meet of CI elements and c cannot be omitted in the intersection above, i.e., $a = c \wedge b \neq b$, and so $c < c \vee b$. Since the cover c^* of c is by definition the least element of the interval $]c, 1]$ of L and $c \vee b \in]c, 1]$, it follows that $c^* \leq c \vee b$.

Using now the fact that L is a modular lattice, we deduce that

$$(c \vee (c^* \wedge b))/c \simeq (c^* \wedge b)/(c \wedge (c^* \wedge b)) = (c^* \wedge b)/(c \wedge b) = (c^* \wedge b)/a.$$

On the other hand, again by modularity, we have $c \vee (c^* \wedge b) = c^* \wedge (c \vee b) = c^*$. Thus $(c^* \wedge b)/a \simeq c^*/c$, and so, the interval $(c^* \wedge b)/a$ is simple, in particular $c^* \wedge b \neq a$. Since $c \wedge (c^* \wedge b) = c \wedge b = a$, we deduce that c is not essential in $1/a$, as desired.

“ \impliedby ”: If c is not essential in $1/a$, then there exists $d \in]a, 1]$ with $c \wedge d = a$. Since L is a lattice with CID by hypothesis, we can write d as an intersection $d = \bigwedge_{j \in J} d_j$ of CI elements d_j , $j \in J$. Now, observe that c is relevant in the intersection $a = c \wedge (\bigwedge_{j \in J} d_j)$ since $a \neq d$, and we are done. \square

Corollary 1.20. Let $L \in \mathcal{M}$ be a lattice with CID, and let $a \leq c$ in L . If $c \in \mathcal{I}^c(L)$, then the cover c^* of c is an essential element in the interval $1/a$.

Proof. If c is not an RCID of a , then c is essential in $1/a$ by Proposition 1.19, so c^* is also essential in $1/a$.

Assume now that c is an RCID of a . Then we can write $a = c \wedge b$, where b is a meet of CI elements and $c \wedge b \neq b$.

If $a = c$, then c^* is essential in $1/c = 1/a$. So, we may assume that $a < c$. In order to show that c^* is essential in $1/a$ we have to prove that $x \wedge c^* \neq a$ for every $x > a$.

If $x \leq c$, then $x \wedge c^* = x \neq a$. If $x \not\leq c$, then $c < c \vee x$, so $c^* \leq c \vee x$. It follows that

$$c \neq (c \vee x) \wedge c^* = c \vee (x \wedge c^*),$$

and hence $x \wedge c^* \neq a$, as desired. \square

Corollary 1.21. Let $L \in \mathcal{U} \cap \mathcal{M}$ be a lattice with CID, and let $1 \neq a \in L$. Then there exists an RCID of a if and only if $\text{Soc}(1/a) \neq a$.

Proof. “ \implies ”: If $c \geq a$ is an RCID of a , then we can write $a = c \wedge b$, where b is a meet of CI elements and $c \wedge b \neq b$.

If $c = a$, then $\text{Soc}(1/a) = \text{Soc}(1/c) = c^* \succ a$. If $c > a$, then $b > a$, and as in the proof of the implication “ \implies ” in Proposition 1.19, we have $(c^* \wedge b)/a \simeq c^*/c$, and so, the interval $(c^* \wedge b)/a$ is simple. Therefore $a \prec (c^* \wedge b) \leq \text{Soc}(1/a)$, as desired.

“ \impliedby ”: If $\text{Soc}(1/a) \neq a$, let $s \succ a$. Since $s \not\leq a$, the set $C := \{c \mid c \geq a, s \not\leq c\}$ is not empty. Now observe that $c \in C \iff s \wedge c = a$ because $s \succ a$. Using now the upper continuity of L , we deduce that the set C is inductive, so it has a maximal element c_0 by Zorn’s Lemma. By Proposition 0.7, it follows that c_0 is CI in $1/a$, so also in L . Since L is a lattice with CID, we can write s as a meet of CI elements, and c_0 is clearly relevant in the intersection $a = c_0 \wedge s$, which finishes the proof. \square

Lemma 1.22. *Let $L \in \mathcal{U} \cap \mathcal{M}$, and let $x = \bigwedge_{i \in I} x_i$ be an ICID of $x \neq 1$. For every $i \in I$ let $\bar{x}_i := \bigwedge_{j \in I \setminus \{i\}} x_j$. Then, the following statements hold.*

- (1) *For each $i \in I$, the interval \bar{x}_i/x is subdirectly irreducible with $u_i := x_i^* \wedge \bar{x}_i \in]x, \bar{x}_i]$ the unique atom covering x .*
- (2) *In the intersection $x = \bigwedge_{i \in I} x_i$ no x_i can be replaced by a larger element of L and still have the intersection be equal to x .*
- (3) *The family $(u_i)_{i \in I}$ is independent in $1/x$ and $\bigvee_{i \in I} u_i \leq \text{Soc}(1/x)$.*

Proof. (1) Let i be a fixed element of I . As in the proof of the implication (4) \implies (1) in Theorem 1.16, by modularity, we have $\bar{x}_i/x = \bar{x}_i/(x_i \wedge \bar{x}_i) \simeq (x_i \vee \bar{x}_i)/x_i \subseteq 1/x_i$. Since x_i is CI, it follows that $1/x_i$ is SI, and hence, so is also its initial subinterval $(x_i \vee \bar{x}_i)/x_i$. This implies that the interval \bar{x}_i/x is SI. From the proof of the implication “ \implies ” in Proposition 1.19 it follows that the unique atom u_i of \bar{x}_i/x is $x_i^* \wedge \bar{x}_i$.

(2) Let $i \in I$ and $y_i \in L$ with $x_i < y_i$. Then $x_i^* \leq y_i$ since $1/x_i$ is SI, so

$$x = x_i \wedge \bar{x}_i < u_i = x_i^* \wedge \bar{x}_i \leq y_i \wedge \bar{x}_i,$$

as desired.

(3) We have $u_i = x_i^* \wedge \bar{x}_i \leq \bar{x}_i \leq x_j$ for every $j \in I \setminus \{i\}$, and so,

$$x \leq u_i \wedge \left(\bigvee_{j \in I \setminus \{i\}} u_j \right) \leq \bar{x}_i \wedge x_i = x,$$

i.e., $x = u_i \wedge \left(\bigvee_{j \in I \setminus \{i\}} u_j \right)$ for every $i \in I$. This shows that the family $(u_i)_{i \in I}$ is independent in $1/x$ and $\text{Soc}(1/x) \geq \bigvee_{i \in I} u_i$. \square

Proposition 1.23. *Let $L \in \mathcal{U} \cap \mathcal{M}$. Then L is semi-Artinian if and only if every $x \in L$ has an ICID in L .*

Proof. “ \implies ”: If L is semi-Artinian, then so is the interval $1/x$ for every $1 \neq x \in L$. It follows that $1/x$ is a RCC lattice. By Theorem 1.16, x has an ICID $x = \bigwedge_{i \in I} x_i$ in $1/x$, with all $x_i, i \in I$, CI in $1/x$, so also CI in L .

“ \impliedby ”: If every $1 \neq x \in L$ has an ICID in L , then the interval $1/x$ has at least an atom by Lemma 1.22, so, by definition, L is semi-Artinian. \square

Note that the condition that L is modular in Proposition 1.23 can be weakened to “ L satisfies condition (M)”, cf. Walendziak [21, Theorem 4].

Corollary 1.24. (Crawley and Dilworth [6, 6.3]). *If every element of a modular compactly generated lattice L has an ICID in L , then L is strongly atomic.*

Proof. Since any compactly generated lattice is upper continuous (see, e.g., Crawley and Dilworth [6, 2.3]), we can apply Proposition 1.23 to conclude that L is semi-Artinian, i.e., strongly atomic. \square

Remarks 1.25. (1) Any Noetherian lattice $L \in \mathcal{C}$ is a lattice with CID. To show that, we proceed as in the proof of Năstăsescu and Van Oystaeyen [18, Proposition 1.4.4]. Assume that L is not with CID. Then, the set C of all elements $1 \neq x \in L$ that cannot be written as intersections of CI elements of L is nonempty, so C has a maximal element, say m . Clearly m is not CI, so we can write $m = \bigwedge_{i \in I} a_i$ with $a_i > m$ for all $i \in I$. Thus $a_i \in L \setminus C$ for all $i \in I$, and then, each a_i can be written as an intersections of CI elements. This implies that m can be written as an intersections of CI elements, which is a contradiction.

(2) A Noetherian lattice $L \in \mathcal{U} \cap \mathcal{M}$ is not necessarily a lattice with ICID, as the following example shows. Let L be the lattice $\mathcal{L}(\mathbb{Z})$ of all subgroups of the Abelian group \mathbb{Z} . Then L is a Noetherian lattice which is not with ICID since the zero ideal of \mathbb{Z} has no ICID in L by Lemma 1.22. \square

We end this section by mentioning the following two results on the uniqueness and replacement property of completely irreducible decompositions in lattices, that extend the corresponding results of Heinzer and Olberding[16] and Fuchs, Heinzer, and Olberding [10].

Theorem 1.26. (Crawley and Dilworth [6, Theorems 7.1 and 7.2]).

(1) *If L is a complete distributive lattice, then any element $1 \neq a \in L$ has at most one ICID.*

- (2) If L is a complete modular lattice, and $1 \neq a \in L$ has two CID decompositions $a = \bigwedge C = \bigwedge C'$, then the Kurosh - Ore replacement property holds: for every $c \in C$ there exists $c' \in C'$ such that $a = c' \wedge \bigwedge (C \setminus \{c\})$; moreover this resulting decomposition is irredundant if the decomposition $a = \bigwedge C$ is irredundant. \square

2 Applications to Grothendieck categories and torsion theories

In this section we apply the lattice-theoretical results established in the previous sections to Grothendieck categories and module categories equipped with a hereditary torsion theory.

Throughout this section \mathcal{G} will denote a fixed *Grothendieck category*, that is, an Abelian category with exact direct limits and with a generator. For any object $X \in \mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X . It is well-known that $\mathcal{L}(X)$ is an upper continuous modular lattice (see e.g., Stenström [19, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]. For all undefined notation and terminology on Abelian categories the reader is referred to Albu and Năstăsescu [3] and/or Stenström [19].

We say that an object $X \in \mathcal{G}$ is *subdirectly irreducible*, abbreviated SI, if the lattice $\mathcal{L}(X)$ is subdirectly irreducible. More generally, if \mathbb{P} is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has \mathbb{P} if the lattice $\mathcal{L}(X)$ is/has \mathbb{P} . Thus, we obtain the concepts of *coirreducible* (uniform) object, *completely coirreducible* object, *irreducible subobject* of an object, *completely irreducible* (CI) *subobject* of an object, object *rich in completely irreducibles* (RCI), object *rich in coirreducibles* (RC), etc. Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has \mathbb{P} if the element Y of the lattice $\mathcal{L}(X)$ is/has \mathbb{P} .

If we specialize Corollary 0.6, Theorems 1.16, and Proposition 1.23 for $L = \mathcal{L}(X)$, we obtain at once.

Proposition 2.1. *If X is a semi-Artinian object of a Grothendieck category \mathcal{G} , then any irreducible subobject of X is CI.* \square

Theorem 2.2. *The following assertions are equivalent for a nonzero object X of a Grothendieck category \mathcal{G} .*

- (1) X is RC.
- (2) X is an essential extension of a direct sum of coirreducible subobjects of X .
- (3) The injective hull $E(X)$ of X is an essential extension of a direct sum of indecomposable injective objects.

- (4) 0 has an irredundant irreducible decomposition in every nonzero subobjects of X . \square

Theorem 2.3. *The following statements are equivalent for a nonzero object X of a Grothendieck category \mathcal{G} .*

- (1) X is RCC.
- (2) Every subobject of X contains a simple subobject.
- (3) The socle $\text{Soc}(X)$ of X is essential in X .
- (4) For every nonzero subobject Y of X there exists a nonempty set I_Y such that 0 can be written as an irredundant intersection

$$0 = \bigcap_{i \in I_Y} X_i$$

of maximal subobjects X_i of Y , $i \in I_Y$, in other words, the Jacobson radical $J(Y)$ of Y is zero and an irredundant intersection of maximal subobjects. \square

As in Năstăsescu and Popescu [17], a Grothendieck category \mathcal{G} is said to be an *L.C.-category* if each nonzero object X of \mathcal{G} contains a coirreducible subobject, in other words, if the lattice $\mathcal{L}(X)$ is RC for each $0 \neq X \in \mathcal{G}$. The next result is a very particular case of Theorem 2.2.

Corollary 2.4. *The following statements are equivalent for a Grothendieck category \mathcal{G} .*

- (1) \mathcal{G} is an L.C.-category.
- (2) Every nonzero object X of \mathcal{G} is an essential extension of a direct sum of coirreducible subobjects of X .
- (3) For every nonzero object X of \mathcal{G} , the injective hull $E(X)$ of X is an essential extension of a direct sum of indecomposable injective objects.
- (4) For every nonzero object X of \mathcal{G} , 0 has an irredundant irreducible decomposition in every nonzero subobjects of X . \square

Remark 2.5. The equivalencies $(1) \iff (2) \iff (3)$ in Corollary 2.4 are precisely the contents of Năstăsescu and Popescu [17, Proposition 1]. \square

Proposition 2.6. *An object X of a Grothendieck category \mathcal{G} is semi-Artinian if and only if every subobject X has an irredundant completely irreducible decomposition in X . \square*

Proposition 2.7. *Let \mathcal{G} be a Grothendieck category, and let $X \in \mathcal{G}$. If X has Gabriel dimension, then X is RC.*

Proof. Apply Examples 1.3 (3) to the lattice $L = \mathcal{L}(X)$. \square

Recall that the concept of Gabriel dimension of an Abelian category \mathcal{A} , due to Gabriel [11], has been originally defined using a transfinite sequence of localizing subcategories of \mathcal{A} . For a Grothendieck category \mathcal{G} , the fact that \mathcal{G} has Gabriel dimension can be equivalently expressed by saying that \mathcal{G} possess a generator G having Gabriel dimension, that is, the lattice $\mathcal{L}(G)$ of all subobjects of G has Gabriel dimension.

Corollary 2.8. (Năstăsescu and Popescu [17, Remarques 1]). *Any Grothendieck category having Gabriel dimension is an L.C.-category.*

We end this paper by presenting some applications of our lattice theoretical results to module categories equipped with a hereditary torsion theory.

Throughout the remainder of the paper $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on $\text{Mod-}R$, and $\tau(M)$ will denote the τ -torsion submodule of a right R -module M . The set $F_\tau := \{I \leq R_R \mid R/I \in \mathcal{T}\}$ is called the *Gabriel topology* associated with τ .

For any M_R we denote $\text{Sat}_\tau(M) = \{N \mid N \leq M, M/N \in \mathcal{F}\}$, and for any $N \leq M$ we denote by $\overline{N} = \bigcap \{C \mid N \leq C \leq M, M/C \in \mathcal{F}\}$ the τ -closure (or τ -saturation, or τ -purification) of N in M ; N is called τ -closed (or τ -saturated, or τ -pure) if $N = \overline{N}$. Note that $\overline{N}/N = \tau(M/N)$ and $\text{Sat}_\tau(M) = \{N \mid N \leq M, N = \overline{N}\}$. It is known that $\text{Sat}_\tau(M)$ is an upper continuous modular lattice for any M_R (see Stenström [19, Chapter 9, Proposition 4.1]).

Recall that a module M_R is said to be τ -simple if the lattice $\text{Sat}_\tau(M)$ has exactly two elements; i.e., $\text{Sat}_\tau(M) = \{\tau(M), M\}$ and $M \notin \mathcal{T}$. A τ -simple τ -torsionfree module is called τ -cocritical. Note that the atoms of the lattice $\text{Sat}_\tau(M)$ are exactly the τ -closed τ -simple submodules of M . A right ideal I of R is called τ -critical if the right R -module R/I is τ -cocritical. The τ -socle of M is defined by $\text{Soc}_\tau(M) = \overline{\sum \{C \mid C \leq M, C \text{ is } \tau\text{-cocritical}\}}$. Note that, by Albu [1, Proposition 1.15], $\text{Soc}_\tau(M)$ is exactly the socle of the lattice $\text{Sat}_\tau(M)$. A submodule N of M is said to be τ -maximal if the module M/N is τ -cocritical. The meet of all τ -maximal submodules is called the τ -Jacobson radical of M and denoted by $J_\tau(M)$; if M fails to have any τ -maximal submodules then we set $J_\tau(M) = M$.

For all undefined notation and terminology on torsion theories the reader is referred to Albu and Năstăsescu [3], Golan [12], and/or Stenström [19].

As in Albu, Iosif, and Teply [2], a module M_R is said to be τ -subdirectly irreducible, abbreviated τ -SI, if the lattice $\text{Sat}_\tau(M)$ is subdirectly irreducible. More generally, if \mathbb{P} is any property on lattices, we say that a module M_R is/has τ - \mathbb{P} if the lattice $\text{Sat}_\tau(M)$ is/has \mathbb{P} . Since the lattices $\text{Sat}_\tau(M)$ and $\text{Sat}_\tau(M/\tau(M))$ are canonically isomorphic, we deduce that M_R is τ - \mathbb{P} if and only if $M/\tau(M)$ is τ - \mathbb{P} . Thus, we obtain the concepts of a τ -Artinian module,

τ -Noetherian module, τ -semi-Artinian module, τ -coirreducible (uniform) module, τ -completely coirreducible module, module rich in τ -coirreducibles, abbreviated τ -RC, module rich in τ -completely coirreducibles, abbreviated τ -RCC, module rich in τ -completely irreducibles, abbreviated τ -RCI, etc. We say that a submodule N of M_R is/has τ - \mathbb{P} if its closure \overline{N} , which is an element of $\text{Sat}_\tau(M)$, is/has \mathbb{P} . Thus, we obtain the concepts of a τ -irreducible submodule of a module, τ -completely irreducible submodule of a module, abbreviated τ -CI, etc. Since $\overline{\overline{N}} = \overline{N}$, it follows that N is/has τ - \mathbb{P} if and only if \overline{N} is/has τ - \mathbb{P} .

Before giving specializations of the latticial results from the previous section to the lattice $\text{Sat}_\tau(M)$ we will present some intrinsic characterizations, that is, without explicitly referring to the lattice $\text{Sat}_\tau(M)$, of τ -irreducible and τ -completely irreducible submodules of a module.

Proposition 2.9. *The following assertions are equivalent for a submodule N of a module M_R .*

- (1) N is τ -irreducible.
- (2) $M/N \notin \mathcal{T}$ and for any submodules P and Q of M with $N \subseteq P \cap Q$ and $(P \cap Q)/N \in \mathcal{T}$ one has $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$.
- (3) $M/N \notin \mathcal{T}$ and for any submodules P and Q of M with $\overline{N} = P \cap Q$ one has $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$.

Proof. (1) \implies (2): First, note that since N is τ -irreducible, $\overline{N} \neq M$, i.e., $M/N \notin \mathcal{T}$. If $N \subseteq P \cap Q$ and $(P \cap Q)/N \in \mathcal{T}$, then $\overline{N} = \overline{P \cap Q} = \overline{P} \cap \overline{Q}$, hence $\overline{N} = \overline{P}$ or $\overline{N} = \overline{Q}$ because N is τ -irreducible, i.e., \overline{N} is an irreducible element of the lattice $\text{Sat}_\tau(M)$. Thus $P/N \subseteq \overline{P}/N = \overline{N}/N \in \mathcal{T}$ or $Q/N \subseteq \overline{Q}/N = \overline{N}/N \in \mathcal{T}$, and so, $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$, as desired.

(2) \implies (3): Let $P, Q \leq M$ with $\overline{N} = P \cap Q$. Then $\overline{N}/N = (P \cap Q)/N \in \mathcal{T}$, so $P/N \in \mathcal{T}$ or $Q/N \in \mathcal{T}$.

(3) \implies (1): If $\overline{N} = X \cap Y$ with $X, Y \in \text{Sat}_\tau(M)$, then $X/N \in \mathcal{T}$ or $Y/N \in \mathcal{T}$ by hypothesis, and so $\overline{N} = \overline{X} = X$ or $\overline{N} = \overline{Y} = Y$. Now observe that $\overline{N} \neq M$ since $M/N \notin \mathcal{T}$. Consequently \overline{N} is an irreducible element of the lattice $\text{Sat}_\tau(M)$, in other words, N is τ -irreducible. \square

Corollary 2.10. *The following assertions are equivalent for a module M_R .*

- (1) M is τ -coirreducible.
- (2) $M \notin \mathcal{T}$ and for every $A, B \leq M$ with $A \cap B \in \mathcal{T}$ one has $A \in \mathcal{T}$ or $B \in \mathcal{T}$.

In particular, if $M \in \mathcal{F}$, then M is τ -coirreducible $\iff M$ is coirreducible.

Proof. M is τ -coirreducible if and only if 0 is a τ -irreducible submodule of M , so apply Proposition 2.9 for $N = 0$. \square

Remark 2.11. A module $M \in \mathcal{F}$ which is τ -completely coirreducible is not necessarily completely coirreducible. Indeed, consider the torsion theory $\tau_0 = (\mathcal{T}_0, \mathcal{F}_0)$ on the ring $R = \mathbb{Z}$ associated with the Gabriel topology F_0 on \mathbb{Z} consisted of all nonzero ideals of \mathbb{Z} . Note that this is the “localization at 0” Gabriel topology F_0 defined by the prime ideal 0 of \mathbb{Z} , \mathcal{T}_0 is the class of all usual torsion Abelian groups, and \mathcal{F}_0 is the class of all usual torsionfree Abelian groups. Observe that the lattice $\text{Sat}_{\tau_0}(\mathbb{Z}) = \{0, \mathbb{Z}\}$ has a unique atom \mathbb{Z} , so \mathbb{Z} is τ_0 -SI, i.e., τ_0 -completely coirreducible, but it is not completely coirreducible because $\bigcap_{n \in \mathbb{N}^*} n\mathbb{Z} = 0$ and $n\mathbb{Z} \neq 0$ for all $n \in \mathbb{N}^*$. \square

In order to extend the characterization of τ -irreducible submodules in Proposition 2.9 to τ -completely irreducible submodules, we introduce below the following definition.

Definition 2.12. Let M_R be a module. We say that a hereditary torsion theory τ on $\text{Mod-}R$ satisfy the condition (\dagger_M) if the closure operator on the lattice of all submodules $\mathcal{L}(M)$ of M commutes with arbitrary intersections, i.e.,

$$(\dagger_M) \quad \overline{\bigcap_{i \in I} X_i} = \bigcap_{i \in I} \overline{X_i} \text{ for any family } (X_i)_{i \in I} \text{ of submodules of } M. \quad \square$$

Note that in condition (\dagger_M) only the inclusion “ \supseteq ” is necessary since “ \subseteq ” always holds. For a module M_R we set

$$F(M) := \{N \leq M \mid M/N \in \mathcal{T}\}.$$

Observe that for $N \leq M$, one has $N \in F(M) \iff \overline{N} = M$. Clearly, $F(R_R)$ is exactly the Gabriel topology F_τ associated with τ .

Lemma 2.13. If the condition (\dagger_M) is satisfied for a module M_R , then $\bigcap_{N \in F(M)} N \in F(M)$.

Proof. If we consider the family $(N)_{N \in F(M)}$ of all elements of $F(M)$, by condition (\dagger_M) we have

$$M = \bigcap_{N \in F(M)} \overline{N} \subseteq \overline{\bigcap_{N \in F(M)} N},$$

so $\overline{\bigcap_{N \in F(M)} N} = M$, i.e., $\bigcap_{N \in F(M)} N \in F(M)$, as desired. \square

Remark 2.14. We do not know whether $\bigcap_{N \in F(M)} N \in F(M)$ implies the condition (\dagger_M) , but we suspect *no*. \square

Recall that the torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ is called *Jansian* (see Golan [12]) if the Gabriel topology F_τ associated with τ has a basis consisting of an idempotent two-sided ideal, or equivalently, if $\bigcap_{D \in F_\tau} D \in F_\tau$.

Proposition 2.15. (Golan [12, Proposition 6.6]). *A hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ on $\text{Mod-}R$ is Jansian if and only if τ satisfies the condition (\dagger_M) for any module M_R .*

Proof. For the reader's convenience we include the proof. Assume that τ is Jansian. Let M_R be a module, let $(X_i)_{i \in I}$ be a family of submodules of M , and let $x \in \bigcap_{i \in I} \overline{X_i}$. For each $i \in I$ there exists $D_i \in F_\tau$ such that $x D_i \subseteq X_i$. If we set $D := \bigcap_{i \in I} D_i$, then $D \in F_\tau$ since τ is Jansian, so $x D \subseteq X_i$ for all $i \in I$. This shows that $x D \in \bigcap_{i \in I} X_i$, and so, $x \in \overline{\bigcap_{i \in I} X_i}$. Therefore, $\bigcap_{i \in I} \overline{X_i} \subseteq \overline{\bigcap_{i \in I} X_i}$, in other words, τ satisfies the condition (\dagger_M) .

Conversely, if τ satisfies the condition (\dagger_M) for any module M_R , then, in particular (\dagger_R) is satisfied, so $\bigcap_{D \in F(R_R)} D \in F(R_R)$ by Lemma 2.13, i.e., $\bigcap_{D \in F_\tau} D \in F_\tau$, which means exactly that τ is Jansian. \square

Proposition 2.16. *Let N be a submodule of a module M_R , and consider the following assertions:*

- (1) N is τ -CI.
- (2) $M/N \notin \mathcal{T}$ and for any family $(P_i)_{i \in I}$ of submodules of M such that $N \subseteq \bigcap_{i \in I} P_i$ and $(\bigcap_{i \in I} P_i)/N \in \mathcal{T}$, one has $P_i/N \in \mathcal{T}$ for some $i \in I$.
- (3) $M/N \notin \mathcal{T}$ and for any family $(P_i)_{i \in I}$ of submodules of M such that $\overline{N} = \bigcap_{i \in I} P_i$, one has $P_i/N \in \mathcal{T}$ for some $i \in I$.

Then $(2) \implies (3) \implies (1)$, and $(1) \implies (2)$ if the torsion theory τ satisfies the condition (\dagger_M) .

Proof. $(2) \implies (3) \implies (1)$: Proceed as in the proof of Proposition 2.9.

$(1) \implies (2)$: Assume that τ satisfies the condition (\dagger_M) , and let N be as in (2). Then

$$\overline{N} = \overline{\bigcap_{i \in I} P_i} = \bigcap_{i \in I} \overline{P_i},$$

so $\overline{N} = \overline{P_i}$ for some $i \in I$ because \overline{N} is a CI element of the lattice $\text{Sat}_\tau(M)$. Thus $P_i/N \subseteq \overline{P_i}/N = \overline{N}/N \in \mathcal{T}$, and then $P_i/N \in \mathcal{T}$, as desired. \square

Definition 2.17. A submodule N of a module M is called *strongly τ -completely irreducible*, abbreviated *strongly τ -CI*, if $M/N \notin \mathcal{T}$ and for any family $(P_i)_{i \in I}$ of submodules of M such that $N \subseteq \bigcap_{i \in I} P_i$ and $(\bigcap_{i \in I} P_i)/N \in \mathcal{T}$, one has $P_i/N \in \mathcal{T}$ for some $i \in I$. \square

Remarks 2.18. (1) Let $\tau_0 = (\mathcal{T}_0, \mathcal{F}_0)$ be the torsion theory on the ring $R = \mathbb{Z}$ associated with the Gabriel topology \mathcal{F}_0 considered in Remark 2.11. Then it is easy to see that 0 is a τ_0 -CI submodule of $M = \mathbb{Z}$ which is not strongly τ_0 -CI.

(2) Any strongly τ -CI submodule N of M , with $N \in \text{Sat}_\tau(M)$ is a CI submodule of M . Indeed, if $(X_i)_{i \in I}$ is a family of submodules of M with $N = \bigcap_{i \in I} X_i$, then $(\bigcap_{i \in I} X_i)/N = 0 \in \mathcal{T}$, so $X_i/N \in \mathcal{T}$ for some $i \in I$. On the other hand $X_i/N \leq M/N \in \mathcal{F}$, so $X_i/N = 0$, i.e., $N = X_i$, which shows that N is a CI submodule of M .

(3) By Proposition 2.6, any τ -CI-submodule of M_R is strongly τ -CI in the presence of condition (\dagger_M) . \square

We are now going to specialize the latticial results obtained for an arbitrary upper continuous modular lattice to the particular case of the lattice $\text{Sat}_\tau(M)$. We will present only two such specializations. Do do that, we need some preparatory results.

Lemma 2.19. The following assertions hold for a module $M_R \in \mathcal{F}$ and a submodule $N \leq M$.

- (1) If $M/N \in \mathcal{T}$, then N is an essential submodule of M .
- (2) N is an essential submodule of \overline{N} .
- (3) If $N \in \text{Sat}_\tau(M)$, then N is an essential submodule of M if and only if N is an essential element of the lattice $\text{Sat}_\tau(M)$.

Proof. (1) Let $0 \neq x \in M$. Since $M/N \in \mathcal{T}$, there exists $I \in \mathcal{F}_\tau$ such that $xI \subseteq N$. But $xI \neq 0$ because $M \in \mathcal{F}$, so there exists $r \in R$ with $0 \neq xr \in N$, which shows that N is essential in M .

(2) Since $\overline{N}/N \in \mathcal{T}$, we can apply (1) by taking \overline{N} as M .

(3) See the proof of Albu [1, Corollary 1.3]. \square

As we already have indicated, a module M_R is said to be *rich in τ -coirreducibles*, abbreviated τ -RC (resp. *rich in τ -completely coirreducibles*, abbreviated τ -RCC) if the lattice $\text{Sat}_\tau(M)$ is RC (resp. RCC). Also, a module M_R is said to be τ -atomic if the lattice $\text{Sat}_\tau(M)$ is atomic. Note that, by Examples 1.3 (1), M_R is τ -RCC if and only if it is τ -atomic.

Proposition 2.20. A module $M_R \in \mathcal{F}$ is τ -RC (resp. τ -RCC) if and only if $M \neq 0$ and for every $0 \neq X \leq M$ there exists $C \leq X$ which is τ -coirreducible (resp. τ -cocritical).

Proof. One implication is clear. For the other one, assume that M is τ -RC (resp. τ -RCC), and let $0 \neq X \leq M$. Then $0 \neq \overline{X} \in \text{Sat}_\tau(M)$, so, by definition, there exists $D \in \text{Sat}_\tau(M)$ such that $D \leq \overline{X}$ and D is a coirreducible element (resp. atom) of the lattice $\text{Sat}_\tau(M)$, that is, D is τ -coirreducible (resp. τ -cocritical). Now, observe that $D \cap X$ is also τ -coirreducible (resp. τ -cocritical) because X is an essential submodule of \overline{X} by Lemma 2.19 (2). \square

Corollary 2.21. *Let $M_R \in \mathcal{F}$. Then M is τ -RC $\iff M$ is RC.*

Proof. Apply Proposition 2.20 and Corollary 2.10. \square

Lemma 2.22. *Let $M_R \in \mathcal{F}$ be a module, and let $(N_i)_{i \in I}$ be a family of submodules of M . Then $(N_i)_{i \in I}$ is an independent family of submodules of M if and only if $(\overline{N_i})_{i \in I}$ is an independent family of elements of the lattice $\text{Sat}_\tau(M)$.*

Proof. The implication \Leftarrow is clear. Conversely, let $(N_i)_{i \in I}$ be an independent family of submodules of M . In order to prove that $(\overline{N_i})_{i \in I}$ is an independent family of elements of the lattice $\text{Sat}_\tau(M)$, it is sufficient to assume that I is the finite set $\{1, \dots, n\}$ for some $n \in \mathbb{N}$, $n \geq 2$, because the independence is a property of finitary character in any upper continuous lattice, as $\text{Sat}_\tau(M)$ is. Denote by \vee and \wedge the join and meet, respectively, in the lattice $\text{Sat}_\tau(M)$. Then, for each $1 \leq k < n$, we have:

$$\left(\bigvee_{1 \leq i \leq k} \overline{N_i} \right) \wedge \overline{N_{k+1}} = \left(\overline{\sum_{1 \leq i \leq k} N_i} \right) \cap \overline{N_{k+1}} = \overline{\left(\sum_{1 \leq i \leq k} N_i \right) \cap N_{k+1}} = \overline{0} = 0.$$

This proves that $(\overline{N_i})_{1 \leq i \leq n}$ is an independent family of $\text{Sat}_\tau(M)$, as desired. \square

Remark 2.23. The results of Lemma 2.19, Corollary 2.21, and Lemma 2.22 may fail in the absence of the condition “ $M \in \mathcal{F}$ ”. To see that, let R be any ring, let $\chi = (\text{Mod-}R, \{0\})$ be the improper torsion theory on $\text{Mod-}R$, let M be any nonzero module, and let N be any submodule of M which is not essential in M . Then Lemma 2.19 fails in this case. An example of a proper torsion theory enjoying the same property is provided by Albu [1, Examples 1.16].

For the failure of Corollary 2.21, consider the same torsion theory χ and a module M which is not RC. Since $\text{Sat}_\chi(M) = \{M\}$, M is vacuously χ -RC, but it is not RC.

Finally, for the failure of Lemma 2.22, let M be a (direct sum) decomposable module: $M = N_1 + N_2$, $N_1 \neq 0$, $N_2 \neq 0$, $N_1 \cap N_2 = 0$. Then $(N_i)_{i=1,2}$ is an independent family of submodules of M , but $\overline{N_1} = \overline{N_2} = M$ in $\text{Sat}_\chi(M) = \{M\}$, where χ is the torsion theory considered above. \square

Lemma 2.24. *The following statements are equivalent for a module $M_R \in \mathcal{F}$.*

- (1) M is τ -coirreducible (resp. τ -completely coirreducible).

- (2) $E_R(M)$ is an indecomposable module (resp. $E_R(M) \simeq E_R(C)$ for some τ -cocritical module C).
- (3) $E_R(M) \simeq E_R(R/I)$ where I is an irreducible (resp. τ -critical) right ideal of R .

Proof. For τ -coirreducibles use Corollary 2.10 and a well known characterization of coirreducible modules via injective hulls, and for τ -completely coirreducibles use Albu, Iosif, and Teply [2, Proposition 2.2]. \square

If we specialize Theorem 1.16 characterizing RC and RCC lattices L for $L = \text{Sat}_\tau(M)$, we obtain at once the following characterizations of τ -RC and τ -RCC modules M .

Theorem 2.25. *The following statements are equivalent for a module $M_R \notin \mathcal{T}$.*

- (1) M is τ -RC (resp. τ -RCC).
- (2) There exists in the lattice $\text{Sat}_\tau(M)$ an independent family $(N_i)_{i \in I}$ of τ -coirreducible (resp. τ -completely coirreducible) submodules N_i of M , $i \in I$, such that $\overline{\sum_{i \in I} N_i}$ is an essential element in the lattice $\text{Sat}_\tau(M)$.
- (3) For every $\tau(M) \neq N \in \text{Sat}_\tau(M)$ there exists a nonempty set I_N such that $\tau(M)$ can be written as an irredundant intersection

$$\tau(M) = \bigcap_{i \in I_N} N_i$$

of τ -irreducible (resp. τ -completely irreducible) submodules N_i in N , $N_i \in \text{Sat}_\tau(M)$, $i \in I_N$.

Moreover, the equivalent conditions (1)–(3) for a τ -RCC module can be reformulated as follows:

- (1)' Any submodule N of M , $\tau(M) \neq N \in \text{Sat}_\tau(M)$ contains a τ -simple submodule in $\text{Sat}_\tau(M)$.
- (2)' The τ -socle $\text{Soc}_\tau(M)$ of M is an essential element in the lattice $\text{Sat}_\tau(M)$.
- (3)' For every $\tau(M) \neq N \in \text{Sat}_\tau(M)$ there exists a nonempty set I_N such that $\tau(M)$ can be written as an irredundant intersection

$$\tau(M) = \bigcap_{i \in I_N} N_i$$

of τ -maximal submodules N_i of N , $i \in I_N$, in other words, the τ -Jacobson radical $J_\tau(N)$ of N is $\tau(M)$ and an irredundant intersection of τ -maximal submodules of N . \square

In case the given module M_R is τ -torsionfree, then characterizations in Theorem 2.25 have the following more simple form, that involve essentiality and independence in the very familiar lattice $\mathcal{L}(M)$ of all submodules of M instead of the ones in the lattice $\text{Sat}_\tau(M)$ of all τ -closed submodules of M . In this way, one can add, as in the original Fort [8, Théorème 3], a new characterization in terms of injective hulls.

Theorem 2.26. *The following statements are equivalent for a nonzero module $M_R \in \mathcal{F}$.*

- (1) M is τ -RC (resp. τ -RCC).
- (2) There exists a sum of an independent family of coirreducible (resp. τ -completely coirreducible) submodules of M that is essential in M .
- (3) The injective hull $E_R(M)$ of M is an essential extension of a direct sum of (indecomposable) injective modules of type $E_R(C)$ where C are coirreducible (resp. τ -completely coirreducible) modules.
- (4) For every $0 \neq N \in \text{Sat}_\tau(M)$ there exists a nonempty set I_N such that 0 can be written as an irredundant intersection

$$0 = \bigcap_{i \in I_N} N_i$$

of τ -irreducible (resp. τ -completely irreducible) submodules N_i in N , $N_i \in \text{Sat}_\tau(M)$, $i \in I_N$.

Moreover, the equivalent conditions (1)–(4) for a τ -RCC module can be reformulated as follows:

- (1)' Any nonzero submodule of M contains a τ -cocritical submodule.
- (2)' The τ -socle $\text{Soc}_\tau(M)$ of M is essential in M .
- (3)' The injective hull $E_R(M)$ of M is an essential extension of a direct sum of indecomposable injective modules of type $E_R(C)$ where C are τ -cocritical modules.
- (4)' For every $0 \neq N \in \text{Sat}_\tau(M)$ there exists a nonempty set I_N such that 0 can be written as an irredundant intersection

$$0 = \bigcap_{i \in I_N} N_i$$

of τ -maximal submodules N_i of N , $i \in I_N$, in other words, the τ -Jacobson radical $J_\tau(N)$ of N is zero and an irredundant intersection of τ -maximal submodules of N .

Proof. Apply Lemma 2.19 (3), Corollary 2.21, Lemma 2.21, Lemma 2.24, and Theorem 2.25. \square

Since M is τ -RC (resp. τ -RCC) if and only if $M/\tau(M)$ is so, we can of course formulate Theorem 2.25 in terms of essentiality and independence in the lattice $\mathcal{L}(M/\tau(M))$ instead of the ones in the lattice $\text{Sat}_\tau(M)$. For instance, condition (2) can be expressed as:

(2)" *There exists an independent family $(X_i)_{i \in I}$ of τ -coirreducible (resp. τ -completely coirreducible) submodules X_i of $M/\tau(M)$, $i \in I$, such that $\bigoplus_{i \in I} X_i$ is an essential submodule of $M/\tau(M)$.*

Proposition 2.27. *A module M_R is τ -semi-Artinian if and only if every $N \in \text{Sat}_\tau(M)$ has an ICID in $\text{Sat}_\tau(M)$.*

Proof. Apply Proposition 1.23 to the lattice $L = \text{Sat}_\tau(M)$. □

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