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Asymptotic behaviour and gradient representation for cad-lag solutions of SDE

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Abstract

Piecewise continuous cad-lag solutions of SDE driven by nonlinear vector fields and containing switchings and jumps are studied involving Lyapunov exponents, weak asymptotic behaviour (in probability), self-financing (admissible) strategies and gradient representation of cad-lag solutions.

In the first part (Sections 1–3), the analysis reveals a strong connection between the existence of Lyapunov exponents and solving second order differential inequalities when weak asymptotic behaviour and admissible strategies are concerned. In addition, as far as the cad-lag solution is a sum of two components, one continuous and the second a piecewise constant one (including jumps), sufficient conditions for asymptotic stability in $L^2(\Omega; P)$ for the continuous component are given. All these results are presented into the three theorems as solutions for the problems (P1), (P2) and (P3).

The second part of this paper (see Section 4) is meaningful by itself and contains a detailed investigation of cad-lag solutions when admitting nonlinear vector fields in the jump (impulsive) part of SDE and a separation into two components (one continuous and another piecewise constant) is possible. Here, the analysis is focused on getting gradient representation of cad-lag solutions which can be viewed as differential-integral representation of the solution using generalized processes valued

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in a dual space $[\mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})]^*$ of second order differentiable functions (see Definition 4.1). The second view (see Definition 4.2) is necessary and restricting the class of functions $G \in \mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})$ to the finite composition of the global flows generated by the nonlinear vector fields in the impulsive part, we may present the results in a more attractive way (see Lemmas 4.1, 4.2 and Theorems 4.1, 4.2).

1 Introduction

The piecewise continuous and $\mathcal{F}_t$-adapted process $\{z(t,x) \in \mathbb{R}^n : t \geqslant 0, x \in \mathbb{R}^n\}$ under consideration is the unique solution of a stochastic integral equation containing switchings and jumps. The analysis relies on the decomposition $z(t,x) = \widehat{z}(t,x) + y(t)$, $t \geqslant 0$, where the continuous and $\mathcal{F}_t$-adapted process $\{\widehat{z}(t,x) : t \geqslant 0\}$ is the unique solution of the following SDE containing switchings,

$$\mathrm{d}_t\widehat{z} = f_0(\widehat{z} + y(t);\mu(t))\mathrm{d}t + \sum_{j=1}^{m} f_j(\widehat{z} + y(t);\mu(t))\mathrm{d}W_j(t),\ t \geqslant 0,\ \widehat{z}(0) = x.$$

Here, the vector fields $f_j(z;\mu) \in \mathbb{R}^n$, $(z,\mu) \in \mathbb{R}^n \times \mathbb{R}^d$, are nonlinear, $w(t) = (w_1(t),\ldots,w_m(t)) \in \mathbb{R}^m$, $t \geqslant 0$, is a standard Wiener process over $(\Omega, \{\mathcal{F}_t\} \subset \mathcal{F}, \mathrm{P})$ and $\lambda(t) := (y(t),\mu(t)) \in \Lambda$, $\Lambda = \mathbb{R}^n \times \mathbb{R}^d$, is a piecewise continuous and $\mathcal{F}_t$-adapted process. We associate the following problems with direct implications in the describing asymptotic behaviour and constructing admissible (self-financing) strategies.

Problem (P1). Find a constant $\gamma < 0$ such that the scalar process $U_\gamma(t,x) := \exp(\gamma t)\varphi(z(t,x))$, $t \geqslant 0$, is bounded from above by a continuous martingale

$$(\mathrm{A1}) \quad \begin{cases} M_\gamma(t,x) := \|x\|^2 + C_\gamma + \\ \quad + \displaystyle\sum_{j=1}^{m}\int_0^t \exp(\gamma s)\langle \partial_z\varphi(\widehat{z}(s,x)), f_j(z(s,x);\mu(s))\rangle \mathrm{d}W_j(s),\ t \geqslant 0, \end{cases}$$

where C_γ is a constant and $\varphi(z) = \|z\|^2$.

A constant $\gamma < 0$ solving the problem (P1) is called a Lyapunov exponent and it implies an exponential stability in $\mathrm{L}^2(\Omega,\mathrm{P})$ of the piecewise continuous process $\{z_\gamma(t,x) = \exp(\gamma t)z(t,x) : t \geqslant 0\}$.

Problem (P2). Find a constant $\gamma < 0$ such that the scalar process $U_\gamma(t,x) := \exp(\gamma t)\varphi(z(t,x))$, $t \in [0,T]$, is bounded from above by a continuous semi-martingale

$$\text{(A2)}\ \ S_\gamma(t,x) := \|x\|^2 + C_\gamma + \int_0^t \exp(\gamma s)\langle \partial_z(\widehat{z}(s,x)), \mathrm{d}_s\widehat{z}(s,x)\rangle,\ t \in [0,T],$$

where C_γ is a constant, $\varphi(z) = \|z\|^2$ and " $\mathrm{d}_t\widehat{z}(t,x)$" stands for the stochastic differential of the continuous process $\{\widehat{z}(t,x) : t \geqslant 0\}$ (see $z(t,x) = \widehat{z}(t,x) + y(t)$). A constant $\gamma < 0$ solving the problem (P2) lead us to the construction of admissible strategies (self-financing strategies) associated with "option problems" corresponding to a functional $\psi(z) \leqslant \exp(\gamma T)\|z\|^2 + b$, $b \geqslant 0$, and using the given piecewise continuous process $\{z(t,x) : t \in [0,T]\}$. A stability (weak stability) property of the process $\{z(t,x) : t \geqslant 0\}$, without involving a Lyapunov exponent, gets a solution if the following problem is solved.

Problem (P3). Assume

$$\text{(A3)}\ \ \langle [A_0(\widehat{z},t) + A_0^T(\widehat{z},t)]\widehat{z}, \widehat{z}\rangle \leqslant 2\beta\|\widehat{z}\|^2,\ \widehat{z} \in \mathbb{R}^n, t \geqslant 0,$$

where $\beta < 0$ and $A_j(\widehat{z},t) := \int_0^1 [\partial_z f_j(\theta\widehat{z} + y(t); \mu(t))]\mathrm{d}\theta$, $j \in \{0,1,\ldots,m\}$;

$$\text{(B3)}\ \ \lim_{t\to\infty} y(t) = y_\infty \text{ exists in } \mathrm{L}^2(\Omega,\mathrm{P});$$

$$\text{(C3)}\ \ f_j(\lambda(t)) = 0,\ t \geqslant 0,\ j \in \{0,1,\ldots,m\}.$$

Find sufficient conditions on $\beta < 0$ such that $\lim_{t\to\infty} z(t,x) = y_\infty$ in $\mathrm{L}^2(\Omega,\mathrm{P})$ (see $\lim_{t\to\infty} \widehat{z}(t,x) = 0$) for each $x \in \mathbb{R}^n$.

The answer for the problems P1–P3 relies on the possibility of solving the following second order differential inequalities,

$$[\gamma\varphi + L_1(\varphi)](\widehat{z};t) \leqslant 2\sum_{j=0}^m \|f_j(\lambda(t))\|^2,\ \widehat{z} \in \mathbb{R}^n,\ t \geqslant 0 \text{ (see P1)},$$

$$[\gamma\varphi + L_2(\varphi)](\widehat{z};t) \leqslant 2\sum_{j=1}^m \|f_j(\lambda(t))\|^2,\ \widehat{z} \in \mathbb{R}^n,\ t \geqslant 0 \text{ (see P2)},$$

$$[|\beta|\varphi + L_1(\varphi)](\widehat{z};t) \leqslant 0,\ \widehat{z} \in \mathbb{R}^n,\ t \geqslant 0 \text{ (see P3)}.$$

Here, the second order differential operators L_1 and L_2 are coming from the stochastic rule of derivation

$$L_1(\psi)(\widehat{z};t) := \langle \partial_z \psi(\widehat{z}), f_0(\widehat{z}+y(t);\mu(t))\rangle + \frac{1}{2}\sum_{j=1}^{m}\langle \partial_z^2\psi(\widehat{z}) f_j(\widehat{z}+y(t);\mu(t)), f_j(\widehat{z}+y(t);\mu(t))\rangle,$$

$$L_2(\psi)(\widehat{z};t) := \frac{1}{2}\sum_{j=1}^{m}\langle \partial_z^2\psi(\widehat{z}) f_j(\widehat{z}+y(t);\mu(t)), f_j(\widehat{z}+y(t);\mu(t))\rangle.$$

The main results (see Theorems 3.1, 3.2 and 3.3) are dealing with the construction of solutions corresponding to the problems P1–P3 where the special structure $z(t,x) = \widehat{z}(t,x) + y(t)$, $t \geqslant 0$, of the piecewise continuous process has been used. It may occurs that a piecewise constant and bounded process $y(t) \in \mathrm{B}(0,\rho) \subset \mathbb{R}^d$, $y(t) = y(t_k)$, $t \in [t_k, t_{k+1})$, $k \geqslant 0$, is acting in a multiplicative form $z(t,x) = G(y(t);\widehat{z}(t,x))$, where $G(y;z) : \mathrm{B}(0,\rho) \times \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and $\{\widehat{z}(t,x) : t \geqslant 0\}$ stands for a continuous process satisfying an Ito SDE. The analysis can be extended to this new situation provided the corresponding gradient representation (see Section 4) of cad-lag solutions is used. As far as a reduced process $h(t,x) = Q^T z(t,x)$, $t \geqslant 0$, is observed, we need to replace the original integral equation by a corresponding one driving $\{h(t,x) : t \geqslant 0\}$. Assuming $Q := (b_1,\ldots,b_k)$, $b_i \in \mathbb{R}^n$, $1 \leqslant i \leqslant k$, and $Q^T A_i(\widehat{z},t) = B_i(\widehat{z},t)Q^T$, $0 \leqslant i \leqslant m$, we define $g_i(h;\alpha(t)) := B_i(\widehat{z}(t,x),t)h + Q^T f_i(\lambda(t))$, $\alpha(t) = (t,\widehat{z}(t,x),\lambda(t))$. Replace the original equation by the following one

$$h(t,x) = Q^T x + Q^T y(t) + \int_0^t g_0(h(s,x);\alpha(s))\mathrm{d}s + \sum_{j=1}^{m}\int_0^t g_j(h(s,x);\alpha(s))\mathrm{d}W_j(s),\ t \geqslant 0.$$

Using the decomposition $h(t,x) = \widehat{h}(t,x) + Q^T y(t)$, $t \geqslant 0$, where the continuous process $\{\widehat{h}(t,x) : t \geqslant 0\}$ is the unique solutions of the corresponding SDE, we can rewrite the problems P1–P3.

2 Statement of the problems P1-P3; some auxiliary results

Let $\{\Omega, \{\mathcal{F}_t\}_\uparrow \subset \mathcal{F}, \mathrm{P}\}$ be a complete filtered probability space and $w(t) : [0,\infty) \to \mathbb{R}^m$ is the standard $\mathcal{F}_t$-adapted Wiener process. Consider an $\mathcal{F}_t$-adapted piecewise continuous process $\lambda(t) := (y(t), \mu(t)) : [0,\infty) \to \mathbb{R}^n \times \mathbb{R}^d$ and let $\{z(t,x) : t \geqslant 0\}$ be the piecewise continuous process satisfying the following integral equation containing switchings and jumps,

$$(1) \quad z(t,x) = x + y(t) + \int_0^t f_0(z(s,x);\mu(s))\mathrm{d}s + \sum_{j=1}^m \int_0^t f_j(z(s,x);\mu(s))\mathrm{d}W_j(s), \; t \geqslant 0.$$

The nonlinear vector fields $f_j(z;\mu) \in \mathbb{R}^n$, $0 \leqslant j \leqslant m$, are continuous functions of $(z,\mu) \in \mathbb{R}^n \times \mathbb{R}^d$, $f_j(z;\mu)$ are continuously differentiable of $z \in \mathbb{R}^n$ and fulfill

$$(2) \quad \|\partial_z f_j(z;\mu)\| \leqslant C_j, \; (z,\mu) \in \mathbb{R}^n \times \mathbb{R}^d, \; (C_j > 0 \text{ constant}).$$

The piecewise continuous process $\lambda(t) = (y(t), \mu(t))$, $t \geqslant 0$, satisfies

$$(3) \quad \|y(t)\|^2, \|f_j(\lambda(t))\|^2 \leqslant K \exp(\nu t), \; t \geqslant 0, \; 0 \leqslant j \leqslant m,$$

where $K > 0$ and $\nu > 0$ are some constants. Assuming (2) and (3) we get a unique global solution $\{z(t,x) : t \geqslant 0, x \in \mathbb{R}^n\}$ fulfilling (1) and it can be decomposed as follows,

$$(4) \quad z(t,x) = \widehat{z}(t,x) + y(t), \; t \geqslant 0,$$

where the continuous and $\mathcal{F}_t$-adapted process $\{\widehat{z}(t,x) : t \geqslant 0, x \in \mathbb{R}^n\}$ is the unique solution of SDE

$$(5) \quad \mathrm{d}_t\widehat{z} = f_0(\widehat{z} + y(t);\mu(t))\mathrm{d}t + \sum_{j=1}^m f_j(\widehat{z} + y(t);\mu(t))\mathrm{d}W_j(t), \; t \geqslant 0, \; \widehat{z}(0) = x.$$

Both the asymptotic behaviour (see Lyapunov exponents) and self-financing (admissible) strategies corresponding to a piecewise continuous process are analyzed using the functional $\varphi_t(z) := \exp(\gamma t)\|z\|^2$, $t \geqslant 0$. The conclusions will be the same for any other functional satisfying $\psi(z) \leqslant \exp(\gamma T)\|z\|^2 + b$, for some $b \geqslant 0$, provided that admissible strategies $\{(\theta_0(t), \theta(t)) : t \in [0,T]\}$ are involved.

Definition 2.1. *A constant $\gamma < 0$ is a Lyapunov exponent for the piecewise continuous process $\{z(t,x) : t \geqslant 0, x \in \mathbb{R}^n\}$ if the scalar process $\{U_\gamma(t,x) = \exp(\gamma t)\varphi(z(t,x)) : t \geqslant 0\}$ is bounded from above by a martingale $M_\gamma(t,x)$,*

$$U_\gamma(t,x) \leqslant M_\gamma(t,x) := \|x\|^2 + C_\gamma + \sum_{j=1}^{m} \exp(\gamma s)\langle \partial_z \varphi(\widehat{z}(s,x)), f_j(z(s,x);\mu(s))\rangle \mathrm{d}W_j(s),$$

for any $t \geqslant 0$, $x \in \mathbb{R}^n$, where $C_\gamma > 0$ is a constant.

Remark 2.1. Notice that $\gamma < 0$ is a Lyapunov exponent for $\{z(t,x) = \widehat{z}(t,x) + y(t) : t \geqslant 0\}$, if $\gamma < 0$ is a Lyapunov exponent for its continuous component $\{\widehat{z}(t,x) : t \geqslant 0\}$ and, in addition, $|\gamma| > \nu$, where $\nu > 0$ is given in (3). On the other hand, an admissible strategy $\{(\theta_0(t), \theta(t)) \in \mathbb{R}^{n+1} : t \in [0,T]\}$ involving $\{z(t,x) : t \in [0,T]\}$ and the functional $\psi(z) = \exp(\gamma t)\|z\|^2$ is found using the value function $\bar{V}_\rho(t,x) = \exp(-\rho t)V_\rho(t,x)$, $t \geqslant 0$,

$$(6) \quad V_\rho(t,x) = \exp(\rho t)\theta_0(t) + \langle \theta(t), z(t,x)\rangle, \; \bar{V}_\rho(t,x) = \theta_0(t) + \langle \theta(t), z_\rho(t,x)\rangle,$$

where $z_\rho(t,x) = \exp(-\rho t)z(t,x)$, and imposing

$$(7) \quad V_\rho(T,x) \geqslant \psi(z(T,x)) = \exp(\gamma T)\varphi(z(T,x)),$$

or

$$\bar{V}_\rho(t,x) \geqslant \exp(-\rho T)\psi(z(T,x)) = \exp(\gamma_1 T)\varphi(z_\rho(T,x)),$$

for $\varphi(z) := \|z\|^2$, $\gamma_1 = \gamma + \rho$. By definition, $z_\rho(t,x) = \bar{z}_\rho(t,x) + y_\rho(t)$, and notice that $\bar{V}_\rho(t,x) = \widehat{V}_\rho(t,x) + \langle \theta(t), y_\rho(t)\rangle$, where $\widehat{V}_\rho(t,x) := \theta_0(t) + \langle \theta(t), \widehat{z}_\rho(t,x)\rangle$, $\widehat{z}_\rho(t,x) = \exp(-\rho t)\widehat{z}(t,x)$ and $y_\rho(t)) = \exp(-\rho t)y(t)$. The so called self-financing equation involves the continuous process $\{\widehat{V}_\rho(t,x) : t \in [0,T]\}$ and it is expressed as an integral equation

$$(8) \quad \widehat{V}_\rho(t,x) = \widehat{V}_\rho(0,x) + \int_0^t \langle \theta(s), \mathrm{d}_s \widehat{z}_\rho(s,x)\rangle, \; t \in [0,T],$$

where $\widehat{V}_\rho(0,x) = \theta_0(0) + \langle \theta(0), x\rangle$ is a constant. The constant $\widehat{V}_\rho(0,x)$ and the main part $\{\theta(t) : t \in [0,T]\}$ of the admissible strategy must be determined such that

$$(9) \quad \bar{V}_\rho(t,x) := \widehat{V}_\rho(t,x) + \langle \theta(t), y_\rho(t)\rangle \geqslant \exp(\gamma_1 t)\varphi(z_\rho(t,x)), \; t \in [0,T],$$

where $\varphi(z) = \|z\|^2$, $\gamma_1 = \gamma + \rho$ and $\gamma < 0$ must be determined to accomplish the following task. Notice that the integral inequality (9) for $t = T$ shows that (7) is satisfied. Using $\varphi(z_\rho(t,x)) = \varphi(\widehat{z}_\rho(t,x)) + 2\langle\widehat{z}_\rho(t,x), y_\rho(t)\rangle\rangle + \|y_\rho(t)\|^2$, $t \in [0,T]$, we need to represent the continuous scalar process $\{\widehat{U}_{\gamma_1}(t,x) = \exp(\gamma_1 t)\varphi(\widehat{z}_\rho(t,x)) : t \in [0,T]\}$ such that the following integral inequality

$$(10)\quad \widehat{U}_{\gamma_1}(t,x) \leqslant \|x\|^2 + K_\gamma + \int_0^t \exp(\gamma_1 s)\langle\partial_z\varphi(\widehat{z}_\rho(s,x)), \mathrm{d}_s\widehat{z}_\rho(s,x)\rangle$$

is fulfilled for any $t \in [0,T]$, where $K_\gamma > 0$ is a constant. Inserting (10) into (9) we get necessary $\theta(t) = \exp(\gamma_1 t)\partial_z\varphi(\widehat{z}_\rho(t,x))$, $t \in [0,T]$, provided that (8) is used and the constant $\widehat{V}_\rho(0,x) = \theta_0(0) + 2\|x\|^2$ has to be taken such that $\widehat{V}_\rho(0,x) \geqslant \|x\|^2 + K_\gamma + \|y_\rho(t)\|^2$ for any $t \in [0,T]$. It completes the description of an admissible strategy $(\theta_0(t), \theta(t)) \in \mathbb{R}^n$, $t \in [0,T]$, mentioning that $\{\theta_0(t) : t \in [0,T]\}$ is given by the self-financing equation (8). On the other hand, the scalar process $\widehat{U}_\gamma(t,x) := \exp(\gamma t)\varphi(\widehat{z}(t,x))$, $t \geqslant 0$, must be upper bounded by a continuous martingale

$$(11)\quad \widehat{U}_\gamma(t,x) \leqslant \|x\|^2 + C_\gamma + \sum_{j=1}^m \int_0^t \exp(\gamma s)\langle\partial_z\varphi(\widehat{z}(s,x)), f_j(z(s,x);\mu(s))\rangle \mathrm{d}W_j(s),$$

for any $t \geqslant 0$, where $C_\gamma > 0$ is a constant, when a Lyapunov exponent $\gamma < 0$ is involved.

The integral inequalities (10) and (11) will be analyzed in the next two lemmas. In this respect, rewrite the nonlinear vector fields $f_i(\widehat{z}+y;\mu) \in \mathbb{R}^n$, $i \in \{0,1,\dots,m\}$, defining the basic equation (1), as follows

$$(12)\quad f_i(\widehat{z}+y(t);\mu(t)) = f_i(\lambda(t)) + A_i(\widehat{z};t)\widehat{z},\ \lambda(t) = (y(t),\mu(t)) \in \mathbb{R}^n \times \mathbb{R}^d, \widehat{z} \in \mathbb{R}^n$$

where the matrix A_i is given by

$$(13)\quad A_i(\widehat{z};t) := \int_0^1 \partial_z f_i(\theta\widehat{z} + y(t);\mu(t))\mathrm{d}\theta$$

and satisfies $\|A_i(\widehat{z};t)\| \leqslant C_i$, $\forall(\widehat{z},t) \in \mathbb{R}^n \times \mathbb{R}_+$, $i \in \{0,1,\dots,m\}$, provided that the hypothesis (2) is assumed. Using $A_i(\widehat{z};t)$ define a symmetric bounded matrix $A(\widehat{z};t)$ and a bounded vector field $F(\widehat{z};t) \in \mathbb{R}$ as follows

$$(14) \quad \begin{cases} A(\widehat{z};t) = \sum_{j=1}^{m} A_j^T(\widehat{z};t)A_j(\widehat{z};t), \ \widehat{z} \in \mathbb{R}^n, t \geqslant 0, \\ F(\widehat{z};t) = \sum_{j=1}^{m} A_j^T(\widehat{z};t)f_j(\lambda(t)), \ \widehat{z} \in \mathbb{R}^n, t \geqslant 0, \end{cases}$$

where $\{\lambda(t) = (y(t), \mu(t)) : t \geqslant 0\}$ fulfils the hypothesis (3).

Lemma 2.1. *Assume the hypotheses* (2) *and* (3) *are satisfied and let* $\gamma < 0$ *be a constant such that*

$$(15) \quad |\gamma| > \sum_{j=1}^{m} C_j^2,$$

where C_j, $j \in \{1, \dots, m\}$, *are given in* (2). *Define* $A(\widehat{z};t)$, $F(\widehat{z};t)$ *as in* (14). *Then the matrices* $Q_\gamma(\widehat{z};t) := [|\gamma| I_n - A(\widehat{z};t)]$, $P_\gamma(\widehat{z};t) = [Q_\gamma(\widehat{z};t)]^{\frac{1}{2}}$ *and* $R_\gamma(\widehat{z};t) = [P_\gamma(\widehat{z};t)]^{-1}$ *are strictly positive definite and bounded for any* $(\widehat{z}, t) \in \mathbb{R}^n \times \mathbb{R}_+$. *In addition, we get*

$$(16) \quad \begin{aligned} \exp(\gamma t)\|\widehat{z}(t,x)\|^2 = \|x\|^2 + 2\int_0^t \exp(\gamma s)\langle \widehat{z}(s,x), \mathrm{d}_s\widehat{z}(s,x)\rangle \\ + \int_0^t \exp(\gamma s)[\|R_\gamma(\widehat{z}(s,x);s)F(\widehat{z}(s,x);s)\|^2 + \sum_{j=1}^{m} \|f_j(\lambda(s))\|^2]\mathrm{d}s \\ - \int_0^t \exp(\gamma s)N_\gamma(s,x)\mathrm{d}s, \ t \geqslant 0 \end{aligned}$$

where $\{\widehat{z}(t,x) : t \geqslant 0\}$ *fulfils* (5) *and*

$$(17) \quad N_\gamma(t,x) := \|P_\gamma(\widehat{z}(t,x);t)\widehat{z}(t,x) - R_\gamma(\widehat{z}(t,x);t)F(\widehat{z}(t,x);t)\|^2 \geqslant 0, \ t \geqslant 0.$$

Proof. Denote $U_\gamma(t,x) = \exp(\gamma t)\varphi(\widehat{z}(t,x))$, $\varphi(z) = \|z\|^2$, and applying the standard rule of stochastic derivation we get

$$(18) \quad \begin{cases} \mathrm{d}_t U_\gamma(t,x) = \exp(\gamma t)[\gamma\varphi + L(\varphi)](\widehat{z}(t,x);t)\mathrm{d}t \\ \qquad + \sum_{j=1}^{m} \exp(\gamma t)\langle \partial_z \varphi(\widehat{z}(t,x)), f_j(z(t,x);\mu(t))\rangle \mathrm{d}W_j(t), \ t \geqslant 0 \\ U_\gamma(0,x) = \|x\|^2. \end{cases}$$

Here, the second order differential operator $L(\varphi)$ is computed as in

(19) $$L(\varphi)(\widehat{z},t) = \langle \partial_z\varphi(\widehat{z}), f_0(\widehat{z}+y(t),\mu(t))\rangle + \sum_{j=1}^{m} \|f_j(\widehat{z}+y(t);\mu(t))\|^2.$$

With the same notations as in (14), we rewrite (19) as follows

(20) $$\sum_{j=1}^{m} \|f_j(\widehat{z}+y(t);\mu(t))\|^2 = \langle A(\widehat{z};t)\widehat{z},\widehat{z}\rangle + 2\langle F(\widehat{z};t),\widehat{z}\rangle + \sum_{j=1}^{m} \|f_j(\lambda(t))\|^2.$$

Using (19) and (20) we get $[\gamma\varphi + L(\varphi)](\widehat{z},t)$ used in (18)

(21) $$\begin{aligned}[\gamma\varphi + L(\varphi)](\widehat{z},t) = &- \langle[|\gamma|I_n - A(\widehat{z},t)]\widehat{z},\widehat{z}\rangle + 2\langle F(\widehat{z};t),\widehat{z}\rangle \\ &+ \sum_{j=1}^{m} \|f_j(\lambda(t))\|^2 + \langle \partial_z\varphi(\widehat{z}), f_0(\widehat{z}+y(t);\mu(t))\rangle.\end{aligned}$$

If $|\gamma| > \sum_{j=1}^{m} C_j^2$ then the matrices $Q_\gamma(\widehat{z};t) := [|\gamma|I_n - A(\widehat{z};t)]$ and

(22) $$P_\gamma(\widehat{z};t) = [Q_\gamma(\widehat{z};t)]^{\frac{1}{2}},\ R_\gamma(\widehat{z};t) = [P_\gamma(\widehat{z};t)]^{-1},\ \widehat{z}\in\mathbb{R}^n, t\geqslant 0,$$

are strictly positive definite and bounded. This claim is proved using an orthogonal matrix $H(\widehat{z};t)$ $(H^{-1} = H^T)$ which implies the diagonal form of the symmetric matrix $A(\widehat{z};t)$,

(23) $$A(\widehat{z};t) = H(\widehat{z};t)[\mathrm{diag}(\gamma_1(\widehat{z},t),\ldots,\gamma_n(\widehat{z},t))]H^{-1}(\widehat{z};t),$$

where $H(\widehat{z};t) = [e_1(\widehat{z};t),\ldots,e_n(\widehat{z};t)]$ and

(24) $$A(\widehat{z};t)e_k(\widehat{z};t) = \gamma(\widehat{z},t)e_k(\widehat{z};t),\ \gamma(\widehat{z},t)\geqslant 0$$

and bounded for any $(\widehat{z},t)\subset\mathbb{R}^n\times\mathbb{R}_+$, $k=1,\ldots,n$. A direct computation shows that

(25) $$\begin{cases} \|A(\widehat{z};t)e_k(\widehat{z};t)\| = \gamma(\widehat{z},t) \leqslant \|A(\widehat{z};t)\|,\ k\in\{1,\ldots,n\}, \\ \|A(\widehat{z};t)\| \leqslant \sum_{j=1}^{m} C_j^2,\ \forall(\widehat{z},t)\in\mathbb{R}\times\mathbb{R}_+. \end{cases}$$

Using (23) write

(26) $$Q_\gamma(\widehat{z};t) := [|\gamma|I_n - A(\widehat{z};t)] = H(\widehat{z};t)D_\gamma(\widehat{z};t)H^{-1}(\widehat{z};t),$$

where the diagonal matrix $D_\gamma(\widehat{z};t) := \operatorname{diag}(|\gamma|-\gamma_1(\widehat{z},t),\ldots,|\gamma|-\gamma_n(\widehat{z},t))$ is strictly positive definite and bounded for any $(\widehat{z},t) \in \mathbb{R}^n \times \mathbb{R}_+$, provided that $|\gamma| > \sum_{j=1}^m C_j^2$ (see $0 \leqslant \gamma_k(\widehat{z},t) \leqslant \sum_{j=1}^m C_j^2$). In addition, the square root of the matrices Q_γ, $Q\gamma^{-1}$ exist and are defined as bounded matrices by

$$(27)\quad \begin{cases} P_\gamma(\widehat{z},t) = [Q_\gamma(\widehat{z};t)]^{1/2} = H(\widehat{z};t)D_\gamma^{1/2}(\widehat{z};t)H^{-1}(\widehat{z};t), \\ R_\gamma(\widehat{z};t) = [Q_\gamma(\widehat{z};t)]^{-1/2} = H(\widehat{z};t)D_\gamma^{-1/2}(\widehat{z};t)H^{-1}(\widehat{z};t), \end{cases}$$

where the diagonal matrices $D_\gamma^{1/2}$ and $D_\gamma^{-1/2}$ are strictly positive defined and bounded. It completes the proof of the first conclusion in Lemma 2.1. To get the second conclusion (see (16)) we rewrite (21) as

$$(28)\quad \begin{aligned} [\gamma\varphi + L(\varphi)](\widehat{z};t) = &- [\langle Q_\gamma(\widehat{z};t)\widehat{z},\widehat{z}\rangle - 2\langle F(\widehat{z};t),\widehat{z}\rangle] \\ &+ \sum_{j=1}^m \|f_j(\lambda(t))\|^2 + \langle \partial_z\varphi(\widehat{z}), f_0(\widehat{z}+y(t);\mu(t))\rangle \end{aligned}$$

and using (27) we represent the first term in (28) as

$$(29)\quad \langle Q_\gamma(\widehat{z};t)\widehat{z},\widehat{z}\rangle - 2\langle F(\widehat{z};t),\widehat{z}\rangle = \|P_\gamma(\widehat{z};t)\widehat{z} - R_\gamma(\widehat{z},t)F(\widehat{z};t)\|^2 - \|R_\gamma(\widehat{z};t)F(\widehat{z};t)\|^2.$$

Insert (28) and (29) into (18) and we get

$$(30)\quad \begin{aligned} \mathrm{d}_tU_\gamma(t,x) = &\exp(\gamma t)[\|R_\gamma(\widehat{z}(t,x);t)F(\widehat{z}(t,x);t)\|^2 + \sum_{j=1}^m \|f_j(\lambda(t))\|^2]\mathrm{d} \\ &+ \exp(\gamma t)\langle \partial_z\varphi(\widehat{z}(t,x)), \mathrm{d}_t\widehat{z}(t,x)\rangle - \exp(\gamma t)N_\gamma(t,x),\ t \geqslant 0, \end{aligned}$$

where $N_\gamma(t,x) \geqslant 0$ is given in (17). Integrating (30) with $U_\gamma(0,x) = \|x\|^2$ we get the second conclusion of Lemma 2.1 and the proof is complete. □

Remark 2.2. As far as Lyapunov exponent for the piecewise continuous process $\{z(t,x) : t \geqslant 0\}$ is concerned we need to represent

$$(31)\quad \widehat{U}_\gamma(t,x) := \exp(\gamma t)\|\widehat{z}(t,x)\|^2,\ t \geqslant 0,$$

as a semimartingale

$$(32)\quad \widehat{U}_\gamma(t,x) = D_\gamma(t,x) + M_\gamma(t,x),$$

where the drift part $D_\gamma(t,x)$ is bounded from above using a positive bounded process

$$(33)\quad D_\gamma(t,x) \leqslant b_\gamma(t,x),\ 0 \leqslant b_\gamma(t,x) \leqslant K_\gamma(x),\ t \geqslant 0$$

for some constant $K_\gamma(x) = \|x\|^2 + \widehat{C}_\gamma$, $\widehat{C}_\gamma > 0$. The martingale part $M_\gamma(t,x)$ is defined explicitely in Lemma 2.2 (see (39))

$$(34)\quad M_\gamma(t,x) = 2\sum_{j=1}^{m}\int_0^t \exp(\gamma s)\langle \widehat{z}(s,x), f_j(z(s,x);\mu(s))\rangle \mathrm{d}W_j,\ t \geqslant 0,$$

where

$$(35)\quad z(t,x) = \widehat{z}(t,x) + y(t),\ t \geqslant 0.$$

It will analyzed in the next Lemma using the bounded matrices $A_i(\widehat{z};t)$, $i \in \{0,1,\ldots,m\}$. Write

$$(36)\quad \begin{cases} \widehat{A}(\widehat{z};t) = [A_0(\widehat{z};t) + A_0^T(\widehat{z};t)] + A(\widehat{z};t),\ A(\widehat{z};t) := \sum_{j=1}^{m} A_j^T(\widehat{z};t)A_j(\widehat{z};t), \\ \widehat{F}(\widehat{z};t) = A_0^T(\widehat{z};t)f_0(\lambda(t)) + F(\widehat{z};t),\ F(\widehat{z};t) := \sum_{j=1}^{m} A_j^T(\widehat{z};t)f_j(\lambda(t)). \end{cases}$$

Consider the following matrices (they will exist as symmetric matrices)

$$(37)\quad \widehat{Q}_\gamma(\widehat{z};t) = [|\gamma|I_n - \widehat{A}(\widehat{z};t)],\ \widehat{P}_\gamma(\widehat{z};t) = [\widehat{Q}_\gamma(\widehat{z};t)]^{1/2}\ \widehat{R}_\gamma(\widehat{z};t) = \widehat{P}_\gamma^{-1}(\widehat{z};t).$$

Lemma 2.2. *Assume that the hypotheses* (2) *and* (3) *are satisfied and let* $\gamma < 0$ *be a constant such that*

$$(38)\quad |\gamma| > 2C_0 + \sum_{j=1}^{m} C_j^2,\ |\gamma| > \nu,$$

where $C_i > 0$, $0 \leqslant i \leqslant m$, *are given in* (2) *and* $\nu > 0$ *in* (3). *Then the matrices* $\widehat{Q}_\gamma(\widehat{z};t)$, $\widehat{P}_\gamma(\widehat{z};t)$ *and* $\widehat{R}_\gamma(\widehat{z};t)$ *are strictly positive definite and bounded for any* $(\widehat{z};t) \in \mathbb{R}^n \times \mathbb{R}_+$. *In addition, the following SDE is valid,*

$$(39)\quad \begin{aligned} \exp(\gamma t)\|\widehat{z}(t,x)\|^2 = b_\gamma(t,x) &+ 2\sum_{j=1}^{m}\int_0^t \exp(\gamma s)\langle \widehat{z}(s,x), f_j(z(s,x);\mu(s))\rangle \mathrm{d}W_j(s) \\ &- \int_0^t \exp(\gamma s)N_\gamma(s,x)\mathrm{d}s,\ t \geqslant 0, \end{aligned}$$

where $\{b_\gamma(t,x) : t \geqslant 0\}$ *is a bounded process verifying*

$$(40)\quad \begin{aligned} 0 \leqslant b_\gamma(t,x) :=& \|x\|^2 + \int_0^t \exp(\gamma s)[\|\widehat{R}_\gamma(\widehat{z}(s,x);s)\widehat{F}(\widehat{z}(s,x);s)\|^2 \\ &+ \sum_{j=1}^m \|f_j(\lambda(s))\|^2]\mathrm{d}W_j(s) \leqslant \|x\|^2 + C_\gamma,\ t \geqslant 0\,(C_\gamma > 0), \end{aligned}$$

and $\{N_\gamma(t,x) : t \geqslant 0\}$ *is a positive process fulfilling*

$$(41)\quad N_\gamma(t,x) := \|\widehat{P}_\gamma(\widehat{z}(t,x);t)\widehat{z}(t,x) - \widehat{R}_\gamma(\widehat{z}(t,x);t)\widehat{F}(\widehat{z}(t,x);t)\|^2 \geqslant 0,\ t \geqslant 0,\ x \in \mathbb{R}^n.$$

Proof. The hypotheses and computations given in Lemma 2.1 are valid and applying the standard rule of stochastic derivation we deduce the stochastic differential equations satisfied by the scalar continuous process $\{\widehat{U}_\gamma(t,x) := \exp(\gamma t)\varphi(\widehat{z}(t,x)) : t \geqslant 0\}$, where $\varphi(z) = \|z\|^2$, we get

$$(42)\quad \begin{aligned} \mathrm{d}_t\widehat{U}_\gamma(t,x) =& \exp(\gamma t)[\gamma\varphi + L(\varphi)](\widehat{z}(t,x);t)\mathrm{d}t \\ &+ 2\sum_{j=1}^m \exp(\gamma t)\langle \widehat{z}(t,x), f_j(\widehat{z}(t,x) + y(t);\mu(t))\rangle \mathrm{d}W_j(s),\ t \geqslant 0, \end{aligned}$$

where $\varphi(z) = \|z\|^2$. Notice $f_i(\widehat{z} + y(t);\mu(t)) = f_i(\lambda(t)) + A_i(\widehat{z};t)\widehat{z}$, $i \in \{0,1,\ldots,m\}$, and rewrite the drift of (42)

$$(43)\quad \begin{aligned} [\gamma\varphi + L(\varphi)](\widehat{z}(t,x);t) =& -\langle \widehat{Q}_\gamma(\widehat{z}(t,x);t)\widehat{z}(t,x), \widehat{z}(t,x)\rangle + 2\langle \widehat{F}(\widehat{z}(t,x);t), \widehat{z}(t,x)\rangle \\ &+ \sum_{j=1}^m \|f_j(\lambda(t))\|^2,\ t \geqslant 0,\ x \in \mathbb{R}^n, \end{aligned}$$

where the bounded matrix $\widehat{Q}_\gamma(\widehat{z};t)$ and the vector $\widehat{F}(\widehat{z};t) \in \mathbb{R}^n$ are defined in (36) and (37). Using an orthogonal matrix $\widehat{H}(\widehat{z};t)$ $(\widehat{H}^T(\widehat{z};t) = \widehat{H}^{-1}(\widehat{z};t))$ we get the diagonal matrix associated with $\widehat{Q}_\gamma$,

$$(44)\quad \widehat{Q}_\gamma(\widehat{z};t) = \widehat{H}(\widehat{z};t)D_\gamma(\widehat{z};t)\widehat{H}^{-1}(\widehat{z};t),\ \widehat{Q}_\gamma(\widehat{z};t) = [|\gamma|I_n - \widehat{A}(\widehat{z};t)],$$

where $D_\gamma(\widehat{z};t) = \mathrm{diag}(|\gamma| - \gamma_1(\widehat{z},t),\ldots,|\gamma| - \gamma_n(\widehat{z},t))$, $\widehat{H}(\widehat{z};t) = [e_1(\widehat{z},t),\ldots,e_n(\widehat{z};t)]$ and

$$(45)\quad \widehat{A}(\widehat{z};t)e_k(\widehat{z},t) = \gamma_k(\widehat{z},t)e_k(\widehat{z},t),\ \|e_k(\widehat{z},t)\| = 1,\ k \in \{1,\ldots,n\}.$$

By a direct computation, we get

$$
(46) \quad \begin{cases} \|\widehat{A}(\widehat{z};t)e_k(\widehat{z},t)\| = |\gamma_k(\widehat{z},t)|, \\ \|\widehat{A}(\widehat{z};t)\| = \max_{\|e\|=1} \|\widehat{A}(\widehat{z};t)e\| \geqslant |\gamma_k(\widehat{z},t)|, \ k \in \{1,\dots,n\} \end{cases}
$$

where

$$
(47) \quad \|\widehat{A}(\widehat{z};t)\| \leqslant 2C_0 + \sum_{j=1}^{m} C_j^2,
$$

(see (36) and assumption (2)) for any $\widehat{z} \in \mathbb{R}^n$, $t \geqslant 0$. Using the hypothesis (38) we notice $|\gamma| > 2C_0 + \sum_{j=1}^{m} C_j^2 \geqslant |\gamma_k(\widehat{z},t)|$ for any $k \in \{1,\dots,n\}$ and $\widehat{z} \in \mathbb{R}^n$, $t \geqslant 0$. Therefore, the diagonal matrix $D_\gamma(\widehat{z},t)$ has a bounded strictly positive inverse

$$
(48) \quad D_\gamma^{-1}(\widehat{z};t) = \text{diag}\{(|\gamma| - \gamma_1(\widehat{z},t))^{-1}, \dots, (|\gamma| - \gamma_n(\widehat{z},t))^{-1}\},
$$

and it allows to take square root of the positive definite and bounded matrices

$$
(49) \quad \begin{cases} \widehat{P}_\gamma(\widehat{z};t) = [\widehat{Q}_\gamma(\widehat{z};t)]^{1/2} = \widehat{H}(\widehat{z};t)[D_\gamma(\widehat{z};t)]^{1/2}\widehat{H}^{-1}(\widehat{z};t), \\ \widehat{R}_\gamma(\widehat{z};t) = \widehat{P}_\gamma^{-1}(\widehat{z};t) = \widehat{H}(\widehat{z};t)[D_\gamma(\widehat{z};t)]^{-1/2}\widehat{H}(\widehat{z};t). \end{cases}
$$

Using (49) we rewrite (43) as follows,

$$
(50) \quad \begin{aligned} [\gamma\varphi + L(\varphi)](\widehat{z}(t,x);t) = & - \|\widehat{P}_\gamma(\widehat{z}(t,x);t)\widehat{z}(t,x) - \widehat{R}_\gamma(\widehat{z}(t,x);t)\widehat{F}(\widehat{z}(t,x);t)\|^2 \\ & + \|\widehat{R}_\gamma(\widehat{z}(t,x);t)\widehat{F}(\widehat{z}(t,x);t)\|^2 + \sum_{j=1}^{m} \|f_j(\lambda(t))\|^2, \ t \geqslant 0. \end{aligned}
$$

Inserting (50) into (42) we get the integral equation (39) and the proof is complet. □

Remark 2.3. The conclusions of the Lemma (2.2) lead us to a positive answer for the following problem:

Find a constant $\gamma < 0$ *such that* $\widehat{U}_\gamma(t,x) := \exp(\gamma t)\varphi(\widehat{z}(t,x))$, $t \geqslant 0$, *is bounded from above by a martingale*

$$
(51) \quad \widehat{M_\gamma}(t,x) = K_\gamma(x) + \sum_{j=1}^{m} \int_0^t \langle \partial_z\varphi(\widehat{z}(s,x)), f_j(\widehat{z}(s,x) + y(s);\mu(s))\rangle \mathrm{d}W_j(s), \ t \geqslant 0,
$$

where $\varphi(z) = \|z\|^2$ *and* $0 < K_\gamma(x) = \|x\|^2 + C_\gamma$ *is a constant.*

Answer: If $|\gamma > 2C_0 + \sum_{j=1}^m C_j^2|$ and $|\gamma > \mu|$ (see (38)) then the conclusion (39) is valid, where the continuous and bounded process $\{b_\gamma(t,x) : t \geqslant 0\}$ satisfies (40). It implies directly

$$(52)\quad \exp(\gamma t)\varphi(\widehat{z}(t,x)) \leqslant M_\gamma(t,x),$$

for any $t \geqslant 0$ and each $x \in \mathbb{R}^n$, where the martingale $\{M_\gamma(t,x) : t \geqslant 0\}$ is defined in (51).

Remark 2.4. There is a particular situation when the asymptotic behaviour of the continuous process $\{\widehat{z}(t,x) : t \geqslant 0, x \in \mathbb{R}^n\}$ can be described without using a Lyapunov exponent $\gamma < 0$. In this respect, assume the hypothesis (2) and, in addition,

$$(53)\quad \begin{cases} f_i(\lambda(t)) = 0,\ i \in \{0, 1, \dots, m\},\ t \geqslant 0, \\ \langle [A_0(\widehat{z}; t) + A_0^T(\widehat{z}; t)]\widehat{z}, \widehat{z}\rangle \leqslant 2\beta|\widehat{z}|^2,\ t \geqslant 0,\ \widehat{z} \in \mathbb{R}^n \end{cases}$$

With the same notations as in Lemmas 2.1 and 2.2 we state

Lemma 2.3. *Assume the conditions* (2) *and* (53) *are fulfilled, where the constant* $\beta < 0$ *satisfies*

$$(54)\quad |\beta| > \sum_{j=1}^m C_j^2$$

and $C_j > 0$, $j \in \{1, \dots, m\}$, *are given in* (2). *Then the continuous process* $\{\widehat{z}(t,x) : t \geqslant 0\}$ *verifies the following scalar integral equation*

$$(55)\quad \begin{aligned} \|\widehat{z}(t,x)\|^2 &= \|x\|^2 + \beta \int_0^t \|\widehat{z}(s,x)\|^2 \mathrm{d}s - \int_0^t N_\beta(s,x)\mathrm{d}s \\ &\quad + 2\sum_{j=1}^m \int_0^t \langle \widehat{z}(s,x), f_j(z(s,x); \mu(s))\rangle \mathrm{d}W_j(s),\ t \geqslant 0, \end{aligned}$$

where

$$(56)\quad N_\beta(t,x) := \langle \{Q_\beta(t,x) + [2\beta I_n - (A_0^T + A_0)(t, \widehat{z}(t,x))]\}\widehat{z}(t,x), \widehat{z}(t,x)\rangle \geqslant 0,$$

and $Q_\beta(t,x) := |\beta| I_n - A(t; \widehat{z}(t,x)) \geqslant 0$ *for any* $t \geqslant 0$ *and* $x \in \mathbb{R}^n$.

Proof. We apply the standard rule of stochastic derivation associated with the continuous process $\{\widehat{z}(t,x) : t \geqslant 0\}$ and using the functional $\varphi(z) = \|z\|^2$, $z \in \mathbb{R}^n$. It implies

$$(57)\quad \mathrm{d}_t[\varphi(\widehat{z}(t,x))] = L_1(\varphi)(\widehat{z}(t,x);t)\mathrm{d}t + \sum_{j=1}^{m} \langle \partial_z \varphi(\widehat{z}(t,x)), f_j(z(t,x);\mu(t))\rangle \mathrm{d}W_j(t),\ t \geqslant 0,$$

where the second order differential operator $L_1(\varphi)(\widehat{z};t)$ satisfies

$$(58)\quad L_1(\varphi)(\widehat{z};t) = \langle \widehat{A}(\widehat{z};t)\widehat{z},\widehat{z}\rangle + 2\langle \widehat{F}(\widehat{z};t),\widehat{z}\rangle + \sum_{j=1}^{m} \|f_j(\lambda(t))\|^2.$$

Here, the symmetric matrix $\widehat{A}(\widehat{z};t)$ and vector field $\widehat{F}(\widehat{z};\lambda(t)) \in \mathbb{R}^n$ are defined in (36). Using conditions (53) we rewrite the drift in (57) as follows,

$$(59)\quad L_1(\varphi)(\widehat{z};t) = \langle \widehat{A}(\widehat{z};t)\widehat{z},\widehat{z}\rangle = \beta\varphi(\widehat{z}) - \langle Q_\beta(t;\widehat{z})\widehat{z},\widehat{z}\rangle - \langle [2\beta I_n - (A_0^T + A_0)(\widehat{z};t)]\widehat{z},\widehat{z}\rangle,$$

where $Q_\beta(t,\widehat{z}) = |\beta| I_n - A(\widehat{z};t)$ and the symmetric matrix $A(\widehat{z};t)$ is defined by

$$(60)\quad A(\widehat{z};t) := \sum_{j=1}^{m} A_j^T(\widehat{z};t)A_j(\widehat{z};t).$$

Using an orthogonal matrix $H(\widehat{z};t)$ $(H^T = H^{-1})$ we get the diagonal form of $A(z;t)$ satisfying

$$(61)\quad \begin{cases} A(\widehat{z};t) = H(\widehat{z};t)D(\widehat{z};t)H^{-1}(\widehat{z};t), \\ D(\widehat{z};t) = \mathrm{diag}(\gamma_1(\widehat{z};t),\ldots,\gamma_n(\widehat{z};t)), \end{cases}$$

where $0 \leqslant \gamma_k(\widehat{z};t) \leqslant \sum_{j=1}^m \|A_j(\widehat{z};t)\|^2 \leqslant \sum_{j=1}^m C_j^2$ for any $t \geqslant 0$ and $\widehat{z} \in \mathbb{R}^n$ provided the hypothesis (2) is assumed. Using (61) into (59) we rewrite (57) as the scalar equation

$$(62)\quad \begin{cases} \mathrm{d}_t[\varphi(\widehat{z}(t,x))] = \beta\varphi(\widehat{z}(t,x))\mathrm{d}t - N_\beta(t,x)\mathrm{d}t \\ \qquad\qquad + \sum_{j=1}^{m} \langle \partial_z \varphi(\widehat{z}(t,x)), f_j(z(t,x);\mu(t))\rangle \mathrm{d}W_j(t), \\ \varphi(\widehat{z}(0,x)) = \|x\|^2, \end{cases}$$

where $N_\beta(t,x) \geqslant 0$ is defined in (56) and the matrix $Q_\beta(t,x) = |\beta| I_n - A(\widehat{z};t) \geqslant 0$ for any $t \geqslant 0$, $\widehat{z} \in \mathbb{R}^n$, provided $|\beta| > \sum_{j=1}^m C_j^2$. We notice that integrating (62) we get the integral equation (55) and the proof is complete. □

3 Main results (Problems P1–P3)

Here we shall present those conclusions which are more or less direct implications of Lemmas 2.1, 2.2 and 2.3 regarding Lyapunov exponents, admissible strategies and convergence in $L_2(\Omega, P)$ of $\{z(t,x) : t \nearrow \infty\}$ satisfying the integral equation (1).

Theorem 3.1. *Assume that the vector fields $f_i(z;\mu)$, $i \in \{0,1,\ldots,m\}$, and the piecewise continuous process $\{\lambda(t) = (y(t), \mu(t)) : t \geqslant 0\}$ fulfill conditions* (2) *and* (3). *Let $\gamma < 0$ be a constant such that*

$$(63)\quad |\gamma| > 2C_0 + \sum_{j=1}^{m} C_j^2,\ |\gamma| > \nu,$$

where $\{C_0, C_1, \ldots, C_m\}$ are given in (2) *and $\nu > 0$ in* (3). *Then $|\gamma| < 0$ satisfying* (63) *is a Lyapunov exponent for the piecewise continuous solution $\{z(t,x) : t \geqslant 0\}$ verifying* (1) *and the following estimate is valid,*

$$(64)\quad (E\|z_\gamma(t,x)\|^2)^{1/2} \leqslant \exp(\gamma t)(E\|z(t,x)\|^2)^{1/2} \leqslant \exp(\alpha t) L_\gamma(x),\ t \geqslant 0,$$

where $\alpha = \max(\frac{\gamma}{2}, \frac{\nu}{2} - |\gamma|) < 0$ and $L_\gamma(x) > 0$ is a constant.

Proof. By definition, $z_\gamma(t,x) = \exp(\gamma t) z(t,x)$, $t \geqslant 0$, satisfies

$$z_\gamma(t,x) = \exp(\gamma t)\widehat{z}(t,x) + \exp(\gamma t) y(t)$$

and

$$(65)\quad [E\|z_\gamma(t,x)\|^2]^{1/2} \leqslant \exp(\gamma t)[E\|\widehat{z}(t,x)\|^2]^{1/2} + \sqrt{K}[\exp(\frac{\nu}{2} - |\gamma|)t],\ t \geqslant 0,$$

where $K > 0$ and $\nu > 0$ are given in the hypothesis (3). Assuming (63), the conditions of Lemma 2.2 are fulfilled and from the conclusion (39) we obtain

$$(66)\quad \begin{cases} E[\exp(\gamma t)\|\widehat{z}(t,x)\|^2] \leqslant K_\gamma(x),\ t \geqslant 0, \\ [E\|\widehat{z}_\gamma(t,x)\|^2]^{1/2} \leqslant \exp(\frac{\gamma}{2} t)\sqrt{K_\gamma(x)},\ t \geqslant 0, \end{cases}$$

provided $|\gamma| > \nu$ (see (39)), where $K_\gamma(x) > 0$ is a constant and $\widehat{z}_\gamma(t,x) := \exp(\gamma t)\widehat{z}(t,x)$, $t \geqslant 0$. Using (66), write (65) as follows,

(67) $[E\|z_\gamma(t,x)\|^2]^{1/2} \leqslant L_\gamma(x)\exp(\alpha t),\ t \geqslant 0,$

where $L_\gamma(x) = \sqrt{K} + \sqrt{K_\gamma(x)} > 0$ and $\alpha = \max(\frac{\gamma}{2}, \frac{\nu}{2} - |\gamma|)$. The proof is complete. □

As is mentioned in Remark 2.1, the construction of an admissible strategy $(\theta_0(t), \theta(t)) \in \mathbb{R}^{n+1}$ uses the value function $\bar{V}_\rho(t,x)$ and the piecewise continuous process $z_\rho(t,x)$,

(68) $$\begin{cases} \bar{V}_\rho(t,x) := \theta_0(t) + \langle\theta(t), z_\rho(t,x)\rangle,\ t \in [0,T],\ (\bar{V}_\rho(t,x) = \exp(-\rho t)V_\rho(t,x)), \\ z_\rho(t,x) := \exp(-\rho t)z(t,x) = \widehat{z}_\rho(t,x) + y_\rho(t),\ t \in [0,T],\ \rho \geqslant 0, \end{cases}$$

where $\{z(t,x) : t \geqslant 0\}$ is the solution of the integral equation (1). Write $\bar{V}_\rho(t,x) = \widehat{V}_\rho(t,x) + \langle\theta(t), y_\rho(t)\rangle$, where the continuous scalar process $\{\widehat{V}_\rho(t,x) := \theta_0(t) + \langle\theta(t), \widehat{z}_\rho(t,x)\rangle : t \in [0,T]\}$ must satisfy the so called "self-finnancing equation"

(69) $\widehat{V}_\rho(t,x) = \widehat{V}_\rho(0,x) + \int_0^t \langle\theta(s), \mathrm{d}_s\widehat{z}_\rho(s,x)\rangle,\ t \in [0,T],$

where the constant $\widehat{V}_\rho(0,x) = \theta_0(0) + \langle\theta(0), x\rangle$ and the continuous process $\{\theta(t) : t \in [0,T]\}$ are determined such that (using $\widehat{V}_\rho(t,x)$ as in (69))

(70) $\bar{V}_\rho(t,x) := \widehat{V}_\rho(t,x) + \langle\theta(t), y_\rho(t)\rangle \geqslant \exp(\gamma_1 t)\varphi(z_\rho(t,x)),\ t \in [0,T],$

where $\varphi(z) = \|z\|^2$, $\gamma_1 = \gamma + \rho$ and $\gamma < 0$ (see $\gamma_1 < 0$) is not fixed. An admissible strategy $\{\theta_0(t), \theta(t) : t \in [0,T]\}$ must satisfy (69) and (70).

Theorem 3.2. *Assume that $f_i(z,\mu)$, $i \in \{0,1,\ldots,m\}$, and the process $\{\lambda(t) = (y(t),\mu(t)) : t \geqslant 0\}$ fulfill the hypothesis (2) and (3). Let $\gamma_1 = \gamma + \rho < 0$ (see $\gamma < 0$) be fixed such that*

(71) $|\gamma_1| > \sum_{j=1}^m C_j^2$ (*see* $\gamma < 0,\ |\gamma| > \rho + \sum_{j=1}^m C_j^2$),

where $C_j > 0$ are given in (2). Then $\{\widehat{\theta}(t) = 2\exp(\gamma t)\widehat{z}(t,x) : t \in [0,T]\}$ and $\{\widehat{\theta}_0(t) : t \in [0,T]\}$ fulfilling the integral equation

(72) $\widehat{\theta}_0(t) + \langle\widehat{\theta}(t), \widehat{z}_\rho(t,x)\rangle = \widehat{\theta}_0(0) + 2\|x\|^2 + \int_0^t \langle\widehat{\theta}(s), \mathrm{d}_s\widehat{z}_\rho(s,x)\rangle,\ t \in [0,T],$

is an admissible strategy satisfying (69) *and* (70) *provided the constant* $\widehat{\theta}_0(0) \in \mathbb{R}_+$ *is taken such that*

(73) $\widehat{\theta}_0(0) + \|x\|^2 \geqslant K + C_\gamma$, ($K > 0$ *is given in* (3)),

where the constant $C_\gamma \geqslant a_{\gamma_1}$ *and*

$$a_{\gamma_1} := \int_0^T \exp(\gamma_1 t)[\|R_{\gamma_1}(\widehat{z}_\rho(t,x);t)F(\widehat{z}_\rho(t,x);t)\|^2 + \sum_{j=1}^m \|f_j(\lambda(t))\|^2]\mathrm{d}t$$

is defined in Lemma 2.1 corresponding to $\gamma_1 < 0$ *and* $\{\widehat{z}_\rho(t,x) : t \geqslant 0\}$.

Proof. By hypothesis, the conclusions of Lemma 1 are valid and write the conclusion (16) as follows,

(74) $$\exp(\gamma_1 t)\|\widehat{z}_\rho(t,x)\|^2 \leqslant \|x\|^2 + \widehat{a}^t_{\gamma_1}(x) + \int_0^t 2\exp(\gamma_1 s)\langle \widehat{z}_\rho(s,x), \mathrm{d}_s\widehat{z}_\rho(s,x)\rangle,\ t \geqslant 0,$$

where

(75) $$\widehat{a}^t_{\gamma_1}(x) := \int_0^t \exp(\gamma_1 s)[\|R_{\gamma_1}(\widehat{z}_\rho(s,x);s)F(\widehat{z}_\rho(s,x);s)\|^2 + \sum_{j=1}^m \|f_j(\lambda(s))\|^2]\mathrm{d}s,\ t \in [0,T]$$

is increasing and bounded for any $x \in \mathbb{R}^n$. Notice that the right hand side of (70) fulfils

(76) $$\exp(\gamma_1 t)\varphi(z_\rho(t,x)) = \exp(\gamma_1 t)[\|\widehat{z}_\rho(t,x)\|^2 + 2\langle \widehat{z}_\rho(t,x), y_\rho(t)\rangle + \|y_\rho(t)\|^2],\ t \in [0,T],$$

and using (74) and (69) we replace (70) by the following inequality

(77) $$\begin{aligned}\widehat{\theta}_0(0) + \langle \widehat{\theta}(0), x\rangle + \int_0^t \langle \widehat{\theta}(s), \mathrm{d}_s\widehat{z}_\rho(s,x)\rangle + \langle \widehat{\theta}(t), y_\rho(t)\rangle &= \bar{V}_\rho(t,x)\\ \geqslant \|x\|^2 + \widehat{a}^t_{\gamma_1}(x) + \int_0^t 2\exp(\gamma_1 s)\langle \widehat{z}_\rho(s,x), \mathrm{d}_s\widehat{z}_\rho(s,x)\rangle&\\ + 2\exp(\gamma_1 t)\langle \widehat{z}_\rho(t,x), y_\rho(t)\rangle + \exp(\gamma_1 t)\|y_\rho(t)\|^2,\ t \in [0,T].&\end{aligned}$$

Choosing $\widehat{\theta}(t) = 2\exp(\gamma_1 t)\widehat{z}_\rho(t,x) = 2\exp(\gamma t)\widehat{z}(t,x)$, $t \geqslant 0$, from (77) we get the condition for $\widehat{\theta}_0(0) \in \mathbb{R}_+$,

(78) $$\widehat{\theta}_0(0) + 2\|x\|^2 \geqslant \|x\|^2 + \widehat{a}^T_{\gamma_1}(x) + \exp((\gamma - \rho)t)\|y(t)\|^2,\ t \in [0,T]$$

which is satisfied provided (73) is fulfilled. Using the constant $\widehat{\theta}_0(0)$ and $\{\widehat{\theta}(t) = 2\exp(\gamma t)\widehat{z}(t,x) : t \in [0,T]\}$ we get the continuous scalar process $\{\widehat{\theta}_0(t) : t \in [0,T]\}$ as the solution of the integral equation (72) and the proof is complete. □

Theorem 3.3. *Assume the hypothesis* (2) *and conditions* (53) *are satisfied for some constant* $\beta < 0$ *verifying*

(79) $$|\beta| > \sum_{j=1}^{m} C_j^2,$$

where $C_j > 0$, $j \in \{1, \ldots, m\}$, *are given in* (2). *Then*

(80) $$E\|\widehat{z}(t,x)\|^2 \leqslant \|x\|^2 \exp(\beta t), \ \forall t \geqslant 0, \ x \in \mathbb{R}^n,$$

where $\{\widehat{z}(t,x) : t \geqslant 0\}$ *is the continuous process fulfilling* (5).

In addition, assume $\lim_{t\to\infty} y(t) = y_\infty$ *in* $\mathrm{L}_2(\Omega; \mathrm{P})$. *Then*

(81) $$\begin{cases} \lim\limits_{t\to\infty} z(t,x) = \lim\limits_{t\to\infty} y(t) := y_\infty \in \mathrm{L}_2(\Omega, \mathrm{P}), \\ \lim\limits_{t\to\infty} \displaystyle\int_{\mathbb{R}^n} \psi(z) P(t,x; \mathrm{d}z) = \int_{\mathbb{R}^n} \psi(z) P_\infty(\mathrm{d}z), \ \forall \psi \in \mathrm{C}_b(\mathbb{R}^n), \end{cases}$$

where $P(t,x;\cdot)$ *and* $P_\infty(\cdot)$ *are the probability measures on* $\mathbb{R}^n$ *generated by* $\{z(t,x)\}$ *and corresponding* $\{y_\infty\}$.

Proof. By hypothesis, the conclusions of Lemma 2.3 are valid and usig (55), (56) we get the corresponding integral deterministic equation satisfied by $h(t,x) = E\|\widehat{z}(t,x)\|^2$, $t \geqslant 0$. A standard representation formula lead us to (80) and (81). □

4 Gradient representation of cad-lag solutions for SDE

The main part of the SDE under consideration is driven by a standard Wiener process $w(t) = (w_1(t), \ldots, w_m(t)) \in \mathbb{R}^m$, $t \geqslant 0$, over a filtered probability space $\{\Omega, \{\mathcal{F}_t\}_\uparrow \subset \mathcal{F}, \mathrm{P}\}$. The switchings and jumps of the SDE are determined by a piecewise constant, bounded and $\mathcal{F}_t$-adapted process

$$\{\lambda(t) = (y(t), \mu(t)) \in \mathbb{R}^d \times \mathbb{R}^l(\Lambda) : \lambda(t) = \lambda(t_j), t \in [t_j, t_{j+1}), t_0 = 0, j \geqslant 0\}.$$

We are looking for piecewise continuous processes which are right continuous and possessing left hand limits in $\mathrm{L}^2(\Omega;\mathbb{R}^n)$ at each $t=t_j$, $j\geqslant 0$ (cad-lag solutions) such that the following system of SDE is satisfied,

$$(1)\quad \begin{cases} \mathrm{d}_t z(t) = f_0(z(t);\mu(t_j))\mathrm{d}t + \sum_{i=1}^m f_i(z(t);\mu(t_j))\mathrm{d}W_i(t) \\ \qquad + \sum_{k=1}^d g_k(z(t_j-);\mu(t_j-))[y_k(t_j)-y_k(t_j-)],\ \forall t\in[t_j,t_{j+1}),\ j\geqslant 0, \\ z(0)=x\in\mathbb{R}^n. \end{cases}$$

where $v(t-) := \lim_{s\nearrow t} v(s)$. Here, the vector fields $h(z;\mu) = f_i(z;\mu),\ g_k(z;\mu)\in\mathbb{R}^n$, $i\in\{0,1,\ldots,m\}$, $k\in\{1,\ldots,d\}$, are continuous functions of $(z,\mu)\in\mathbb{R}^n\times\mathbb{R}^l$ and Lipschitz continuous of $z\in\mathbb{R}^n$, i.e.

$$(2)\quad |h(z';\mu(t)) - h(z'';\mu(t))| \leqslant C_1|z'-z''|,\ \forall z',z''\in\mathbb{R}^n,\ t\geqslant 0,$$

for some constant $C_1>0$. Under the assumption (2), a unique (cad-lag) solution of (1) exists and the first integral representation of it can be described as

$$\{z(t) = z_j(t) : t\in[t_j,t_{j+1}), j\geqslant 0\}$$

where $\{z_j(t) : t\in[t_j,t_{j+1})\}$ is a continuous process for each $j\geqslant 0$ fulfilling the following integral equations,

$$(3)\quad z_0(t) = x + \int_0^t f_0(z_0(s);\mu(0))\mathrm{d}s + \sum_{i=1}^m \int_0^t f_i(z_0(s);\mu(0))\mathrm{d}W_i(s),\ t\in[0,t_1),$$

$$(4)\quad \begin{aligned} z_j(t) = {} & z_{j-1}(t_j-) + b(\Delta y(t_j), z_{j-1}(t_j-), \mu(t_j-)) + \int_{t_j}^t f_0(z_j(s);\mu(t_j))\mathrm{d}s \\ & + \sum_{i=1}^m \int_{t_j}^t f_i(z_j(s);\mu(t_j))\mathrm{d}W_i(s),\ \forall t\in[t_j,t_{j+1}),\ j\geqslant 1. \end{aligned}$$

Here, $v(t_j-) := \lim_{t\nearrow t_j} v(t)$ in $\mathrm{L}^2(\Omega;\mathbb{R}^n)$, $\Delta y(t_j) := y(t_j) - y(t_j-)$ and

$$b(y,z,\mu) := \sum_{k=1}^d g_k(z;\mu)y_k,\ y\in\mathbb{R}^d,\ z\in\mathbb{R}^n,\ \mu\in\mathbb{R}^l.$$

4.1 A stochastic rule of derivation and the corresponding decomposition formula

A $\varphi \in C_p^{1,2}([0,\infty) \times \mathbb{R}^n)$ means a continuous function $\varphi(t,z) : [0,\infty) \times \mathbb{R}^n \to R$ which is continuously differentiable of first order with respect to t and second order with respect to $z \in \mathbb{R}^n$ satisfying a polinomial growth condition for $z \in \mathbb{R}^n$, i.e. $\forall z \in \mathbb{R}^n$, $t \in [0,T]$, $T > 0$, $i, j \in \{1, \dots, n\}$,

$$(5) \quad |\varphi(t,z)|,\ |\partial_t\varphi(t,z)|,\ |\partial_i\varphi(t,z)|,\ |\partial^2_{ij}\varphi(t,z)| \leqslant C_T(1+|z|^N),$$

where the natural number $N \geqslant 1$ and the constant (depending on T) $C_T > 0$ are fixed.

Proposition 4.1. *Let $\{z(t) = z_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ be the (cad-lag) solution verifying SDE* (1) *where the vector fields $h = f_i$, g_k fulfill the hypothesis* (2). *Consider a $\varphi \in C_p^{1,2}([0,\infty) \times \mathbb{R}^n$ and define $\{\varphi(t, z(t)) = \varphi(t, z_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0)\}$. The following integral equation is valid,*

$$(6) \quad \varphi(t,z(t)) = \varphi(0,x) + \int_0^t (L\varphi)(s, z(s); \mu(s))\mathrm{d}s + M(t) + \sum_{0<t_j\leqslant t} [\varphi(t_j, z(t_j)) - \varphi(t_j, z(t_j-))],$$

for any $t \geqslant 0$, where the parabolic operator L is defined by

$$(7) \quad (L\varphi)(t,z,\mu) := \partial_z\varphi(t,z) + \langle \partial_z\varphi(t,z), f_0(z;\mu)\rangle + \frac{1}{2}\sum_{i=1}^m \langle [\partial_z^2\varphi(t,z)] f_i(z;\mu), f_i(z;\mu)\rangle$$

and the continuous martingale $M(t)$, $t \geqslant 0$ is given by

$$(8) \quad M(t) := \sum_{i=1}^m \int_0^t \langle \partial_z\varphi(s, z(s)), f_i(z(s), \mu(s))\rangle \mathrm{d}W_i(s),$$

fulfilling $EM(t) = 0$ for any $t \geqslant 0$.

Proof. We apply the standard rule of stochastic derivation on each $t \in [t_j, t_{j+1})$ and get

$$(9) \quad \begin{aligned} \varphi(t, z(t)) = & \varphi(t_j, z(t_j)) + \int_{t_j}^t (L\varphi)(s, z(s), \mu(s))\mathrm{d}s \\ & + \sum_{i=1}^m \int_{t_j}^t \langle \partial_z\varphi(s, z(s)), f_i(z(s); \mu(s))\rangle \mathrm{d}W_i(s),\ \forall t \in [t_j, t_{j+1}), \end{aligned}$$

where the parabolic operator L is given in (7). Using (9), we obtain

$$\lim_{t \nearrow t_{j+1}} \varphi(t, z(t)) = \varphi(t_{j+1}, z(t_{j+1}-)) \in \mathrm{L}_2(\Omega, \mathrm{P}), \ j \geqslant 0.$$

In particular, rewrite

$$\varphi(t_j, z(t_j)) = \varphi(t_j, z(t_j-)) + [\varphi(t_j, z(t_j)) - \varphi(t_j, z(t_j-))], \tag{10}$$

and

$$\begin{aligned} \varphi(t_j, z(t_j-)) = \lim_{t \nearrow t_j} \varphi(t, z(t)) =& \varphi(t_{j-1}, z(t_{j-1})) + \int_{t_{j-1}}^{t_j} (L\varphi)(s, z(s); \mu(s)) \mathrm{d}s \\ &+ \sum_{i=1}^{m} \int_{t_{j-1}}^{t_j} \langle \partial_z \varphi(s, z(s)), f_i(z(s); \mu(s)) \rangle \mathrm{d}W_i(s). \end{aligned} \tag{11}$$

Insert (10) and (11) into (9) and we obtain

$$\begin{aligned} \varphi(t, z(t)) =& \varphi(t_{j-1}, z(t_{j-1})) + \int_{t_{j-1}}^{t} (L\varphi)(s, z(s); \mu(s)) \mathrm{d}s \\ &+ \sum_{i=1}^{m} \int_{t_{j-1}}^{t} \langle \partial_z \varphi(s, z(s)), f_i(z(s); \mu(s)) \rangle \mathrm{d}W_i(s) \\ &+ [\varphi(t_j, z(t_j)) - \varphi(t_j, z(t_j-))], \ t \in [t_{j-1}, t_{j+1}). \end{aligned} \tag{12}$$

Using the induction argument we see easily that (12) can be extended to $t \in [0, t_{j+1})$ by adding the corresponding piecewise constant components and we get

$$\begin{aligned} \varphi(t, z(t)) = \varphi(0, z(0)) + &\int_0^t (L\varphi)(s, z(s); \mu(s)) \mathrm{d}s + M(t) \\ &+ \sum_{0 < t_j \leqslant t} [\varphi(t_j, z(t_j)) - \varphi(t_j, z(t_j-))] \end{aligned} \tag{13}$$

which stands for the conclusion (6) of Proposition 4.1 where $z(0) = x$ is used. The property $EM(t) = 0$, $\forall t \geqslant 0$, mentioned in (8) is a direct consequence of (5) and

$$|h(z; \mu(t))| \leqslant C_2(1 + |z|), \ \forall t \geqslant 0, \ z \in \mathbb{R}^n \ (h = f_i, \ g_k),$$

which lead us to the conclusion that the integrands

$$\{h_i(t) := \langle \partial_z \varphi(t, z(t)), f_i(z(t); \mu(t)) \rangle, \ t \in [0, T]\}, \ i \in \{1, \ldots, m\}, \tag{14}$$

under stochastic integration in the martingale part $\{M(t) : t \geqslant 0\}$ are in $\mathrm{L}^2([0, T] \infty \omega)$, $\int_0^T E|h_i(t)|^2 \mathrm{d}t < \infty$, and it implies $EM(t) = 0$ for any $t \geqslant 0$. The proof is complete. □

Remark 4.1. Using the stochastic rule of derivation given in Proposition 4.1 we obtain the first decomposition formula of the (cad-lag) solution $\{z(t) = z(t_j) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ satisfying (1) into a continuous process and piecewise constant process. In this respect, take $\varphi_i(t,z) = z_i$, $i \in \{1,\dots,n\}$, and the conclusion (6) written for all $i \in \{1,\dots,n\}$ lead us to the following expression,

$$\begin{aligned} z(t) &= x + \int_0^t f_0(z(s);\mu(s))\mathrm{d}s + \sum_{i=1}^m \int_0^t f_i(z(s),\mu(s))\mathrm{d}W_i(s) \\ &+ \sum_{0<t_j\leqslant t} [z(t_j) - z(t_j-)],\ t \geqslant 0. \end{aligned} \tag{15}$$

Using the integral equations in (3) and (9) we can write

$$\begin{aligned} z(t_j) - z(t_j-) &= z_j(t_j) - z_{j-1}(t_j-) = b(\Delta y(t_j), z_{j-1}(t_j-), \mu(t_j-)) \\ &= \sum_{1\leqslant k\leqslant d} g_k(z_{j-1}(t_j-);\mu(t_j-))[y_k(t_j) - y_k(t_j-)],\ j \geqslant 1, \end{aligned} \tag{16}$$

which allow to define the piecewise constant component as follows,

$$z_d(t) := \sum_{1\leqslant k\leqslant d} [\sum_{0<t_j\leqslant t} g_k(z_{j-1}(t_j-);\mu(t_j-))(y_k(t_j) - y_k(t_j-))].$$

In the case that the vector fields $g_k(z;\mu)$ do not depend on $z \in \mathbb{R}^n$, for any $k \in \{1,\dots,d\}$, then the equations (15) stand for the decomposition $z(t) = z_c(t) + z_d(t)$, $t \geqslant 0$, where the continuous component $\{z_c(t) : t \geqslant 0\}$ is the unique solution of an integral equation,

$$z_c(t) = x + \int_0^t f_0(z_c(s) + z_d(s);\mu(s))\mathrm{d}s + \sum_{i=1}^m \int_0^t f_i(z_c(s) + z_d(s);\mu(s))\mathrm{d}W_i(s).$$

Remark 4.2. We shall focus on the gradient representation of (cad-lag) solutions in (1) as a source for getting a decomposition of the solution when g_k, $k \in \{1,\dots,d\}$, depend only on $z \in \mathbb{R}^n$.

4.2 Definition of the gradient representation for (cad-lag) solutions

Denote by $\mathrm{Z} = \mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^n)$ the space of second order differentiable functions $f(z) : \mathbb{R}^n \to \mathbb{R}^n$ and let $\mathrm{C}^2_g(\mathbb{R}^d;\mathrm{Z}) \subset \mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^n)$ be the subspace of functions $G(y;z) : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$

fulfilling

(a) $G(0;z)=z$ and there exists $H(y;\cdot):=[G(y;\cdot)]^{-1}$ for each $y\in B(0,R)\subset\mathbb{R}^d$.

Consider a pair of piecewise constant and bounded processes

$$\{(p(t),\widehat{p}(t))=(p(t_j),\widehat{p}(t_j))\in B(0,R)\times B(0,R):t\in[t_j,t_{j+1}),j\geqslant 0\}$$

and let $\{\widehat{z}(t,x):t\geqslant 0,\widehat{z}(0,x)=x\in\mathbb{R}^n\}$ be a continuous process, where $\{t_j\}_{j\geqslant 0}$ is used in SDE (1). Here an abuse is done by not mentioning the explicit dependence on parameter $\omega\in\Omega$ of the above given process. Associate a new piecewise continuous process valued in the dual space $[\mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})]^*$, for each $\omega\in\Omega$, satisfying

(b)
$$\begin{cases}\{[p^1(t),z^1(t,x)]=[p^1(t_j),z^1(t,x)]\in[\mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})]^*:t\in[t_j,t_{j+1}),j\geqslant 0]\},\\ [p^1(t_j),z^1(t,x)](G(y;z)):=G(\widehat{p}(t_j-);\widehat{z}(t,x))+\partial_y G(\widehat{p}(t_j-);\widehat{z}(t_j,x))[p(t_j)-p(t_j-)],\end{cases}$$

for any $t\in[t_j,t_{j+1})$, $j\geqslant 0$ and $G\in\mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})$.

Denote $\{\widehat{z}_j(t)=\widehat{z}(t,x):t\in[t_j,t_{j+1}]\}$ for each $j\geqslant 0$, where $\{\widehat{z}(t,x):t\geqslant 0\}$ is the fixed continuous process.

Definition 4.1. *We say that $\widehat{G}\in\mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})$, a pair of piecewise constant processes $\{(p(t),\widehat{p}(t))=(p(t_j),\widehat{p}(t_j)):t\in[t_j,t_{j+1}),j\geqslant 0\}$ and a continuous process $\{\widehat{z}(t,x):t\geqslant 0,\widehat{z}(0,x)=x\in\mathbb{R}^n\}$ define a gradient representation for (cad-lag) solution $\{z(t,x)=z_j(t):t\in[t_j,t_{j+1}),j\geqslant 0\}$ satisfying (1) if $z_j(t)=[p^1(t),z^1(t,x)](\widehat{G}(y;z))$, $t\in[t_j,t_{j+1})$, $j\geqslant 0$, where the distribution valued process $[p^1(t),z^1(t,x)]$ is defined in (b).*

The above given definition is too abstarct for constructing the unknown entering the gradient representation and we need to mention the real constraints which are contained in it. First of all we notice that in the analysis which follows only the particular $\widehat{G}\in\mathrm{C}_g^2(\mathbb{R}^d;\mathrm{Z})$,

(c) $\widehat{G}(y;z):=G_1(y_1)\circ\cdots\circ G_d(y_d)(z)$, $y=(y_1,\dots,y_d)\in\mathbb{R}^d$, $z\in\mathbb{R}^n$,

is used, where $G_i(y_i)(z)$, $y_i\in\mathbb{R}$, $z\in\mathbb{R}^n$, stands for the global flow generated by the vector field $g_i\in\mathrm{C}^2(\mathbb{R}^n;\mathbb{R}^n)$. In addition, the basic equation in Definition 4.1 can be separated into two parts more suitable to compair the (cad-lag) solution defined in (3) and (4) with the general definition adopted here in (b).

Definition 4.2. *We say that $\widehat{G} \in C_g^2(\mathbb{R}^d; Z)$ (see (c)), a pair of piecewise constant processes $\{(p(t), \widehat{p}(t)) = (p(t_j), \widehat{p}(t_j)) \in B(0,R) \times B(0,R) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ and a continuous process $\{\widehat{z}(t,x) = \widehat{z}_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ define a gradient representation for the (cad-lag) solution $\{z(t,x) = z_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ satisfying (1) if*

$$(17)\quad \begin{cases} z_j(t) = \widehat{G}(\widehat{p}(t_j-); \widehat{z}_j(t)) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{p}(t_j-); \widehat{z}_j(t_j)))(y_k(t_j) - y_k(t_j-)), \\ \partial_y \widehat{G}(\widehat{p}(t_j-); \widehat{z}_j(t_j))[p(t_j) - p(t_j-)] = \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{p}(t_j-); \widehat{z}_j(t_j)))(y_k(t_j) - y_k(t_j-)), \\ t \in [t_j, t_{j+1}),\ j \geqslant 0. \end{cases}$$

Commment on the gradient representation. (see Definition 4.2). There are several constraints which must be satisfied by the piecewise constant processes $(p(t), \widehat{p}(t)) \in B(0,R) \times B(0,R)$ and the continuous process $\{\widehat{z}(t,x) \in \mathbb{R}^n : t \geqslant 0\}$ fulfilling the equations (17). The second equations of (17) writen for $j = 0$ will make sense if we assume that the given piecewise constant process $\{y(t) = y(t_j), t \in [t_j, t_{j+1}), j \geqslant 0\}$ (see (1)) and $\{(p(t), \widehat{p}(t)) = (o(t_j), \widehat{p}(t_j)), t \in [t_j, t_{j+1}), j \geqslant 0\}$ we are looking for verify $y(t) = 0$, $t \in (-t_1, t_1)$, and $(p(t), \widehat{p}(t)) = (0,0) \in \mathbb{R}^d \times \mathbb{R}^d$, for any $t \in (-t_1, t_1)$. In addition, the second equations (17) will lead us to the solution $[p(t_j) - p(t_j-)] \in \mathbb{R}^d$ provided $\widehat{p}(t_j-) \in \mathbb{R}^d$ and $\widehat{z}_j(t_j) \in \mathbb{R}^n$ are known and the smooth mapping $\widehat{G} \in C_g^2(\mathbb{R}^d; Z)$ (see (c)) has the property (A1): *there exist smooth vector fields $\{q_k(p) \in \mathbb{R}^d : p \in \mathbb{R}^d\}$, $q_k(0) = e_k$, $k = \{1, \dots, d\}$, such that $\partial_y \widehat{G}(p; z) q_k(p) = g_k(\widehat{G}(p; z))$, for any $p \in \mathbb{R}^d$, $z \in \mathbb{R}^n$ and $k \in \{1, \dots, d\}$, $q_1(p) = e_1 \in \mathbb{R}^d$.* Assuming (A1), we may and do construct $[p(t_j) - p(t_j-)]$ as follows,

$$(E1)\quad p(t_j) - p(t_j-) = \sum_{k=1}^{d} q_k(\widehat{p}(t_j-))[y_k(t_j) - y_k(t_j-)],\ j \geqslant 0.$$

On the other hand, recalling the definition of a (cad-lag) solution (see (3) and (4)) we rewrite the first equations in (17) at $t = t_j$ and get

$$
\text{(E2)} \quad \begin{aligned} z_{j-1}(t_j-)+&\sum_{k=1}^{d} g_k(z_{j-1}(t_j-))(y_k(t_j) - y_k(t_j-)) = \\ &= \widehat{G}(\widehat{p}(t_j-);\widehat{z}_j(t_j)) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{p}(t_j-);\widehat{z}_j(t_j)))(y_k(t_j) - y_k(t_j-)). \end{aligned}
$$

These equations are solved provided $\widehat{z}_j(t_j)$ is known and find $\widehat{p}(t_j-) = \widehat{p}(t_{j-1}) \in \mathbb{R}^d$ such that

(E3) $\widehat{G}(\widehat{p}(t_j-);\widehat{z}_j(t_j)) = z_{j-1}(t_j-),\ j \geqslant 1.$

This algorithm requires a precise order in solving equations and on the first place is situated the construction of the continuous process $\{\widehat{z}(t) : t \in [0, t_1)\}$ satisfying (17) for $j = 0$ and we get $z_0(t) = \widehat{z}_0(t)$, $t \in [0, t_1)$ (see $\widehat{p}(0-) = 0$, $\widehat{G}(0; z) = z$ and $y(0) = y(0-) = 0$). Then find $\{\widehat{z}_1(t) : t \in [t_1, t_2), \widehat{z}_1(t_1) = \widehat{z}_0(t_1)\}$ such that

(E4) $z_1(t) = \widehat{z}_1(t) + \sum_{k=1}^{d} g_k(\widehat{z}_1(t_1))y_k(t_1),\ t \in [t_1, t_2)$ (see (4) for $j = 1$).

Actualy, (E4) is the first equation we must solve in order to get a solution of (E3) for $j = 2$, $\widehat{p}(t_2-) = \widehat{p}(t_1)$, and then find $[p(t_2) - p(t_1)]$ from (E1) for $j = 2$ (where $p(t_1) \in \mathbb{R}^d$ is known).

4.3 $\{g_1, \ldots, g_d\} \subset Z$ commute and the gradient representation of (cad-lag) solutions

Assume that the piecewise constant process $\{y(t) = y(t_j) : t \in [t_j, t_{j+1}], j \geqslant 0\}$ and the vector fields $g_k \in C_b^2(\mathbb{R}^n; \mathbb{R}^n)$, $k \in \{1, \ldots, d\}$, satisfy

$$
\text{(18)} \quad \begin{cases} |g_k(z)|,\ |\partial_i g_k(z)|,\ |\partial_{ij}^2 g_k(z)| \leqslant C_2,\ \forall z \in \mathbb{R}^n,\ i, j \in \{1, \ldots, n\},\ k \in \{1, \ldots, d\}, \\ V_y := \sum_{j=0}^{\infty} |y(t_{j+1} - y(t_j))| \leqslant \rho,\ y(t) = 0,\ t \in (-t_1, t_1), \end{cases}
$$

where $C_2 > 0$, $\rho > 0$ are some constants, and $\partial_i g = \frac{\partial g}{\partial z_i}$, $\partial^2_{ij} g = \frac{\partial^2 g}{\partial z_i \partial z_j}$. In addition, for the time being consider that

(19) $\{g_1, \dots, g_d\} \subset \mathrm{C}^2_b(\mathbb{R}^n; \mathbb{R}^n)$ are commuting, i.e. Lie brackets $[g_i, g_j] = 0$, $1 \leqslant i, j \leqslant d$.

Associate the reduced jumping system

$$(20)\quad \begin{cases} \mathrm{d}_t h(t; \widehat{z}) = \sum_{k=1}^{d} g_k(h(t-; \widehat{z}))\mathrm{d}y_k(t), \ t \geqslant 0, \\ h(0, \widehat{z}) = \widehat{z} \in \mathbb{R}^n. \end{cases}$$

The (cad-lag) solution of (20) is constructed as a piecewise constant process satisfying $h(t; \widehat{z}) = h(t_j; \widehat{z})$, $t \in [t_j, t_{j+1})$ and

$$(21)\quad \begin{cases} h(t_j; \widehat{z}) = h(t_j-; \widehat{z}) + \sum_{k=1}^{d} g_k(h(t_j-; \widehat{z}))(y_k(t_j) - y_k(t_j-)), \ j \geqslant 0, \\ h(0; \widehat{z}) = \widehat{z} \in \mathbb{R}^n. \end{cases}$$

Using (19) we prove easily that a gradient representation is valid for the piecewise constant process in (21) and in this respect associate $\widehat{G} \in \mathrm{C}^2_g(\mathbb{R}^d; \mathrm{Z})$ (see Definition (4.2))

(22) $\widehat{G}(y; z) := G_1(y_1) \circ \dots \circ G_d(y_d)(z)$, $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ $\widehat{G}(0, z) = z \in \mathbb{R}^n$,

where $G_k(y_k)(z)$ for $y_k \in \mathbb{R}$, $z \in \mathbb{R}^n$, is the global flow generated by the vector field $g_k \in \mathrm{C}^2_b(\mathbb{R}^n; \mathbb{R}^n)$ (see (18)). Using definition, $\widehat{H}(y; \cdot) := [\widehat{G}(y; \cdot)]^{-1} = \widehat{G}(-y; \cdot)$ and

(23) $\partial_y \widehat{G}(y; z) = (g_1(\widehat{G}(y; z)), \dots, g_d(\widehat{G}(y; z)))$, $\forall y \in \mathbb{R}^d$, $z \in \mathbb{R}^n$.

The assumption (A1) associated with Definition 4.2 of a gradient representation is satisfied when (19) is assumed and using (23) we get $q_k(y) = e_k$, $k \in \{1, \dots, d\}$, where $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$ is the canonical basis. It implies that the piecewise constant process $\{p(t) = p(t_j) \in \mathbb{R}^d : t \in [t_j, t_{j=1}), j \geqslant 0\}$ which solves the equations (E1) must satisfy

(24) $p(t_j) - p(t_j-) = y(t_j) - y(t_j-)$, $p(t) = y(t)$, $t \in (-t_1, t_1)$, $j \geqslant 0$.

It shows that $p(t) = y(t)$, for any $t \geqslant 0$, and taking $\widehat{z} \in \mathbb{R}^n$ as a continuous process $\{\widehat{z}(t, \widehat{z}) = \widehat{z} : t \geqslant 0\}$, we need to define a piecewise constant process $\{\widehat{y}(t) = \widehat{y}(t_j) \in \mathbb{R}^d : t \in [t_j, t_{j+1}), j \geqslant 0\}$, $\widehat{y}(t) = 0$, $t \in (-t_1, t_1)$, such that

$$(25)\quad h(t_j; \widehat{z}) = \widehat{G}(\widehat{y}(t_j-); \widehat{z}) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_j-); \widehat{z}))(y_k(t_j) - y_k(t_j-)),\ j \geqslant 0,$$

where $h(t_j; \widehat{z}) \in \mathbb{R}^n$ defined in (21) replaces the continuous process $\{z_j(t) : t \in [t_j, t_{j+1})\}$ in the first equations of (17) from Definition 4.2. The unique solution $\widehat{y}(t_j-) = \widehat{y}(t_{j-1})$ of (25) is given by solving the corresponding equations

$$(26)\quad \widehat{G}(\widehat{y}(t_{j-1}); \widehat{z}) = h(t_{j-1}; \widehat{z}),\ j \geqslant 1,$$

which imply $\widehat{y}(0) = 0$ and $\widehat{G}(\widehat{y}(t_j); \widehat{z}) = h(t_j; \widehat{z})$ for any $j \geqslant 1$. As far as

$$h(t_1; \widehat{z}) = \widehat{z} + \sum_{k=1}^{m} g_k(\widehat{z}) y_k(t_1)$$

and

$$\widehat{G}(\widehat{y}(t_1); \widehat{z}) = \widehat{z} + \widehat{y}(t_1) \int_0^1 \partial_y \widehat{G}(\theta \widehat{y}(t_1); \widehat{z}) d\theta = \widehat{z} + \sum_{k=1}^{d} \widehat{y}_k(t_1) \int_0^1 g_k(\widehat{G}(\theta \widehat{y}(t_1); \widehat{z})) d\theta$$

we need to assume that each $g_k \in C_b^2(\mathbb{R}^n; \mathbb{R}^n)$ has the following structure which agree with (18) and (19),

$$(27)\quad g_k(z) = \alpha_k(z) b_k,\ k \in \{1, \ldots, d\},$$

where $\{b_1, \ldots, b_d\} \subset \mathbb{R}^d$ are fixed and $\alpha_k(\cdot) \in C_b^2(\mathbb{R}^n; R)$ satisfying $0 < \delta \leqslant \alpha_k(z) \leqslant M$, $z \in \mathbb{R}^n$, agree with the commuting property in (19), i.e.

$$\alpha_j(z)\langle \partial_z \alpha_i(z), b_j \rangle - \alpha_i(z)\langle \partial_z \alpha_j(z), b_i \rangle = 0,\ z \in \mathbb{R}^n,\ i, j \in \{1, \ldots, d\}.$$

If the hypothesis (27) is assumed, then the expression in (26) for $j = 1$ and $h(t_1; \widehat{z}) = \widehat{z} + \sum_{k=1}^{d} (\alpha_k(\widehat{z}) y_k(t_1)) b_k$ will allow to solve the equation

$$(28)\quad \widehat{G}(\widehat{y}(t_1); \widehat{z}) = h(t_1; \widehat{z}),\ \forall \widehat{z} \in \mathbb{R}^n$$

using a nonlinear contractive mapping. In this respect, rewrite $\widehat{G}(\widehat{y};\widehat{z})$ as follows

$$(29)\quad \widehat{G}(\widehat{y};\widehat{z}) = \widehat{z} + \sum_{k=1}^{d}(\alpha_k(\widehat{z})\widehat{y}_k)b_k + \sum_{k=1}^{d}\alpha_k(\widehat{z})\widehat{y}_k[\sum_{j=1}^{d}\beta_{kj}(\widehat{y};\widehat{z})\widehat{y}_j]b_k$$

where

$$\beta_{kj}(\widehat{y};\widehat{z}) := \frac{1}{\alpha_k(\widehat{z})}\int_0^1 \theta\theta \int_0^1 \langle\alpha_j(\widehat{G}(\theta_1\theta\widehat{y};\widehat{z}))\partial_z\alpha_k(\widehat{G}(\theta_1\theta\widehat{y};\widehat{z})), b_j\rangle d\theta_1$$

is a continuous bounded function of $(\widehat{y},\widehat{z}) \in \mathbb{R}^d \times \mathbb{R}^n$ fulfilling

$$(30)\quad \begin{cases} |\beta_{kj}(\widehat{y};\widehat{z})| \leqslant \widehat{C},\ \widehat{y} \in \mathbb{R}^d,\ \widehat{z} \in \mathbb{R}^n, \\ |\beta_{kj}(\widehat{y}';\widehat{z}) = \beta_{kj}(\widehat{y}'')| \leqslant \widehat{L}|\widehat{y}' - \widehat{y}''|,\ \forall \widehat{y}', \widehat{y}'' \in \mathbb{R}^d,\ \widehat{z} \in \mathbb{R}^n \end{cases}$$

$k, j \in \{1, \ldots, d\}$, and $\widehat{C} > 0$, $\widehat{L} > 0$ are some constants. Write $L = \max(\widehat{L}, \widehat{C})$, $\rho_1 = \rho\frac{M}{\delta}$ (for $\rho > 0$ see (18) and for $\frac{M}{\delta}$ see (27)) and define $L_{\rho_1} := (1+2\rho_1)L$. Take $\rho > 0$ sufficiently small such that

$$(31)\quad 4\rho_1 L_{\rho_1} = \gamma \leqslant \frac{1}{2},\ \sum_{j=1}^{d}|\beta_{kj}(\widehat{y};\widehat{z})\widehat{y}_j| \leqslant \frac{1}{2},\ \forall \widehat{y} \in B(0, 2\rho_1) \subset \mathbb{R}^d,\ \widehat{z} \in \mathbb{R}^n,\ k \in \{1, \ldots, d\}.$$

Define a mapping $T(\widehat{y}) : B(0, 2\rho_1) \to \mathrm{C}(\mathbb{R}^n; \mathbb{R}^d)$, $T = (T_1, \ldots, T_d)$, by

$$(32)\quad T_k(\widehat{y})(z) = 1 + \sum_{j=1}^{d}\beta_{kj}(\widehat{y}; z)\widehat{y}_j,\ k \in \{1, \ldots, d\}$$

and associate the following nonlinear operator $U(\widehat{y}) : B(0, 2\rho_1) \subset \mathbb{R}^d \to \mathrm{C}(B(0,\rho) \times \mathbb{R}^{2n}; \mathbb{R}^d)$, $U = (U_1, \ldots, U_d)$,

$$(33)\quad U_k(\widehat{y})(y, z) := y_k[T_k(\widehat{y})(z_2)]^{-1}\frac{\alpha_k(z_1)}{\alpha_k(z_2)},\ z = (z_1, z_2) \in \mathbb{R}^{2n}.$$

Using (30) and the second inequality in (31) we notice that

$$(34)\quad \begin{cases} |[T_k(\widehat{y})(z_2)]^{-1}| \leqslant 2,\ \forall \widehat{y} \in B(0, 2\rho_1) \subset \mathbb{R}^d,\ z_2 \in \mathbb{R}^n, \\ |[T_k(\widehat{y}')(z_2)]^{-1} - [T_k(\widehat{y}'')]^{-1}| \leqslant 4|T_k(\widehat{y}')(z_2) - T_k(\widehat{y}'')(z_2)| \\ \qquad \leqslant 4(1+2\rho_1)L|\widehat{y}' - \widehat{y}''| = 4L_{\rho_1}|\widehat{y}' - \widehat{y}''|,\ \forall \widehat{y}', \widehat{y}'' \in B(0, 2\rho_1), \end{cases}$$

for each $k \in \{1, \dots, d\}$, where $L_{\rho_1} := (1 + 2\rho_1)L$. Using (34) and (31) we get that the continuous nonlinear operator $\{U(\widehat{y}) : \widehat{y} \in B(0, 2\rho_1)\}$ is Lipschitz continuous with a constant $4\rho_1 L_{\rho_1} = \gamma_1$, $(\rho_1 = \rho\frac{M}{C})$ and

$$
(35) \qquad \begin{aligned} |U_k(\widehat{y}')(y,z) - U_k(\widehat{y}'')(y,z)| &\leqslant |y_k|\frac{M}{\delta}|[T_k(\widehat{y}')(z_2)]^{-1} - [T_k(\widehat{y}'')(z_2)]^{-1}| \\ &\leqslant \rho_1(4L_{\rho_1})|\widehat{y}' - \widehat{y}''| = \gamma|\widehat{y}' - \widehat{y}''|, \end{aligned}
$$

$\forall \widehat{y}', \widehat{y}'' \in B(0, 2\rho_1) \in \mathbb{R}^d$, $y \in B(0,\rho)$, $z = (z_1, z_2) \in \mathbb{R}^{2n}$, $k \in \{1, \dots, d\}$, where $0 < \gamma \leqslant \frac{1}{2}$ (see (31)). In addition, the equation (28) for the unknown $\widehat{y}(t_1) \in B(0,\rho_1) \in \mathbb{R}^d$, will be replaced by functional nonlinear equations

$$
(36) \quad \widehat{y}_k = U_k(\widehat{y}),\ k\{1, \dots, d\},\ \text{for } \widehat{y} \in \widehat{Y} := \mathrm{C}(B(0,\rho) \times \mathbb{R}^{2n}; B(0, 2\rho_1)).
$$

The unique solution for (36) is constructed as the limit of a Cauchy sequence $\{\widehat{y}^j\}_{j\geqslant 0} \subset \widehat{Y}$ in the complete metric space $\widehat{Y}$,

$$
(37) \quad \begin{cases} \widehat{y}^0 = \{0\},\ \widehat{y}^1 := U(0) = \{(y_1\dfrac{\alpha_1(z_1)}{\alpha_1(z_2)}, \dots, y_d\dfrac{\alpha_d(z_1)}{\alpha_d(z_2)}) : y \in B(0,\rho), (z_1, z_2) \in \mathbb{R}^{2n}\}, \\ \widehat{y}^{j+1} := U(\widehat{y}j),\ \|\widehat{y}^{j+1}(y)\| \leqslant \|\widehat{y}^1(y)\|(1 + \gamma + \dots + \gamma^j) \leqslant |y|\dfrac{M}{\delta}\dfrac{1}{1-\gamma} \leqslant \dfrac{2M}{\gamma}|y|, \end{cases}
$$

for any $y \in B(0,\rho) \subset \mathbb{R}^d$, where $\|\widehat{y}(y)\| := \sup_{z\in\mathbb{R}^{2n}} |\widehat{y}(y,z)|$. Using the metric $d(\widehat{y}', \widehat{y}'') := \sup_{(y,z)\in B(0,\rho)\times\mathbb{R}^{2n}} |\widehat{y}'(y,z) - \widehat{y}''(y,z)|$ and the induction argument we prove that

$$
|\widehat{y}^{j+1}(y,z) - \widehat{y}^j(y,z)| \leqslant |\widehat{y}^1(y,z)|\gamma^j,\ \forall j \geqslant 0.
$$

AS a consequence the following estimate is valid

$$
\|\widehat{y}^{j+1}(y)\| \leqslant \|\widehat{y}^1(y)\|(1 + \gamma + \dots + \gamma^j) \leqslant \frac{M}{\delta}\frac{1}{1-\gamma}|y| \leqslant \frac{2M}{\delta}|y|.
$$

Denote $f(y,z) = \lim_{j\to\infty} \widehat{y}^j(y,z)$, $y \subset B(0,\rho) \in \mathbb{R}^d$, $z = (z_1, z_2) \in \mathbb{R}^{2n}$, the corresponding continuous and bounded function $f(y,z) : B(0,\rho) \times \mathbb{R}^{2n} \to B(0, 2\rho_1) \subset \mathbb{R}^d$. The above given computations are restated as

Lemma 4.1. *Consider the mapping* $\widehat{G}(y; z) : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ *defined in* (22) *where the vector fields* $g_k \in \mathrm{C}_b^2(\mathbb{R}^n; \mathbb{R}^n)$, $k \in \{1, \dots, d\}$ *fulfill* (19) *and* (27). *Associate the following* *equations*

$$\text{(E)}\quad \widehat{G}(\widehat{y}; z_2) = z_2 + \sum_{k=1}^{d} g_k(z_1) y_k,$$

for $y = (y_1, \dots, y_d) \in B(0, \rho) \subset \mathbb{R}^d$, $z_1, z_2 \in \mathbb{R}^n$ *and* $\widehat{y} \in B(0, 2\rho_1) \subset \mathbb{R}^d$, $\rho_1 = \frac{M}{\delta}\rho$ *where* ρ *verifies* (31).

Then there exists a unique continuous and bounded function $\widehat{y} = f(y, z_1, z_2) : B(0, \rho) \times \mathbb{R}^n \times \mathbb{R}^n \to B(0, 2\rho_1)$ *satisfying (E) and*

$$\|f(y)\| = \sup_{z_1, z_2 \in \mathbb{R}^n} |f(y, z_1, z_2)| \leqslant \frac{2M}{\gamma}|y| \leqslant 2\rho_1,\ y \in B(0, \rho).$$

Proof. By hypothesis, the equations (E) of Lemma 4.1 coincide with the functional equations (36) for which a unique solution $f(y, z_1, z_2) = \lim_{j\to\infty} \widehat{y}^j(y, z_1, z_2)$ is constructed using the Cauchy sequence $\{\widehat{y}^j\}_{j \geqslant 0} \subset \widehat{Y}$ described in (37). The proof is complete. □

The equation (28) is a particular case of the equation (E) in Lemma 4.1 and define

$$(38)\quad \widehat{y}(t_1) = f(y(t_1), z_1, z_2),\ y(t_1) \in B(0, \rho),$$

(see $V_y \leqslant \rho$ in (18)) and ρ is sufficiently small such that the inequality (31) are fulfilled. Using an induction argument we prove that (26) are solved and in this respect, assume that $\widehat{y}(t_{j-1}) = \widehat{y}(t_j-)$ fulfils

$$(39)\quad \widehat{G}(\widehat{y}(t_j-); \widehat{z}) = h(t_j-; \widehat{z}),\ \forall \widehat{z} \in \mathbb{R}^n,$$

where $|\widehat{y}(t_k) - \widehat{y}(t_k-)| \leqslant \frac{2M}{\delta}|y(t_k) - y(t_k-)|$ for any $1 \leqslant k \leqslant j-1$, and $j \geqslant 2$ is fixed. It implies that

$$(40)\quad \widehat{G}(\widehat{y}(t_j); \widehat{z}) = h(t_j; \widehat{z})$$

can be solved (see $\widehat{y}(t_j) = \widehat{y}(t_j-) + \Delta\widehat{y}(t_j)$) and

$$(41)\quad |\Delta\widehat{y}(t_j)| := |\widehat{y}(t_j) - \widehat{y}(t_j-)| \leqslant \frac{2M}{\delta}|\Delta y(t_j)|.$$

In this respect, rewrite (40) as follows

(42)

$$\widehat{G}(\widehat{y}(t_j);\widehat{z}) = \widehat{G}([\widehat{y}(t_j)-\widehat{y}(t_j-)];h(t_{j-1};\widehat{z})) = h(t_{j-1};\widehat{z})+\sum_{k=1}^{d} g_k(h(t_{j-1};\widehat{z}))[y_k(t_j)-y_k(t_{j-1})].$$

Now replace the equation (42) for the unknown $\widehat{y} = \widehat{y}(t_j) - \widehat{y}(t_j-)$ and $h(t_{j-1};\widehat{z}) := \widehat{h} \in \mathbb{R}^n$ by the following one

$$(43)\quad \widehat{G}(\widehat{y};\widehat{h}) = \widehat{h} + \sum_{k=1}^{d} g_k(\widehat{h})[y_k(t_j) - y_k(t_j-)],\ \widehat{h} \in \mathbb{R}^n,$$

where $\Delta y(t_j) := y(t_j) - y(t_j-) \in B(0,\rho) \subset \mathbb{R}^d$. Using hypothesis (27) we construct the unique solution of (43)

$$(44)\quad \widehat{y} = f(\Delta y(t_j), \widehat{h}, \widehat{h}),$$

where the continuous and bounded function $f(y, z_1, z_2) : B(0,\rho) \times \mathbb{R}^n \times \mathbb{R}^n \to B(0, 2\rho_1) \subset \mathbb{R}^d$ is the unique solution of the functional equation (36) (see (E) of Lemma 4.1). Write $\widehat{y}(t_j) = \widehat{y}(t_{j-1}) + f(\Delta y(t_j), h(t_{j-1};\widehat{z}), h(t_{j-1};\widehat{z}))$ and it satisfies the equation (40) and the inequalities (41). The above given computations for the reduced jumping system (20) will be reviewed as

Theorem 4.1. *Assume that the piecewise constant process* $\{y(t) = y(t_j) \in \mathbb{R}^d : t \in [t_j, t_{j+1}), j \geqslant 0\}$ *and the vector fields* $g_k \in C_b^2(\mathbb{R}^n;\mathbb{R}^n)$, $k \in \{1,\dots,d\}$, *fulfill the hypothesis* (18), (19) *and* (27).

Then there exists a piecewise constant and bounded process $\{\widehat{y}(t) = \widehat{y}(t_j) \in \mathbb{R}^d : t \in [t_j, t_{j+1}), j \geqslant 0, \widehat{y}(0)\}$ *such that the piecewise constant solution of* (20) $\{h(t,\widehat{z}) = h(t_j,\widehat{z}) \in \mathbb{R}^n : t \in [t_j, t_{j+1}), j \geqslant 0, h(0,\widehat{z}) = \widehat{z}\}$ *defined in* (21) *can be written as follows,*

$$h(t_j;\widehat{z}) = \widehat{G}(\widehat{y}(t_j-);\widehat{z}) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_j-);\widehat{z}))[y_k(t_j) - y_k(t_j)],\ j \geqslant 0,$$

(the gradient representation in Definition 4.2 of (cad-lag) solution (21) *is valid by taking* $p(t) = y(t)$ *and* $\widehat{p}(t) = \widehat{y}(t)$, $t \geqslant 0$*) where* $\widehat{G} \in C_g^2(\mathbb{R}^d; \mathrm{Z})$ *is defined in* (22) *and*

$$V_{\widehat{y}} := \sum_{j=1}^{\infty} |\widehat{y}(t_j) - \widehat{y}(t_j-)| \leqslant \frac{2M}{\delta} V_y \leqslant \frac{2M}{\delta}\rho = 2\rho_1$$

(see (18) *and* (27)*).*

Proof. (Sketch) By hypothesis, the conditions of the fixed point theorem proved in Lemma 4.1 are fulfilled. As a consequence, the equations

$$(45)\quad \widehat{G}(\widehat{y}(t_j-);\widehat{z}) = h(t_j-;\widehat{z}),\ j \geqslant 0$$

for the unknown piecewise constant process $\{\widehat{y}(t) = \widehat{y}(t_j) \in B(0,\rho_1) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ will be solved using the induction argument (see (39) and (40)) and the solution $\widehat{y} = f(y, z_1, z_2) : B(0,\rho) \times \mathbb{R}^n \times \mathbb{R}^n \to B(0, 2\rho_1) \subset \mathbb{R}^d$ of the functional equation

$$(46)\quad \widehat{G}(\widehat{y}; z_2) = z_2 + \sum_{k=1}^{d} g_k(z_1) y_k,\ y = (y_1, \ldots, y_d) \in B(0,\rho),\ z_1, z_2 \in \mathbb{R}^n$$

given in Lemma 4.1 such that

$$(47)\quad \|f(y)\| := \sup_{(z_1,z_2)\in\mathbb{R}^{2n}} |f(y, z_1, z_2)| \leqslant \frac{2M}{\delta}|y|.$$

Define $\widehat{y}(t_j) = \widehat{y}(t_{j-1}) + \Delta\widehat{y}(t_j)$, where $\Delta\widehat{y}(t_j) = \widehat{y}(t_j) - \widehat{y}(t_j-)$ is given by

$$(48)\quad \Delta\widehat{y}(t_j) = f(\Delta y(t_j), h(t_j-;\widehat{z}), h(t_j-;\widehat{z})),\ j \geqslant 1$$

and it implies (40), (41) are satisfied for any $j \geqslant 2$. The proof is complete. □

Our goal is to show that, under the hypothesis (18), (19) and (27), the gradient representation of the (cad-lag) solution in (1) is valid when Definition 4.2 (see equations (17)) is used. The commuting property assumed in (19) lead us directly to the following equations

$$(49)\quad \partial_y \widehat{G}(y;z) = [g_1(\widehat{G}(y;z)), \ldots, g_d(\widehat{G}(y;z))],\ \forall y \in \mathbb{R}^d,\ z \in \mathbb{R}^n,$$

which allow to take the piecewise constant process $p(t) \in \mathbb{R}^d$ verifying

$$(50)\quad p(t) = y(t),\ t \geqslant 0,$$

where $\{y(t) = y(t_j) \in \mathbb{R}^d : t \in [t_j, t_{j+1}), j \geqslant 0\}$ is given in (1). It reduces the unknowns of the equations (17) and we must find a piecewise constant and bounded process

$$(51)\quad \{\widehat{y}(t) = \widehat{y}(t_j) \in B(0,R) \in \mathbb{R}^d : t \in [t_j, t_{j+1}), j \geqslant 0; \widehat{y}(t) = 0, t \in (-t_1, t_1)\}$$

and a continuous process

(52) $\{\widehat{z}(t,x) = \widehat{z}_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0, \widehat{z}(0,x) = x \in \mathbb{R}^n\}$

such that

(53) $z_j(t) = \widehat{G}(\widehat{y}(t_j-); \widehat{z}_j(t)) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_j-); \widehat{z}_j(t_j)))[y_k(t_j) - y_k(t_j-)],\ t \in [t_j, t_{j=1}),$

for any $j \geqslant 0$, where $\{z(t,x) = z_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ is the (cad-lag) solution (see (3), (4) satisfying (1)). The pair $(\widehat{y}(t), \widehat{z}(t))$, $t \geqslant 0$, of a piecewise constant process and a continuous one fulfilling (53) are verified for any $j \geqslant 0$.

Step 1. As far as $\widehat{y}(t) = 0$, $t \in (-t_1, t_1)$, and $\widehat{G}(0; z) = z$, using (53) for $j \in \{0, 1\}$, we find

(54) $$\begin{cases} \widehat{z}_0(t) = z_0(t),\ t \in [0, t_1],\ \widehat{z}_1(t_1) = z_0(t_1-) \\ z_1(t) = \widehat{z}_1(t) + \sum_{k=1}^{d} g_k(\widehat{z}_1(t_1)) y_k(t_1),\ t \in [t_1, t_2). \end{cases}$$

Step 2. The first significant equation for unknown $\widehat{y}(t_j-) \in \mathbb{R}^d$ in (53) appears for $j = 2$, $\widehat{y}(t_2-) = \widehat{y}(t_1)$, and using $\widehat{z}_2(t_2) = \widehat{z}_1(t_2-)$ (see $\widehat{z}_1(t)$, $t \in [t_1, t_2)$ defined in (54)), we write for $j = 2$ and $t = t_2$ as follows

(55) $z_2(t_2) = \widehat{G}(\widehat{y}(t_1); \widehat{z}_2(t_2)) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_1); \widehat{z}_2(t_2)))[y_k(t_2) - y_k(t_2-)],$

where (see (4) for $j = 2$)

(56) $z_2(t_2) = z_1(t_2-) + \sum_{k=1}^{d} g_k(z_1(t_2-))[y_k(t_2) - y_k(t_2-)].$

Both equations (55) and (56) are fulfilled provided $\widehat{y}(t_1) \in \mathbb{R}^d$ is found such that

(57) $$\begin{cases} \widehat{G}(\widehat{y}(t_1); \widehat{z}(t_2)) = z_1(t_2-) \text{ where (see (54) for} t \in [t_1, t_2)) \\ z_1(t_2) = \widehat{z}_1(t_2) + \sum_{k=1}^{d} g_k(\widehat{z}_1(t_1)) y_k(t_1) = \widehat{z}_2(t_2) + \sum_{k=1}^{d} g_k(\widehat{z}_1(t_1)) y_k(t_1). \end{cases}$$

One may notice that (57) are solved if we get the solution of the following equation

(58) $\widehat{G}(\widehat{y}; z_2) = z_2 + \sum_{k=1}^{d} g_k(z_1) y_k(t_1),\ y(t_1) \in B(0, \rho) \subset \mathbb{R}^d,\ z_1, z_2 \in \mathbb{R}^n,$

for the unknown $\widehat{y \in \mathbb{R}^d}$. By hypothesis, the equation (58) fulfils the conditions assumed in Lemma 4.1 (see (E)) and let $\widehat{y} = f(y, z_1, z_2) : B(0,\rho) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ the unique continuous and bounded solution for (58) such that

$$(59) \quad \begin{cases} f(y, z_1, z_2) \in B(0, 2\rho_1) \subset \mathbb{R}^d, \ \forall y \in B(0,\rho), \ z_1, z_2 \in \mathbb{R}^n, \ \rho_1 = \dfrac{M}{\delta}\rho, \\ \|f(y)\| := \sup\limits_{z_1, z_2 \in \mathbb{R}n} |f(y, z_1, z_2)| \leqslant \dfrac{2M}{\delta}|y|, \ y \in B(0,\rho) \subset \mathbb{R}^d. \end{cases}$$

Using (59), define the solution for (57) by

$$(60) \quad \widehat{y}(t_1) := f(y(t_1), \widehat{z}_1(t_1), \widehat{z}_2(t_2)) \in B(0, 2\rho_1) \subset \mathbb{R}^d$$

which satisfies $|\widehat{y}(t_1)| \leqslant \frac{2M}{\delta}|y(t_1)|$ for $y(t_1) \in B(0,\rho)$.

Step 3. With $\widehat{y}(t_1) \in B(0, 2\rho_1) \subset \mathbb{R}^d$ found in (60) and solving the equations (57) we return to the main equation (53), for $j = 2$, and find $\{\widehat{z}_2(t) : t \in [t_2, t_3), \widehat{z}_2(t_2) = \widehat{z}_1(t_2-)\}$ as a continuous process fulfilling (53) for $j = 2$, i.e.

$$(61) \quad \begin{cases} z_2(t) = \widehat{G}(\widehat{y}(t_1); \widehat{z}_2(t)) + \sum\limits_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_1); \widehat{z}_1(t_2)))[y_k(t_2) - y_k(t_2-)], \ t \in [t_2, t_3) \\ \widehat{z}_2(t) = \widehat{G}(-\widehat{y}(t_1); \bar{z}_2(t)), \ t \in [t_2, t_3), \end{cases}$$

where $\widehat{G}(-y; \cdot) = [\widehat{G}(y; \cdot)]^{-1}$ and $\bar{z}_2(t) := z_2(t) - \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_1); \widehat{z}_1(t_2)))[y_k(t_2) - y_k(t_2-)]$, for $t \in [t_2, t_3)$ are used.

Step 4. We are in position to stipulate what is necessary for getting (53) proved, using an induction argument. A verification of (53) for $j = 2$ was done into Step 2 and for some $j \geqslant 2$ the algorithm requires as known the following items: a continuous process $\{\widehat{z}(t,x) = \widehat{z}_k(t) : t \in [t_k, t_{k+1}), 0 \leqslant k \leqslant j\}$ and a piecewise constant process $\{\widehat{y}(t) = \widehat{y}(t_k) : t \in [t_k, t_{k+1}), 0 \leqslant k \leqslant j-1\}$ fulfilling

$$(62) \quad |\Delta\widehat{y}(t_k)| \leqslant \frac{2M}{\delta}|\Delta y(t_k)|, \ 0 \leqslant k \leqslant j-1,$$

where $\Delta v(t_k) := v(t_k) - v(t_k-)$, such that (53) are valid for $t \in [t_k, t_{k+1})$, $1 \leqslant k \leqslant j$.

In particular, the following equations are fulfilled

$$(63) \quad \widehat{G}(\widehat{y}(t_k-); \widehat{z}_{k-1}(t_k)) = z_{k-1}(t_k-), \ 0 \leqslant k \leqslant j.$$

Assuming (62) and (63), we must find $\Delta\widehat{y}(t_j) := \widehat{y}(t_j) - \widehat{y}(t_j-)$ and a continuous process $\{\widehat{z}_{j+1}(t) : t \in [t_{j+1}, t_{j+2}), \widehat{z}_{j+1}(t_{j+1}) = \widehat{z}_j(t_{j+1})\}$ such that

$$(64)\quad |\Delta\widehat{y}(t_j)| \leqslant \frac{2M}{\delta}|\Delta y(t_j)|,$$

and

$$(65)\quad \widehat{G}(\widehat{y}(t_j); \widehat{z}_j(t_{j+1})) = z_j(t_{j+1}-), \text{ see } \widehat{y}(t_j) = \widehat{y}(t_{j+1}-)).$$

Using the equality $\widehat{y}(t_{j+1}-) = \widehat{y}(t_j) = \widehat{y}(t_j-) + \Delta\widehat{y}(t_j)$ found in (65), define $\{\widehat{z}_{j+1}(t) : t \in [t_{j+1}, t_{j+2})\}$ as the solution of the following equation

$$(66)\quad z_{j+1}(t) = \widehat{G}(\widehat{y}(t_j); \widehat{z}_{j+1}(t)) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_j); \widehat{z}_j(t_{j+1})))[y_k(t_{j+1}) - y_k(t_{j+1}-)],$$

for any $t \in [t_{j+1}, t_{j+2})$, (see (53) for $j+1$). We restate the above given remarks as

Lemma 4.2. *Let the piecewise constant process* $\{y(t) = y(t_j) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ *and the vector fields* $\{g_1, g_2, \ldots, g_d\} \subseteq C_b^2(\mathbb{R}^n; \mathbb{R}^n)$ *are given such that* (18), (19) *and* (27) *are fulfilled. Consider the piecewise constant process* $\{\widehat{y}(t) = \widehat{y}(t_k) : t \in [t_k, t_{k+1}), 0 \leqslant k \leqslant j-1\}$ *and the continuous process* $\{\widehat{z}(t,x) = \widehat{z}_k(t) : t \in [t_k, t_{k+1}) : 0 \leqslant k \leqslant j\}$ *are constructed such that they satisfy* (53), (62) *and* (63). *Then there exist* $\widehat{y}(t_j) = \widehat{y}(t_{j+1}-) \in \mathbb{R}^d$ *such that the equations* (64) *and* (65) *are satisfied (see* $|\Delta y(t_j)| < \rho$*) and the continuous process* $\{\widehat{z}_{j+1}(t) : t \in [t_{j+1}, t_{j+2})\}$ *defined in the formula* (66) *which fulfils* $\widehat{z}_{j+1}(t_{j+1}) = \widehat{z}_j(t_{j+1})$.

In addition, the solution given in (65) *and* (66) *agrees with the basic equations* (53) *written for* $t \in [t_{j+1}, t_{j+2})$, *and*

$$(67)\quad |\Delta\widehat{y}(t_j)| \leqslant \frac{2M}{\delta}|\Delta y(t_j)|, \quad \textit{for any } j \geqslant 1,$$

which lead us to the conclusion $V_{\widehat{y}} := \sum_{j=1}^{\infty} |\Delta\widehat{y}(t_j)| \leqslant \frac{2M}{\delta} V_y \leqslant \frac{2M}{\delta}\rho$.

Proof. Using the group property of $\widehat{G}(y; \cdot)$, $y \in \mathbb{R}^d$ we get

$$(68)\quad \widehat{G}(\widehat{y}(t_j); z) = \widehat{G}[\Delta\widehat{y}(t_j); \widehat{G}(\widehat{y}(t_j-); z)], \ \forall z \in \mathbb{R}^n.$$

On the other hand, using (53), we obtain, for $t \in [t_j, t_{j+1})$,

$$(69)\quad z_j(t_{j+1}-) = \widehat{G}(\widehat{y}(t_j-); \widehat{z}_j(t_{j+1})) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_j-); \widehat{z}_j(t_j)))\Delta y_k(t_j).$$

In order to solve (65), we use (68) and (69) and rewrite (65) as

$$(70)\quad \widehat{G}(\widehat{y}(t_j); \widehat{z}_2) = \widehat{z}_2 + \sum_{k=1}^{d} g_k(\widehat{z}_1)\Delta y_k(t_j),$$

where $\widehat{z}_2 := \widehat{G}(\widehat{y}(t_j-); \widehat{z}_j(t_{j+1}))$ and $\widehat{z}_1 := \widehat{G}(\widehat{y}(t_j-); \widehat{z}_j(t_j))$.

By hypothesis, the functional equations (70) fulfill the conditions of Lemma 4.1 (see (E)) and let $\widehat{y} = f(y, z_1, z_2) : B(0,\rho) \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow B(0, 2\rho_1)$ be the unique continuous and bounded solution of (E), i.e.

$$(71)\quad \widehat{G}(\widehat{y}; z_2) = z_2 + \sum_{k=1}^{d} g_k(z_1)y_k,\ \forall z_1, z_2 \in \mathbb{R}^n,\ y \in B(0,\rho) \subset \mathbb{R}^d.$$

Define $\Delta\widehat{y}(t_j) := f(\Delta y(t_j), \widehat{z}_1, \widehat{z}_2)$ and $\Delta y(t_j) := y(t_j) - y(t_j-) \in B(0,\rho)$. In addition,

$$(72)\quad |\Delta y(t_j)| \leqslant \|f(\Delta y(t_j)\| = \sup_{z_1, z_2 \in \mathbb{R}^n} |f(\Delta y(t_j), \widehat{z}_1, \widehat{z}_2)| \leqslant \frac{2M}{\delta}|\Delta y(t_j)|$$

and (64) is verified.

Define a continuous process $\{z_{j+1}(t); t \in [t_{j+1}, t_{j+2})\}$ such that (66) is satisfied. For $t = t_{j+1}$ we get

$$(73)\quad \begin{aligned} z_{j+1}(t_{j+1}) &:= z_j(t_{j+1}-) + \sum_{k=1}^{d} g_k(z_j(t_{j+1}-))[y_k(t_{j+1}) - y_k(t_{j+1}-)] \\ &= \widehat{G}(\widehat{y}(t_j); \widehat{z}_{j+1}(t_{j+1}) + \sum_{k=1}^{d} g_k(z_j(t_{j+1}-))[y_k(t_{j+1}) - y_k(t_{j+1}-)]. \end{aligned}$$

Therefore, $z_j(t_{j+1}-) = \widehat{G}(\widehat{y}(t_j); \widehat{z}_{j+1}(t_{j+1})$ holds and this combined with (65) gives $\widehat{z}_j(t_{j+1}) = \widehat{z}_{j+1}(t_{j+1})$. □

Step 5. The continuous process $\{\widehat{z}(t,x) = \widehat{z}_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ was constructed such that the equations (53) are fulfilled, for any $j \geqslant 0$, where $\widehat{y}(t_j-) = \widehat{y}(t_{j-1})$, $j \geqslant 1$, is known and $\{z(t,x) = z_j(t) : t \in [t_j, t_{j+1}), j \geqslant 0\}$ is the (cad-lag) solution of the SDE (1).

Using (63) for $k = j$ and $\widehat{z}_j(t_j) = \widehat{z}_{j-1}(t_j)$ we rewrite (53) as

$$(74)\quad z_j(t) = \widehat{G}(\widehat{y}(t_{j-1}); \widehat{z}_j(t) + \sum_{k=1}^{d} g_k(z_{j-1}(t_j-))[y_k(t_j) - y_k(t_j-)],\ t \in [t_j, t_{j+1}).$$

Set

$$(75)\quad \bar{z}_j(t) = z_j(t) - b(\Delta y(t_j), z_{j-1}(t_j-)),\ t \in [t_j, t_{j+1}),$$

where $b(y,z) := \sum_{k=1}^{d} g_k(z) y_k$, $y = (y_1, y_2, \ldots, y_d)$, $z \in \mathbb{R}^n$. Using the inverse mapping $[\widehat{G}(y;\cdot)]^{-1} = \widehat{G}(-y;\cdot)$ and the formula (75), we rewrite (74) as follows

$$(76)\quad \widehat{z}_j(t) = \widehat{G}(-\widehat{y}(t_{j-1}); \bar{z}_j(t)),\ t \in [t_j, t_{j+1}),$$

where $\{\bar{z}_j(t) : t \in [t_j, t_{j+1})\}$ is a continuous process fulfilling the following system of SDEs (see equation (4) for $z_j(t)$)

$$(77)\quad \begin{cases} \mathrm{d}_t \bar{z}_j(t) = f_0(\bar{z}_j + b(\Delta y(t_j), z_{j-1}(t_j-)); \mu(t_j))\mathrm{d}t \\ \qquad\qquad + \sum_{i=1}^{m} f_i(\bar{z}_j + b(\Delta y(t_j), z_{j-1}(t_j-)); \mu(t_j))\mathrm{d}W_i(t),\ t \in [t_j, t_{j+1}), \\ \bar{z}_j(t_j) = z_{j-1}(t_j-). \end{cases}$$

Applying the standard rule of stochastic derivation for the test function $\varphi(y,z) = \widehat{G}(-y,z)$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^n$, and the continuous process $\{(y(t), \bar{z}_j(t)) = (-\widehat{y}(t_{j-1}), \bar{z}_j(t)) : t \in [t_j, t_{j+1})\}$, we get the corresponding SDE fulfilled by $\{\widehat{z}_j(t) : t \in [t_j, t_{j+1})\}$

$$(78)\quad \begin{cases} \mathrm{d}_t \widehat{z}_j(t) = h_0(\widehat{z}_j; \widehat{\lambda}_j(t,x))\mathrm{d}t + \sum_{i=1}^{m} h_i(\widehat{z}_j; \widehat{\lambda}_j(t,x))\mathrm{d}W_i(t),\ t \in [t_j, t_{j+1}), \\ \widehat{z}_j(t_j) = \widehat{z}_{j-1}(t_j). \end{cases}$$

Here $\widehat{\lambda}_j(t,x) := (\widehat{y}(t_j-), \lambda(t), z(t,x)) \in \mathbb{R}^d \times \Lambda \times \mathbb{R}^n$ agrees with the given piecewise constant process $\{\lambda(t) = \lambda(t_j) : t \in [t_j, t_{j+1})\}$ given in (1).

In addition, the vector fields $h_i(\widehat{z}_j; \lambda_j(t,x)) \in \mathbb{R}^n$, $\widehat{z}_j \in \mathbb{R}^n$, $i = 0, 1, \ldots, m$, are obtained by a direct inspection of the applied stochastic rule, when the system (77) is used. We thus get

$$(79)\quad \begin{cases} h_i(\widehat{z}_j;\widehat{\lambda}_j(t,x)) =\partial_z\widehat{G}(-\widehat{y}(t_{j-1});\bar{z}_j(t))f_i(\widehat{G}(\widehat{y}(t_{j-1});\widehat{z}_j) \\ \qquad\qquad + b(\Delta y(t_j),z_{j-1}(t_j-));\mu(t_j)),\ 1\leqslant i\leqslant m, \\ h_0(\widehat{z}_j;\widehat{\lambda}_j(t,x)) =\partial_z\widehat{G}(-\widehat{y}(t_{j-1});\bar{z}_j(t))f_0(\widehat{G}(\widehat{y}(t_{j-1});\widehat{z}_j) \\ \qquad\qquad + b(\Delta y(t_j),z_{j-1}(t_j-));\mu(t_j)) + \widehat{h}_0(\widehat{\lambda}_j(t,x)). \end{cases}$$

Here $\widehat{h}_0 = (\widehat{h}_0^1,\dots,\widehat{h}_0^n)$ is obtained as

$$(80)\quad \widehat{h}_0^\gamma := \frac{1}{2}\sum_{i=1}^{m}\langle \partial_z^2\widehat{G}^\gamma(-\widehat{y}(t_{j-1});\bar{z}_j(t))f_i(z_j(t);\mu(t_j)), f_i(z_j(t);\mu(t_j))\rangle,$$

for $\gamma\in\{1,\dots,n\}$ and $\widehat{G}=(\widehat{G}^1,\dots,\widehat{G}^n)$.

Theorem 4.2. *Consider the SDE* (1) *where the vector fields* $\{f_i(z;\mu(t)) : z\in\mathbb{R}^n, t\geqslant 0\}$, $i=1,\dots,m$ *fulfill the hypothesis* (2) *and the vector fields* $\{g_k(z) : z\in\mathbb{R}^n\}$, $k=1,\dots,d$, *satisfy* (18), (19) *and* (27), *where* $V_y := \sum_{j=1}^\infty |\Delta y(t_j)| \leqslant \rho$ *and* ρ *is a sufficient small positive constant such that* (31) *are satisfied (see Lemma* (4.1)*).*

Then there exists a piecewise constant process $\{\widehat{y}(t)=\widehat{y}(t_j) : t\in[t_j,t_{j+1})\}$ *such that* $\widehat{y}(t_j)$ *is* $\mathcal{F}_{t_{j+1}}$*-measurable and a continuous process* $\{\widehat{z}(t,x)=\widehat{z}_j(t) : t\in[t_j,t_{j+1})\}$ *such that the (cad-lag) solution of SDEs has the gradient representation*

$$(81)\quad z(t,x) = \widehat{G}(\widehat{y}(t_{j-1});\widehat{z}(t,x)) + \sum_{k=1}^{d} g_k(\widehat{G}(\widehat{y}(t_{j-1});\widehat{z}(t_j;x))[y_k(t_j)-y_k(t_j-)],$$

for any $t\in[t_j,t_{j+1})$, $j\geqslant 0$,

$$(82)\quad V_{\widehat{y}} := \sum_{j=1}^{\infty}|\Delta\widehat{y}(t_j)| \leqslant \frac{2M}{\delta}\sum_{j=1}^{\infty}|\Delta y(t_j)| = \frac{2M}{\delta}V_y \leqslant \frac{2M}{\delta}\rho.$$

In addition, $\{\widehat{z}(t,x)\}$ *fulfil the SDE* (78), *for any* $j\geqslant 1$ *and* $\widehat{z}_0(t)=\widehat{z}(t,x)=z_0(t)$, *for* $t\in[0,t_1)$.

Proof. One may easily see that (81) stands for the gradient representation associated with the cad-lag solution $\{z(t,x)\}$ of system (1). By hypothesis, the conditions of Lemma 4.2 stand in force and one may recognize that the conclusions (81) and (82) are obtained via this lemma. The last statement of the theorem describes the dynamical system satisfied by each component $\{\widehat{z}_j(t,x)\}$ of the continuous process $\{\widehat{z}(t,x)\}$. □

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