INSTITUTUL DE MATEMATICA
"SIMION STOILOW"
AL ACADEMIEI ROMANE

PREPRINT SERIES OF THE INSTITUTE OF MATHEMATICS
OF THE ROMANIAN ACADEMY

ISSN 0250 3638

Open discrete mappings having local ACL^n inverses

by

Mihai Cristea*

Preprint nr.6/2008

BUCURESTI

Open discrete mappings having local ACL^n inverses

by

Mihai Cristea[1]

Preprint nr.6/2008

Iulie, 2008

---

1 University of Bucharest Faculty of Mathematics and Computer Sciences,
Str. Academiei 14, R-010014, Bucharest, Romania,
E-mail: mcristea@fmi.unibuc. ro

Open discrete mappings having local ACL^n inverses

Mihai Cristea*

Abstract. We consider open, discrete mappings between domains from $\mathbf{R}^n$ satisfying condition (N), having local ACL^n inverses on $D \setminus B_f$, so that $\mu_n(B_f) = 0, H^*(\cdot, f) < \infty$ on B_f and $K_I(f) \in L^1_{loc}(D)$. For this class of mappings (or even for larger classes of open, discrete mappings) we generalize the important modular inequality of Poleckii. Using also the modular estimates of the spherical rings from [5], we continue the work from [5] of generalizing some basic facts from the theory of quasiregular mappings. We give equicontinuity results, Picard, Montel and Liouville type theorems, estimates of the modulus of continuity, analogues of Schwarz's lemma, eliminability results and boundary extensions theorems. Together with the multiple extensions of Zoric's theorem from [5], we establish strong generalizations of one of the most important theorems from the theory of quasiregular mappings. We also extend similar results given in some recent classes of functions larger than the class of quasiregular functions, as the class of mappings of finite distortion and satisfying condition $(\mathcal{A})$, or the class of mappings of finite dilatation with dilatation in the BMO class.

AMS 2000 Classification NO: 30C65.

Keywords: Mappings of finite distortion, Poleckii's modular inequality, equicontinuity results, boundary extension results.

1. Introduction

Throughout this paper we shall work with domains D from $\mathbf{R}^n$ and mappings $f : D \to \mathbf{R}^n$ and we denote by $B_f = \{x \in D | f \text{ is not a local homeomorphism in } x\}$. Such a map is said to be of finite distortion (dilatation) if $f \in W^{1,1}_{loc}(D, \mathbf{R}^n)$ ($f \in W^{1,n}_{loc}(D, \mathbf{R}^n)$), $J_f \in L^1_{loc}(D)$, and there exists $K : D \to [0, \infty]$ measurable and finite a.e. so that $|f'(x)|^n \le K(x) J_f(x)$ a.e. in D. If $K \in L^p_{loc}(D)$ for some $p > n - 1$, we see from [16] that f is open, discrete and if $K \in L^\infty(D)$, then we obtain the known class of quasiregular mappings. If $f : D \to \mathbf{R}^n$ is a.e. differentiable with $J_f(x) \neq 0$ a.e. we define the outer dilatation $K_0(f)$, the inner dilatation $K_I(f)$ and the dilatation $K(f)$ by $K_0(f)(x) = \frac{|f'(x)|^n}{|J_f(x)|}$, $K_I(f)(x) = \frac{|J_f(x)|}{l(f'(x))^n}$, $K_f(x) = \frac{|f'(x)|}{l(f'(x))}$ for $x \in D$ so that $J_f(x) \neq 0$. Here, we put $|A| = \sup_{|h|=1} |A(h)|$, $l(A) = \inf_{|h|=1} |A(h)|$ if $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$.

If $f : D \to \mathbf{R}^n$ is of finite dilatation and there exists $Q \in BMO(D)$ so that $max\{K_I(f)(x), K_0(f)(x)\} \le Q(x)$ a.e. in D, we say as in [19] that f is a Q-map. If $f : D \to \mathbf{R}^n$ is of finite distortion and there exists $\mathcal{A} : [0, \infty) \to [0, \infty)$ smooth, strictly increasing, with $\mathcal{A}(0) = 0$, $\lim_{t\to\infty} \mathcal{A}(t) = \infty$, $\exp(\mathcal{A} \circ K_0(f)) \in L^1_{loc}(D)$, $\int_1^\infty \frac{\mathcal{A}'(t)}{t} dt = \infty$, and there exists $t_0 > 0$ so that the function $t \to t\mathcal{A}'(t)$ increases to infinity for $t \ge t_0$, we say as in [14] that f is a map of finite distortion and satisfying condition $\mathcal{A}$.

Two modular inequalities are important tools in the well known theory of quasiregular mappings:

(a) $M(f(\Gamma)) \le M_{K_I(f)}(\Gamma)$ for every path family Γ from D (Poleckii's inequality).

(b) $M_{K_0(f)^{n-1}}(\Delta(\bar{B}(x, r), CB(x, R), (B(x, R) \setminus \bar{B}(x, r)))) \to 0$ when $r \to 0$ and $R > 0$ is kept fixed if $x \in D$ and $\bar{B}(x, R) \subset D$.

Here, if Γ is a path family in D we define $F(\Gamma) = \{\rho : \overline{\mathbf{R}}^n \to [0, \infty]$ Borel maps $| \int_\gamma \rho ds \ge 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$ and if $p > 0$ and $\omega : D \to [0, \infty]$ is measurable and finite a.e.,

* This research is supported by CEEX Program, Project 2- CEex-10/25.07.0

we put $M^p_\omega(\Gamma)=\inf_{\rho\in F(\Gamma)}\int_{\mathbf{R}^n}\rho^p(x)\omega(x)dx$. If $p=n$, we put $M^n_\omega(\Gamma)=M_\omega(\Gamma)$, and for $\omega=1$, we obtain the usual modulus $M(\Gamma)$. If Γ_1,Γ_2 are path families in $\bar{\mathbf{R}}^n$, we say that $\Gamma_1>\Gamma_2$ if every path from Γ_1 has a subpath in Γ_2,and as in the classical case, we prove that if $\Gamma_1>\Gamma_2$, then $M_\omega(\Gamma_1)\le M_\omega(\Gamma_2)$. Also, $M_\omega(\bigcup_{i=1}^{\infty}\Gamma_i)\le\sum_{i=1}^{\infty}M_\omega(\Gamma_i)$ and if $\omega_1\le\omega_2$, then $M_{\omega_1}(\Gamma)\le M_{\omega_2}(\Gamma)$. We say that $E=(A,C)$ is a condenser if $C\subset A\subset\mathbf{R}^n$, with C compact and A open, and we define $cap^p_\omega E=\inf\int_{\mathbf{R}^n}|\nabla u|^p(x)\omega(x)dx$, the ω-capacity of E, where $u\in C_0^\infty(A)$ and $u\ge 1$ on C and $p>1$. If $p=n$, we set $cap_\omega E=cap^n_\omega(E)$ and if $p=n$ and $\omega=1$ we obtain the usual capacity $capE$. If u is a test function for $cap^p_\omega E$, then $\rho=|\nabla u|\in F(\Gamma_E)$, hence $M^p_\omega(\Gamma_E)\le cap^p_\omega(E)$. Here, if $q:[0,1)\to\mathbf{R}^n$ is a path and $x\in\bar{\mathbf{R}}^n$, we say that x is a limit point of q if there exists $t_p\to 1$ so that $q(t_p)\to x$ and if $E=(A,C)$ is a condenser, we set $\Gamma_E=\{\gamma:[0,1)\to A \text{ path } |\gamma(0)\in C \text{ and } \gamma \text{ has a limit point in } \partial A\}$. We see from Prop. 10.2, page 54 in [23] that $capE=M(\Gamma_E)$ for every condenser $E=(A,C)$. If $C\subset\mathbf{R}^n$ is compact, we say that $capC=0$ if $cap(A,C)=0$ for some open set $C\subset A\subset\mathbf{R}^n$, and from Lemma 2.2, page 64 in [24], we see that the definition is independent on the open set A so that $C\subset A\subset\mathbf{R}^n$. If $C\subset\mathbf{R}^n$ is arbitrary, we say that $capC=0$ if $capK=0$ for every $K\subset C$ compact. If $\omega:D\to[0,\infty]$ is measurable and finite a.e. and $A\subset D$, we say that A is of zero ω-modulus (we write $M_\omega(A)=0$) if the ω-modulus of all path having some limit points in A is zero. If $\omega\ge 1$, then $M(\Gamma)\le M_\omega(\Gamma)$, hence, if $M_\omega(A)=0$, then $capA=0$.

The modular inequalities (a) and (b) are first systematically used in [19], [20], [25], [26], [27] for Q-homeomorphisms, but in a non explicit form. In [10], [25], [26] are considered some classes of non-injective Q-mappings for which the modular inequality (a) holds (the so called FLD - mappings), basically open, discrete mappings $f:D\to\mathbf{R}^n$ and path families Γ from D so that $m_1(Im\alpha\cap B_f)=0$ for a.e. paths $\beta=f\circ\alpha$ with $\alpha\in\Gamma$.

The methods and the techniques of the modular inequalities (a) and (b) were also later considered in [12], [13], [14], [3], [4] for mappings of finite distortion and satisfying condition $(\mathcal{A})$, and in [14] is shown that (a) and (b) hold in this class of mappings.

The basic result from this paper is Theorem 3. We show that if $f:D\to\mathbf{R}^n$ is continuous, open, discrete so that there exists $K\subset D$ closed in D so that $\mu_n(K)=0$ and

(a_1) $\mu_n(B_f)=0$, f satisfies condition (N) and has local ACL^n inverses on $D\setminus(K\cup B_f)$.

(a_2) $K_I(f)\in L^1_{loc}(D)$.

(a_3) There exists $E\subset K\cup B_f$ so that $f(E)$ is of σ - finite $(n-1)$ - dimensional measure, $F\subset K\cup B_f$ so that $m_1(F)=0$ and for every $x\in(K\cup B_f)\setminus(E\cup F)$ there exists $\alpha_x>0$, $H_x>0$ so that $\limsup_{r\to 0}\frac{d(U(x,f,\alpha_x r))^n}{\mu_n(U(x,f,r))}\le H_x$.

Then f satisfies the modular inequality of Poleckii (a) for every path family Γ from D.

Here we say that f satisfies condition (N) if $\mu_n(f(A))=0$ for every $A\subset D$ with $\mu_n(A)=0$ (we denote by μ_n the Lebesgue measure in $\mathbf{R}^n$). We say that f is open if $f(Q)$ is open for every $Q\subset D$ open, we say that f is discrete if $f^{-1}(y)$ is empty or discrete, and we say that f is light if for every $x\in D$ and every $V\in\mathcal{V}(x)$ there exists $U\in\mathcal{V}(x)$ so that $UsubsetV$ and $\partial U\cap f^{-1}(f(x))=\phi$. If f is continuous, open, discrete, then $dimB_f\le n-2$, $dimf(B_f)\le n-2$ and let $U(x,f,r)$ be the component of $f^{-1}(B(f(x),r))$ containing x. Then, for every $x\in D$, there exists $r_x>0$ so that $f|U(x,f,r):U(x,f,r)\to B(f(x),r)$ lifts the paths, $f(\partial U(x,f,r))=S(f(x),r)$ and $d(f,U(x,f,r),f(x))=i(f,x)$. We used here the notations and the notions from the topological degree theory from the book of Lloyd [15]. Also, if $U\subset D$ is a domain so that $V=f(U)$, $\partial V=f(\partial U)$ (such a domain is called a normal domain), $m=d(f,U,V)$, the map

$f_V : V \to \mathbf{R}^n$ defined by $f_V(y) = \sum_{x \in f^{-1}(y) \cap U} \frac{xi(f,x)}{m}$ for $y \in V$ is called a quasiinverse of f, and we know from [24] that f_V is continuous on V.

If E, F are Hansdorff spaces, $f : E \to F$ is a map, we say that f lifts the paths if for every path $p : [0,1] \to F$ and every $x \in E$ so that $p(0) = f(x)$, there exists a path $q : [0,1] \to E$ so that $q(0) = x$ and $f \circ q = p$. If $p : [0,1] \to F$ is a path, $x \in E$ is so that $f(x) = p(0), 0 < a \leq 1$, we say that the path $q : [0, a) \to E$ is a maximal lifting of p from x by the map f, if $q(0) = x, f \circ q = p|[0, a)$ and q is maximal with this property.

We say that $f : E \to F$ locally lifts the paths if for every $x \in E$, there exits $U \in \mathcal{V}(x), V \in \mathcal{V}(f(x))$ so that $f|U : U \to V$ lifts the paths. If $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ is continuous, open and light. then f locally lifts the paths and for such a map, if $\beta : [0,1] \to \mathbf{R}^n$ is a map and $x \in D$ is so that $f(x) = \beta(0)$, we can always find a maximal lifting of β starting from x.

If $A \subset \mathbf{R}^n$, $p, t > 0$, we set $m_p^t(A) = \inf \sum_{i=1}^{\infty} d(A_i)^p$, where the infimum is taken over all coverings $A \subset \bigcup_{i=1}^{\infty} A_i$ so that $d(A_i) < t$ for every $i \in \mathbf{N}$ and we put $m_p^*(A) = \lim_{t \to 0} m_p^t(A)$. Then m_p^* is an outer measure on $\mathbf{R}^n$ and we obtain the class of p - Hansdorff measurable sets, which contains all Borel sets from $\mathbf{R}^n$. We say that a set $A \subset \mathbf{R}^n$ is of σ - finite p - dimensional measure if $A = \bigcup_{i=1}^{\infty} A_i$ and $m_p^*(A_i) < \infty$ for every $i \in \mathbf{N}$.

We see from Prop. 1 in [5] that condition (a_1) implies that f is a.e. differentiable and $J_f(x) \neq 0$ a.e. in D. An important condition which ensures that a continuous, open, discrete map has ACL^n inverses on $D \setminus B_f$ is that $f \in W^{1,n}_{loc}(D \setminus B_f, \mathbf{R}^n)$ and $K_0(f) \in L^{n-1}_{loc}(D \setminus B_f)$ (see Theorem 6.1 in [8]).

We denote by $L^*(x,f,r) = \sup_{y \in \partial U(x,f,r)} |y - x|$, $l^*(x,f,r) = \inf_{y \in \partial U(x,f,r)} |y - x|$ and we define the inverse linear dilatation of f in x by $H^*(x,f) = \limsup_{r \to 0} \frac{L^*(x,f,r)}{l^*(x,f,r)}$. If $V_n = \mu_n(B^n)$, then $\limsup_{r \to 0} \frac{d(U(x,f,r))^n}{\mu_n(U(x,f,r))} \leq \frac{2^n}{V_n} H^*(x,f)$ for every $x \in D$. Here $B^n = \{x \in \mathbf{R}^n \,|\, |x| < 1\}$ and $S^n = \{x \in \mathbf{R}^n \,|\, |x| = 1\}$.

We conjecture that condition (a) (the modular inequality of Poleckii) holds only if conditions (a_1) and (a_2) are satisfied. However, condition (a_3), with $K = \phi$, can hold in some important cases:

3.1) $f(B_f)$ is of $(n-1)$ - dimensional measure.

3.2) $m_1(B_f) = 0$.

3.3) $H^*(x,f) < \infty$ on B_f.

If f is quasiregular, then f is either constant, or open, discrete and $H^*(x,f)$ is locally bounded in D (see Theorem 4.4, page 37 in [24] or [18]), hence our class of mappings extends the known class of quasiregular mappings. We also see from Theorem 3 in [6] that a nonconstant map f is quasiregular if and only if is open, discrete and there exist $\alpha, \beta > 0$ so that $\limsup_{r \to 0} \frac{d(U(x,f,\alpha r))^n}{\mu_n(U(x,f,r))} \leq \beta$ for every $x \in D$.

If f is a Q-map or a map of finite distortion and satisfying condition $(\mathcal{A})$, then f satisfies conditions (a_1) and (a_2) (in fact $K_I(f) \in L^p_{loc}(D)$ for every $p > 0$). It results that at least in the class of open, discrete mappings satisfying conditions (a_1), (a_2), (a_3), we can strictly extend the similar results established in the class of Q-mappings or in the class of mappings of finite distortion, and satisfying condition $(\mathcal{A})$. An example showing that our extension is sharp even in the class of homeomorphisms was given ini [5]. Since our class of mappings is a large enough class of continuous, open, discrete mappings satisfying condition (a_1) and (a_2), it results that many of the assumptions from the theory of Q - maps, or from the theory of mappings of finite distortion and satisfying condition $(\mathcal{A})$ are redundant in order to extend some basic properties of quasiregular mappings. The classes of Q mappings and of mappings of finite distortion and satisfying condition $(\mathcal{A})$ were intensively studied in the last 10 years.

We denote for E, F subsets from $\bar{D}$ by $\Delta(E,F,D) = \{\gamma : [0,1] \to D \text{ path } | \gamma(0) \in E, \gamma(1) \in F\}$, and we set for $x \in \bar{D}$ and $0 < r < a$ $\Gamma_{x,r,a,D} = \Delta(\bar{B}(x,r) \cap D, \mathcal{C}B(x,a) \cap D, (B(x,a) \setminus \bar{B}(x,r)) \cap D)$ and if D is unbounded and $0 < r < s$ we set $\Gamma_{\infty,r,s,D} = \Delta(\bar{B}(0,r) \cap D, \mathcal{C}B(0,s) \cap D, (B(0,s) \setminus \bar{B}(0,r)) \cap D)$. We see from Lemma 7 that if $K_I(f) \in L^1_{loc}(D)$, then $M_{K_I(f)}(x) = 0$ if and only if $\lim_{r \to 0}(\Gamma_{x,r,a,D} = 0$ for $x \in \bar{D}$ and $a > 0$ kept fixed. We also see from Lemma 10 that $M_{K_I(f)}(\infty) = 0$ if and only if $\lim_{s \to \infty} M_{K_I(f)}(\Gamma_\infty, r, s, D) = 0$ for $x = \infty$ and $r > 0$ kept fixed. We shall use a more general form of (b), namely we say that f satisfies condition (c) in a point $x \in \bar{D}$ if

(c) $\lim_{r \to 0} M_{K_I(f)}(\Gamma_{x,r,a,D}) = 0$ if $x \in \bar{D}$, $0 < r < a$ and a is kept fixed.

$\lim_{s \to \infty} M_{K_I(f)}(\Gamma_{\infty,r,s,D}) = 0$ if $x = \infty$, $0 < r < s$ and $r > 0$ is kept fixed.

The condition (c) in $x = \infty$ (i.e. the condition $M_{K_I(f)}(\infty) = 0$) was systematically used in [5] for proving Zoric's type theorems and eliminability results. We see from Lemma 2 and Lemma 3 in [5] that condition (c) in ∞ holds if

(c^*) D is unbounded and there exists $0 \le \alpha < n-1$ so that

$$\limsup_{r \to 0} (r^n/(\ln r)^\alpha) \int_{\mathcal{C}B(0,r) \cap D} \frac{K_I(f)(y)}{|y|^{2n}} dy < \infty.$$

(c^{**}) D is unbounded and there exists $0 \le \alpha < n-1$ so that

$$\limsup_{r \to 0} \int_{B(0,r) \cap D} \frac{K_I(f)(x)}{r^n (\ln r)^\alpha} dx < \infty.$$

Theorem 2 in [5] shows that condition (c) holds in some point $x \in \bar{D} \subset \mathbf{R}^n$ if one of the following conditions holds:

(c_1) There exists $a > 0$ and an Orlicz map $\mathcal{A}$ so that $\int_{B(x,r) \cap D} \exp(\mathcal{A} \circ K_I(f)) dz < \infty$ for $0 < r < a$.

(c_2) There exists $a > 0, 0 \le \alpha < n-1$ and $M > 0$ so that $\int_{B(x,r) \cap D} K_I(f)(z) dz \le M \cdot \mu_n(B(x,r))(\ln(\frac{a}{r}))^\alpha$ for every $0 < r < a$.

(c_3) $n \ge 3$ and there exists $a > 0, M > 0, 0 \le \alpha < n-2$ and $Q \in L^1(D \cap B(x,a))$ with $K_I(f) \le Q$ and $⨍_{B(x,r) \cap D} |Q(z) - Q_{B(x,r) \cap D}| dz \le M (\ln \frac{a}{r})^\alpha$ for every $0 < r < a$, where $x \in \bar{D}$ is a φ-point in D, with $\varphi : (0, \frac{a}{e}) \to (0, \infty)$ decreasing and so that $l_2 = \sum_{k=1}^{\infty} \frac{\varphi(ae^{-k})}{k^{n-\alpha-1}} < \infty$.

(c_4) $n = 2$ and there exists $Q \in L^1(D \cap B(x,a))$ with $K_I(f) \le Q$ and $M > 0$ so that $⨍_{B(x,r) \cap D} |Q(z) - Q_{B(x,r) \cap D}| dz \le M$ for every $0 < r < a$, and x is φ-point of $\bar{D}$, with $\varphi = c$.

If $f \in L^1(A)$ for every bounded set $A \subset D$, we set $f_A = \int_A \frac{f(x)dx}{\mu_n(A)}$ and we write $f_A = ⨍_A f(x)dx$. We say as in [9] and [10] that f is of finite mean oscillation in a point $x \in D$ if

$\limsup_{r\to 0} \fint_{B(x,r)\cap D} |f(z) - f_{B(x,r)\cap D}|dz < \infty$, and we say that f is of bounded mean oscillation (we write $f \in BMO(D)$), if there exists $M > 0$ so that $\fint_B |f(z) - f_B|dz < M$ for every ball $B \subset\subset D$. If $x \in \bar{D}$, $a > 0$, and $\varphi : (0, \frac{a}{e}) \to (0, \infty)$ is a map, we say that x is a φ-point of D if $\mu_n(B(x, er) \cap D) \leq \varphi(r)\mu_n(B(x,r) \cap D)$ for every $0 < r < \frac{a}{e}$. If $\varphi(t) = c$ for $t > 0$ small enough, we say as in [10] that D satisfies a doubling condition in x. If $x \in IntD$, then x is a φ-point of D, with $\varphi = e^n$.

We remark that if $E \subset \bar{D}$ is at most countable and in each point of E is satisfied the condition $M_{K_I(f)}(x) = 0$ (and this happens if one of the conditions (c_1), (c_2), (c_3), (c_4), (c^*), (c^{**}) holds in x), then $M_{K_I(f)}(E) = 0$, and this thing can be elementary proved. We shall need the condition "$M_{K_I(f)}(E) = 0$" for some "singular" sets $E \subset \bar{D}$ in some theorems from this paper.

We could have used the method from [5] and [22] to extend the definition of continuous, open, discrete mappings satisfying conditions (a_1), (a_2), (a_3) to $\bar{\mathbf{R}}^n$ valued mappings. This means that such a map $f : D \to \bar{\mathbf{R}}^n$ would have the property that for every $x \in D$ would have existed $U \in \mathcal{V}(x)$ and a Möbius map $g : \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ so that $g \circ (f|U) : U \to \mathbf{R}^n$ would have been continuous, open, discrete and satisfying conditions (a_1), (a_2), (a_3) and also the set $f^{-1}(\infty)$ would have been closed and discrete in D. We prefer not to impose any condition around the points $x \in D$ so that $\lim_{y\to x} f(y) = \infty$. A motivation is that if $\Gamma_0 = \{\gamma : [0,1) \to D$ path $|\gamma$ has some limit point in $f^{-1}(\infty)\}$, then $M(f(\Gamma_0)) = 0$, and (a) holds for every path family Γ from D if and only if it holds for every path family Γ in D so that $\Gamma \cap \Gamma_0 = \phi$. In fact, if $E \subset D$ is so that $capf(E) = 0$ and $\tilde{\Gamma} = \{\gamma : [0,1) \to D$ path $|\gamma$ has some limit point in $E\}$ then $M(f(\tilde{\Gamma})) = 0$ and (a) holds for every path family in D if and only if it holds for every path family Γ in D so that $\Gamma \cap \tilde{\Gamma} = \phi$.

If $D \subset \mathbf{R}^n$ is a domain, we shall say that a map $f : D \to \bar{\mathbf{R}}^n$ is continuous, open, (light) discrete if there exists a closed, discrete set $M \subset D$ so that $f|D \setminus M : D \setminus M \to \mathbf{R}^n$ is continuous, open, (light) discrete and $\lim_{y\to x} f(y) = \infty$ for every $x \in M$. If we consider the chordal metric on $\bar{\mathbf{R}}^n$, we see immediately that $f : D \to \bar{\mathbf{R}}^n$ is continuous, open, (light) discrete and the construction is similar to the extension of the class of analytic functions to the class of meromorphic functions. It results that if $f : D \to \bar{\mathbf{R}}^n$ is continuous, open, discrete and satisfies conditions (a_1), (a_2), (a_3) on $D \setminus f^{-1}(\infty)$, then $M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma)$ for every path family Γ in D, i.e. f satisfies condition (a) on D.

If $D \subset \bar{\mathbf{R}}^n$ is a domain, $f|D \setminus \{x\} : D \setminus \{x\} \to \bar{\mathbf{R}}^n$ is continuous, open, (light) discrete we say that x is an essential singularity of f if the limit $\lim_{y\to x} f(y)$ does not exists in $\bar{\mathbf{R}}^n$. We shall say that a closed set $E \subset D$ with $\mu_n(E) = 0$ is eliminable for a continuous, open, discrete mapping $f : D \setminus E \to \bar{\mathbf{R}}^n$ satisfying conditions (a_1), (a_2), (a_3) on $D \setminus (E \cup f^{-1}(\infty))$ if there exists a continuous, open, discrete mapping $F : D \to \bar{\mathbf{R}}^n$ satisfying conditions (a_1), (a_2), (a_3) on $D \setminus F^{-1}(\infty)$.

We continue in this paper the work from [5], where we proved (a) for local homeomorphisms $f : D \to \bar{\mathbf{R}}^n$ satisfying condition (N) and having local ACL^n inverses (the case $B_f = \phi, K = \phi$ in conditions (a_1) and (a_3)) and we gave multiple extensions to the known theorem of Zoric from the theory of quasiregular mappings. In the light of our new improvements of the hypothesis for which the modular inequality of Poleckii (a) holds, we see that the theorems from [5] concerning the extensions given to Zoric's theorem, or ragarding the eliminability of the sets of null $K_I(f)$-modulus for local homeomorphisms, can hold now for some larger classes of local homeomorphisms $f : D \subset \mathbf{R}^n \to \bar{\mathbf{R}}^n$ satisfying condition (N). Indeed, we can have a "singular" closed set $K \subset D$ with $\mu_n(K) = 0$ so that f has local ACL^n inverses only on $D \setminus K$ and so that condition (a_3) holds for the set K and $B_f = \phi$. Also, we don't need in the points $x \in f^{-1}(\infty)$ the existence of a neighborhood $U \in \mathcal{V}(x)$ and of a Möbius map $g : \bar{\mathbf{R}}^n \to \bar{\mathbf{R}}^n$ so that $g \circ (f|U) : U \to \mathbf{R}^n$ is a local homeomorphisms having local ACL^n inverses.

We give now in the class of continuous, open, discrete mappings $f : D \to \bar{\mathbf{R}}^n$ satisfying conditions (a_1), (a_2), (a_3) on $D \setminus f^{-1}(\infty)$, equicontinuity results, Picard, Montel and Liouville type theorems, analogous of Schwarz's lemma, estimates of the modulus of continuity, eliminability results, boundary extension results.

We shall, in fact, enounce the results for continuous, open, light, mappings $f : D \to \bar{\mathbf{R}}^n$ for which conditions (a) (and sometimes (c)) hold. Also, in Theorem 14, 17 and 18, we shall need the condition $M_{K_I(f)}(\Gamma) = M_{K_I(f)}(\Gamma^r)$ (which is satisfied if $K_I(f) \in L^1_{loc}(D)$), and we shall presume that f is a.e. differentiable with $J_f(x) \neq 0$ a.e. in D, in order to define the inner dilatation $K_I(f)$ a.e. in D, even if this dilatation can be defined for mappings from $W^{1,1}_{loc}(D, \mathbf{R}^n)$ with the distributional jacobian $J_f(x) \neq)$ a.e. in D.

Here, if Γ is a path family in D, we denote by $\Gamma^r = \{\gamma \in \Gamma | \gamma \text{ is rectifiable }\}$.

We shall have in mind that all these theorems are given for continuous, open, discrete mappings $f : D \to \bar{\mathbf{R}}^n$ satisfying conditions (a_1), (a_2), (a_3) on $D \setminus f^{-1}(\infty)$, if f satisfies condition (a) on D and that one of the conditions (c_1), (c_2), (c_3), (c_4) or (c^*), (c^{**}) holds in some points b from $\bar{D}$ where the condition $M_{K_I(f)}(b) = 0$ is satisfied.

2. Preliminaries

We denote by $W^{1,p}_{loc}(D, \mathbf{R}^n)$ the Sobolev space of all functions $f : D \to \mathbf{R}^n$ which are locally in L^p together with their first order derivatives. We say that f is ACL if for every cube $Q \subset D$ with the sides parallel to coordinate axes and every face S of Q, it results that $f | P_S^{-1}(y) \cap Q : P_S^{-1}(y) \cap Q \to \mathbf{R}^n$ is absolutely continuous for a.e. $y \in S$, where $P_S : \mathbf{R}^n \to S$ is the projection on S. An ACL map has a.e. partial derivatives and if $p > 0$, we say that f is ACL^p if the partial derivatives are locally on L^p. We see from Prop 1.2, page 6 in [24] that if $p > 1$ and $f \in C(D, \mathbf{R}^n)$, then f is ACL^p if and only if $f \in W^{1,p}_{loc}(D, \mathbf{R}^n)$. If $A \subset D$ and $y \in \mathbf{R}^n$, we put $N(y, f, A) = Card f^{-1}(y) \cap A$ and $N(f, A) = \sup_{y \in \mathbf{R}^n} N(y, f, A)$. If $E \subset \mathbf{R}^n$ and $w \in \mathbf{R}^n$, we set $M(E, w) = \{z \in \mathbf{R}^n | \text{ there exists } y \in E \text{ and } t \geq 0 \text{ so that } z = w + t(y - w)\}$.

If $\alpha : [a, b] \to \mathbf{R}^n$ is a rectifiable path, we denote by $s_\alpha : [a, b] \to [0, l(\alpha)]$ its length function, and if α^0 is the normal representation of α (see [30], page 5), we have $\alpha = \alpha^\circ \circ s_\alpha$. If f is continuous and light, $\alpha : [a, b] \to D$ is a path and $\beta = f \circ \alpha$ is rectifiable, we define the path $\alpha^* = \alpha \circ s_\beta$ and the definition is correct due to the lightness of f. We also have $\beta^0 = f \circ \alpha^*$, $(\alpha^*)^\circ = \alpha^\circ$.

If $b \in \partial D$, we set $C(f, b) = \{w \in \bar{\mathbf{R}}^n | \text{ there exists } b_p \in D, b_p \neq b, b_p \to b \text{ so that } f(b_p) \to w\}$ and for $B \subset \partial D$, we put $C(f, b) = \bigcup_{b \in B} C(f, b)$.

If $K \subset \bar{D}$ and $b \in K'$, we define $F : \bar{D} \to \mathcal{P}(\bar{\mathbf{R}}^n)$ by $F(x) = f(x)$ if $x \in D$, $F(x) = C(f, x)$ if $x \in \partial D$, and if $(U_m)_{m \in N}$ is a fundamental system of neighborhoods of b so that $U_{m+1} \subset U_m$ for every $m \in N$, we put $C(f, b, K) = \bigcap_{m=1}^{\infty} \overline{F(U_m \cap (K \setminus \{b\}))}$.

If $D \subset \mathbf{R}^n$ is a domain and $b \in \partial D$, we say that D is locally connected in b if there exists $\mathcal{U}$, a fundamental system of neighborhoods of b so that $U \cap D$ is connected for every $U \in \mathcal{U}$. Following [30], page 54, we say that D has property P_2 in b if for every $b_1 \neq b$, $b_1 \in \partial D$, there exists $\delta > 0$ and $F \subset D$ compact so that $M(\Delta(E, F, D)) > \delta$ for every $E \subset D$ connected so that $b \in \bar{E}$, $b_1 \in \bar{E}$. If $b \in \partial D$, $\gamma : [0,1) \to D$ is a path, $\lim\limits_{t \to 1} \gamma(t) = b$ and $w \in \bar{\mathbf{R}}^n$ is so that $w = \lim\limits_{t \to 1} f(\gamma(t))$, we say that w is an asymptotic value of f in b, and we set $A(f,b) = \{w \in \bar{\mathbf{R}}^n | w$ is an asymptotic value of f in $b\}$.

We denote by q the chordal metric in $\bar{\mathbf{R}}^n$ given by $q(a,b) = |a-b|(1+|a|^2)^{-\frac{1}{2}}(1+|b|^2)^{-\frac{1}{2}}$ if $a \neq b$, $a, b \in \mathbf{R}^n$, $q(a,\infty) = (1+|a|^2)^{-\frac{1}{2}}$ if $a \in \mathbf{R}^n$, where $|a-b|$ is the euclidian distance between a and b if $a, b \in \mathbf{R}^n$. We denote by $B_q(x,r)$ ($S_q(x,r)$) the ball of center x and radius r (the sphere of center x and radius r) for $x \in \bar{\mathbf{R}}^n$ and $r > 0$ and we consider the chordal metric on $\bar{\mathbf{R}}^n$. If $A \subset \bar{\mathbf{R}}^n$, we consider $q(A)$ the diameter of A in the chordal metric.

If X is a separable metric space and $A = (A_i)_{i \in I}$ is a collection of sets, we define $\limsup A_i = \{x \in X |$ every neighborhood of x contains points from infinitely many sets $A_i\}$ and $\liminf A_i = \{x \in X |$ every neighborhood of x contains points of all but a finite number of sets $A_i\}$. If $\liminf A_i = \limsup A_i = A$, we say that the sequence of sets $(A_i)_{i \in I}$ is convergent and we put $A = \lim A_i$.

If W is a family of mappings $f : D \to \mathbf{R}^n$, we say that W is bounded if for every $K \subset D$ compact there exists $M(K) > 0$ so that $|f(x)| \leq M(K)$ for every $x \in K$ and every $f \in W$. If X, Y are metric spaces and W is a family of mappings $f : X \to Y$, we say that W is equicontinuous in a point $x \in X$ if for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $d(f(x), f(y)) \leq \epsilon$ if $d(x,y) \leq \delta_\epsilon$, for every $f \in W$. We say that the family W is equicontinuous if it is equicontinuous in every point $x \in X$.

If $D \subset \mathbf{R}^n$ is open, $\omega : D \to [0,\infty]$ is measurable and finite a.e., $\omega > 0$ a.e. and $p > 0$, we set $L^p_\omega = \{f : \mathbf{R}^n \to \mathbf{R} | \int\limits_{\mathbf{R}^n} \omega(x)|f(x)|^p dx < \infty\}$.

Lemma 1. L^p_ω is a Banach space.

Proof. The proof is standard, and the norm is $||f||^p_\omega = (\int\limits_{\mathbf{R}^n} \omega(x)|f(x)|^p dx)^{\frac{1}{p}}$.

Lemma 2. (Ziemer's lemma) Let $p \geq 2$, $D \subset \mathbf{R}^n$ be open, $\omega : D \to [0,\infty]$ be measurable and finite a.e. so that $\omega > 0$ a.e., and let Γ_m be path families in $\mathbf{R}^n$ so that $\Gamma_m \subset \Gamma_{m+1}$ for every $m \in N$ and let $\Gamma = \bigcup\limits_{m=1}^{\infty} \Gamma_m$. Then $M^p_\omega(\Gamma) = \lim\limits_{m \to \infty} M^p_\omega(\Gamma_m)$.

Proof. We follow the classical proof from [36]. We see that $M^p_\omega(\Gamma_m) \leq M^p_\omega(\Gamma_{m+1}) \leq M^p_\omega(\Gamma)$ for every $m \in N$, hence there exists $l = \lim\limits_{m \to \infty} M^p_\omega(\Gamma_m)$ and $l \leq M^p_\omega(\Gamma)$. We show that $M^p_\omega(\Gamma) \leq l$, and we can suppose that $l < \infty$. Let $\rho_m : \bar{\mathbf{R}}^n \to [0,\infty]$ be Borel mappings so that $\rho_m \in F(\Gamma_m)$ and $\int\limits_{\mathbf{R}^n} \omega(x)\rho_m(x)^p dx \leq M^p_\omega(\Gamma_m) + \frac{1}{2^m}$ for every $m \in N$.

Using Clarkson's inequality, we have:

$$(||\frac{\rho_i + \rho_j}{2}||^p_\omega)^p + (||\frac{\rho_i - \rho_j}{2}||^p_\omega)^p \leq \frac{1}{2}((||\rho_i||^p_\omega)^p) + (||\rho_j||^p_\omega)^p) \tag{1}$$

Also, if $i > j$, then $\frac{\rho_i + \rho_j}{2} \in F(\Gamma_j)$. Using (1), we have

$$M^p_\omega(\Gamma_j) + \int\limits_{\mathbf{R}^n} \omega(x)||\frac{\rho_i - \rho_j}{2}||^p(x)dx \leq \int\limits_{\mathbf{R}^n} \omega(x)||\frac{\rho_i + \rho_j}{2}||^p(x)dx + \int\limits_{\mathbf{R}^n} \omega(x)||\frac{\rho_i - \rho_j}{2}||^p(x)dx \leq$$

$$\frac{1}{2}(\int\limits_{\mathbf{R}^n} \omega(x)\rho_i(x)^p dx + \int\limits_{\mathbf{R}^n} \omega(x)\rho_j(x)^p dx) \leq \frac{1}{2}(M^p_\omega(\Gamma_i) + \frac{1}{2^i} + M^p_\omega(\Gamma_j) + \frac{1}{2^j})$$

for $i > j$.

Since $M^p_\omega(\Gamma_j) \leq l < \infty$, we have

$$[||\rho_i - \rho_j||^p_\omega]^p \leq 2^{p-1}(M^p_\omega(\Gamma_i) - M^p_\omega(\Gamma_j) + \frac{1}{2^i} + \frac{1}{2^j}) \tag{2}$$

for $i > j$.

Since $M^p_\omega(\Gamma_i) \nearrow l < \infty$, we see from (2) that $(\rho_i)_{i\in N}$ is a Cauchy sequence from the Banach space L^p_ω, hence there exists $\rho \in L^p_\omega$ so that $||\rho_i - \rho||^p_\omega \to 0$. As in [30], page 94, we can find $(\rho_{i_k})_{k\in N}$ a subsequence of $(\rho_i)_{i\in N}$ and a path family $\tilde{\Gamma} \subset \Gamma$ with $M^p_\omega(\tilde{\Gamma}) = 0$ and so that $\int_\gamma |\rho_{i_k} - \rho| ds \to 0$ for every $\gamma \in \Gamma \setminus \tilde{\Gamma}$. Let $\gamma \in \Gamma \setminus \tilde{\Gamma}$. There exists $m_\epsilon \in N$ so that $\int_\gamma |\rho_{i_k} - \rho| ds \leq \epsilon$ for $k \geq m_\epsilon$. Let $k_\epsilon \geq m_\epsilon$ be so that $\gamma \in \Gamma_{i_k}$ for $k \geq k_\epsilon$. Then $\int_\gamma \rho ds \geq \int_\gamma \rho_{i_k} ds - \int_\gamma |\rho - \rho_{i_k}| ds \geq 1 - \epsilon$, and letting $\epsilon \to 0$, we see that $\int_\gamma \rho ds \geq 1$. Since $\gamma \in \Gamma \setminus \tilde{\Gamma}$ was chosen arbitrarily, we proved that $\rho \in F(\Gamma \setminus \tilde{\Gamma})$. Then $M^p_\omega(\Gamma) \leq M^p_\omega(\Gamma \setminus \tilde{\Gamma}) + M^p_\omega(\tilde{\Gamma}) = M^p_\omega(\Gamma \setminus \tilde{\Gamma}) \leq \int_{\mathbf{R}^n} \omega(x)\rho^p(x)dx = (||\rho||^p_\omega)^p \leq (||\rho_i||^p_\omega + ||\rho - \rho_i||^p_\omega)^p \leq ((M^p_\omega(\Gamma_i) + \frac{1}{2^i})^{\frac{1}{p}} + ||\rho - \rho_i||^p_\omega)^p$.

Letting $i \to \infty$, we obtain that $M^p_\omega(\Gamma) \leq l$. We therefore proved that $M^p_\omega(\Gamma) = l$.

Lemma 3. Let $D \subset \mathbf{R}^n$ be open, $p \geq 2, \omega \in L^1_{loc}(D)$ and Γ a path family in D. Then $M^p_\omega(\Gamma) = M^p_\omega(\Gamma^r)$.

Proof. Let $D_m \subset\subset D$ be open, $D_m \nearrow D$ and $\Gamma_m = \{\gamma \in \Gamma | Im\gamma \subset D_m\}$ for $m \in N$. We fix $m \in N$ and let $\epsilon > 0$ and $\rho \in F(\Gamma^r_m)$. Let $\rho_m : \mathbf{R}^n \to [0, \infty)$, $\rho_m = \mathcal{X}_{D_m}$ and $\rho_{\epsilon,m} = (\rho^p + \epsilon^p \rho^p_m)^{\frac{1}{p}}$ for $m \in N$. If $\gamma \in \Gamma^r_m$, then $1 \leq \int_\gamma \rho ds \leq \int_\gamma \rho_{\epsilon,m} ds$ and if $\gamma \in \Gamma_m \setminus \Gamma^r_m$ is locally rectifiable, then $1 \leq \infty = \epsilon \int_\gamma \rho_m ds \leq \int_\gamma \rho_{\epsilon,m} ds$. This shows that $\rho_{\epsilon,m} \in F(\Gamma_m)$, hence $M^p_\omega(\Gamma_m) \leq \int_\gamma \omega(x)\rho_{\epsilon,m}(x)^p dx = \int_{\mathbf{R}^n} \omega(x)\rho(x)^p dx + \epsilon^p \int_{D_m} \omega(x)dx$. Letting $\epsilon \to 0$ and using the fact that $\int_{D_m} \omega(x)dx < \infty$ and $\rho \in F(\Gamma^r_m)$ was choosed arbitrarily, we see that $M^p_\omega(\Gamma_m) \leq M^p_\omega(\Gamma^r_m)$. We also have that $M^p_\omega(\Gamma^r_m) \leq M^p_\omega(\Gamma_m)$, hence $M^p_\omega(\Gamma_m) = M^p_\omega(\Gamma^r_m)$ for every $m \in N$.

Since $\Gamma_m \nearrow \Gamma$, $\Gamma^r_m \nearrow \Gamma^r$ and using Ziemer's lemma, we have that $M^p_\omega(\Gamma) = \lim_{m\to\infty} M^p_\omega(\Gamma_m) = \lim_{m\to\infty} M^p_\omega(\Gamma^r_m) = M^p_\omega(\Gamma^r)$.

Lemma 4. Let $D \subset \mathbf{R}^n$ be open, $\omega \in L^1_{loc}(D), p > 1$ and Γ a path family in D. Then for every $\epsilon > 0$ and every $\rho \in F(\Gamma)$, there exists $\rho \leq \eta$ lower semicontinuous so that $\int_{\mathbf{R}^n} \omega(x)\eta^p(x) \leq \int_{\mathbf{R}^n} \omega(x)\rho(x)^p dx + \epsilon$.

Proof. We write $\rho^p = \sum_{j=1}^{\infty} c_j \mathcal{X}_{E_j}$, where $c_j \geq 0$ and E_j are measurable and bounded for every $j \in N$, and the sets $E_2, ..., E_i, ...$ are not necessary disjoint.

Since the sets E_j are bounded and $\omega \in L^1_{loc}(D)$, we can find $E_j \subset V_j$ open sets so that $0 \leq \int_{V_j} \omega(x)dx - \int_{E_j} \omega(x)dx \leq \frac{\epsilon}{2^{j+1}c_j}$ for every $j \in N$. Let $\eta^p = \sum_{j=1}^{\infty} c_j \mathcal{X}_{V_j}$. Then $\rho \leq \eta$ and η is lower semicontinuous and $0 \leq \int_{\mathbf{R}^n} \omega(x)\eta(x)^p dx - \int_{\mathbf{R}^n} \omega(x)\rho(x)^p dx = \sum_{j=1}^{\infty} c_j \int_{V_j} \omega(x)dx - \sum_{j=1}^{\infty} c_j \int_{E_j} \omega(x)dx = \sum_{j=1}^{\infty} c_j (\int_{V_j} \omega(x)dx - \int_{E_j} \omega(x)dx) \leq \sum_{j=1}^{\infty} c_j \frac{\epsilon}{c_j 2^{j+1}} = \epsilon$.

Remark 1. The proof of the preceding lemma follows the classical proof of Vitali - Caratheodory theorem (see [28], Theorem 2.24, page 57).

Lemma 5. Let $D \subset \mathbf{R}^n$ be open, $p > 1$, $\rho > 0$, $G \subset\subset D$, G open, Γ a path family in G so that $l(\gamma) \geq \delta$ for every $\gamma \in \Gamma$ and let $\omega : D \to [0, \infty]$ measurable and finite a.e. so that $\int_{\bar{G}} \omega(x)dx < \infty$. Then $M_\omega^p(\Gamma) \leq \frac{1}{\delta^p} \int_{\bar{G}} \omega(x)dx < \infty$.

Proof. Let $\rho : \mathbf{R}^n \to [0, \infty)$, $\rho(x) = \frac{1}{\delta}$ for $x \in \bar{G}, \rho(x) = 0$ if $x \notin \bar{G}$. Then $\rho \in F(\Gamma)$, hence $M_\omega^p(\Gamma) \leq \int_{\mathbf{R}^n} \omega(x)\rho^p(x)dx = \frac{1}{\rho^p} \int_{\bar{G}} \omega(x)dx < \infty$.

Lemma 6. Let C_0, C_1 be closed, disjoint sets in $\mathbf{R}^n$ so that $d(C_0, C_1) = r > 0$ and the set $G = \mathbf{R}^n \setminus (C_0 \cup C_1)$ is bounded, $D \subset G$ open, $\omega \in L^1_{loc}(D)$, $p \geq 2$, $\Gamma = \Delta(C_0, C_1, D)$, $\Gamma_\delta = \Delta(C_0 + \delta B^n, C_1 + \delta B^n, D)$ for $0 < \delta < \frac{r}{2}$. Then $M_\omega^p(\Gamma) = \lim_{\delta \to 0} M_\omega^p(\Gamma_\delta)$.

Proof. Let $G_m = \{x \in G | d(x, C_0 \cup C_1) \geq \frac{1}{m}\}$ for $m \in N$. Then G_m are compact sets and $G_m \nearrow G$. Let $\Gamma_m = \{\beta$ path$|$ there exists $\gamma : [0, 1] \to D, \gamma \in \Gamma$ locally rectifiable, $0 \leq \alpha_\gamma < \beta_\gamma \leq 1$ so that $\beta = \gamma | [\alpha_\gamma, \beta_\gamma], \gamma(\alpha_\gamma) \in \partial G_m, \gamma(\beta_\gamma) \in \partial G_m$ and $Im \gamma \subset \bar{G}_m\}$ for $m \in N$. Then $\Gamma > \Gamma_{m+1} > \Gamma_m$ for every $m \in N$, hence $M_\omega^p(\Gamma) \leq M_\omega^p(\Gamma_{m+1}) \leq M_\omega^p(\Gamma_m)$ for every $m \in N$, and it results that there exists $\lim_{m \to \infty} M_\omega^p(\Gamma_m) \geq M_\omega^p(\Gamma)$. Let $\epsilon > 0$. Using Lemma 4, we can find $\eta \in F(\Gamma)$ lower semicontinuous so that $\int_{\mathbf{R}^n} \omega(x)\eta(x)^p dx \leq M_\omega^p(\Gamma) + \frac{\epsilon}{2}$. Let $\lambda_m = \sup\{\lambda > 0 | \int_\gamma \eta ds \geq \lambda$ for every $\gamma \in \Gamma_m\}$ for $m \in N$. Then $\lambda_{m+1} \geq \lambda_m$ for very $m \in N$, hence there exists $\lambda = \lim_{m \to \infty} \lambda_m$. We show that $\lambda \geq 1$.

Indeed, suppose that $\lambda < 1$, and let $0 < \lambda < \lambda' < 1$. We can find paths $\gamma_m \in \Gamma_m$ so that $\int_{\gamma_m} \eta ds < \lambda'$ for every $m \in N$. Let $\theta_m \in \Gamma, \theta_m = \theta_m^0$ be so that $\gamma_m = \theta_m | [\alpha_m, \beta_m]$ for every $m \in N$. Let $\theta_{qm} : [0, \infty] \to G_q$ be defined in the following way: Let $0 \leq \alpha_m \leq s_{qm} < t_{qm} \leq \beta_m \leq l(\theta_m)$ be the greatest, respectively the least $t \in [0, l(\theta_m)]$ for which $\theta_m(t) \in \partial G_q$, for $q = 1, ..., m$, $m \in N$. We set $\theta_{qm} = \theta_m | [s_{qm}, t_{qm}]$ and θ_{qm} is constant on $[0, s_{qm}]$ and on $[t_{qm}, \infty]$ for $q = 1, ..., m$, $m \in N$. We see that θ_{km} is a subpath of θ_{lm} for $1 \leq k \leq l \leq m$ and θ_{km} is a subpath of γ_m for $k = 1, ...,, m \in N$.

The family $(\theta_{1m})_{m \in N}$ is a 1-lipschitzian family, hence it is an equicontinuous family, and since it is also an uniformly bounded family, we use Ascoli's theorem to obtain a sequence $(\theta_{1m_j})_{j \in N}$ converging to a path $\beta^1 : [0, \infty] \to G_1$. Taking a subsequence, we can presume that $s_{1m_j} \to a_1, t_{1m_j} \to b_1$ and β^1 is constant outside $[a_1, b_1]$. We set $J_1 = \{m_1, m_2, ..., m_j, ..., \}$. The family $(\theta_{2l})_{l \in J_1}$ is equicontinuous and uniformly bounded and using again Ascoli's theorem, we obtain a set $J_2 \subset J_1$ of increasing natural numbers so that the family $(\theta_{2l})_{l \in J_2}$ converges to a path $\beta^2 : [0, \infty] \to G_2$. We can suppose that the first number from J_2 is the first number from J_1, that $s_{2l} \to a_2, t_{2l} \to b_2$ for $l \to \infty$, $l \in J_2$, with $a_2 \leq a_1 \leq b_1 \leq b_2$ and that β^2 is constant outside $[a_2, b_2]$. We continue the process at infinite. At Step k, we find a set $J_k \subset J_{k-1} \subset ... \subset J_2 \subset J_1$ of increasing natural numbers so that the first $k - 1$ numbers from J_k are the first $k - 1$ numbers from J_{k-1} and so that the family $(\theta_{kl})_{l \in J_k}$ converges to a path $\beta^k : [0, \infty] \to G_k$. We can also suppose that $s_{kl} \to a_k, t_{kl} \to b_k$ for $l \to \infty$, $l \in J_k$, that β^k is constant outside $[a_k, b_k]$ and that $a_k \leq a_{k-1} \leq ... \leq a_1 \leq b_1 \leq ... \leq b_{k-1} \leq b_k$.

Let l_k be the k^{th} term from J_k and $J = \{l_1, l_2, ..., l_k, ...\}$. Then $J \subset J_k$ for every $k \in N$, $\theta_{kl} \to \beta^k$ uniformly on every compact interval $I \subset [a_k, b_k]$ for $l \to \infty$, $l \in J$ and $\beta^k | I = \beta^{k+1} | I$, $k \in N$. We can correctly define now $\beta : [0, \infty] \to D$, $\beta | [a_k, b_k] = \beta^k | [a_k, b_k]$ for $k \in N$. Then β is 1 - lipschitzian, hence $|\beta'(t)| \leq 1$ a.e. and β is absolutely continuous and hence

$\int\limits_{\beta|I} \eta ds = \int\limits_I \eta(\beta(t))|\beta'(t)|dt$ for every compact interval $I \subset [0, \infty]$. We see that $a_k \to a, b_k \to b$ and let $a < a' < b' < b$. We can also suppose that $a_k < a' < b' < b_k$ for every $k \in N$. Then $\theta_{kl_k}|[a', b'] = \theta_{1l_k}|[a', b'] \to \beta^1|[a', b'] = \beta|[a', b']$ and since every path θ_{kl_k} is the restriction on G_k of a path γ_{l_k}, we see that $\int\limits_{\theta_k l_k} \eta ds \leq \int\limits_{\gamma_{l_k}} \eta ds \leq \lambda'$ for every $k \in N$. Using Fatou's lemma and the lower semicontinuity of the maps η, we obtain:

$$\int_{a'}^{b'} \eta(\beta(t))|\beta'(t)|dt \leq \int_{a'}^{b'} \eta(\beta(t))dt = \int_{a'}^{b'} \eta(\lim_{k\to\infty} \theta_{kl_k})dt \leq \liminf_{k\to\infty} \int_{a'}^{b'} \eta(\theta_{kl_k}(t))dt =$$

$$\liminf_{k\to\infty} \int_{a'}^{b'} \eta(\theta^0_{kl_k}(t))dt \leq \liminf_{k\to\infty} \int\limits_{\theta_k l_k} \eta ds \leq l' < 1.$$

Letting $a' \to a, b' \to b$, and using the fact that $\beta \in \Gamma$ we see that $1 \leq \int\limits_\beta \eta ds = \int_a^b \eta$ $(\beta(t))|\beta'(t)|dt \leq \lambda' < 1$, and we reached a contradiction.

We showed that $\lambda \geq 1$.

Let now compact sets $D_q \subset D$ so that $D_q \nearrow D$, $D_{qm} = D_q \cap G_m$, $\Gamma_{qm} = \{\gamma \in \Gamma_m | Im\gamma \subset D_q\}$ for $m, q \in N$. Let $\rho_{qm} \in F(\Gamma_{qm})$ be so that $\int\limits_{\mathbf{R}^n} \omega(x)\rho_{qm}(x)^p dx \leq M^p_\omega(\Gamma_{qm}) + \frac{1}{2^m}$ for every $m, q \in N$. Since $\frac{\eta}{\lambda_m} \in F(\Gamma_m)$, we see that $\frac{\eta}{\lambda_m} \in F(\Gamma_{qm})$, hence $\frac{1}{2}(\frac{\eta}{\lambda_m} + \rho_{qm}) \in F(\Gamma_{qm})$ for every $q, m \in N$. We suppose that $\frac{1}{m} < \frac{r}{4}$ for every $m \in N$ and using Clarkson's inequality, we have:

$$\int\limits_{\mathbf{R}^n} \omega(x)[\frac{1}{2}(\frac{\eta}{\lambda_m} + \rho_{qm})]^p(x)dx + \int\limits_{\mathbf{R}^n} \omega(x)[\frac{1}{2}|\frac{\eta}{\lambda_m} - \rho_{qm}|]^p(x)dx \leq \frac{1}{2}(\frac{1}{\lambda^p_m} \int\limits_{\mathbf{R}^n} \omega(x)\eta^p(x)dx +$$

$$\int\limits_{\mathbf{R}^n} \omega(x)\rho_{qm}(x)^p dx) \leq \frac{1}{2}(\frac{1}{\lambda^p_m}(M^p_\omega(\Gamma) + \frac{\epsilon}{2}) + M^p_\omega(\Gamma_{qm}) + \frac{1}{2^m})$$

for $m \in N$.

Then

$$M^p_\omega(\Gamma_{qm}) + \int\limits_{\mathbf{R}^n} \omega(x)[\frac{1}{2}|\frac{\eta}{\lambda_m} - \rho_{qm}|]^p(x)dx \leq \int\limits_{\mathbf{R}^n} \omega(x)[\frac{1}{2}(\frac{\eta}{\lambda_m} + \rho_{qm})]^p(x)dx +$$

$$\int\limits_{\mathbf{R}^n} \omega(x)[\frac{1}{2}|\frac{\eta}{\lambda_m} - \rho_{qm}|]^p(x)dx \leq \frac{1}{2}(\frac{1}{\lambda^p_m}(M^p_\omega(\Gamma) + \frac{\epsilon}{2}) + M^P_\omega(\Gamma_{qm}) + \frac{1}{2^m})$$

for $m \in N$.

We see from Lemma 5 that $M^p_\omega(\Gamma_{qm}) \leq (\frac{2}{r})^p \int\limits_{D_q} \omega(x)dx < \infty$, hence $0 \leq \int\limits_{\mathbf{R}^n} \omega(x)[\frac{1}{2}|\frac{\eta}{\lambda_m} - \rho_{qm}|]^p(x)dx \leq \frac{1}{2}(\frac{1}{\lambda^p_m}(M^p_\omega(\Gamma) + \frac{\epsilon}{2}) - M^p_\omega(\Gamma_{qm}) + \frac{1}{2^m})$ for every $m, q \in N$. Let $m_\epsilon \in N$ be so that $\frac{1}{2^m} \leq \frac{\epsilon}{2\lambda^p_m}$ for $m \geq m_\epsilon$. Then $\lambda^p_m M^p_\omega(\Gamma_{qm}) \leq M^p_\omega(\Gamma) + \epsilon$ for every $m \geq m_\epsilon$ and every $q \in N$.

We fix $m \geq m_\epsilon$ and we let $q \to \infty$. Since $\Gamma_{qm} \nearrow \Gamma_m$, we see from Lemma 2 that $\lambda^p_m M^p_\omega(\Gamma_m) \leq M^p_\omega(\Gamma) + \epsilon$ for every $m \geq m_\epsilon$.

Letting now $m \to \infty$, we see that $\lim_{m\to\infty} M^p_\omega(\Gamma_m) \leq M^p_\omega(\Gamma) + \epsilon$, and letting $\epsilon \to 0$, we see that $\lim_{m\to\infty} M^p_\omega(\Gamma_m) \leq M^p_\omega(\Gamma)$.

We therefore proved that $\lim_{m\to\infty} M^p_\omega(\Gamma_m) = M^p_\omega(\Gamma)$.

Remark 2. The preceding lemma generalizes Theorem 37.1, page 123 in [30] and Lemma 6 from [3]. If $\Delta_\delta = \Delta(C_0, (C_1 + \delta B^n) \cap D, D)$, then we also have that $M^p_\omega(\Gamma) = \lim_{\delta\to 0} M^p_\omega(\Delta_\delta)$.

Lemma 7. Let $D \subset \mathbf{R}^n$ be open, $\omega \in L^1_{loc}(D)$, $p \geq 2, x \in \bar{D}$ and $0 < a$. Then $M^p_\omega(x) = 0$ if and only if $\lim_{r\to 0} M^p_\omega(\Gamma_{x,r,a,D}) = 0$.

Proof. We take $C_0 = \{x\}$, $C_1 = \mathcal{C}B(x,a)$ and we apply Lemma 6 and Remark 2.

Lemma 8. Let $D, D^{'}$ be open in $\mathbf{R}^n$, $g : D \to D^{'}$ conformal, Γ a path of family in D, $\omega : D^{'} \to [0,\infty]$ measurable and finite a.e., $\Gamma^{'} = g(\Gamma)$. Then $M_\omega(\Gamma^{'}) = M_{\omega o g}(\Gamma)$.

Proof. Let $\rho^{'} \in F(\Gamma^{'})$ and $\rho : \mathbf{R}^n \to [0,\infty]$, $\rho(x) = \rho^{'}(g(x))|g^{'}(x)|$ if $x \in D, \rho(x) = 0$ if $x \notin D$. Then $\rho \in F(\Gamma)$, hence $M_{\omega o g}(\Gamma) \leq \int_{\mathbf{R}^n} \omega(g(x))\rho^n(x)dx = \int_D \rho^{'n}(g(x))\omega(g(x))|J_g(x)|dx = \int_{D^{'}} \rho^{'n}(y)\omega(y)dy \leq \int_{\mathbf{R}^n} \omega(y)\rho^{'n}(y)dy$. Since $\rho^{'} \in F(\Gamma^{'})$ was choosed arbitrary, it results that $M_{\omega o g}(\Gamma) \leq M_\omega(\Gamma^{'})$. Since $g^{-1} : D^{'} \to D$ is also conformal, we have $M_\omega(\Gamma^{'}) = M_{\omega o g o g^{-1}}(\Gamma^{'}) \leq M_{\omega o g}(g^{-1}(\Gamma^{'})) = M_{\omega o g}(\Gamma)$. We proved that $M_\omega(\Gamma^{'}) = M_{\omega o g}(\Gamma)$.

Lemma 9. Let C_0, C_1 be closed and disjoint sets in $\mathbf{R}^n$ so that $IntC_0 \neq \phi$, $D \subset \mathbf{R}^n \backslash (C_0 \cup C_1)$ be open, $\omega \in L^1_{loc}(D)$, $\Gamma = \Delta(C_0, C_1, D)$, $\Gamma_\delta = \Delta((C_0 + \delta B^n) \cap D, (C_1 + \delta B^n) \cap D, D)$ for $\delta > 0$. Then $M_\omega(\Gamma) = \lim_{\delta\to 0} M_\omega(\Gamma_\delta)$.

Proof. Let $G = \mathbf{R}^n \setminus (C_0 \cup C_1)$. If G is bounded, we use Lemma 6. If G is unbounded, we take $x \in IntC_0$ and $r > 0$ so that $\bar{B}(x,r) \subset C_0$ and let $g : \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$, $g(z) = x + r^2 \frac{z-x}{|z-x|^2}$, $z \in \overline{\mathbf{R}}^n$. Then g is conformal, $g(C_0)$ and $g(C_1)$ are closed, disjoint, $G \cap \bar{B}(x,r) = \phi$ and hence $G^{'} = g(G)$ is bounded. We use now Lemma 6 and Lemma 8.

Lemma 10. Let $D \subset \mathbf{R}^n$ be open and unbounded, $\omega \in L^1_{loc}(D)$ and $0 < r < s < \infty$. Then $M_\omega(\infty) = 0$ if and only if $\lim_{s\to\infty} (\Gamma_{\infty,r,s,D}) = 0$ for $r > 0$ kept fixed.

Using Theorem 6.3, page 107 in [11] and Vitali's covering theorem, we have the following change of variable formulae, which is valid in our class of mappings:

Proposition 1. Let $n \geq 2$, $D \subset \mathbf{R}^n$ be open, $f : D \to \mathbf{R}^n$ be continuous and satisfying condition (N), $E \subset D$ closed ni D so that $\mu_n(E) = 0$ and let $f \in W^{1,1}_{loc}(D \setminus E, \mathbf{R}^n)$. Then, if $g : \mathbf{R}^n \to [0,\infty]$ is measurable and finite a.e., it results that $\int_A g(f(x))|J_f(x)|dx = \int_{\mathbf{R}^n} g(y)N(y,f,A)dy$ for every $A \subset D$ measurable.

3. Poleckii's modular inequality

Theorem 1. Let $f : D \to \mathbf{R}^n$ be continuous, open, discrete and satisfying conditions $(a_1), (a_2), (a_3)$, $U \subset\subset D$ a normal domain and $V = f(U)$. Then the map $f_V : V \to \mathbf{R}^n$ is ACL^n.

Proof. We can suppose that the set K from conditions (a_1) and (a_3) is so that $K = \phi$. Let $e_1, ..., e_n$ be the canonic base from $\mathbf{R}^n$ and let $Q \subset\subset V$ be an open cube with the sides parallel to the coordinate axes and of side r_0. We fix a face Q_0 of the cube Q and suppose that Q_0 is perpendicular on e_j. We define for $y \in Q_0$ the path $\beta_y : [0,1] \to Q$ by $\beta_y(t) = y + tr_0e_j$ for $t \in [0,1]$ and let $J_y = Im\beta_y$. Let P be the projection on Q_0 and let $E_A = P^{-1}(A) \cap Q$ for $A \in \mathcal{B}(Q_0)$. We define a set function $\varphi : \mathcal{B}(Q_0) \to \mathbf{R}_+$ by $\varphi(A) = \mu_n(f^{-1}(E_A) \cap U)$ for $A \in \mathcal{B}(Q_0)$. Then there exists $K_0 \subset Q_0$ with $m_{n-1}(K_0) = 0$ so that $\varphi^{'}(y) < \infty$ for every $y \in Q_0 \backslash K_0$. Since $f(E)$ is of σ-finite $(n-1)$-dimensional measure, we use a theorem of Gross (see

Theorem 30.14, page 103 in [30]) to see that $J_y \cap f(E)$ is at most countable with the possible exception of a set $K_1 \subset Q_0$ with $m_{n-1}(K_1) = 0$. Let $q = |d(f, U, V)|$. Using Besicowitch's theorem, we can cover $Q \setminus f(B_f \cap \bar{U})$ with domains V_i so that every point $z \in Q \setminus f(B_f \cap \bar{U})$ belongs to at most $N(n)$ sets V_i and $f^{-1}(V_i) = \bigcup_{j=1}^{q} U_{ij}$ with $U_{ij} \subset\subset U$ disjoint domains so that $f(\partial U_{ij}) = \partial V_i$ and $f|U_{ij} : U_{ij} \to V_i$ is a homeomorphism for every $i \in N$ and $j = 1, ..., q$. Let $h_{ij} : V_i \to U_{ij}$ be their inverses for $i \in N$, $j = 1, ..., q$, $A_{ij} = \{z \in V_i | h_{ij}$ is not differentiable in $z\}$, $Z_{ij} = \{z \in V_i \setminus A_{ij} | J_{h_{ij}}(z) = 0\}$ for $i \in N$, $j = 1, ..., q$. We see from [23], page 110 that $\mu_n(A_{ij}) = 0$, and from [1] we see that $\mu_n(h_{ij}(Z_{ij})) = 0$, $i \in N$, $j = 1, ..., q$. Since f satisfies condition (N), we see that $\mu_n(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{q} (A_{ij} \cup Z_{ij} \cup f(B_f \cap \bar{U}))) = 0$. Let $\Delta = \{\beta_y^0 : [0, l(\beta_y)] \to Q | y \in Q_0$ and $m_1(\{t \in [0, l(\beta_y)] | \beta_y^0(t) \in (\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{q} Z_{ij} \cup A_{ij} \cup f(B_f \cap \bar{U}))\}) > 0\}$ We see from Theorem 33.1, page 111 in [30] that $M(\Delta) = 0$, hence we can find a set $K_2 \subset Q_0$ with $m_{n-1}(K_2) = 0$ and so that $m_1(\{t \in [0, l(\beta_y)] | \beta_y^0(t) \in (\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{q} A_{ij} \cup Z_{ij} \cup f(B_f \cap \bar{U}))\}) = 0$ for every $y \in Q_0 \setminus K_2$. We also see that $m_1(f(B_f \cap \bar{U}) \cap J_y) = 0$ for every $y \in Q_o \setminus K_2$.

We define now a Borel map $\rho : \mathbf{R}^n \to [0, \infty]$ by $\rho(y) = \sup_{j=1,...,q} |h'_{ij}(y)|$ for $y \in V_i \setminus \bigcup_{j=1}^{q} (A_{ij} \cup Z_{ij})$, $i \in N$, $\rho(y) = 0$ otherwise and the definition is correct and does not depends on the domains V_i so that $y \in V_i$, since $\rho(y) = \sup_{x \in f^{-1}(y) \cap U} |f'(x)^{-1}|$ if $y \in Q \setminus f(B_f \cap \bar{U})$. Since h_{ij} are ACL^n homeomorphisms, are a.e. differentiable mappings satisfying condition (N) and we see from the change of variable formulae in Theorem 5.23, page 132 in [7] that

$$\int_{\mathbf{R}^n} \rho^n(y)dy \leq \sum_{i=1}^{\infty} \int_{V_i} \rho^n(y)dy \leq \sum_{i=1}^{\infty} \sum_{j=1}^{q} \int_{V_i} |h'_{ij}(y)|^n dy = \sum_{i=1}^{\infty} \sum_{j=1}^{q} \int_{V_i} K_I(f)(f^{-1}(y)) |J_{f^{-1}}(y)| dy =$$

$$\sum_{i=1}^{\infty} \sum_{j=1}^{q} \int_{U_{ij}} K_I(f)(x)dx \leq N(n) \int_{U} K_I(f)(x)dx < \infty.$$

Let $\Delta_0 = \{\beta_y | y \in Q_0$ and $\int_{\beta_y} \rho ds = \infty\}$. Then $\frac{\rho}{k} \in F(\Delta_0)$ for every $k \in N$, hence $M(\Delta_0) \leq \frac{1}{k^n} \int_{\mathbf{R}^n} \rho^n(y)dy$ for $k \in N$. Letting $k \to \infty$, we see that $M(\Delta_0) = 0$, hence we can find a set $K_3 \subset Q_0$ with $m_{n-1}(K_3) = 0$ and so that $\int_{\beta_y} \rho ds < \infty$ for every $y \in Q_0 \setminus K_3$.

Let now $\Gamma_{ij} = \{\beta_y | y \in Q_0 | J_y \cap V_i \neq \phi$ and there exists $j \in \{1, ..., q\}$ and an interval $J \subset J_y \cap V_i$ so that $h_{ij} \circ \beta_y^0$ is not absolutely continuous on $J\}$ for $i \in N, j = 1, ..., q$. Using Fuglede's theorem (see Theorem 28.2, page 95 in [30]) and the fact that h_{ij} are ACL^n maps, we see that $M(\Gamma_{ij}) = 0$ for $i \in N$ and $j = 1, ..., q$. It results that there exists a set $K_4 \subset Q_0$ with $m_{n-1}(K_4) = 0$ and so that $h_{ij} \circ \beta_y^0$ is absolutely continuous on every interval $J \subset J_y \cap V_i$ for every $i \in N, j = 1, ..., q$ and every $y \in Q_0 \setminus K_4$. Let $K_5 = \bigcup_{i=0}^{4} K_i$.

We fix $y \in Q_0 \setminus K_5$ and let $\alpha : [0, 1] \to U$ be a path so that $f \circ \alpha = \beta_y$. We show that α^* is absolutely continuous. Let $F_0 = J_y \cap f(B_f \cap Im\alpha^*)$ and $E_0 = (\beta_y^0)^{-1}(F_0)$. Then F_0 is compact and $m_1(F_0) = 0$ and also E_0 is compact and $m_1(E_0) = 0$, hence $(0, l(\beta_y)) \setminus E_0 = \bigcup_{l=1}^{\infty} I_l$, with

$I_l \subset (0, l(\beta_y))$ open intervals for $l \in N$. Let $l \in N$ and $I \subset I_l$ be a maximal open interval so that $\beta_y^0(I) \subset V_i$. Since $f \circ \alpha^* = \beta_y^0$ we can find $j \in \{1, ..., q\}$ so that $\alpha^*|I = h_{ij} \circ \beta_y^0|I$, hence α^* is absolutely continuous on I and this implies that α^* is absolutely continuous on every interval I_l for $l \in N$.

We show that $\int_0^{l(\beta_y)} |\alpha^{*'}(t)|dt < \infty$. Let $B = \{t \in [0, l(\beta(y))] | \beta_y^0(t) \in (\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{q} A_{ij} \cup Z_{ij} \cup f(B_f \cap \bar{U}))\}$ and $C = \{t \in [0, l(\beta_y)] | \beta_y^0$ and α^* are not differentiable in $t\}$. Them $m_1(B \cup C) = 0$ and let $t \in [0, l(\beta_y)] \setminus (B \cup C)$. Since $E_0 \subset B$, we can find $i, l \in N, j \in \{1, ..., q\}$ and an interval $I \subset I_l$ so that $t \in I$ and $\alpha^*|I = h_{ij} \circ \beta_y^0$. Then $|\alpha^{*'}(t)| = |h_{ij}^{'}(\beta_y^0(t))||\beta_y^{0'}(t)| \leq h_{ij}^{'}(\beta_y^0(t)) \leq \rho(\beta_y^0(t))$, and this implies that $|\alpha^{*'}(t)| \leq \rho(\beta_y^0(t))$ a.e. in $[0, l(\beta_y)]$. We see now that

$$\int_0^{l(\beta_y)} |\alpha^{*'}(t)|dt \leq \int_0^{l(\beta_y)} \rho(\beta_y^0(t))dt = \int_{\beta_y} \rho ds < \infty.$$

We show now that $m_1(\alpha^*(A)) = 0$ if $A \subset [0, l(\beta_y)]$ is so that $m_1(A) = 0$. We denote by $F_{m,p} = \{x \in Im\alpha^* | \limsup_{r \to 0} d(U(x, f, \frac{r}{p}))^n / \mu_n(U(x, f, r) \leq m\}$, by $F_{k,m,p} = \{x \in Im\alpha^* | d(U(x, f, \frac{r}{p}))^n \leq m\mu_n(U(x, f, r))$ for $0 < r < \frac{1}{k}\}$ and by $F_{m,p}^r = \{x \in Im\alpha^* | d(U(x, f, \frac{r}{p}))^n \leq m\mu_n (U(x, f, r))\}$ for $k, m, p \in N$ and $r > 0$.

Let $r > 0$ be fixed and take $x_l \in Im\alpha^*, x_l \to x$ and $0 < r < \rho$. There exists $l_0 \in N$ so that $B(f(x_l), r) \subset B(f(x), \rho)$ for $l \geq l_0$, hence $U(x_l, f, r) \subset U(x, f, \rho)$ for $l \geq l_0$ and hence $\limsup_{l \to \infty} \mu_n(U(x_l, f, r)) \leq \mu_n(U(x, f, \rho))$. Letting $\rho \searrow r$ we see that $\limsup_{l \to \infty} \mu_n(U(x_l, f, r)) \leq \mu_n(\bar{U}(x, f, r)) = \mu_n(U(x, f, r))$, hence the map $x \to \mu_n(U(x, f, r))$ is upper semicontinuous on $Im\alpha^*$. Also, if $x_l \in Im\alpha^*$, $x_l \to x$ and $0 < \rho < r$, we see that there exists $l_1 \in N$ so that $U(x, f, \rho) \subset U(x_l, f, r)$ for $l \geq l_1$ and hence $d(U(x, f, \rho))^n \leq \liminf_{l \to \infty} d(U(x_l, f, r))^n$. Letting $\rho \nearrow r$, we see that $d(U(x, f, r))^n \leq \liminf_{l \to \infty} d(U(x_l, f, r))^n$, hence the map $x \to d(U(x, f, r))^n$ is lower semicontinuous on $Im\alpha^*$. Since the sum of two lower semicontinuous mappings is a lower semicontinuous mapping, it results that the map $x \to d(U(x, f, \frac{r}{p}))^n - m\mu_n(U(x, f, r))$ is lower semicontinuous on $Im\alpha^*$, and hence the sets $F_{m,p}^r$ are compact on $Im\alpha^*$ for every $m, p \in N$ and every $r > 0$. Since $F_{k,m,p} = \bigcap_{0<r<\frac{1}{k}} F_{m,p}^r$, it results that $F_{k,m,p}$ are compact in $Im\alpha^*$ for $k, m, p \in N$. Now $F_{k,m,p} \nearrow F_{m,p}$, and this implies that $F_{m,p}$ are Borel subsets of $Im\alpha^*$ for every $m, p \in N$.

Let $E_{k,m,p} = (\alpha^*)^{-1}(F_{k,m,p} \cap B_f)$, $E_{m,p} = (\alpha^*)^{-1}(F_{m,p} \cap B_f)$ for $k, m, p \in N$. Then $\alpha^*(E_{m,k,p}) = F_{k,m,p}$, $E_{k,m,p} \nearrow E_{m,p}$. $E_{k,m,p} \subset E_0$ are compact for $k, m, p \in N$ and hence $m_1(E_{k,m,p}) = 0$ for $k, m, p \in N$. Let us fix $k, m, p \in N$. Let $\epsilon, t > 0$. We show that $m_1(\alpha^*(E_{k,m,p})) = 0$. We denote by $B^{n-1}(y, \lambda)$ the ball of center y and radius λ from Q_0 if $y \in Q_0$. Using Lemma 31.1 page 106 in [30], we find $0 < \delta < \frac{1}{k}$ so that for every $0 < r < \delta$, the subset $f(F_{k,m,p} \cap B_f)$ of J_y is covered by open intervals $\Delta_i \subset J_y$ of length $\frac{2r}{p}$ and centered in some points $y_i = f(x_i) \in J_y$, with $x_i \in F_{k,m,p} \cap B_f$, so that every point from J_y belongs to at most two intervals $\Delta_i, i = 1, ..., l$ and $\frac{lr}{p} \leq m_1(f(F_{k,m,p} \cap B_f)) + \epsilon$. Let $B_i = B(y_i, \frac{r}{p}), i = 1, ..., l$.

Then $f(F_{k,m,p} \cap B_f) \subset \bigcup_{i=1}^{l} B_i$ and we choose $0 < r < \delta$ so that $|\alpha^*(t^{'}) - \alpha^*(t^{''})| \leq t$ for every $t^{'}, t^{''} \in [0, l(\beta_y)]$ so that $|\beta_y^0(t^{'}) - \beta_y^0(t^{''})| < \frac{2r}{p}$.

Let $W_i = U(x_i, f, \frac{r}{p}) \cap Im\alpha^*$, $Q_i = U(x_i, f, r)$ for $i = 1, ..., l$. Then $d(W_i) \leq t$ for $i = 1, ..., l$

and we show that $F_{k,m,p} \cap B_f \subset \bigcup_{i=1}^{l} W_i$. Indeed, let $z \in F_{k,m,p} \cap B_f$. Then $z \in Im\alpha^*$ and $f(z) = w \in f(F_{k,m,p} \cap B_f) \subset J_y \cap (\bigcup_{i=1}^{l} B_i)$. Let $i \in \{1, ..., l\}$ be so that $w \in B_i \cap f(F_{k,m,p} \cap B_f)$ and let $J_i \subset [0, l(\beta_y)]$ be an interval so that $\beta_y^0(J_i) = B_i \cap Im\beta_y^0$. Take now $t_i \in J_i$ so that $y_i = \beta_y^0(t_i)$. Since $\beta_y^0 = f \circ \alpha^*$, f is injective on $Im\alpha^*$ and $y_i = f(x_i)$, $y_i = \beta_y^0(t_i) = f(\alpha^*(t_i))$, we see that $x_i = \alpha^*(t_i)$, and since $f(\alpha^*(J_i)) = \beta_y^0(J_i) = B_i \cap Im\beta_y^0$, and $\alpha^*(J_i)$ is connected so that $x_i \in \alpha^*(J_i)$, it results that $\alpha^*(J_i) \subset W_i$. Since $w \in B_i \cap Im\beta_y^0 = \beta_y^0(J_i) = f(\alpha^*(J_i))$, we can find $a_i \in J_i$ so that $w = f(\alpha^*(a_i))$. Now $z \in Im\alpha^*$, $f(z) = w = f(\alpha^*(a_i))$ and f is injective on $Im\alpha^*$, hence $z = \alpha^*(a_i) \in \alpha^*(J_i) \subset W_i \subset \bigcup_{k=1}^{l} W_k$. We proved that $\alpha^*(E_{k,m,p}) = F_{k,m,p} \cap B_f \subset \bigcup_{i=1}^{l} W_l$.

Since the balls $\frac{1}{2}B_i$ are disjoint for $i = 1, ..., l$, we see that every point from $E_{B^{n-1}(y,r)}$ belongs to at most $6p$ balls $pB_i, i = 1, ..., l$. Now, every value $w \in V$ is taken by $f|U : U \to V$ by at most q times, hence every point from $\mathbf{R}^n$ belongs to at most $6pq$ sets $Q_i, i = 1, ..., l$. Then $\sum_{i=1}^{l} \mu_n(Q_i) \le 6pq\mu_n(\bigcup_{i=1}^{l} Q_i) \le 6pq\mu_n(f^{-1}(E_{B^{h-1}(y,r)}) \cap U) = 6pq\varphi(B^{n-1}(y, r))$. We have

$$(m_1^t(\alpha^*(E_{k,m,p})))^n \le (\sum_{i=1}^{l} d(W_i))^n \le l^{n-1}(\sum_{i=1}^{l} d(W_i)^n) \le$$

$$p^{n-1}\frac{(m_1(f(F_{k,m,p} \cap B_f)) + \epsilon)^{n-1}}{r^{n-1}} \cdot m \cdot (\sum_{i=1}^{l} \mu_n(Q_i)) \le$$

$$6p^n qmV_{n-1}(m_1(f(F_{k,m,p}) \cap B_f)) + \epsilon)^{n-1} \frac{\varphi(B^{n-1}(y, r))}{\mu_{n-1}(B^{n-1}(y, r))}.$$

Letting first $r \to 0$, then $\epsilon \to 0$ and then $t \to 0$, we find that $(m_1^*(\alpha^*(E_{k,m,p})))^n \le 6p^n qmV_{n-1}\varphi'(y)(m_1(f(F_{k,m,p} \cap B_f))^{n-1} \le 6p^n qmV_{n-1}\varphi'(y)m_1(F_0)^{n-1} = 0$.

We found that $m_1^*(\alpha^*(E_{k,m,p})) = 0$ for every $k, m, p \in N$ and since $E_{k,m,p} \nearrow E_{m,p}$, we see that $m_1(\alpha^*(E_{m,p})) = 0$ for every $m, p \in N$.

Let $G = Im\alpha^* \cap B_f \cap E$, $H = (\alpha^*)^{-1}(G)$. Since $J_y \cap f(E)$ is at most countable, the sets G and H are at most countable and we have that $B_f \cap Im\alpha^* = \bigcup_{m,p=1}^{\infty} F_{m,p} \cup G \cup F$, hence $E_0 \subset \bigcup_{m,p=1}^{\infty} E_{m,p} \cup H \cup (\alpha^*)^{-1}(F)$.

From what we have proved before, we see that $m_1(\alpha^*(E_0)) = 0$. Since α^* is absolutely continuous on each interval I_l, we see that $m_1(\alpha^*(A)) = 0$ for every $A \subset I_l$ with $m_1(A) = 0$ and every $l \in N$, and this implies that $m_1(\alpha^*(A)) = 0$ for every $A \subset [0, l(\beta_y)]$ so that $m_1(A) = 0$.

Let $\alpha_1^*, ..., \alpha_n^*$ be the components of α^*. We proved that the components $\alpha_i^* : [0, l(\beta_y)] \to \mathbf{R}^n$ satisfy condition (N) and $\int_0^{l(\beta_y)} |\alpha_i^{*''}(t)|dt < \infty$ for every $i = 1, ..., n$, and using a theorem of Bary (see [29], page 285) we see that $\alpha_1^*, , \alpha_n^*$ are absolutely continuous, and hence α^* is absolutely continuous.

We proved that if $y \in Q_0 \setminus K_5$ and $\alpha : [0, 1] \to U$ is a path so that $f \circ \alpha = \beta_y$, it results that α^* is absolutely continuous. Let now $y \in Q_0 \setminus K_5$ and $\alpha_1, ..., \alpha_q$ be all the paths so that $f \circ \alpha_i = \beta_y, i = 1, ..., q$. Then $f_V \circ \beta_y^0 = \frac{1}{q}\sum_{i=1}^{q} \alpha_i^*$ is absolutely continuous. Since the face Q_0 of Q was arbitrarily chosen, we proved that f_V is ACL.

Let us show now that the map f_V is ACL^n. Let Q and $V_i, i \in N$ as before. Then $f_V(y) = \frac{1}{q}\sum_{j=1}^{q} h_{ij}(y)$ for every $y \in V_i, i \in N$, hence $|f_V^{'}(y)|^n \le \frac{1}{q}\sum_{j=1}^{q} |h_{ij}^{'}(y)|^n$ for $y \in V_i, i \in N$. Let $k \in \{1,...,n\}$. Then $\int_Q |\partial_k f_V^{'}(z)|^n dz \le \int_Q |f_V^{'}(z)|^n dz \le \sum_{i=1}^{\infty}\int_{V_i} |f_V^{'}(z)|^n dz \le \frac{1}{q}\sum_{i=1}^{\infty}\sum_{j=1}^{q}\int_{V_i} |h_{ij}^{'}(z)|^n dz = \frac{1}{q}\sum_{i=1}^{\infty}\sum_{j=1}^{q}\int_{V_i} K_I(f)(f^{-1}(z))|J_{f^{-1}}(z)|dz = \frac{1}{q}\sum_{i=1}^{\infty}\sum_{j=1}^{q}\int_{U_{ij}} K_I(f)(x)dx \le N(n)\int_U K_I(f)(x)dx < \infty$.

We therefore proved that the map f_V is ACL^n on V.

Theorem 2. Let $f : D \to \mathbf{R}^n$ be continuous and light, satisfying conditions (a_1) and (a_2) and let Γ be a path family in D. Then

1) There exists $\Gamma^0 \subset \Gamma$ so that $M(f(\Gamma^0)) = 0$ and for every $\alpha \in \Gamma \setminus \Gamma^0$ it results that the path $\beta = f \circ \alpha$ is rectifiable, α^* is a.e. differentiable and $\int_0^{l(\beta)} |\alpha^{*'}(t)|dt < \infty$.

2) If $\Gamma^1 = \{\alpha \in \Gamma | \beta = f \circ \alpha$ is rectifiable and $m_1(Im\alpha^* \cap (B_f \cup K)) = 0\}$, it results that $M(f(\Gamma^1)) \le M_{K_I(f)}(\Gamma)$, where K is the set from condition (a_1).

Proof. We can suppose that $K = \phi$. Let $D_k \subset D$ be domains so that $D_k \nearrow D$ and let $\Gamma_k = \{\alpha \in \Gamma | Im\alpha \subset D_k\}$ for $k \in N$. Let us fix $k \in N$. Using Vitali's covering theorem, we can find disjoint balls V_i and a set $L \subset \mathbf{R}^n$ so that $\mu_n(L) = 0$ and $f(D_k) \setminus (L \cup f(B_f \cap \bar{D}_k)) = \bigcup_{i=1}^{\infty} V_i$, $f^{-1}(V_i) = \bigcup_{j=1}^{j(i)} U_{ij}$, $f|U_{ij} : U_{ij} \to V_i$ are homeomorphisms having ACL^n inverses $h_{ij} : V_i \to U_{ij}$ for $i \in N$, $j = 1,...,j(i)$. Let $A_{ij} = \{z \in V_i | h_{ij}$ is not differentiable in $z\}$ and $Z_{ij} = \{z \in V_i \setminus A_{ij} | J_{h_{ij}(z)} = 0\}$ for $i \in N, j = 1,...,j(i)$ and let $A = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{j(i)} h_{ij}(A_{ij} \cup Z_{ij})$. Since h_{ij} are ACL^n mappings, they are a.e. differentiable and satisfy condition (N), and using Sard's lemma from [1], we see that $\mu_n(A) = 0$. Since f satisfies condition (N), we see that $\mu_n(f(A)) = 0$, hence, if $B = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{j(i)} A_{ij} \cup Z_{ij}$ and $C = B \cup f(B_f \cap \bar{D}_k) \cup L$, it results that $\mu_n(C) = 0$. Let $\Gamma_k^1 = \Gamma_k \cap \Gamma^1$. We set $\Gamma_k^{ij} = \{\alpha \in \Gamma_k | \beta = f \circ \alpha$ is rectifiable and there exists an interval $J \subset [0, l(\beta)]$ so that $\beta^0(J) \subset V_i$ and $h_{ij} \circ \beta^0$ is not absolutely continuous on $J\}$ for $i \in N$, $j = 1,...,j(i)$. Using Fuglede's theorem, we see that $M(f(\Gamma_k^{ij})) = 0$ for $i \in N, j = 1,...,j(i)$. Let $\Gamma_k^3 = \{\alpha \in \Gamma_k | \beta = f \circ \alpha$ is rectifiable and $m_1(\{t \in [0, l(\beta)] | \beta^0(t) \in C\} > 0)$ and $\Gamma_k^4 = \{\alpha \in \Gamma_k$ and $\beta = f \circ \alpha$ is not rectifiable $\}$ for $k \in N$. Then also $M(f(\Gamma_k^3)) = 0$ and $M(f(\Gamma_k^4)) = 0$.

We define now a Borel map $\eta : \mathbf{R}^n \to [0, \infty]$, $\eta(y) = \sup_{j=1,...,j(i)} |h_{ij}^{'}(y)|$ for $y \in V_i \setminus \bigcup_{j=1}^{j(i)} (A_{ij} \cup Z_{ij})$, $i \in N$, $\eta(y) = 0$ otherwise. Using the change of variable formulae from Theorem 5.23, page 132, in [7], we have

$$\int_{\mathbf{R}^n} \eta^n(y)dy \le \sum_{i=1}^{\infty}\int_{V_i} \eta^n dy \le \sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{V_i} |h_{ij}^{'}(y)|^n dy = \sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{V_i} K_I(f)(f^{-1}(y))|J_{f^{-1}}(y)|dy =$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{U_{ij}} K_I(f)(x)dx \le \int_{\bar{D}_k} K_I(f)(x)dx < \infty.$$

Let $\Gamma_k^5 = \{\alpha \in \Gamma_k | \beta = f \circ \alpha$ is locally rectifiable and $\int\limits_\beta \eta ds = \infty\}$. As in Theorem 1, we see that $M(f(\Gamma_k^5)) = 0$ and let $\Gamma_k^0 = \Gamma_k^3 \cup \Gamma_k^4 \cup \Gamma_k^5 \cup \bigcup\limits_{i=1}^{\infty} \sum\limits_{j=1}^{j(i)} \Gamma_k^{ij}$. Then $M(f(\Gamma_k^0)) = 0$.

Let now $\alpha \in \Gamma_k \setminus \Gamma_k^0$ and let $\beta = f \circ \alpha$. Then β is rectifiable and let $E_0 = \{t \in [0, l(\beta)] \beta^0(t) \in f(B_f \cap Im\alpha^*)\}$, $C_0 = \{t \in [0, l(\beta)] | \beta^0(t) \in C\}$ and $B_0 = \{t \in [0, l(\beta)] | \beta^0$ or α^* is not differentiable in $t\}$. Then $E_0 \subset C_0$, $m_1(B_0 \cup C_0) = 0$ and $(0, l(\beta)) \setminus E_0 = \bigcup\limits_{l=1}^{\infty} I_l$, with $I_l \subset (0, l(\beta))$ open intervals so that $l(\beta) = \sum\limits_{l=1}^{\infty} l(I_l)$. We denote for $i, l \in N$ by I_{lj}, $j \in J_i$ all open maximal intervals I_{lj} from I_l so that $\beta^0(I_{lj}) \subset V_i$ for $j \in J_i$. Then $I_l = \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{j \in J_i} I_{lj}$ and for each such interval I_{lj} we can find $k \in \{1, ..., j(i)\}$ so that $\alpha^* | I_{lj} = h_{ik} o \beta^0 | I_{lj}$.

This implies that α^* is a.e. differentiable on each interval I_{lj}, hence is a.e. differentiable on each interval I_l, $l \in N$ and hence is a.e. differentiable on $[0, l(\beta)]$. If $t \in [0, l(\beta)] \setminus (B_0 \cup C_0)$, we can find $i, l \in N$, $j \in J_i$, $k \in \{1, ..., j(i)\}$ so that $t \in I_{lj}$ and $\alpha^* | I_{lj} = h_{ik} \circ \beta^0 | I_{lj}$. Differentiating in t, we see that $\alpha^{*'}(t)| = |h'_{ik}(\beta^0(t)) \circ \beta^{0'}(t)| \leq |h'_{ik}(\beta^0(t)| \leq \eta(\beta^0(t))$. It results that $|\alpha^{*'}(t)| \leq \eta(\beta^0(t))$ a.e. in $[0, l(\beta)]$ and hence $\int\limits_0^{l(\beta)} |\alpha^{*'}(t)| dt \leq \int\limits_0^{l(\beta)} \eta(\beta^0(t)) dt = \int\limits_\beta \eta ds < \infty$. We also have that $\alpha^{*'}(t) = h'_{ik}(\beta^0(t))(\beta^{0'}(t)) = [f'(\alpha^*(t))]^{-1}(\beta^{0'}(t))$, hence $|\alpha^{*'}(t)| l(f'(\alpha^*(t))) \leq |f'(\alpha^*(t))(\alpha^{*'}(t))| = |\beta^{0'}(t)| = 1$. We proved also that

$$l(f'(\alpha^*(t)) |\alpha^{*'}(t)| \leq 1 \qquad (1)$$

a.e. in $[0, l(\beta)]$.

Suppose now in addition that $\alpha \in \Gamma_k^1 \setminus \Gamma_k^0$. Since $\alpha^*(E_0) \subset B_f \cap Im\alpha^*$, we see that $m_1(\alpha^*(E_0)) = 0$ and since α^* is absolutely continuous on each interval I_l, $l \in N$, we see that $m_1(\alpha^*(G)) = 0$ for every $G \subset [0, l(\beta)]$ with $m_1(G) = 0$. Let $\alpha_1^*, ..., \alpha_n^*$ be the components of α^*. Using Bary's theorem again, we see that $\alpha_1^*, ..., \alpha_n^*$ are absolutely continuous. We proved that if $\alpha \in \Gamma_k^1 \setminus \Gamma_k^0$, then $\beta = f \circ \alpha$ is rectifiable and α^* is absolutely continuous.

Let now $\rho \in F(\Gamma_k^1 \setminus \Gamma_k^0)$. We define $\rho' : \mathbf{R}^n \to [0, \infty]$, $\rho'(y) = \sup\limits_{x \in f^{-1}(y) \cap \bar{D}_k} \frac{\rho(x)}{l(f'(x))}$ if $y \in f(D_k) \setminus C$, $\rho'(y) = 0$ otherwise. Then $\rho'(y) = \sup\limits_{j=1,...,j(i)} \rho(h_{ij}(y)) |h'_{ij}(y)|$ for every $y \in V_i \setminus C$ and every $i \in N$. Since V_i and C are Borel sets, h_{ij} and h'_{ij} are Borel maps on $V_i \setminus C$ for $i \in N$, $j = 1, ..., j(i)$, we see that ρ' is a Borel map, and let us show that $\rho' \in F(f(\Gamma_k^1 \setminus \Gamma_k^0))$. Let $\alpha \in \Gamma_k^1 \setminus \Gamma_k^0$ and $\beta = f \circ \alpha$. Since α^* is absolutely continuous, we see that also s_{α^*} is absolutely continuous and using (1) and a change of variable formulae for absolutely continuous and increasing real functions, we see that

$$\int\limits_\beta \rho' ds = \int\limits_0^{l(\beta)} \rho'(\beta^0(t)) dt = \int\limits_0^{l(\beta)} \rho'(f(\alpha^*(t))) dt \geq \int\limits_0^{l(\beta)} \rho(\alpha^*(t)) / l(f'(\alpha^*(t))) dt \geq$$

$$\int\limits_0^{l(\beta)} \rho(\alpha^*(t)) |\alpha^{*'}(t)| dt = \int\limits_0^{l(\beta)} (\rho \circ (\alpha^*)^0 \circ s_{\alpha^*})(t) s'_{\alpha^*}(t) dt =$$

$$\int_0^{l(\beta)}(\rho\circ\alpha^0\circ s_{\alpha^*})(t)s^{'}_{\alpha^*}(t)dt=\int_0^{l(\alpha)}\rho(\alpha^0(t))dt=\int_\alpha\rho ds\geq 1.$$

We proved that $\rho^{'}\in F(f(\Gamma_k^1\setminus\Gamma_k^0))$. We have now that

$$M(f(\Gamma_k^1\setminus\Gamma_k^0))\leq\int_{\mathbf{R}^n}\rho^{'n}(y)dy=\sum_{i=1}^{\infty}\int_{V_i}\rho^{'n}(y)dy\leq$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{V_i}\rho^n(h_{ij}(y))|h^{'}_{ij}(y)|^n dy=\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{V_i}\rho^n(f^{-1}(y))K_I(f)(f^{-1}(y))|J_{f-1}(y)|dy=$$

$$\sum_{i=1}^{\infty}\sum_{j=1}^{j(i)}\int_{U_{ij}}\rho^n(x)K_I(f)(x)dx\leq\int_{\mathbf{R}^n}\rho^n(x)K_I(f)(x)dx.$$

Since $\rho\in F(\Gamma_k^1\setminus\Gamma_k^0)$ was arbitrary chosen, we see that $M(f(\Gamma_k^1\setminus\Gamma_k^0))\leq M_{K_I(f)}(\Gamma_k^1\setminus\Gamma_k^0))\leq M_{K_I(f)}(\Gamma)$. We take now $\Gamma^0=\bigcup_{k=1}^{\infty}\Gamma_k^0$. Then $M(f(\Gamma^0))=0$ and since $\Gamma_k^1\setminus\Gamma_k^0\nearrow\Gamma^1\setminus\Gamma^0$, we use Ziemer's lemma to see that $M(f(\Gamma^1\setminus\Gamma^0))=\lim_{k\to\infty}M(f(\Gamma_k^1\setminus\Gamma_k^0))\leq M_{K_I(f)}(\Gamma)$. We have now that $M(f(\Gamma^1))\leq M(f(\Gamma^1\setminus\Gamma^0))+M(f(\Gamma^0))=M(f(\Gamma^1\setminus\Gamma^0))\leq M_{K_I(f)}(\Gamma)$.

Corollary 1. Let $D\subset\mathbf{R}^2$ be open, $f:D\to\mathbf{R}^2$ continuous and light so that f satisfies condition (N), $K_I(f)\in L^1_{loc}(D)$ and f has local ACL^n inverses on $D\setminus B_f$. Then $M(f(\Gamma))\leq M_{K_I(f)}(\Gamma)$ for every path family Γ from D.

Remark 3. If $f:D\to\mathbf{R}^n$ is continuous and light and f satisfies conditions (a_1) and (a_2) (with $K=\phi$) and $m_1(B_f)=0$, then $m_1(B_f\cap Im\alpha)=0$ for every path family Γ from D.

Remark 4. Let S be a surface in $\mathbf{R}^n$ or S^n and let $\Gamma^{'}=\{\gamma_y:[0,\infty)\to\mathbf{R}^n||\gamma_y(t)=y+te_n,t\geq 0,y\in S\}$ or $\Gamma^{'}=\{\gamma_y:[0,\infty)\to\mathbf{R}^n|\gamma_y(t)=(1+t)y,t\geq 0,y\in S^n\}$. Then $\Gamma^{'}$ is a line family and let $f:D\to\mathbf{R}^n$ be continuous and light and satisfying conditions $(a_1),(a_2),(a_3)$. As in Theorem 1, we show that for a.e. paths $\beta\in\Gamma^{'}$, every path α in D so that $\beta=f\circ\alpha$ is so that α^* is absolutely continuous. If in addition f is open, then f locally lifts the paths, hence if $\beta:[0,1]\to\mathbf{R}^n$ is a path and $x\in D$ is so that $f(x)=\beta(0)$, then there exists a maximal lifting of β starting from x. This implies that if $\Gamma^{'}$ is the same line family as before and Γ is the family of all maximal liftings of the path from $\Gamma^{'}$ starting from some points $a_y\in D$ so that $f(a_y)=y,y\in S$, then $\Gamma^{'}>f(\Gamma)$ and if $\Gamma^{'}=\{\beta_y:[0,1]\to\mathbf{R}^n|y\in S\}$, $\Gamma=\{\alpha_y:[0,c_y)\to D$ path $|0\leq c_y\leq 1,\alpha_y(0)=a_y,f\circ\alpha_y=\beta_y|[0,c_y),y\in S\}$, then α_y^* is absolutely continuous for a.e. $y\in S$. Then $m_1(B_f\cap Im\alpha^*)=0$ for a.e. $y\in S$ and from Theorem 2 we see that $M(\Gamma^{'})\leq M_{K_I(f)}(\Gamma)$.

It results that for some special line families $\Gamma^{'}$ and paths families Γ of maximal liftings of the paths from $\Gamma^{'}$, the modular inequality of Poleckii "$M(\Gamma^{'})\leq M_{K_I(f)}(\Gamma)$" holds for continuous, open, light mappings satisfying conditions $(a_1),(a_2),(a_3)$. Note that a result of Wilson [35] shows that there exist continuous, open, light mappings $f:D\to\mathbf{R}^n$ so that $B_f=D$, hence the class of continuous, open, discrete mappings $f:D\to\mathbf{R}^n$ is strictly included in the class of continuous, open, light mappings $f:D\to\mathbf{R}^n$.

Theorem 3. (Modular inequality of Poleckii) Let $f:D\to\mathbf{R}^n$ be continuous, open, discrete, satisfying conditions $(a_1),(a_2),(a_3)$ on D. Then $M(f(\Gamma))\leq M_{K_I(f)}(\Gamma)$ for every path family Γ from D.

Proof. We can take $K = \phi$ in conditions (a_1) and (a_3) and we can suppose that f is sense preserving on D. Let $B_k = \{x \in B_f | i(f,x) \geq k\}$ for $k \in N, k \geq 2$. If $x \in B_k$ and $U \subset\subset D$ is a normal domain so that $\bar{U} \cap f^{-1}(f(x)) = \{x\}$ and $V = f(U)$, then $d(f, U, V) = k$, hence $\bar{U} \cap B_k = f^{-1}(f(\bar{U} \cap B_k))$ and this implies that f is injective on $\bar{U} \cap B_k$. We cover now each set B_k with normal domains U_{kj} so that $U_{kj} \cap B_k \neq \phi$, and let $V_{kj} = f(U_{kj})$ for $k \geq 2, j \in J_k$. Let Γ be a path family in D. Let $\Gamma_0 = \{\alpha \in \Gamma | \beta = f \circ \alpha$ is not rectifiable $\}$, $\Gamma_1 = \{\alpha \in \Gamma | \beta = f \circ \alpha$ is rectifiable and $m_1(\{t \in [0, l(\beta)] | \beta^0(t) \in f(B_f)\}) > 0\}$, $\Gamma_2^{kj} = \{\alpha \in \Gamma | \beta = f \circ \alpha$ is rectifiable and there exists an interval $J \subset [0, l(\beta)]$ so that $\beta^0(J) \subset V_{kj}$ and $f_{V_{kj}} \circ \beta^0$ is not absolutely continuous on $J\}$ for $k \geq 2, j \in J_k$. Let $\tilde{\Gamma} = \Gamma_0 \cup \Gamma_1 \cup \bigcup_{k=2}^{\infty} \bigcup_{j \in J_k} \Gamma_2^{kj}$. Using Theorem 1 and Fuglede's theorem, we see that $M(f(\tilde{\Gamma})) = 0$.

Let $\alpha \in \Gamma \setminus \tilde{\Gamma}, \beta = f \circ \alpha$ and let us fix $k \geq 2$. Then $(\alpha^*)^{-1}(B_k \cap Im\alpha^* \cap U_{kj}) \subset (\beta^0)^{-1}(f(B_k \cap Im\alpha^* \cap U_{kj})) \subset (\beta^0)^{-1}(f(B_f))$, and since $m_1((\beta^0)^{-1}(f(B_f))) = 0$, we see that $m_1((\alpha^*)^{-1}(B_k \cap Im\alpha^* \cap U_{kj})) = 0$ for $j \in J_k$. Now, the map $f_{V_{kj}} \circ \beta^0$ is absolutely continuous on each interval $J \subset [0, l(\beta)]$ so that $\beta^0(J) \subset V_{kj}$, so that if A is a subset of such an interval J and $m_1(A) = 0$, it results that $m_1((f_{V_{kj}} \circ \beta^0)(A)) = 0$. Since $B_k \cap Im\alpha^* \cap U_{kj} = (f_{V_{kj}} \circ f)(B_k \cap Im\alpha^* \cap U_{kj}) = (f_{V_{kj}} \circ f \circ \alpha^*)((\alpha^*)^{-1}(B_k \cap Im\alpha^* \cap U_{kj})) = (f_{V_{kj}} \circ \beta^0)((\alpha^*)^{-1}(B_k \cap Im\alpha^* \cap U_{kj}))$ for $j \in J_k$, this implies that $m_1(B_k \cap Im\alpha^* \cap U_{kj}) = 0$ for every $k \geq 2$ and every $j \in J_k$. Since $B_f \subset \bigcup_{k=2}^{\infty} B_k \subset \bigcup_{k=2}^{\infty} \bigcup_{j \in J_k} (U_{kj} \cap B_k)$, we proved that $m_1(B_f \cap Im\alpha^*) = 0$ for every path $\alpha \in \Gamma \setminus \tilde{\Gamma}$. Using Theorem 2, we see that

$$M(f(\Gamma)) \leq M(f(\Gamma \setminus \tilde{\Gamma})) + M(f(\tilde{\Gamma})) = M(f(\Gamma \setminus \{\tilde{\Gamma})) \leq M_{K_I(f)}(\Gamma).$$

Proposition 2. Let $D \subset \mathbf{R}^n$ be open, $p > 1$, $\omega \in L^1_{loc}(D)$ and $E = (A, C)$ be a condenser in $\mathbf{R}^n$. Then $M^p_\omega(\Gamma_E) = cap^p_\omega(E)$.

Proof. If u is admissible for $cap^p_\omega(E)$, then $\rho = |\nabla u| \in F(\Gamma_E)$, hence $M^p_\omega(\Gamma_E) \leq \int_{\mathbf{R}^n} \omega(x)|\nabla u|^p(x)dx$, and this implies that $M^p_\omega(\Gamma_E) \leq cap^p_\omega(E)$.

We show now that $cap^p_\omega(E) \leq M^p_\omega(\Gamma_E)$. We can presume that $M^p_\omega(\Gamma_E) < \infty$ and let $\epsilon > 0$. Using Lemma 4, we can find $\rho \in F(\Gamma_E)$ lower semicontinuous so that $[||\rho||^p_\omega]^p \leq M^p_\omega(\Gamma_E) + \epsilon$. We take $A_k \subset\subset A$ open sets so that $C \subset A_k$ for every $k \in N$ and $A_k \nearrow A$ and we set $\rho_k = \min\{\rho\chi_{A_k}, k\}$ for $k \in \mathbf{N}$. We define $u_k : A \to \mathbf{R}$ by $u_k(x) = \inf \int_\alpha \rho_k ds$, where the infimum is taken over all rectifiable paths $\alpha : [0, 1) \to A$ so that $\alpha(0) = x$ and α has at least a limit point in ∂A, for $x \in A$ and $k \in N$. We see from Proposition 10.2, page 54 in [24] that $|\nabla u_k| \leq \rho_k$ a.e.

Let $d_k = \inf_{x \in C} u_k(x)$ for $k \in N$. We see from the proof of Proposition 10.2, page 54, in [24] that $\liminf_{k \to \infty} d_k \geq 1$. Then $v_k = \frac{u_k}{d_k}$ is admissible for $cap^p_\omega(E)$ for every $k \in N$, hence $cap^p_\omega(E) \leq \liminf_{k \to \infty} \int_{\mathbf{R}^n} \omega(x)|\nabla v_k|^p(x)dx \leq \liminf_{k \to \infty} \frac{1}{d_k^p} \int_{\mathbf{R}^n} \omega(x)\rho_k(x)^p dx \leq \int_{\mathbf{R}^n} \omega(x)\rho^p(x)dx \leq M^p_\omega(\Gamma_E) + \epsilon$. Letting $\epsilon \to 0$, we find that $cap^p_\omega(E) \leq M^p_\omega(\Gamma_E)$.

Corollary 2. Let $f : D \to \mathbf{R}^n$ be continuous, open, light, satisfying condition (a) and let $E = (A, C)$ be a condenser in D. Then $capf(E) \leq cap_{K_I(f)}(E)$.

Proof. Since f is an open map, we see that $f(E) = (f(A), f(C))$ is a condenser and since f locally lifts the maps, we see that $\Gamma_{f(E)} > f(\Gamma_E)$. Then $capf(E) = M(\Gamma_{f(E)}) \leq M(f(\Gamma_E)) \leq M_{K_I(f)}(\Gamma_E) = cap_{K_I(f)}(E)$.

4. Geometric properties of open, discrete mappings having local ACL^n inverses

Theorem 4. (Schwarz's lemma) Let $n \geq 2$, $f : B^n \to B^n$ with $f(0) = 0$, f continuous, open, light, a.e differentiable with $J_f(x) \neq 0$ a.e. and satisfying condition (a). Suppose that

1) There exists $0 \leq \alpha < n-1$ and $M > 0$ so that $\int_{B(0,r)} K_I(f)(x)dx \leq M(\ln(\frac{1}{r}))^\alpha$ for every $0 < r < 1$.

Then there exists $\varphi : (0,1) \to (0,1)$ continuous, increasing, so that $\lim_{t \to 0} \varphi(t) = 0$, $\lim_{t \to 1} \varphi(t) = 1$ and $|f(x)| \leq \varphi(|x|)$ for every $x \in B^n$.

Proof. Let $\nu_n : (0,1) \to (0,1)$ be defined by $\nu_n(r) = cap(B^n, [0, re_1])$ for $0 < r \leq 1$. We see from Lemma 1.2, page 60 in [24] that ν_n is strictly increasing and $\lim_{r \to 0} \nu_n(r) = 0$, $\lim_{r \to 1} \nu_n(r) = \infty$. Let $x \in B^n$, $E = (B^n, [0,x])$ and $E^{‘} = f(E) = (f(B^n), f([0,x]))$. Since f is an open map, $E^{‘}$ is also a condenser, and let $\Gamma^{‘} = \{\gamma : [0,1] \to f(B^n)$ path $|\gamma(0) \in f([0,x])$ and γ has at least a limit point in $\partial f(B^n)\}$. Then $M(\Gamma^{‘}) = capE^{‘}$ and let Γ be the family of all maximal liftings of the paths from $\Gamma^{‘}$ starting from the points of $[0,x]$. Then $\Gamma^{‘} > f(\Gamma)$ and every path $\gamma \in \Gamma$ has at least a limit point in S^n. Using condition (a) and Theorem 2 from [5], we have that $\nu_n(|f(x)|) \leq capE^{‘} = M(\Gamma^{‘}) \leq M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma) \leq (\frac{C(n)}{\ln \ln(\frac{e}{|x|})})^n$, where $C(n)$ is a constant depending only on n. We take now $\varphi : (0,1) \to (0,1)$ defined by $\varphi(t) = \nu_n^{-1}(\frac{C(n)}{(\ln \ln(\frac{e}{t}))^n})$ for $t \in (0,1)$.

Remark 1. Condition 1) from Theorem 4 is just condition $c_2)$ for $x = 0$ and $a = 1$. We can replace this condition by one of the conditions $c_1), c_3), c_4)$ for $x = 0$ and $a = 1$, obtaining a different function $\varphi : (0,1) \to (0,1)$ with the properties from Theorem 4.

Theorem 5. (Modulus of continuity) Let $n \geq 2, x \in D$, $r_0 > 0$ so that $\bar{B}(x, r_0) \subset D$, $f : D \to \mathbf{R}^n$ be continuous, open, light a.e. differentiable with $J_f(x) \neq 0$ a.e. in D, satisfying condition (a) and suppose that one of the conditions $c_1), c_2), c_3), c_4)$ hold in x for $a = r_0$. Then $|f(y) - f(x)| \leq d(f(\bar{B}(x, r_0))\varphi(\frac{|y-x|}{r_0})$ for every $y \in \bar{B}(x, r_0)$, where φ is the function from Theorem 4.

Proof. We apply Theorem 4 to the function $g : B^n \to B^n$ defined by $g(z) = \frac{f(x+r_0 z) - f(x)}{d(f(B(x,r_0))}$ for $z \in B^n$.

Theorem 6. (Liouville's theorem) Let $n \geq 2, f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. in D, satisfying condition (a) and suppose that one of the conditions $c_1), c_2), c_3), c_4)$ holds for $x = 0$ and every $a > 0$. Let $M(r) = \sup_{x \in \bar{B}(0,r)} |f(x) - f(0)|$, $\lambda : (0, \infty) \to (0, \infty)$ be so that $r < \lambda(r)$ for every $r > 0$ and $\lim_{r \to \infty} \frac{r}{\lambda(r)} = 0$ and suppose that $\lim_{r \to \infty} M(\lambda(r))\varphi(\frac{r}{\lambda(r)}) = 0$. Then f is constant on $\mathbf{R}^n$, hence, if f is bounded on $\mathbf{R}^n$, it results that f is constant on $\mathbf{R}^n$.

Proof. Let $x \in \mathbf{R}^n$ be fixed and $r > 0$ so that $|x| < r < \lambda(r)$. Then $|f(x) - f(0)| \leq M(\lambda(r))\varphi(\frac{|x|}{\lambda(r)}) \leq M(\lambda(r))\varphi(\frac{r}{\lambda(r)}) \to 0$ if $r \to \infty$. It results that $f(x) = f(0)$ for every $x \in \mathbf{R}^n$. If f is bounded on $\mathbf{R}^n$, we take $\lambda(r) = r^2$.

Theorem 7. (Picard's theorem) Let $n \geq 2$, $E \subset \mathbf{R}^n$ be closed, $f : \mathbf{R}^n \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, light a.e. differentiable with $J_f(x) \neq 0$ a.e. in D, satisfying condition (a) and so that $M_{K_I(f)}(E \cup \{\infty\}) = 0$. Then $cap(\mathcal{C}f(\mathbf{R}^n \setminus E)) = 0$.

Proof. Suppose that $cap(\mathcal{C}f(\mathbf{R}^n \setminus E)) > 0$ and let $K \subset \mathbf{R}^n \setminus E$ be compact so that $CardK > 1$. Then $\Delta = (f(\mathbf{R}^n \setminus E), f(K))$ is a condenser and we see from Lemma 2.6, page 65 in [24] that there exists $\delta > 0$ so that $\delta < cap\Delta$. Let $\Gamma' = \Delta(f(K), (f(\mathbf{R}^n \setminus E), \overline{\mathbf{R}}^n)$ and let let Γ be the family of all maximal liftings of the path from Γ' starting from K. Then $\Gamma' > f(\Gamma)$, $M(\Gamma') = cap\Delta$ and every path $\gamma \in \Gamma$ has at least a limit point in $E \cup \{\infty\}$. Then $M_{K_I(f)}(\Gamma) = 0$ and since f satisfies condition (a), we have $\delta \leq cap\Delta = M(\Gamma') \leq M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma) = 0$, and we reached a contradiction.

Theorem 8. (Equicontinuity result) Let $n \geq 2$, $M \subset \mathbf{R}^n$ with $cap\bar{M} > 0$ and let W be a family of continuous, open, light mappings $f : D \to \overline{\mathbf{R}}^n \setminus M$, a.e. differentiable with $J_f(x) \neq 0$ a.e. on D and satisfying condition (a). Suppose that there exists $\omega \in L^1_{loc}(D)$ so that $K_I(f) \leq \omega$ for every $f \in W$ and let $x \in D$ be so that $M_\omega(x) = 0$. Then the family W is equicontinuous in x, and we take the Euclidian distance on D and the chordal distance on $\overline{\mathbf{R}}^n$.

Proof. Let $\epsilon > 0$ be so that $\bar{B}(x, \epsilon) \subset D$. Suppose that there exists $\rho > 0$, $r_p \to 0$ and $f_p \in W$ so that $q(f_p(\bar{B}(x, r_p))) \geq \rho$ for every $p \in N$. Since $Imf_p \cap M = \phi$ and Imf_p are open sets, we see that $Imf_p \cap \bar{M} = \phi$ for every $p \in N$, hence $f_p(\bar{B}(x, r_p)) \cap \bar{M} = \phi$ for every $p \in N$. Let $\Gamma'_p = \Delta(f_p(\bar{B}(x, r_p), \partial f_p(B(x, \epsilon)), f_p(B(x, \epsilon)) \setminus f_p(\bar{B}(x, r_p)))$ and Γ_p the family of all maximal liftings of the paths from Γ'_p starting from the points of $\bar{B}(x, r_p)$ for every $p \in N$. Then $\Gamma'_p > f(\Gamma_p)$ and we see from Lemma 2.6, page 65 in [24] that there exists $\delta > 0$ so that $\delta \leq cap(C\bar{M}, f_p(\bar{B}(x, r_p))$ for every $p \in N$. Let $\Delta_p = \Delta(\bar{B}(x, r_p), S(x, \epsilon), (B(x, \epsilon) \setminus \bar{B}(x, r_p))$ for $p \in N$. Since every path $\gamma \in \Delta_p$ has at least a limit point outside $B(x, \epsilon)$, we see that $\Gamma_p > \Delta_p$ for every $p \in N$, and since $M_\omega(x) = 0$, we see from Lemma 7 that $\lim_{p \to \infty} M_\omega(\Delta_p) = 0$. We have

$$\delta \leq cap(C\bar{M}, f_p(\bar{B}(x, r_p))) \leq cap(f_p(B(x, \epsilon)), f_p(\bar{B}(x, r_p))) =$$

$$M(\Gamma'_p) \leq M(f(\Gamma_p)) \leq M_{K_I(f)}(\Gamma_p) \leq M_{K_I(f)}(\Delta_p) \leq M_\omega(\Delta_p) \to 0$$

for $p \to \infty$.

We reached a contradiction, hence the family W is equicontinuous in x.

Remark 6. The preceding theorem extends a classical result from the theory of quasiregular mappings (see Corollary 2.7, page 66 in [24]), and brings something new even in the case when all the mappings from the family W are quasiregular mappings, since the exceptional set M which is avoided by every map $f \in W$ can be chosen at most countable and so that $cap\bar{M} > 0$.

Theorem 9. (Montel) Let $n \geq 2$, W be a bounded family of continuous, open, light mappings $f : D \to \mathbf{R}^n$, a.e. differentiable with $J_f(x) \neq 0$ a.e. on D, satisfying condition (a) and so that there exists $\omega \in L^1_{loc}(D)$ so that $K_I(f) \leq \omega$ for every $f \in W$. Then, if $M_\omega(x) = 0$ for every $x \in D$, it results that W is a normal family.

The following eliminability result extends a classical one from the theory of quasiregular mappings (see Theorem 2.9, page 66 in [24]) and a result from [3] established for mappings of finite distortion and satisfying condition $(\mathcal{A})$.

Theorem 10. (Eliminability result) Let $n \geq 2$, $E \subset D$ be closed in D, $x \in E$, $f : D \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a), so that $K_I(f) \in L^1_{loc}(D \setminus f^{-1}(\infty))$ and $M_{K_I(f)}(E) = 0$. Suppose that there exists $r_x > 0$ so that $\bar{B}(x, r_x) \subset D$ and $cap(\mathcal{C}(f(B(x, r_x) \setminus E))) > 0$. Then we can extend f by continuity in x.

Proof. Suppose that f is not continuous in x. Then $x \notin f^{-1}(\infty)$ and we can find $x_j \to x, y_j \to x$, $b_1 \neq b_2$ so that $f(x_j) \to b_1, f(y_j) \to b_2$, $x_j, y_j \in D \setminus E$ for $j \in N$, and let $r_j = \max\{2|x - x_j|, 2|y - x_j|\}$ for $j \in N$. Since $capE = 0$, it results that E is nowhere disconnecting and let C_j be compact and connected joining x_j with y_j in $B(x, r_j) \setminus E$ for $j \in N$. Let $E_j = (B(x, r_x) \setminus E, C_j), E'_j = f(E_j) = (f(B(x, r_x) \setminus E), f(C_j))$, $\Gamma'_j = \{\gamma : [0, 1) \to f(B(x, r_x) \setminus E)$ path $| \gamma(0) \in f(C_j)$ and γ has at least a limit point in $\partial f(B(x, r_x) \setminus E)\}$, and let

Γ_j be the family of all maximal liftings of the paths from $\Gamma_j^{'}$ starting from the points from C_j, for $j \in N$. Let $\Gamma_{1j} = \{\gamma \in \Gamma_j |$ has at least a limit point in $E\}$ and $\Gamma_{2j} = \{\gamma \in \Gamma_j | \gamma$ has at least a limit point outside $S(x, r_x)\}$ for $j \in N$. Then $capE_j^{'} = M(\Gamma_j^{'}), \Gamma_j^{'} > f(\Gamma_j)$, $\Gamma_j = \Gamma_{1j} \cup \Gamma_{2j}$, $M_{K_I(f)}(\Gamma_{1j}) = 0$ and we see from Lemma 2.6, page 65 in [24] that there exists $\delta > 0$ so that $\delta \leq capE_j^{'}$ for every $j \in N$. Let $\Delta_j = \Delta(\bar{B}(x, r_j), S(x, r_x), (B(x, r_x) \setminus \bar{B}(x, r_j))$ for $j \in N$. Then $\Gamma_{2j} > \Delta_j$ for $j \in N$ and since $M_{K_I(f)}(x) = 0$, we see from Lemma 7 that $M_{K_I(f)}(\Delta_j) \to 0$ for $j \to \infty$. We have

$$\delta \leq capE_j^{'} = M(\Gamma_j^{'}) \leq M(f(\Gamma_j)) \leq M_{K_I(f)}(\Gamma_j) = M_{K_I(f)}(\Gamma_{1j} \cup \Gamma_{2j}) \leq$$

$$M_{K_I(f)}(\Gamma_{1j}) + M_{K_I(f)}(\Gamma_{2j}) = M_{K_I(f)}(\Gamma_{2j}) \leq M_{K_I(f)}(\Delta_j) \to 0$$

if $j \to \infty$.

We reached a contradiction, hence we can extend f continuously in x.

Theorem 11. Let $n \geq 2$, $E \subset D$ closed in D, $x \in E$, $f : D \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a), so that $K_I(f) \in L^1_{loc}(D \setminus f^{-1}(\infty))$ and $M_{K_I(f)}(E) = 0$. Suppose that $\lim_{y \to x} f(y)$ does not exists in $\overline{\mathbf{R}}^n$ (i.e. x is an essential singularity of f). Then $cap(\mathcal{C}(f(B(x, r) \setminus E)) = 0$ for every $r > 0$.

Theorem 12. Let $n \geq 2$, $E \subset D$ closed in D, $f : D \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, discrete, a.e. differentiable with $J_f(x) \neq 0$, a.e. on $D \setminus E$, satisfying condition (a) and so that $M_{K_I(f)}(E) = 0$. Suppose that for every $x \in E$ there exists $r_x > 0$ so that $\bar{B}(x, r_x) \subset D$, $K_I(f) \in L^1_{loc}(B(x, r_x) \setminus f^{-1}(\infty))$ and $cap(\mathcal{C}f(B(x, r_x) \setminus E)) > 0$. Then f extends continuously to a map $F : D \to \overline{\mathbf{R}}^n$ and if $\mu_n(F(E)) = 0$, then F is open, discrete on D.

Proof. We see from Theorem 10 that f extends to a continuous map $F : D \to \overline{\mathbf{R}}^n$. Since $capE = 0$ and $(D \setminus E) \cap F^{-1}(\infty)$ is at most countable, we see that F is a light map, that $D \setminus (E \cup F^{-1}(\infty))$ is a domain and $i(F, \cdot)$ is constant on $D \setminus (E \cup F^{-1}(\infty))$. We use now Theorem 1 from [2] to see that F is open and discrete on D.

Theorem 13. Let $n \geq 2$, $E_0 \subset D$ be closed in D, $f : D \setminus E_0 \to \overline{\mathbf{R}}^n$ be continuous, open, discrete, satisfying conditions $(a_1), (a_2), (a_3)$ on $D \setminus (E_0 \cup f^{-1}(\infty))$ and so that $M_{K_I(f)}(E_0) = 0$. Suppose that for every $x \in E_0$, there exists $r_x > 0$ so that $\bar{B}(x, r_x) \subset D$, $K_I(f) \in L^1_{loc}(B(x, r_x) \setminus f^{-1}(\infty))$ and $cap(\mathcal{C}f(B(x, r_x) \setminus E_0)) > 0$. Then f extends continuously to a map $f_0 : D \to \overline{\mathbf{R}}^n$, and if $\mu_n(f_0(E_0)) = 0$, then f_0 is open, discrete and the set E_0 is eliminable for f.

Proof. We see from Theorem 12 that there exists $f_0 : D \to \overline{\mathbf{R}}^n$ continuous so that $f_0 | D \setminus E_0 = f$, and if $\mu_n(f_0(E_0)) = 0$, then f_0 is open and discrete on D. If K and F are the sets from conditions (a_1) and (a_3) for the map f, then, since $m_1(E_0) = 0$, we can replace them by the sets $K \cup E_0$, respectively $F \cup E_0$, and we see that the map f_0 satisfies conditions $(a_1), (a_2), (a_3)$ on $D \setminus f^{-1}(\infty)$ with these replaced sets.

Remark 7. The eliminability of the "singular" set E_0 implies that f_0, the continuous extension of f on D, satisfies the modular inequality pf Poleckii "$M(f(\Gamma)) \leq M_{K_I(f)}(\Gamma)$" for every path family Γ from D.

We begin now to study the boundary behavior of our new class of mappings satisfying the modular inequality of Poleckii. We shall see as in the classical theory that this is an important tool in establishing some basic properties of this type. We first give an extension to a result proved by M. Vuorinen in [32] for closed quasiregular mappings and by M. Cristea in [3] for mappings of finite distortion and satisfying condition $(\mathcal{A})$.

Theorem 14. Let $n \geq 2$, $D, D^{'}$ be domains in $\mathbf{R}^n$, $f : D \to D^{'}$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on D, satisfying condition (a) and so that $K_I(f) \in L^1_{loc}(D)$. Let $b \in \partial D$ be so that D is locally connected in b, $M_{K_I(f)}(b) = 0$, $C(f, b) \subset D^{'}$ and

$C(f,b)$ has property P_2 in at least one of his points, and suppose that there exists $\rho > 0$ so that $A(f,x) \subset \partial D'$ for every $x \in B(b,\rho) \cap \partial D$. Then f extends continuously in b.

Proof. Suppose that there exists $b_1, b_2 \in C(f,b)$, $b_1 \neq b_2$, and D' has property P_2 in b_1. Let $r_j \to 0$, $U_j \in \mathcal{V}(b)$ be so that $U_{j+1} \subset U_j \subset B(b,r_j)$ and $U_j \cap D$ is connected for every $j \in N$. Let $F \subset D'$ be compact. Since $C(f,b) \subset \partial D'$, we can suppose that $f^{-1}(F) \cap B(b,\rho) \neq \phi$ and we can suppose that $\overline{f(U_j \cap D)} \cap F = \phi$ for every $j \in N$. Let $\Gamma'_j = \{\gamma : [0,1) \to D$ path $|\gamma(0) \in f(U_j \cap D)$ and γ has at least a limit point in $F\}$ and let Γ_j be the family of all maximal liftings of the paths from Γ'_j starting from the points of $U_j \cap D$ for $j \in N$. Let $\Gamma_{1j} = \{\gamma \in \Gamma_j | \gamma$ has at least a limit point in $B(0,\rho) \cap \partial D\}$ and $\Gamma_{2j} = \{\gamma \in \Gamma_j | Im\gamma \cap S(b,\rho) \neq \phi\}$ for $j \in N$. Then $\Gamma_j = \Gamma_{1j} \cup \Gamma_{2j}$, $\Gamma'_j > f(\Gamma_j)$ and since D' has property P_2 in b_1 and $\overline{f(U_j \cap D)}$ is connected, we can find $\delta > 0$ so that $\delta \leq M(\Gamma'_j)$ for $j \in N$. Let now $\gamma : [0,1) \to D, \gamma \in \Gamma^r_{1j}$. Since γ is rectifiable, there exists $x = \lim_{t \to 1} \gamma(t) \in B(b,\rho) \cap \partial D$, and let $l \in A(f,x)$. Since $A(f,x) \subset \partial D'$ for every $x \in B(b,\rho) \cap \partial D$, we see that $l \in \partial D'$. On the other side, $f \circ \gamma$ is a subpath of a path from Γ'_j, hence $l \in D'$ and we reached a contradiction. It results that $\Gamma^r_{1j} = \phi$ for $j \in N$, and from Lemma 3 we see that $M_{K_I(f)}(\Gamma_{1j}) = 0$ for every $j \in N$. Let $\Delta_j = \Delta(\bar{B}(b,r_j) \cap D, S(b,\rho) \cap D, (B(b,\rho) \setminus (\bar{B}(b,r_j)) \cap D)$ for $j \in N$. Then $\Gamma_{2j} > \Delta_j$ for $j \in N$ and since $M_{K_I(f)}(b) = 0$, we see from Lemma 7 that $\lim_{j \to \infty} M_{K_I(f)}(\Delta_j) = 0$. We have

$$\delta \leq M(\Gamma'_j) \leq M(f(\Gamma_j)) \leq M_{K_I(f)}(\Gamma_j) =$$

$$M_{K_I(f)}(\Gamma_{1j} \cup \Gamma_{2j}) \leq M_{K_I(f)}(\Gamma_{1j}) + M_{K_I(f)}(\Gamma_{2j}) = M_{K_I(f)}(\Gamma_{2j}) \leq M_{K_I(f)}(\Delta_j) \to 0$$

if $j \to \infty$.

We reached a contradiction. It results that f extends continuously in x.

The following result was proved for plane meromorphic functions by K. Noshiro [21], for quasiregular mappings by O. Martio and S. Rickman in [17], and for mappings of finite distortion and satisfying condition $(\mathcal{A})$ by M. Cristea in [3].

Theorem 15. Let $n \geq 2$, $E \subset \partial D$, $f : D \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on D, satisfying condition (a) and so that $M_{K_I(f)}(E) = 0$. Let $x \in (\partial D \setminus E)'$ and $z \in (C(f,x) \setminus (C(f,x,\partial D \setminus E) \cup (\bigcap_{r>0} f(B(x,r) \cap D)))$. Then either $x \in E$ and $z \in A(f,x)$, or there exists $x_k \in E$, $x_k \to x$ so that $z \in A(f,x_k)$ for every $k \in N$.

Proof. We can suppose that $z \in \mathbf{R}^n$ and let $r_k \searrow 0$ be so that $S(x,r_x) \cap E = \phi$ for every $k \in N$. Let $F_k = C(f, \bar{B}(x,r_k) \cap ((\partial D \setminus E) \setminus \{x\}))$ for $k \in N$. Then $F_{k+1} \subset F_k$ for every $k \in N$ and $C(f,x,\partial D \setminus E) = \bigcap_{k \in N} \bar{F}_k$. Since $2\alpha = d(z, C(f,x,\partial D \setminus E)) > 0$, we can suppose that $\alpha < d(z,\bar{F}_k)$ for every $k \in N$. Let $\rho_k = d(z, f(S(x,r_k) \cap D)$ for $k \in N$. Let $k \in N$ be fixed and suppose that $\rho_k = 0$. We can find $a_{kj} \in S(x,r_k) \cap D$ so that $f(a_{kj} \to z$ and extracting if necessary a subsequence, we can presume that there exists $a_k \in S(x,r_k)$ so that $a_{kj} \to a_k$. If $a_k \in S(x,r_k) \cap \partial D$, then $a_k \in S(x,r_k) \cap (\partial D \setminus E)$, hence $z \in C(f,a_k) \subset F_k$, which contradicts the fact that $d(z,\bar{F}_k) > \alpha > 0$. It results that $a_k \in S(x,r_k) \cap D)$ and hence $f(a_k) = z$. Since $z \not\in \bigcap_{r>0} f(B(x,r) \cap D)$ and $a_k \to x$, we see that there exists $k_0 \in N$ so that $\rho_k > 0$ for every $k \geq k_0$, and we can suppose that $\rho_k > 0$ for every $k \in N$. Since $z \in C(f,x)$, there exists $\alpha_k \in B(x,r_k) \cap D$ so that $f(\alpha_k) \to z$, and since f is an open map, we can find $0 < r'_k < \rho_k$, $C_k \subset S(z,r'_k)$ a cap of the sphere $S(z,r'_k)$ and $Q_k \subset B(x,r_k) \cap D$ connected so that $f(Q_k) = C_k$ for $k \in N$.

Let us fix $k \in N$. We denote for $y \in C_k$ and $i \in N$ by $\gamma_{yi} : [0, 1-\frac{1}{i}] \to B(z,r'_k)$ the path defined by $\gamma_{yi}(t) = (1-t)y + tz$ for $t \in [0, 1-\frac{1}{i}]$. Let $A_i = \{y \in C_k | \gamma_{yi}$ cannot be lifted from every point from $Q_k\}$, $\Gamma_i^{'} = \{\gamma_{yi} | y \in A_i\}$ and let Γ_i be the family of all maximal liftings of the paths from $\Gamma_i^{'}$ starting from the points of Q_k, for $i \in N$. We see that a path $\gamma \in \Gamma_i$ cannot have any limit point in $(D \cap S(x, r_k) \cup B(x, r_k) \cap (\partial D \setminus E))$, hence $Im\gamma \subset B(x, r_k) \cap D$ and hence γ has at least a limit point in E. Since $M_{K_I(f)}(E) = 0$, this implies that $M_{K_I(f)}(\Gamma_i) = 0$, and since $\Gamma_i^{'} > f(\Gamma_i)$, we see that $M(\Gamma_i^{'}) \leq M(f(\Gamma_i)) \leq M_{K_I(f)}(\Gamma_i) = 0$. Then $m_{n-1}(A_i) = 0$ for $i \in N$ and let $A_k = \bigcup_{i=1}^{\infty} A_i$.

Let $y_k \in C_k \setminus A_k$ and $\gamma_{y_k} : [0, 1) \to B(z, r_k^{'})$, $\gamma_{y_k}(t) = (1 - t)y_k + t \cdot z$ for $t \in [0, 1)$. Since $y_k \not\in A_k$, there exists $q_k : [0, 1) \to B(x, r_k) \cap D$ a path so that $q_k(0) \in Q_k$ and $f \circ q_k = \gamma_{y_k}$. Let B_k be the set of all limit points of q_k. We see that if $w_k \in B_k \cap B(x, r_k) \cap D$, then $f(w_k) = z$, hence, since $z \not\in \bigcap_{r>0} f(B(x, r) \cap D)$, there exists $k_0 \in N$ so that $B_k \subset \partial D \cap \bar{B}(x, r_k)$ for every $k \geq k_0$. If there exists $k \geq k_0$ so that $CardB_k > 1$, then, since B_k is connected and $capE = 0$, we can find a point $b_k \in (B_k \setminus E) \cap \partial D \cap \bar{B}(x, r_k)$, hence $z \in C(f, b_k) \subset \bar{F}_k$. On the other side, $\alpha < d(z, \bar{F}_k)$, and we reached a contradiction. It results that $CardB_k = 1$ for $k \geq k_0$, hence there exists $x_k = \lim_{t \to 1} q_k(t) \in \bar{B}(x, r_k) \cap E$ for $k \geq k_0$. If there exists $k \geq k_0$, so that $x_k = x$, then $x \in E$ and $z \in A(f, x)$, and if $x_k \neq x$ for every $k \geq k_0$, then $x_k \to x$, $x_k \in E$ and $z \in A(f, x_k)$ for every $k \geq k_0$.

Remark 8. Using Remark 4, we see that condition (a) can hold in Theorem 15 if conditions $(a_1), (a_2), (a_3)$ are satisfied and f is only a light map, since the modular inequality "$M(f(\Gamma_i)) \leq M_{K_I(f)}(\Gamma_i)$" from Theorem 15 remains valid for light mappings which are not necessarily discrete.

We extend now a result which for plane meromorphic functions is known as Iversen's theorem and Cartwright's theorem. Our result also extends some theorems from [17] and [24], page 170, established for quasiregular mappings and some results from [3] established for mappings of finite distortion and satisfying condition $(\mathcal{A})$.

Theorem 16. Let $n \geq 2$, $E \subset D$ closed in D, $f : D \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on D, satisfying condition (a) on $D \setminus E$, so that $K_I(f) \in L^1_{loc}(D \setminus f^{-1}(\infty))$, $M_{K_I(f)}(E) = 0$, and let $x \in E$ be an essential singularity of f. Then, if x is an isolated point of E, it results that $\overline{\mathbf{R}}^n \setminus \bigcap_{r>0} f(B(x, r) \setminus E) \subset A(f, x)$ and in the general case, there exists $x_k \in E$, $x_k \neq x$, $x_k \to x$ and so that $\overline{\mathbf{R}}^n \setminus \bigcap_{r>0} f(B(x, r) \setminus E) \subset A(f, x_k)$ for every $k \in N$.

Proof. Since $x \in IntD$, we see that $B(x, r) \cap \partial D = \phi$ for $r > 0$ small enough, hence $C(f, x, \partial D \setminus E) = \phi$. Since x is an essential singularity of f, we see that $cap(\mathcal{C}f(B(x, r) \setminus E)) = 0$ for every $r > 0$, hence $C(f, x) = \overline{\mathbf{R}}^n$. Let $z \in \overline{\mathbf{R}}^n \setminus \bigcap_{r>0} f(B(x, r) \setminus E)$. Then $z \in C(f, x) \setminus (C(f, x, \partial D \setminus E) \cup \bigcap_{r>0} f(B(x, r) \setminus E))$ and we apply now Theorem 15.

The next theorem extends a result of O. Martio and S. Rickman from [17] concerning the density of the points $b \in S^n$ at which a quasiregular map $f : B^n \to \mathbf{R}^n$ with $cap(\mathcal{C}f(B^n)) > 0$ has some asymptotic values. A version for mappings of finite distortion and satisfying condition $(\mathcal{A})$ was given in [3] by M. Cristea.

Theorem 17. Let $n \geq 2$, $B = \{b \in \partial D |$ there exists $\gamma : [0, 1) \to D$ a path so that $\lim_{t \to 1} \gamma(t) = b\}$, let $f : D \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on D, satisfying condition (a), and let $E = \{b \in B |$ there exists $\gamma : [0, 1) \to D$ a path so that $\lim_{t \to 1} \gamma(t) = b$ and there exists $\lim_{t \to 1} f(\gamma(t)) = l \in \overline{\mathbf{R}}^n\}$. Suppose that $K_I(f) \in L^1_{loc}(D)$

and $M_{K_I(f)}(b) = 0$ for every $b \in B \setminus E$, that $capCf(D \cap B(b,\epsilon)) > 0$ for every $b \in B \setminus E$ and every $\epsilon > 0$ and that $M_{K_I(f)}(B \cap B(b,\epsilon)) > 0$ for every $b \in B$ and every $\epsilon > 0$. Then $M_{K_I(f)}(E \cap B(b,\epsilon)) > 0$ for every $b \in B$ and every $\epsilon > 0$, hence E is densely in B.

Proof. Suppose that there exists $b \in B$ and $\epsilon > 0$ so that $M_{K_I(f)}(E \cap B(b,\epsilon)) = 0$. Since $M_{K_I(f)}(B \cap B(b,\frac{\epsilon}{2})) > 0$, we can find a point $y \in (B \setminus E) \cap B(b,\frac{\epsilon}{2})$. Let $\beta : [0,1) \to B(y,\frac{\epsilon}{2}) \cap D$ be a path so that $\lim_{t\to 1} \beta(t) = y$ and $\lim_{t\to 1} f(\beta(t))$ does not exists. Let $u \neq v$ and $s_m \nearrow 1$ be so that $\lim_{m\to\infty} f(\beta(s_{2m})) = u$, $\lim_{m\to\infty} f(\beta(s_{2m+1})) = v$ and let $F_m = f(\beta([s_{2m}, s_{2m+1}]))$ for $m \in N$. Let $r_m \to 0$ be so that $\beta([s_{2m}, s_{2m+1}]) \subset B(y, r_m)$ for every $m \in N$. We can suppose that there exists $r > 0$ so that $q(F_m) \geq r$ and $0 < r_m < \frac{\epsilon}{2}$ for every $m \in N$.

Let $\Gamma'_m = \{\gamma : [0,1) \to \mathbf{R}^n \text{ path } | \gamma(0) \in F_m, \gamma(1) \in Cf(B(y,\frac{\epsilon}{2}) \cap D)\}$ and let Γ_m be the family of all maximal liftings of the paths from Γ'_m starting from the points of $\beta([s_{2m}, s_{2m+1}])$ for $m \in N$. We see that $\Gamma'_m > f(\Gamma_m)$ and from Lemma 2.6, page 65 in [24], we can find $\delta > 0$ so that $\delta < M(\Gamma'_m)$ for every $m \in N$. Let $\Gamma_{m1} = \{\gamma : [0,1) \to D \cap B(y,\frac{\epsilon}{2}) | \gamma \in \Gamma_m$ and γ has at least a limit point in $\partial D \cap B(y,\frac{\epsilon}{2})\}$, $\Gamma_{m2} = \{\gamma \in \Gamma_m | Im\gamma \cap S(y,\frac{\epsilon}{2}) \neq \phi\}$ and let $\Delta_m = \Delta(\bar{B}(y,r_m) \cap D, S(y,\frac{\epsilon}{2}) \cap D, (B(y,\frac{\epsilon}{2}) \setminus B(y,r_m)) \cap D)$ for $m \in N$. Then $\Gamma_m = \Gamma_{m1} \cup \Gamma_{m2}$, $\Gamma_{m2} > \Delta_m$ for $m \in N$ and since $M_{K_I(f)}(y) = 0$, we see from Lemma 7 that $\lim_{m\to\infty} M_{K_I(f)}(\Delta_m) = 0$.

Let now $\gamma : [0,1) \to D \cap B(y,\frac{\epsilon}{2})$, $\gamma \in \Gamma^r_{m1}$. Then there exists $\beta_\gamma = \lim_{t\to 1} \gamma(t) \in \partial D$ and since $\lim_{t\to 1} f(\gamma(t))$ obviously exists, it results that $\beta_\gamma \in E$. We see that $E \cap B(y,\frac{\epsilon}{2}) \subset E \cap B(b,\epsilon)$ and $M_{K_I(f)}(E \cap B(b,\epsilon)) = 0$, hence $M_{K_I(f)}(E \cap B(y,\frac{\epsilon}{2})) = 0$ and this implies that $M_{K_I(f)}(\Gamma^r_{m1}) = 0$. We use now Lemma 3 to see that $M_{K_I(f)}(\Gamma_{m1}) = 0$ for every $m \in N$. We have

$$\delta \leq M(\Gamma'_m) \leq M(f(\Gamma_m)) \leq M_{K_I(f)}(\Gamma_m) =$$

$$M_{K_I(f)}(\Gamma_{m1} \cup \Gamma_{m2}) \leq M_{K_I(f)}(\Gamma_{m1}) + M_{K_I(f)}(\Gamma_{m2}) = M_{K_I(f)}(\Gamma_{m2}) \leq M_{K_I(f)}(\Delta_m) \to 0$$

if $m \to \infty$.

We reached a contradiction. We therefore proved that $M_{K_I(f)}(E \cap B(b,\epsilon)) > 0$ for every $b \in B$ and every $\epsilon > 0$.

We say that a domain $D \subset \overline{\mathbf{R}}^n$ has the continuum property if for every two compact, disjoint subsets K and M so that K is connected, $CardK > 1$ and $capM > 0$, it results that $M(\Delta(K, M, D)) > 0$.

Theorem 18. Let $n \geq 2$, $E \subset \bar{D}$ so that $D \setminus E$ is open, $f : D \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a) on $D \setminus E$, so that $M_{K_I(f)}(E) = 0$ and $K_I(f) \in L^1_{loc}(D \setminus E)$. Let $b \in \partial D \cap (\partial D \setminus E)'$ be so that $M_{K_I(f)}(b) = 0$, $M = C \bigcap_{r>0} f(B(b,r) \setminus E) \cap D)$ and suppose that there exists a closed set C so that $\overline{\mathbf{R}}^n \setminus C$ has the continuous property and $C(f, b, \partial D \setminus E) \subset C$. Then either $cap(M \setminus C) = 0$, or $C(f, b) \subset C$.

Proof. Suppose that $cap(M \setminus C) > 0$ and that $C(f,b) \not\subset C$. Let $y_1 \in C(f,b) \setminus C$. Since $M = \bigcup_{r>0} Cf(B(b,r) \setminus E) \cap D)$, we can find $M_1 \subset M$ compact so that $y_1 \notin M_1$, $M_1 \cap C = \phi$, $capM_1 > 0$ and there exists $\rho > 0$ so that $M_1 \subset Cf(B(b,\rho) \setminus E) \cap D)$. Now either $y_1 \in \mathbf{R}^n$, or $y_1 = \infty$ and C is compact and in both cases we see that $r = q(y_1, M_1 \cup C) > 0$. Let $D_m = B(b,\frac{1}{m}) \cap D$, $K_m = B(b,\frac{1}{m}) \cap \partial D$ for $m \in N$, and let $z_m \in K_m \setminus (E \cup \{b\})$ and $x_m \in D_m$ be so that $f(x_m) \to y_1$. Since $capE = 0$, we see that $m_\alpha(E) = 0$ for every $\alpha > 0$ and from Lemma 7 in [3], we see that $m_p(M(E, x_m) \cup M(E, z_m)) = 0$ for every $p > 1$ and every $m \in N$. We can find a point $w_m \in B(b,\frac{1}{m}) \setminus (M(E,x_m)) \cup M(E,z_m))$ and a path $q_m : [0,1] \to B(b,\frac{1}{m}) \setminus E$ so that $q_m(0) = x_m$, $q_m(1) = z_m$ and $Imq_m = [x_m, w_m] \cup [w_m, z_m]$ for $m \in N$. We take now

$t_m = \sup\{t \in [0,1] | q_m(t) \in D_m\}$, $\lambda_m = q_m|[0, t_m]$ for every $m \in N$. Let now $F : \partial D \to \mathcal{P}(\overline{\mathbf{R}}^n)$, $F(x) = C(f, x)$ for $x \in \partial D$. Then $C(f, b, \partial D \setminus E) = \bigcap_{m=1}^{\infty} \overline{F(K_m \setminus (E \cup \{b\}))}$ and let $m_0 \in N$ be so that $F(K_m \setminus (E \cup \{b\})) \subset B_q(C, \frac{r}{5})$ for every $m \geq m_0$.

Since $C(f, \lambda_m(t_m), Im\lambda_m) \subset F(K_m \setminus (E \cup \{b\})) \subset B_q(C, \frac{r}{5})$ for $m \geq m_0$ and $f(x_m) \to y_1$ we see that there exists $m_1 \geq m_0$ so that $Imf \circ \lambda_m \cap B_q(C, \frac{r}{5}) \neq \phi$ and $f(x_m) \in B_q(y_1, \frac{r}{5})$ for $m \geq m_1$.

We can therefore find closed subpaths α_m of λ_m so that $Im\alpha_m \subset D_m \setminus E$, $Imf \circ \alpha_m \subset \bar{B}_q(y_1, \frac{2r}{5}) \setminus B_q(y_1, \frac{r}{5})$, $Imf \circ \alpha_m \cap S_q(y_1, \frac{r}{5}) \neq \phi$, $Imf \circ \alpha_m \cap S_q(y_1, \frac{2r}{5}) \neq \phi$ for every $m \geq m_1$. Let $H_m = Im\alpha_m$, $Q_m = Imf \circ \alpha_m$ for $m \geq m_1$. Then Q_m are compact, connected subsets from $\mathbf{R}^n \cap (\bar{B}_q(y_1, \frac{2r}{5}) \setminus B_q(y_1, \frac{r}{5}))$, $q(Q_m) \geq r$ and $q(Q_m, C \cup B_q(M_1, \frac{r}{5})) \geq \frac{2r}{5}$ for every $m \geq m_1$. Using Theorem 7.1 page 11 in [34], we can suppose that there exists $Q = \lim_{m\to\infty} Q_m$. Then Q is compact and connected in $\mathbf{R}^n \cap (\bar{B}_q(y_1, \frac{2r}{5}) \setminus B_q(y_1, \frac{r}{5}))$, $q(Q) \geq r$ and $q(Q, C \cup B_q(M_1, \frac{r}{5})) \geq \frac{2r}{5}$.

Let $\Delta = \Delta(Q, M_1, \mathbf{R}^n \setminus C)$, $\Delta_m = \Delta(Q, M_1, \overline{\mathbf{R}}^n \setminus \bar{B}_q(C, \frac{1}{m}))$ for $m \in N$. Since $\overline{\mathbf{R}}^n \setminus C$ has the continuum property, we can find $\delta > 0$ so that $4\delta < M(\Delta)$. Since $\Delta_m \nearrow \Delta$, we use Ziemer's lemma from [36] to see that $M(\Delta_m) \nearrow M(\Delta)$, hence we cand find $m_2 \geq m_1$ so that $2\delta < M(\Delta_{m_2})$ and $\frac{1}{m_2} < \rho$. Let $U = \bar{B}_1(C, \frac{1}{m_2})$, $\Gamma'_m = \Delta(Q_m, M_1, \overline{\mathbf{R}}^n \setminus U)$ for $m \geq m_2$. Using Lemma 6 in [3], we see that $\lim_{m\to\infty} M(\Gamma'_m) = M(\Delta_{m_2})$, and let $m_3 \geq m_2$ be so that $\delta < M(\Gamma'_m)$ and $F(K_m \setminus (E \cup \{b\}) \subset U$ for $m \geq m_3$. Let $\Lambda_m = \Delta(\bar{B}(b, \frac{1}{m}) \cap D, CB(b, \frac{1}{m_3}) \cap D, (B(b, \frac{1}{m_3}) \setminus B(b, \frac{1}{m})) \cap D)$ for m$\geq m_3$ and let Γ_m be the family of all maximal liftings by the map $g = f|D \setminus E$ of the paths from Γ'_m, starting from some points of H_m, for $m \geq m_3$. Let $\Gamma_{m1} = \{\gamma \in \Gamma^r_m | \gamma$ has at least a limit point in $E\}$, $\Gamma_{m2} = \{\gamma \in \Gamma^r_m | \gamma$ has at least a limit point in $K_{m_3} \setminus E\}$, $\Gamma_{m3} = \{\gamma \in \Gamma^r_m | Im\gamma \cap S(b, \frac{1}{m_3}) \neq \phi\}$ for $m > m_3$. Since $\partial D_{m_3} \subset E \cup (K_{m_3} \setminus E) \cup S(b, \frac{1}{m_3})$ and $M_1 \subset Cf(B(b, \frac{1}{m}) \setminus E) \cap D)$ for $m \geq m_3$, we see that $\Gamma^r_m = \Gamma_{m1} \cup \Gamma_{m2} \cup \Gamma_{m3}$ for $m > m_3$, and also $\Gamma'_m > f(\Gamma_m)$ for $m > m_3$. Let now $\gamma : [0,1] \to K_{m_3} \setminus (E \cup \{b\}))$, $\gamma \in \Gamma_{m2}$. Since γ is rectifiable, there exists $w = \lim_{t\to 1} \gamma(t)$ and $w \in K_{m_3} \setminus (E \cup \{b\})$. Since $f \circ \gamma$ is a subpath of a path from Γ'_m, there exists $l = \lim_{t\to 1} f(\gamma(t)) \in C\bar{U}$. On the other side, $l \in C(f, w, Im\alpha) \subset F(K_{m_3} \setminus (E \cup \{b\})) \subset U$ and we reached a contradiction. It results that $\Gamma_{m2} = \phi$, hence $\Gamma^r_m = \Gamma_{m1} \cup \Gamma_{m3}$ for $m > m_3$. Also, $\Gamma_{m3} > \Lambda_m$ and from Lemma 3, we see that $M_{K_I(f)}(\Gamma_m) = M_{K_I(f)}(\Gamma^r_m)$ for $m > m_3$. Using Lemma 7 and the fact that $M_{K_I(f)}(b) = 0$, we see that $\lim_{m\to\infty} M_{K_I(f)}(\Lambda_m) = 0$. Since Γ_m is a path family from $D \setminus E$, and f satisfies condition (a) on $D \setminus E$, we have for $m > m_3$ that

$$\delta \leq M(\Gamma'_m) \leq M(f(\Gamma_m)) \leq M_{K_I(f)}(\Gamma_m) = M_{K_I(f)}(\Gamma^r_m) =$$

$$M_{K_I(f)}(\Gamma_{m1} \cup \Gamma_{m3}) \leq M_{K_I(f)}(\Gamma_{m1}) + M_{K_I(f)}(\Gamma_{m3}) = M_{K_I(f)}(\Gamma_{m3}) \leq M_{K_I(f)}(\Lambda_m) \to 0$$

is $m \to \infty$.

We obtained a contradiction. We proved that if $cap(M \setminus C) > 0$, then $C(f, b) \subset C$.

Remark 9. We see from Lemma 2.6, page 65 in [24] that $\overline{\mathbf{R}}^n$ has the continuum property. We also see from [31] that if $F \subset \overline{\mathbf{R}}^n$ is closed and $m_{n-1}(F) = 0$, then $M(\Delta(A, B, \overline{\mathbf{R}}^n \setminus F)) = M(\Delta(A, B, \overline{\mathbf{R}}^n))$ for every $A, B \subset \overline{\mathbf{R}}^n \setminus F$ disjoint sets, and this implies that if $F \subset \overline{\mathbf{R}}^n$ is closed and $m_{n-1}(F) = 0$, then $\overline{\mathbf{R}}^n \setminus F$ has the continuum property. Using Lemma 4 in [3], we see that $\overline{\mathbf{R}}^n \setminus B$ has the continuum property for every ball $B \subset \mathbf{R}^n$. Also, if $D_p \nearrow D$ are domains having the continuum property for every $p \in N$, then D has the continuum property. Now, if $B \subset \mathbf{R}^n$ is a ball, $G = \mathbf{R}^n \setminus B$ and $f : D \to G$ is a homeomorphism which is locally quasiconformal, it results that D has the continuum property and such an important case holds when f is a C^1 diffeomorphism. In this way we see that if $H \subset \mathbf{R}^n$ is a half space, then $\mathbf{R}^n \setminus \bar{H}$ has the continuum property. Let now $C \subset \mathbf{R}^n$ be a Jordan domain so that there exists a C^1 diffeomorphism $f : \partial C \to S^n$ and C is starlike with respect to a point $a \in C$, and let $D = \mathbf{R}^n \setminus \bar{C}$. Taking $F : \mathbf{R}^n \to \mathbf{R}^n$, $F((1-t)a + tx) = \frac{tx}{|x|}$ for $t \geq 0$ and $x \in \partial C$, we see that $F : \mathbf{R}^n \to \mathbf{R}^n$ is a diffeomorphism which maps D ono $\mathbf{R}^n \setminus \bar{B}$, and this implies that D has the continuum property.

We see now that we can take the set C from Theorem 18 a ball, or a half space, so that $m_{n-1}(C) = 0$ or so that $C = \bigcap_{p \in N} \bar{D}_p$, where D_p are starlike Jordan domains so that ∂D_p is diffeomorphic to S^n for $p \in N$. An important case holds if $m_{n-1}(C(f,b,\partial D \setminus E)) = 0$, and in this case we extend a result from [3] given for mappings of finite distortion and satisfying condition $(\mathcal{A})$. When C is a ball from $\mathbf{R}^n$ in Theorem 18, we obtain a "maximum principle" which extends a similar one from [3] given for mappings of finite distortion and satisfying condition $(\mathcal{A})$.

If $m_{n-1}(C(f,b,\partial D \setminus E)) = 0$ and the set M from Theorem 18 is so that $m_{n-1}(M) > 0$, then the condition "$cap(M \setminus C(f,b,\partial D \setminus E)) > 0$" obviously holds, and we obtain that $C(f,b,\partial D \setminus E) = C(f,b)$. Essentially, the result says that if the cluster set $C(f,b,\partial D \setminus E)$ is small enough and the set M is great enough, then we have the equality "$C(f,b,\partial D \setminus E) = C(f,b)$". An important case when the condition "$cap(M \setminus C(f,b,\partial D \setminus E)) > 0$" holds is when f is bounded near b and $m_{n-1}(C(f,b,\partial D \setminus E)) = 0$.

In the case $n = 2$, we obtain a special case of Theorem 18.

Theorem 19. Let $n = 2$, $E \subset \bar{D}$ be so that $D \setminus E$ is open, $b \in \partial D$, $f : D \setminus E \to \overline{\mathbf{R}}^2$ continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a) on $D \setminus E$, so that $M_{K_I(f)}(E) = 0$, $M_{K_I(f)}(b) = 0$, $K_I(f) \in L^1_{loc}(D \setminus E)$. Let $G \subset D$ be a Jordan domain so that there exists $\gamma_k : [0,1] \to \bar{D}$ arcs, $k = 1,2,3$ so that $Im\gamma_3 \subset D$, $\gamma_k([0,1)) \subset D$, $\lim_{t \to 1} \gamma_k(t) = b$, $C(f,b,Im\gamma_k) = \{c\}$, $k = 1,2$ and $\partial G = Im(\gamma_1 \bigvee \gamma_2^- \bigvee \gamma_3)$, let $M = \mathcal{C} \bigcap_{r>0} f(B(b,r) \setminus E) \cap G)$ and suppose that $capM > 0$. Then $\lim_{\substack{z \to b \\ z \in G \setminus E}} f(z) = c$.

Proof. Since $capM > 0$ and $C(f,b,\partial G \setminus E) = \{c\}$, it results that $cap(M \setminus C(f,b,\partial G \setminus E)) > 0$. We see now from Theorem 18 that $C(f,b,G \setminus E) = C(f,b,\partial G \setminus E) = \{c\}$.

A theorem of Lindelöf says that if $f : B^2 \to \mathbf{C}$ is meromorphic and admits two distinct asymptotic values at some point $b \in S^2$, then f assumes infinitely often in any neighborhood of b all values of the extended plane, with at most two possible exceptions. We use now Theorem 18 to extend Lindelöf's theorem and a result from[3].

Theorem 20. Let $n = 2$, $E \subset \bar{D}$ so that $D \setminus E$ is open, $b \in \partial D$, $f : D \setminus E \to \overline{\mathbf{R}}^2$ continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a) on $D \setminus E$ so that $M_{K_I(f)}(E) = 0$, $M_{K_I(f)}(b) = 0$ and $K_I(f) \in L^1_{loc}(D \setminus E)$. Suppose that f admits two distinct asymptotic values in b. Then there exists $M \subset \mathbf{R}^2$ with $capM = 0$ and so that $\mathbf{R}^2 \setminus M \subset f((U \setminus E) \cap D)$ for every $U \in \mathcal{V}(b)$.

Proof. We know that the locus of a path is also the locus of an arc, hence we can suppose that there exists arcs $\gamma_k : [0,1] \to \bar{D}$ with $\gamma_k([0,1)) \subset D$, $\gamma_k(1) = b$, $\lim_{t \to 1} f(\gamma_k(t)) = b_k$, $k = 1,2$, with $b_1 \neq b_2$, and we can suppose that $Im\gamma_1 \cap Im\gamma_2 = \{b\}$. We can also find an arc $\gamma_3 : [0,1] \to D$ and a Jordan domain $G \subset D$ so that $\partial G = Im(\gamma_1 \bigvee \gamma_2^- \bigvee \gamma_3)$. Let $M = \mathcal{C} \bigcap_{r>0} f(B(b,r) \setminus E) \cap G)$. Since $capE = 0$, we see that $b \in \partial G \cap (\partial G \setminus E)'$ and suppose that $capM > 0$. Since $C(f,b,\partial G \setminus E) = \{b_1, b_2\}$, it results that $cap(M \setminus C(f,b,\partial G \setminus E)) > 0$, and from Theorem 18 we see that $C(f,b,G \setminus E) = C(f,b,\partial G \setminus E) = \{b_1, b_2\}$. On the other side, since

G is a Jordan domain, is locally connected in b. Since E is nowhere disconnecting, we see that also $G \setminus E$ is locally connected in b and this implies that $C(f, b, G \setminus E)$ is connected. We reached a contradiction. It results that $capM = 0$ and we see that $\mathbf{R}^2 \setminus M = \bigcap_{r>0} f(B(b,r) \setminus E) \cap G) \subset f((U \setminus E) \cap D)$ for every $U \in \mathcal{V}(b)$.

We extend now another theorem of Lindelöf given for bounded analytic functions and generalized by M. Vuorinen in [33] for quasimeromorphic functions.

Theorem 21. Let $n \geq 2$, $E \subset \bar{D}$ be so that $D \setminus E$ is open, $f : \bar{D} \setminus E \to \overline{\mathbf{R}}^n$ be continuous on $\bar{D} \setminus E$, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$ satisfying condition (a) on $D \setminus E$, so that $M_{K_I(f)}(E) = 0$ and $K_I(f) \in L^1_{loc}(D \setminus E)$. Let $b \in E \cap (\partial D \setminus E)'$, $M = C \bigcap_{r>0} f(B(b,r) \setminus E) \cap D)$ and suppose that $capM > 0$ and there exists $\lim_{\substack{x \to b \\ x \in \partial D \setminus E}} f(x) = \alpha$. Then there exists $\lim_{\substack{x \to b \\ x \in \bar{D} \setminus E}} f(x)$ and equals α.

Proof. We see that $C(f, b, \partial D \setminus E) = \{\alpha\}$, hence $cap(M \setminus C(f, b, \partial D \setminus E)) > 0$. We use now Theorem 18 to see that $C(f, b) = C(f, b, \partial D \setminus E) = \{\alpha\}$, hence $\lim_{\substack{x \to b \\ x \in \bar{D} \setminus E}} f(x) = \alpha$.

We remarked that if the set C from Theorem 18 is a ball we obtain a "maximum principle". We use now this maximum principle to extend a similar one established in [17] for quasiregular mappings and in [3] for mappings of finite distortion and satisfying condition $(\mathcal{A})$:

Theorem 22. Let $n \geq 2$, $E \subset \bar{D}$ so that $D \setminus E$ is open and $\partial D \setminus (E \cap \partial D)$ is densely in ∂D, $f : D \to \mathbf{R}^n$ continuous and open on D, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a) on $D \setminus E$, so that $M_{K_I(f)}(E) = 0$ and $K_I(f) \in L^1_{loc}(D \setminus E)$. Suppose that there exists $L > 0$ so that $\limsup_{y \to x} |f(y)| \leq L$ for every $x \in \partial D \setminus E$ and let $M_x = C \bigcap_{r>0} f(B(x,r) \setminus E) \cap D)$ for $x \in E$. Then, if $cap(M_x \setminus \bar{B}(0, L)) > 0$ for every $x \in E$, it results that $|f(x)| \leq L$ for every $x \in D$.

Proof. We see that $E \cap \partial D \subset (\partial D \setminus E)'$, that $C(f, x, D \setminus E) \subset \bar{B}(0, L)$ for every $x \in \partial D \setminus E$ and hence $C(f, x, \partial D \setminus E) \subset \bar{B}(0, L)$ for every $x \in E \cap \partial D$. Taking the set $C = \bar{B}(0, L)$ in Theorem 18, we see that $C(f, x, D \setminus E) \subset \bar{B}(0, L)$ for every $x \in E \cap \partial D$. Since $D \setminus E$ is densely in D and f is continuous on D, we see that $C(f, x) \subset \bar{B}(o, L)$ for every $x \in \partial D$. We use now the openness of the map f on D and the fact that f takes finite values on D to see that $|f(x)| \leq L$ for every $x \in D$.

We extend now a theorem which is known for plane meromorphic functions as Iversen-Tsuji's theorem. A version for quasimeromorphic mappings can be found in [17] and [33] and a version for mappings of finite distortion and satisfying condition $(\mathcal{A})$ was given in [3].

Theorem 23. Let $n \geq 2$, $E \subset \bar{D}$ be so that $D \setminus E$ is open, $b \in (\partial D \setminus E)'$, $f : D \setminus E \to \overline{\mathbf{R}}^n$ be continuous, open, light, a.e. differentiable with $J_f(x) \neq 0$ a.e. on $D \setminus E$, satisfying condition (a) on $D \setminus E$, so that $M_{K_I(f)}(E) = 0$ and $K_I(f) \in L^1_{loc}(D \setminus E)$. Let $M = C \bigcap_{r>0} f(B(b,r) \setminus E) \cap D)$ and suppose that $cap(M \setminus B(0,r)) > 0$ for every $r > 0$. Then $\limsup_{x \to b} |f(x)| = \lim_{\substack{z \to b \\ z \in \partial D \setminus E}} (\limsup_{x \to z} |f(x)|)$.

Proof. Let $\alpha = \limsup_{x \to b} |f(x)|$ and let $\psi : \partial D \setminus E \to \mathbf{R}_+$ be defined by $\psi(z) = \limsup_{x \to z} |f(x)|$ for $z \in \partial D \setminus E$. Let $\beta = \lim_{\substack{z \to b \\ z \in \partial D \setminus E}} \psi(z)$. Then $\beta \leq \alpha$ and we show that $\alpha \leq \beta$. We can suppose that $\beta < \infty$ and let $\epsilon > 0$. We can find $\delta_\epsilon > 0$ so that $C(f, z) \subset \bar{B}(0, \beta + \epsilon)$ for every $z \in (\partial D \setminus E) \cap B(b, \delta_\epsilon)$, hence $C(f, b, \partial D \setminus E) \subset \bar{B}(0, \beta + \epsilon)$. If $b \in \partial D \setminus E$, then $C(f, b) \subset \bar{B}(0, \beta + \epsilon)$, and if $b \in E$, we see from Theorem 18 that $C(f, b) \subset \bar{B}(0, \beta + \epsilon)$. It results that $C(f,b) \subset \bar{B}(0,\beta+\epsilon)$ and letting $\epsilon \to 0$, we see that $C(f,b) \subset \bar{B}(0,\beta)$ and hence that $\alpha \leq \beta$.

Remark 10. We don't need the "singular" set E from Theorems 15-17, 20-23 to be compact as in [17], [24], [33]. Also, if we can continuously extend the map f on $D \cap E$ to a map $F : D \to \overline{\mathbf{R}^n}$ in Theorems 18, 19, 20, 21, 23, the conclusions of these theorems will hold for the new cluster set $C(F,b)$ instead of $C(f,b)$. We can use for instance the conditions from Theorem 13 to extend f continuously ono $D \cap E$.

References

1) M. Cristea, A generalization of Sard's theorem. A Jacobian condition for interiority, Demonstratio Mathematica, 21 (1988), 399-405.

2) M. Cristea, A generalization of the theorem on the univalence on the boundary, Rev. Roumaine Math. Pures Appl, 40, 5-6 (1995), 435-448.

3) M. Cristea, Mappings of finite distortion: Boundary extension, Rev. Roumaine Math. Pures Appl., 51, 5-6 (2006), 607 - 631.

4) M. Cristea, Mappings of finite distortion: Zoric's theorem and equicontinuity results, Rev. Roumaine Math. Pures Appl., 52, 5(2007), 539-554.

5) M. Cristea, Local homeomorphism having local ACL^n inverses, Complex Variables and Elliptic Equations, 53, 1(2008), 77-99.

6) M. Cristea, Some conditions of quasiregularity (II), Rev. Roumaine Math. Pures Appl., 39, 6(1994), 599-609.

7) I. Fonseca and W. Gangbo, Degree Theory in Analysis and Appl., Oxford Univ. Press, 1995.

8) J. Heinonen and P. Koskela, Sobolev mappings with integrable dilatations, Arch. Rat. Mech. Anal., 125 (1993), 81-97.

9) A. Ignatiev and V. Ryazanov, On the boundary behavior of space mappings, Preprint 350, 2003, Reports Math. Dept. Univ. of Helsinki.

10) A. Ignatiev and V. Ryazanov, Finite mean oscillation in the mapping theory, Preprint 322, 2002, Reports Math. Dept. Univ. of Helsinki.

11) T. Iwaniecz and G. Martin, Geometric function theory and nonlinear analysis, Oxford Math. Monogr., 2001.

12) J. Kauhanen, P. Koskela, J. Malý, J. Onninen and X. Zhong, Mappings of finite distortion: Sharp Orlicz conditions, JEMS, 5(2003), 95-105.

13) P. Koskela, J. Onninen and K. Rajala, Mappings of finite distortion: Injectivity radius of local homeomorphisms, Future Trends in Geometric Function Theory, RNC Workshop Jyväskylä, 2003, Rep. Univ. Jyväskylä, Dept. Mat. Stat., 92 (2003), 169-174.

14) P. Koskela and J. Onninen, Mappings of finite distortion: Capacity and modulus inequalities, Preprint 257, Univ. of Jyväskylä, 2002.

15) N. G. Lloyd, Degree Theory, Cambridge Univ. Press, Cambridge, 1978.

16) J. Manfredi and E. Villamor, An extension of Reshetnyak's theorem, Indiana Univ. Math. J., 43, 3 (1998), 49-69.

17) O. Martio and S. Rickman, Boundary behavior of quasiregular mappings, Ann. Acad. Sci. Fenn., Ser AI, Math., 507 (1978), 1-17.

18) O. Martio, S. Rickman and J. Väisälä, Definitions for quasiregular mappings, Ann. Acad. Sci. Fenn, Ser AI, Math., 448 (1969), 1-40.

19) O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, On Q-homeomorphisms, Ann. Acad. Sci. Fenn., Math., 30(2005), 49-69.

20) O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, Mappings of finite length distortion, J. D'. Analysé Math., 93(2004), 215-236.

21) K. Noshiro, Cluster sets, Springer - Verlag, 1960.

22) K. Rajala, Mappings of finite distortion: Removable singularities for locally homeomorphic mappings, Proc. AMS, 123(2004), 3251-3258.

23) Y. G. Reshetnyak, Space Mappings of Bounded Distortion, Transl. of Math. Monogr., AMS, Providence, RI, 73(1989).

24) S. Rickman, Quasiregular mappings, Ergebnise der Mathematik und ihrer Grenzgebiete, 26, Springer - Verlag, 1993.

25) V. Ryazanov, U. Srebro and E. Yakubov, To the theory of BMO - quasiregular mappings, Dokl. Akad. Nauk. Rossi, 369 (1999), 13-15.

26) V. Ryazanov, U. Srebro and E. Yakubov, Plane mappings with dilatations dominated by functions of bounded mean oscillation, Siberian Adv. Math., 11, (2001), 94-130.

27) V. Ryazanov, U. Srebro and E. Yakubov, BMO - quasiconformal mappings, J. D'. Analysé Math., 83(2001), 1-20.

28) W. Rudin, Real and Complex Analysis, Mc Graw - Hill, New York, 1970.

29) S. Saks, Theory of the integral, Dover Publications, New York, 1964.

30) J. Väisälä, Lectures on n - dimensional quasiconoformal mappings, Lecture Notes in Math., 229, Springer - Verlag, 1971.

31) J. Väisälä, On the null sets for extremal distances, Ann. Acad. Sci. Fenn., Ser AI, Math., 322 (1962), 1-11.

32) M. Vuorinen, Exceptional sets and boundarý behavior of quasiregular mappings in n - space, Ann. Acad. Sci. Fenn., Ser AI, Math., Dissertationes, 1976.

33) M. Vuorinen, On the Iversen - Tsuji theorem for quasiregular mappings, Math. Scand., 41 (1977), 90-98.

34) G. T. Whyburn, Analytic Topology, Colloq. Publ., 28, AMS, Providence, RI, 1942.

35) D. Wilson, Open mappings on manifolds and a counter - example to the Whyburn conjecture, Duke Math. J., 40 (1973), 705-716.

36) W. P. Ziemer, Extremal length and p - capacity, Michigan Math. J., 16 (1969), 43-51.

University of Bucharest
Faculty of Mathematics and Computer Sciences,
Str. Academiei 14, R-010014,
Bucharest, Romania,
Email: mcristea@fmi.unibuc.ro