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I. Molnar and C. Varsan

I. Two problems for stochastic flows associated with nonlinear parabolic equations

1 Introduction

Consider that $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T]\}$ is the unique solution of SDE driven by complete vector fields $f \in (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$ and $g \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$,

$$(1.1) \quad \begin{cases} d_t\widehat{x} = \varphi(\lambda)f(\widehat{x})dt + g(\widehat{x}) \circ dw(t), \; t \in [0,T], \; x \in \mathbb{R}^n, \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n, \end{cases}$$

where $\varphi \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ and $w(t) \in \mathbb{R}$ is a scalar Wiener process over a complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}_t\}, P\}$. We recall that Fisk-Stratonovich integral "$\circ$" in (1.1) is computed by

$$g(x) \circ dw(t) = g(x) \cdot dw(t) + \frac{1}{2}\partial_x g(x) \cdot g(x)dt,$$

using Ito stochastic integral "$\cdot$".

We are going to introduce some nonlinear SPDE or PDE of parabolic type which describe the evolution of stochastic functionals $u(t,x) := h(\psi(t,x))$, or $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, for a fixed $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$. Here $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ is the unique solution satisfying integral equations

$$(1.2) \quad \widehat{x}_\varphi(t;\lambda) = x \in \mathbb{R}^n, \; t \in [0,T].$$

The evolution of $\{S(t,x) : t \in [0,T], x \in R^n\}$ will be defined by some nonlinear backward parabolic equation considering that $\{\widehat{x}_\psi(s;t,x) : s \in [t,T], x \in R^n\}$ is the unique solution of SDE

$$\begin{cases} d_s\widehat{x} = \varphi(\psi(t,x))f(\widehat{x})ds + g(\widehat{x}) \circ dw(s), \; s \in [t,T], \\ \widehat{x}(t) = x \in \mathbb{R}^n. \end{cases}$$

2 Some problems and their solutions

Problem (P1). Assume that g and f commute using Lie bracket, i.e.

(2.1) $[g,f](x) = 0, \; x \in \mathbb{R}^n,$

where $[g,f](x) := [\partial_x g(x)]f(x) - [\partial_x f(x)]g(x),$

(2.2) $TVK = \rho \in [0,1),$

where $V = \sup\{|\partial_x \varphi(x)| : x \in \mathbb{R}\}^n$ and $K = \sup\{|f(x)|; x \in \mathbb{R}^n\}$.

Under the hypotheses (2.1) and (2.2), find the nonlinear SPDE of parabolic type satisfied by $\{u(t,x) = h(\psi(t,x)) :\in [0,T], x \in \mathbb{R}^n\}$, $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$, where $\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}_t$-adapted solution of the integral equation (1.2).

Problem (P2). Using $\{\lambda = \psi(t,x)\}$ found in (P1), describe the evolution of a functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$ using backward parabolic equations, where $\{\widehat{x}_\psi(s;t,x) : s \in [t,T]\}$ is the unique solution of SDE

$$(2.3) \quad \begin{cases} d_s\widehat{x} = \varphi(\psi(t,x))f(\widehat{x})ds + g(\widehat{x}) \circ dw(s), \; s \in [t,T] \\ \widehat{x}(t) = x \in \mathbb{R}^n. \end{cases}$$

2.1 Solution for the Problem (P1)

Remark 2.1. Under the hypotheses (2.1) and (2.2) of (P1), the unique solution of integral equations (1.2) will be found as a composition

(2.4) $\psi(t,x) = \widehat{\psi}(t, \widehat{z}(t,x)),$

where $\widehat{z}(t,x) := G(-w(t))[x]$ and $\lambda = \widehat{\psi}(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}^n$, is the unique deterministic solution satisfying integral equations

(2.5) $\lambda = F(-\theta(t;\lambda))[z] =: \widehat{V}(t,z;\lambda)$, $t \in [0,T]$, $z \in \mathbb{R}^n$.

Here $F(\sigma)[z]$ and $G(\tau)[z]$, $\sigma, \tau \in \mathbb{R}$, are the global flows generated by complete vector fields f and g correspondingly, and $\theta(t;\lambda) = t\varphi(\lambda)$. The unique solution of (2.5) is constructed in the following

Lemma 2.1. *Assume that* (2.2) *is fulfilled. Then there exists a unique smooth deterministic mapping* $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], x \in \mathbb{R}^n\}$ *solving integral equations* (2.5) *such that*

$$(2.6)\quad \begin{cases} F(\theta(t;\widehat{\psi}(t,z)))[\widehat{\psi}(t,z)] = z \in \mathbb{R}^n,\ t \in [0,T], \\ |\widehat{\psi}(t,z) - z| \leqslant R(T,z) := \dfrac{r(T,z)}{1-\rho},\ t \in [0,T],\ \textit{where } r(T,z) = TK|\varphi(z)|, \end{cases}$$

$$(2.7)\quad \begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)f(z)\varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ x \in \mathbb{R}^n, \\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n. \end{cases}$$

Proof. The mapping $\widehat{V}(t,z;\lambda)$ (see (2.5)) is a contractive application with respect to $\lambda \in \mathbb{R}^n$, uniformly of $(t,z) \in [0,T] \times \mathbb{R}^n$ which allows us to get the unique solution of (2.5) using a standard procedure (Banach theorem). By a direct computation, we get

(2.8) $|\partial_\lambda \widehat{V}(t,z;\lambda)| = |f(\widehat{V}(t,z;\lambda))\partial_\lambda\theta(t;\lambda)| \leqslant TVK = \rho \in [0,1)$,

for any $t \in [0,T]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, where $\partial_\lambda\theta(t;\lambda)$ is a row vector. The corresponding convergent sequence $\{\lambda_k(t,z) : t \in [0,T], z \in \mathbb{R}^n\}_{k\geqslant 0}$ is constructed fulfilling

(2.9) $\lambda_0(t,z) = z$, $\lambda_{k+1}(t,z) = \widehat{V}(t,z;\lambda_k(t,z))$, $t \geqslant 0$,

$$(2.10)\quad \begin{cases} |\lambda_{k+1}(t,z) - \lambda_k(t,z)| \leqslant \rho^k|\lambda_1(t,z) - \lambda_0(t,z)|,\ k \geqslant 0, \\ |\lambda_1(t,z) - \lambda_0(t,z)| \leqslant |\widehat{V}(t,z;z) - z| \leqslant TK|\varphi(z)| =: r(T,z). \end{cases}$$

Using (2.10) we obtain that $\{\lambda_k(t,z)\}_{k\geqslant 0}$ is convergent and

$$(2.11)\quad \widehat{\psi}(t,z) = \lim_{k\to\infty} \lambda_k(t,z),\ |\widehat{\psi}(t,z) - z| \leqslant \frac{r(T,z)}{1-\rho} =: R(T,z),\ t \in [0,T].$$

Passing $k \to \infty$ into (2.9) and using (2.11) we get the first conclusion (2.6). On the other hand, notice that $\{\widehat{V}(t,z;\lambda) : t \in [0,T], z \in \mathbb{R}^n\}$ of (2.5) fulfils

$$(2.12)\quad \widehat{V}(t,\widehat{y}(t,\lambda);\lambda) = \lambda,\ t \in [0,T],\ \text{where } \widehat{y}(t,\lambda) = F(\theta(t;\lambda))[\lambda].$$

This shows that all the components of $\widehat{V}(t,z;\lambda) \in \mathbb{R}^n$ are first integrals associated with the vector field $f_\lambda(z) = \varphi(\lambda)f(z)$, $z \in \mathbb{R}^n$, for each $\lambda \in \mathbb{R}^n$, i.e.

$$(2.13)\quad \partial_t\widehat{V}(t,\widehat{y}(t,\lambda);\lambda) + [\partial_z\widehat{V}(t,\widehat{y}(t,\lambda);\lambda)]f(\widehat{y}(t,\lambda))\varphi(\lambda) = 0,\ t \in [0,T]$$

is valid for each $\lambda \in \mathbb{R}^n$. In particular, for $\lambda = \widehat{\psi(t,z)}$ we get $\widehat{y}(t,\widehat{\psi}(t,z)) = z$ and (2.13) becomes (H-J)-equation

$$(2.14)\quad \partial_t\widehat{V}(t,z;\widehat{\psi}(t,z)) + [\partial_z\widehat{V}(t,z;\widehat{\psi}(t,z))]f(z)\varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ z \in \mathbb{R}^n.$$

Combining (2.5) and (2.14), by direct computation, we convince ourselves that $\lambda = \widehat{\psi}(t,z)$ fulfils the following nonlinear (H-J)-equation (see (2.7))

$$(2.15)\quad \begin{cases} \partial_t\widehat{\psi}(t,z) + [\partial_z\widehat{\psi}(t,z)]f(z)\varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ z \in \mathbb{R}^n, \\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n, \end{cases}$$

and the proof is complete. □

Remark 2.2. Under the hypothesis (2.1), the stochastic flow $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$ generated by SDE (1.1) can be represented as follows

$$(2.16)\quad \widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F(\theta(t;\lambda))[\lambda] = H(t,w(t);\lambda),\ t \in [0,T],\ \lambda \in \mathbb{R}^n$$

where $\theta(t;\lambda) = t\varphi(\lambda)$.

Lemma 2.2. *Assume that* (2.1) *and* (2.2) *are satisfied and consider* $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ *found in Lemma 2.1. Then the stochastic flow generated by SDE* (1.1) *fulfils*

(2.17) $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$ *can be represented as in* (2.16),

(2.18) $\psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x))$ *is the unique solution of integral equations* (1.2), *where* $\widehat{z}(t,x) = G(-w(t))[x]$.

Proof. Using the hypothesis (2.1), we see easily that

$$(2.19)\quad y(\theta,\sigma)[\lambda] := G(\sigma)\circ F(\theta)[\lambda],\ \theta,\sigma\in\mathbb{R},\ \lambda\in\mathbb{R}^n$$

is the unique solution of the gradient system

$$(2.20)\quad \begin{cases} \partial_\theta y(\theta,\sigma)[\lambda] = f(y(\theta,\sigma)[\lambda]),\ \partial_\sigma y(\theta,\sigma)[\lambda] = g(y(\theta,\sigma)[\lambda]), \\ y(0,0)[\lambda] = \lambda \end{cases}$$

Applying the standard rule of stochastic derivation associated with the smooth mapping $\varphi(\theta,\sigma) := y(\theta,\sigma)[\lambda]$ and the continuous process $\theta = \theta(t;\lambda) = t\varphi(\lambda)$, $\sigma = w(t)$, we get that $\widehat{y}_\varphi(t;\lambda) = y(\theta(t;\lambda), w(t))$, $t\in[0,T]$, fulfils SDE (1.1), i.e.

$$(2.21)\quad \begin{cases} d_t\widehat{y}_\varphi(t;\lambda) = \varphi(\lambda) f(\widehat{y}_\varphi(t;x))dt + g(\widehat{y}_\varphi(t;\lambda))\circ dw(t),\ t\in[0,T], \\ \widehat{y}_\varphi(0;\lambda) = \lambda. \end{cases}$$

On the other hand, the unicity of the solution satisfying (1.1) lead us to the conclusion that $\widehat{x}_\varphi(t;\lambda) = \widehat{y}_\varphi(t;\lambda)$, $t\in[0,T]$, and (2.17) is proved. The conclusion (2.18) is a direct consequence of (2.17) combined with $\{\lambda = \widehat{\psi}(t,z) : t\in[0,T], z\in\mathbb{R}^n\}$ is the solution defined in Lemma 2.1. The proof is complete. □

Lemma 2.3. *Under the hypotheses in Lemma 2.2, consider the continuous and $\mathcal{F}_t$-adapted process $\widehat{z}(t,x) = G(-w(t))[x]$, $t\in[0,T]$, $x\in\mathbb{R}^n$. Then the following SPDE of parabolic type is valid*

$$(2.22)\quad \begin{cases} d_t\widehat{z}(t,x) + \partial_x\widehat{z}(t,x)g(x)\widehat{\circ}dw(t) = 0,\ t\in[0,T], x\, x\in\mathbb{R}^n, \\ \widehat{z}(0,x) = x \end{cases}$$

where the Fisk-Stratonovich integral "$\widehat{\circ}$" is computed by

$$h(t,x)\widehat{\circ}dw(t) = h(t,x)\cdot dw(t) - \frac{1}{2}\partial_x h(t,x)g(x)dt,$$

using Ito stochastic integral "$\cdot$".

Proof. The conclusion (2.22) is a direct consequence of applying standard rule of stochastic derivation associated with $\sigma = w(t)$ and smooth deterministic mapping $H(\sigma)[x] :=$

$G(-\sigma)[x]$. In this respect, using $H(\sigma)\circ G(\sigma)[\lambda]=\lambda\in\mathbb{R}^n$ for any $x=G(\sigma)[\lambda]$, we get

$$(2.23)\quad \begin{cases} \partial_\sigma\{H(\sigma)[x]\}=-\partial_x\{H(\sigma)[x]\}\cdot g(x),\ \sigma\in\mathbb{R},\ x\in\mathbb{R}^n,\\ \partial_\sigma^2\{H(\sigma)[x]\}=\partial_\sigma\{\partial_\sigma\{H(\sigma)[x]\}\}=\partial_\sigma\{-\partial_x\{H(\sigma)[x]\}\cdot g(x)\}\\ \qquad\qquad =\partial_x\{\partial_x\{H(\sigma)[x]\}\cdot g(x)\}\cdot g(x),\ \sigma\in\mathbb{R},\ x\in\mathbb{R}^n. \end{cases}$$

The standard rule of stochastic derivation lead us to SDE

$$(2.24)\quad d_t\hat{z}(t,x)=\partial_\sigma\{H(\sigma)[x]\}_{\sigma=w(t)}\cdot dw(t)+\frac{1}{2}\partial_\sigma^2\{H(\sigma)[x]\}_{\sigma=w(t)}dt,\ t\in[0,T],$$

and rewritting the right hand side of (2.24) (see (2.23)) we get SPDE of parabolic type given in (2.22). The proof is complete. □

Lemma 2.4. *Assume the hypotheses (2.1) and (2.2) are fulfilled and consider* $\{\lambda=\psi(t,x):t\in[0,T],x\in\mathbb{R}^n\}$ *defined in Lemma (2.2). Then* $u(t,x):=h(\psi(t,x))$, $t\in[0,T]$, $x\in\mathbb{R}^n$, $h\in(C_b^1\cap C^2)(\mathbb{R}^n)$, *satisfies the following nonlinear SPDE of parabolic type*

$$(2.25)\quad \begin{cases} d_tu(t,x)+\langle\partial_xu(t,x),f(x)\rangle\varphi(\psi(t,x))dt+\langle\partial_xu(t,x),g(x)\rangle\hat{\circ}dw(t)=0\\ u(0,x)=h(x),\ t\in[0,T],\ x\in\mathbb{R}^n, \end{cases}$$

where the Fisk-Stratonovich integral "$\hat{\circ}$" is computed by

$$h(t,x)\hat{\circ}dw(t)=h(t,x)\cdot dw(t)-\frac{1}{2}\partial_xh(t,x)g(x)dt.$$

Proof. By definition (see Lemma (2.2)), $\psi(t,x)=\hat{\psi}(t,\hat{z}(t,x))$, $t\in[0,T]$, where $\hat{z}(,x)=G(-w(t))[x]$ and $\{\hat{\psi}(t,z)\in\mathbb{R}^n:t\in[0,T],z\in\mathbb{R}^n\}$ satisfies nonlinear (H-J)-equations (2.7) of Lemma 2.1. In addition $\{\hat{z}(t,x)\in\mathbb{R}^n:t\in[0,T],x\in\mathbb{R}^n\}$ fulfils SPDE (2.22) in Lemma 2.3, i.e.

$$(2.26)\quad d_t\hat{z}(t,x)+\partial_x\hat{z}(t,x)\hat{\circ}dw(t)=0,\ t\in[0,T],\ x\in\mathbb{R}^n.$$

Applying the standard rule of stochastic derivation associated with the smooth mapping $\{\lambda=\hat{\psi}(t,z):t\in[0,T],z\in\mathbb{R}^n\}$ and stochastic process $\hat{z}(t,x):=G(-w(t))[x]=:H(w(t))[x]$, $t\in[0,T]$, we get the following nonlinear SPDE

$$(2.27)\quad \begin{cases} d_t\psi(t,x)+\partial_x\psi(t,x)f(x)\varphi(\psi(t,x))dt+\partial_x\psi(t,x)g(x)\hat{\circ}dw(t)=0,\\ \psi(0,x)=x,\ t\in[0,T]. \end{cases}$$

In addition, the functional $u(t,x) = h(\psi(t,x))$ can be rewritten $u(t,x) = \widehat{u}(t,\widehat{z}(t,x))$, where $\widehat{u}(t,z) := h(\widehat{\psi}(t,z))$ is a smooth deterministic functional satisfying nonlinear (H-J)-equations (see (2.7) of Lemma 2.1)

$$(2.28)\quad \begin{cases} \partial_t \widehat{u}(t,z) + \langle \partial_z \widehat{u}(t,z), f(z)\rangle \varphi(\widehat{\psi}(t,z)) = 0,\ t \in [0,T],\ z \in \mathbb{R}^n, \\ \widehat{u}(0,z) = h(z). \end{cases}$$

Using (2.26)and (2.28) we obtain SDPE fulfilled by $\{u(t,x)\}$,

$$(2.29)\quad \begin{cases} d_t u(t,x) + \langle \partial_z \widehat{u}(t,\widehat{z}(t,x)), f(\widehat{z}(t,x))\rangle \varphi(\psi(t,x))dt + \langle \partial_x u(t,x), g(x)\rangle \widehat{\circ} dw(t) = 0, \\ u(0,x) = h(x),\ t \in [0,T],\ x \in \mathbb{R}^n. \end{cases}$$

The hypothesis (2.1) allows us to write

$$(2.30)\quad \begin{aligned} \langle \partial_z \widehat{u}(t,\widehat{z}(t,x)), f(\widehat{z}(t,x))\rangle &= \partial_z \widehat{u}(t,\widehat{z}(t,x))[\partial_x \widehat{z}(t,x)][\partial_x \widehat{z}(t,x)]^{-1} f(\widehat{z}(t,x)) \\ &= \langle \partial_x u(t,x), f(x)\rangle,\ t \in [0,T],\ x \in \mathbb{R}^n, \end{aligned}$$

and using (2.30) into (2.29) we get the conclusion (2.25),

$$(2.31)\quad \begin{cases} \partial_t u(t,x) + \langle \partial_x u(t,x), f(x)\rangle \varphi(\psi(t,x))dt + \langle \partial_x u(t,x), g(x)\rangle \widehat{\circ} dw(t) = 0, \\ u(0,x) = h(x),\ t \in [0,T], x \in \mathbb{R}^n, \end{cases}$$

where the Fisk-Stratonovich integral "$\widehat{\circ}$" is computed by

$$(2.32)\quad h(t,x)\widehat{\circ} dw(t) = -\frac{1}{2}\partial_x h(t,x) g(x) dt + h(t,x) \cdot dw(t),$$

using Ito integral "$\cdot$". The proof is complete. □

Remark 2.3. The complete solution of Problem (P1) is contained in Lemmas 2.1–2.4. We shall rewrite them as a theorem.

Theorem 2.1. *Assume that the vector fields* $f \in (C_b \cap C_b^1 \cap C^2)(\mathbb{R}^n;\mathbb{R}^n)$, $g \in (C_b^1 \cap C^2)(\mathbb{R}^n;\mathbb{R}^n)$, *and scalar function* $\varphi \in (C_b^1 \cap C^2)(\mathbb{R}^n)$ *fulfil the hypotheses* (2.1) *and* (2.2). *Consider the continuous and* $\mathcal{F}_t$*-adapted process* $\{\lambda = \psi(t, x \in \mathbb{R}^n) : t \in [0,T], x \in \mathbb{R}^n\}$ *satisfying integral equations* (1.2). *Then* $u(t,x) := h(\psi(t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, *fulfils nonlinear SPDE of parabolic type* (2.25) *(see Lemma 2.4), for each* $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$.

2.2 Solution for the Problem (P2)

Using the same notations as in subsection 2.1, we consider the unique solution $\{\widehat{x}_\psi(s;t,x):$ $s\in[t,T]\}$ satisfying SDE (2.3) for each $0\leqslant t<T$ and $x\in\mathbb{R}^n$. As far as SDE (2.3) is a non-markovian system, the evolution of a functional $S(t,x):=Eh(\widehat{x}_\psi(T;t,x))$, $t\in[0,T]$, $x\in\mathbb{R}^n$, $h\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$, will be described using the pathwise representation of the conditional mean values functional

(2.33) $v(t,x)=E\{h(\widehat{x}_\psi(T;t,x))\mid\psi(t,x)\}$, $0\leqslant t<T$, $x\in\mathbb{R}^n$.

Assuming the hypotheses (2.1) and (2.2) we may and do write the following integral representation

(2.34) $\widehat{x}_\psi(T;t,x)=G(w(T)-w(t))\circ F[(T-t)\varphi(\psi(t,x))][x]$, $0\leqslant t<T$, $x\in\mathbb{R}^n$,

for a solution of SDE (2.3), where $G(\sigma)[z]$ and $F(\tau)[z]$, $\sigma,\tau\in\mathbb{R}$, $z\in\mathbb{R}^n$, are the global flows generated by $g,f\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$. The right side hand of (2.34) is a continuous mapping of the two independent random variables, $z_1=[w(T)-w(t)]\in\mathbb{R}$ and $z_2=\psi(t,x)\in\mathbb{R}^n$ ($\mathcal{F}_t$-measurable) for each $0\leqslant t<T$, $x\in\mathbb{R}^n$. A direct consequence of this remark is to use a parameterized random variable

(2.35) $y(t,x;\lambda)=G(w(T)-w(t))\circ F[(T-t)\varphi(\lambda)][x]$, $0\leqslant t<T$,

and to compute the conditional mean values (2.33) by

(2.36) $v(t,x)=[Eh(y(t,x;\lambda))](\lambda=\psi(t,x))$.

Here the functional

(2.37) $u(t,x;\lambda):=Eh(y(t,x;\lambda))$, $t\in[0,T]$, $x\in\mathbb{R}^n$,

satisfies a backward parabolic equation (Kolmogorov's equation) for each $\lambda\in\mathbb{R}^n$ and rewrite (2.36) as follows,

(2.38) $v(t,x)=u(t,x;\psi(t,x))$, $0\leqslant t<T$, $x\in\mathbb{R}^n$.

In conclusion, the functional $\{S(t,x)\}$ can be written as

(2.39) $S(t,x)=E[E\{h(\widehat{x}_\psi(T;t,x))\mid\psi(t,x)\}=Eu(t,x;\psi(t,x))$, $0\leqslant t<T$, $x\in\mathbb{R}^n$,

where $\{u(t,x;\lambda) : t \in [0,T], x \in \mathbb{R}^n\}$ satisfies the corresponding backward parabolic equations with parameter $\lambda \in \mathbb{R}^n$,

$$(2.40) \quad \begin{cases} \partial_t u(t,x;\lambda) + \langle \partial_x u(t,x;\lambda), f(x,\lambda)\rangle + \frac{1}{2}\langle \partial_x^2 u(t,x;\lambda) g(x), g(x)\rangle = 0, \\ u(T,x;\lambda) = h(x),\ f(x,\lambda) := \varphi(\lambda) f(x) + \frac{1}{2}[\partial_x g(x)] g(x). \end{cases}$$

We conclude these remarks by a theorem.

Theorem 2.2. *Assume that the vector fields f, g and the scalar function φ of SDE* (2.3) *fulfil the hypotheses* (2.1) (2.2), *where the continuous and $\mathcal{F}_t$-adapted process $\{\psi(t,x) \in \mathbb{R}^n : t \in [0,T]\}$ is defined in Theorem 2.1. Then the evolution of the functional*

$$(2.41) \quad S(t,x) := Eh(\widehat{x}_\psi(T;t,x)),\ t \in [0,T],\ x \in \mathbb{R}^n,\ h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$$

can be described as in (2.39), *where $\{u(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ satisfies linear backward parabolic equations* (2.40) *for each $\lambda \in \mathbb{R}^n$.*

Remark 2.4. Consider the case of several vector fields defining both the drift and diffusion of SDE (1.1), i.e.

$$(2.42) \quad \begin{cases} d_t\widehat{x} = [\sum_{i=1}^{m} \varphi_i(\lambda) f_i(\widehat{x})]dt + \sum_{i=1}^{m} g_i(\widehat{x}) \circ dw_i(t),\ t \in [0,T], \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n. \end{cases}$$

We notice that the analysis presented in Theorems 2.1 and 2.2 can be extended to this multiple vector fields case (see next section).

3 Multiple vector fields case

We are given two finite sets of vector fields $\{f_1, \ldots, f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ and $\{g_1, \ldots, g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n; \mathbb{R}^n)$ and consider the unique solution $\{\widehat{x}_\varphi(t,\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$ of SDE

$$(3.1) \quad \begin{cases} d_t\widehat{x} = [\sum_{i=1}^{m} \varphi_i(\lambda) f_i(\widehat{x})]dt + \sum_{i=1}^{m} g_i(\widehat{x}) \circ dw_i(t),\ t \in [0,T],\ \widehat{x} \in \mathbb{R}^n, \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n \end{cases}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m) \subset (C_b^1 \cap C^2)$ are fixed and $w = (w_1(t), \ldots, w_m(t)) \in \mathbb{R}^m$ is a standard Wiener process over a complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}_t\}, P\}$. Each Fisk-Stratonovich integral "$\circ$" in (3.1) is computed by

(3.2) $g_i(x) \circ dw_i(t) = g_i(x) \cdot dw_i(t) + \frac{1}{2}[\partial_x g_i(x)] g_i(x) dt,$

using Ito integral "$\cdot$".

Assume that $\{\lambda = \psi(t, x) \in \mathbb{R}^n : t \in [0, T], x \in \mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}_t$-adapted solution satisfying integral equations

(3.3) $\widehat{x}_\varphi(t; \lambda) = x \in \mathbb{R}^n,\ t \in [0, T].$

For each $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$, associate stochastic functionals $\{u(t, x) = h(\psi(t, x) : t \in [0, T], x \in \mathbb{R}^n)\}$ and deterministic mappings $\{S(t, x) = Eh(\widehat{x}_\psi(T; t, x)) : t \in [0, T], x \in \mathbb{R}^n\}$, where $\{\widehat{x}_\psi(s; t, x) : s \in [t, T], x \in \mathbb{R}^n\}$ satisfies the following SDE

$$\begin{cases} d_s\widehat{x} = [\sum_{i=1}^{m} \varphi_i(\psi(t, x)) f_i(\widehat{x})] ds + \sum_{i=1}^{m} g_i(\widehat{x}) \circ dw_i(t),\ s \in [t, T], \\ \widehat{x}(t) = x. \end{cases}$$

Problem (P1). Assume that

(3.4) $\begin{cases} M = \{f_1, \ldots, f_m, g_1, \ldots, g_m\} \text{ are muttualy commuting using Lie bracket i.e.} \\ [X_1, X_2](x) = 0 \text{ for any pair } X_1, X_2 \in M \end{cases}$

(3.5) $TV_i K_i = \rho_i \in [0, \frac{1}{m}),$

where $V_i := \sup\{|\partial_x \varphi_i(x)| : x \in \mathbb{R}^n\}$ and $K_i = \{|f_i(x)| : x \in \mathbb{R}^n\}$, $i = 1, \ldots, m$.

Under the hypotheses (3.4) and (3.5), find the nonlinear SPDE of parabolic type satisfied by $\{u(t, x) = h(\psi(t, x)), t \in [0, T], x \in \mathbb{R}^n\}$, $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$, where $\{\lambda = \psi(t, x) \in \mathbb{R}^n : t \in [0, T], x \in \mathbb{R}^n\}$ is the unique continuous and $\mathcal{F}_t$-adapted solution of integral equations (3.3).

Problem (P2). Using $\{\lambda = \psi(t, x) \in \mathbb{R}^n : t \in [0, T], x \in \mathbb{R}^n\}$ found in (P1), describe the evolution of a functional $S(t, x) = Eh(\widehat{x}_\psi(T; t, x))$ using backward parabolic equations, where $\{\widehat{x}_{\psi}(s;t,x) : s \in [t,T]\}$ is the unique solution of SDE

$$(3.6) \quad \begin{cases} d_s\widehat{x} = [\sum_{i=1}^{m} \varphi_i(\psi(t,x)) f_i(\widehat{x})]ds + \sum_{i=1}^{m} g_i(\widehat{x}) \circ dw_i(s), \; s \in [t,T], \\ \widehat{x}(t) = \widehat{x} \in \mathbb{R}^n. \end{cases}$$

3.1 Solution for (P1)

Under the hypotheses (3.4) and (3.5), the unique solution of SPDE (3.1) can be represented by

$$(3.7) \quad \widehat{x}_{\varphi}(t;\lambda) = G(w(t)) \circ F(\theta(t;\lambda))[\lambda] =: H(t, w(t); \lambda)$$

where

$$G(\sigma)[z] = G_1(\sigma_1) \circ \cdots \circ G_m(\sigma_m)[z], \; \sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m,$$

$$F(\sigma)[z] = F_1(\sigma_1) \circ \cdots \circ F_m(\sigma_m)[z], \; \theta(t;\lambda) = (t\varphi_1(\lambda), \ldots, t\varphi_m(\lambda)) \in \mathbb{R}^m \text{ and}$$

$$\{(F_i(\sigma_i)[z], G_i(\sigma_i)[z]) : \sigma_i \in \mathbb{R}, z \in \mathbb{R}^n\}$$

are the global flows generated by (f_i, g_i), $i \in \{1, \ldots, m\}$.

The arguments for solving (P1) in the case of one pair (f, g) of vector fields (see subsection (2.1)) can be used also here and we get the following similar results. Under the representation (3.7), the unique continuous and $\mathcal{F}_t$-adapted solution $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ solving equations

$$(3.8) \quad \widehat{x}_{\varphi}(t;\lambda) = x \in \mathbb{R}^n, \; t \in [0,T]$$

will be found as a composition

$$(3.9) \quad \psi(t,x) = \widehat{\psi}(t, \widehat{z}(t,x)), \; \widehat{z}(t,x) := G(-w(t))[x].$$

Here $\lambda = \widehat{\psi}(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}^n$ is the unique solution satisfying deterministic integral eqations

$$(3.10) \quad \lambda = F(-\theta(t;\lambda))[z] =: \widehat{V}(t,z;\lambda), \; t \in [0,T], \; z \in \mathbb{R}^n.$$

Lemma 3.1. *Asume that* (3.4) *and* (3.5) *is fulfilled. Then there exists a unique smooth mapping* $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ *solving deterministic integral equations* (3.10) *such that*

$$(3.11)\quad \begin{cases} F(\theta(t;\widehat{\psi}(t,z)))[\widehat{\psi}(t,z)] = z \in \mathbb{R}^n,\ t \in [0,T], \\ |\widehat{\psi}(t,z) - z| \leqslant R(T,z) := \dfrac{r(T,z)}{1-\rho},\ t \in [0,T],\ z \in \mathbb{R}^n, \end{cases}$$

where $\rho = \rho_1 + \cdots + \rho_m \in [0,1)$ *and* $r(T,z) = T\sum_{i=1}^m K_i|\varphi_i(z)|$.

In addition, the following nonlinear (H-J)-equation is valid

$$(3.12)\quad \begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)[\sum_{i=1}^m \varphi_i(\widehat{\psi}(t,z))f_i(z)] = 0,\ t \in [0,T],\ z \in \mathbb{R}^n, \\ \widehat{\psi}(0,z) = z. \end{cases}$$

The proof is based on the arguments of Lemma 2.1 in subsection 2.1.

Lemma 3.2. *Assume that* (3.4) *and* (3.5) *are satisfied and consider* $\{\lambda = \widehat{\psi}(t,z) \in \mathbb{R}^n : t \in [0,T], z \in \mathbb{R}^n\}$ *found in Lemma* (3.1). *Then the stochastic flow generated by SDE* (3.1) *fulfils*

(3.13) $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$ *can be represented as in* (3.7),

$$(3.14)\quad \psi(t,x) = \widehat{\psi}(t,\widehat{z}(t,x)),\ \textit{is the unique solution of } (3.8), \qquad \textit{where } \widehat{z}(t,x) = G(-w(t))[x].$$

The proof follows the arguments used in Lemma 2.2 of section 2.1.

Lemma 3.3. *Under the hypothesis* (3.4), *consider the continuous and* $\mathcal{F}_t$*-adapted process* $\widehat{z}(t,x) = G(-w(t))[x]$, $t \in [0,T]$, $x \in \mathbb{R}^n$. *Then the following SPDE of parabolic type is valid*

$$(3.15)\quad \begin{cases} d_t\widehat{z}(t,x) + \sum_{i=1}^m \partial_x\widehat{z}(t,x)g_i(x)\widehat{\circ}dw_i(t) = 0,\ t \in [0,T],\ x \in \mathbb{R}^n \\ \widehat{z}(0,x) = x, \end{cases}$$

where the Fisk-Stratonovich integral "$\widehat{\circ}$*" is computed by*

$$h_i(t,x)\widehat{\circ}dw_i(t) = h_i(t,x)\cdot dw_i(t) - \frac{1}{2}\partial_x h_i(t,x)g_i(x)dt$$

using Ito stochastic integral "$\cdot$*".*

Proof. The conclusion (3.15) is a direct consequence of applying standard rule of stochastic derivation associated with $\sigma = w(t) \in \mathbb{R}^m$ and smooth deterministic mapping $H(\sigma)[x] = G(-\sigma)[x]$. In this respect, using $H(\sigma) \circ G(\sigma)[\lambda] = \lambda \in \mathbb{R}^n$ for any $x = G(\sigma)[\lambda]$, we get

$$(3.16) \quad \begin{cases} \partial_{\sigma_i} H(\sigma)[x] = -\partial_x\{H(\sigma)[x]\}g_i(x),\ \sigma = (\sigma_1, \dots, \sigma_m) \in \mathbb{R}^m,\ x \in \mathbb{R}^n, \\ \partial^2_{\sigma_i}\{H(\sigma)[x]\} = \partial_{\sigma_i}\{\partial_{\sigma_i}\{H(\sigma)[x]\}\} = \partial_{\sigma_i}\{-\partial_x\{H(\sigma)[x]\}g_i(x)\} \\ \qquad = \partial_x\{\partial_x\{H(\sigma)[x]\}g_i(x)\}g_i(x),\ \sigma \in \mathbb{R}^m,\ x \in \mathbb{R}^n \end{cases}$$

for each $i \in \{1, \dots, m\}$. Recall that the standard rule of stachastic derivation lead us to SDE

$$(3.17) \quad d_t\widehat{z}(t,x) = \sum_{i=1}^{m} \partial_{\sigma_i}\{H(\sigma)[x]\}_{(\sigma=w(t))} \cdot dw_i(t) + \frac{1}{2}\sum_{i=1}^{m} \partial^2_{\sigma_i}\{H(\sigma)[x]\}_{(\sigma=w(t))}dt,$$

for any $t \in [0,T]$, $x \in \mathbb{R}^n$. Rewritting the right hand side of (3.17) (see (3.16)) we get SPDE of parabolic type given in (3.15). □

Lemma 3.4. *Assume the hypotheses* (3.4) *and* (3.5) *are fulfilled and consider* $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ *defined in Lemma 3.2. Then* $u(t,x) := h(\psi(t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$, *satisfies the following nonlinear SPDE*

$$(3.18) \quad \begin{cases} d_t u(t,x) + \langle \partial_x u(t,x), \sum_{i=1}^{m} \varphi_i(\psi(t,x) f_i(x))\rangle dt \\ \qquad + \sum_{i=1}^{m} \langle \partial_x u(t,x), g_i(x)\rangle \widehat{\circ} dw_i(t) = 0,\ t \in [0,T] \\ u(0,x) = h(x) \end{cases}$$

where the nonstandard Fisk-Stratonovich integral "$\widehat{\circ}$" is computed by

$$h_i(t,x)\widehat{\circ}dw_i(t) = h_i(t,x) \cdot dw_i(t) - \frac{1}{2}\partial_x h_i(t,x) g_i(x) dt.$$

The proof uses the same arguments as in Lemma 2.4 of section 2.1.

Theorem 3.3. *Assume that the vector fields* $\{f_1, \dots, f_m\} \subset (C_b \cap C_b^1 \cap C^2)(\mathbb{R}^n; \mathbb{R}^n)$, $\{g_1, \dots, g_m\} \subset (C_b^1 \cap C^2)(\mathbb{R}^n; \mathbb{R}^n)$ *and scalar functions* $\{\varphi_1, \dots, \varphi_m\} \subset (C_b^1 \cap C^2)(\mathbb{R}^n)$ *fulfil the hypotheses 3.4 and 3.5*

Consider the continuous and $\mathcal{F}_t$-adapted process $\{\lambda = \psi(t,x) \in \mathbb{R}^n : t \in [0,T], x \in \mathbb{R}^n\}$ satisfying integral equations (3.8) (see Lemma 3.2). Then $\{u(t,x) := h(\psi(t,x)) : t \in [0,T], x \in \mathbb{R}^n\}$ fulfils nonlinear SPDE of parabolic type (3.18) (see Lemma 3.4) for each $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$.

3.2 Solution for (P2)

As far as SDE (3.6) is a non-markovian system, the evolution of a functional $S(t,x) := Eh(\widehat{x}_\psi(T;t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, for each $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ will be described using the pathwise representation of the conditioned mean values functional

(3.19) $v(t,x) := E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\}$, $0 \leqslant t < T$, $x \in \mathbb{R}^n$.

Here $\widehat{x}_\psi(T;t,x)$ can be expressed using the following integral representation

(3.20) $\widehat{x}_\psi(T;t,x) = G(w(T) - w(t)) \circ F[(T-t)\varphi(\psi(t,x))](x)$, $0 \leqslant t < T$,

where $G(\sigma)[z]$ and $F(\sigma)[z]$, $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, are defined in (P1) (see (3.7)) for $\varphi := (\varphi_1, \ldots, \varphi_m)$. The right hand side of (3.20) is a continuous mapping of the two independent random variables, $z_1 = w(T) - w(t) \in \mathbb{R}^m$ and $z_2 = \psi(t,x) \in \mathbb{R}^n$ ($\mathcal{F}_t$-measurable) for each $0 \leqslant t < T$, $x \in \mathbb{R}^n$.

Using the parameterized random variable

(3.21) $y(t,x;\lambda) = G(w(T) - w(t)) \circ F[(T-t)\varphi(\lambda)](x)$, $0 \leqslant t < T$

we may and do compute the functional $v(t,x)$ in (3.19) by

(3.22) $v(t,x) = [Eh(y(t,x;\lambda))](\lambda = \psi(t,x))$, $0 \leqslant t < T$, $x \in \mathbb{R}^n$.

Here, the functional

(3.23) $u(t,x;\lambda) = Eh(y(t,x;\lambda))$, $t \in [0,T]$, $x \in \mathbb{R}^n$,

satisfies a backward parabolic equation (Kolmogorov's equation) for each $\lambda \in \mathbb{R}^n$ and rewrite (3.22) as follows,

(3.24) $v(t,x) = u(t,x;\psi(t,x))$, $0 \leqslant t < T$, $x \in \mathbb{R}^n$.

In conclusion, the functional $S(t,x) = Eh(\widehat{x}_\psi(T;t,x))$ can be reprezented by

$$(3.25)\quad S(t,x) = E[E\{h(\widehat{x}_\psi(T;t,x)) \mid \psi(t,x)\}] = Eu(t,x;\psi(t,x))$$

for any $0 \leqslant t < T$, $x \in \mathbb{R}^n$, where $\{u(t,x;\lambda) : t \in [0,T], x \in \mathbb{R}^n\}$ satisfies the corresponding backward parabolic equations with parameter $\lambda \in \mathbb{R}^n$,

$$(3.26)\quad \begin{cases} \partial_t u(t,x;\lambda) + \langle \partial_x u(t,x;\lambda), f(x,\lambda)\rangle + \dfrac{1}{2}\displaystyle\sum_{i=1}^{m} \langle \partial_x^2 u(t,x;\lambda) g_i(x), g_i(x)\rangle = 0, \\ u(T,x;\lambda) = h(x),\ f(x,\lambda) = \displaystyle\sum_{i=1}^{m} \varphi_i(\lambda) f_i(x) + \dfrac{1}{2}\sum_{i=1}^{m} [\partial_x g_i(x)] g_i(x). \end{cases}$$

We conclude these remarks by a theorem.

Theorem 3.4. *Assume that the vector fields* $\{f_1,\ldots,f_m\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, $\{g_1,\ldots,g_m\} \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$, *and scalar functions* $\varphi = (\varphi_1,\ldots,\varphi_m) \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ *of SDE* (3.6) *fulfil the hypotheses* (3.4) *and* (3.5)*. Then the evolution of the functional*

$$(3.27)\quad S(t,x) := Eh(\widehat{x}_\psi(T;t,x)),\ t \in [0,T],\ x \in \mathbb{R}^n,\ h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$$

can be described as in (3.25)*, where* $\{u(t,x;\lambda) : t \in [0,T], x \in \mathbb{R}^n\}$ *satisfies linear backward parabolic equations* (3.26)*, for each* $\lambda \in \mathbb{R}^n$.

Final remark. One may wonder about the meaning of the martingale representation associated with the non-markovian functionals $h(\widehat{x}_\psi(T;t,x))$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$. In this respect, we may use the parameterized functional $\{u(t,x;\lambda) : t \in [0,T], x \in \mathbb{R}^n\}$ fulfilling backward parabolic equations (3.26). Write

$$(3.28)\quad h(\widehat{x}_\psi(T;t,x)) = u(T,\widehat{x}_\psi(T;t,x);\widehat{\lambda} = \psi(t,x))$$

and apply the standard rule of stochastic derivation associated with smooth mapping $\{u(s,x;\widehat{\lambda}) : s \in [0,T], x \in \mathbb{R}^n\}$ and stochastic process $\{\widehat{x}_\psi(s;t,x) : s \in [t,T]\}$. We get

$$(3.29)\quad \begin{aligned} h(\widehat{x}_\psi(T;t,x)) =& u(t,x;\widehat{\lambda}) + \int_t^T (\partial_s + L_{\widehat{\lambda}})(u)(s,\widehat{x}_\psi(s;t,x);\widehat{\lambda}) ds \\ &+ \sum_{i=1}^{m} \int_t^T \langle \partial_x u(s,\widehat{x}_\psi(s;t,x);\widehat{\lambda}), g_i(x)\rangle dw_i(s), \end{aligned}$$

where $L_{\widehat{\lambda}}(u)(s,x;\widehat{\lambda}) := \langle \partial_x u(s,x;\widehat{\lambda}), f(x,\widehat{\lambda})\rangle + \frac{1}{2}\sum_{i=1}^{m}\langle \partial_x^2 u(s,x;\widehat{\lambda})g_i(x), g_i(x)\rangle$ coincides with parabolic operator in PDE (3.26). Using (3.26) for $\widehat{\lambda} = \psi(t,x)$, we obtain the following martingale representation

$$(3.30)\quad h(\widehat{x}_\psi(T;t,x)) = u(t,x;\psi(t,x)) + \sum_{i=1}^{m}\int_t^T \langle \partial_x u(s,\widehat{x}_\psi(s;t,x);\widehat{\lambda}), g_i(x)\rangle \cdot dw_i(s),$$

which shows that the standard constant in the markovian case is replaced by a $\mathcal{F}_t$-measurable random variable $u(t,x;\psi(t,x))$. In addition, the backward evolution of stochastic functional $\{Q(t,x) := h(\widehat{x}_\psi(T;t,x)) : t \in [0,T], x \in \mathbb{R}^n\}$ given in (3.30) depends essentially on the forward evolution process $\{\psi(t,x)\}$ for each $t \in [0,T]$ and $x \in \mathbb{R}^n$.

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II. Functionals associated with gradient stochastic flows and nonlinear SPDEs

1 Introduction

The investigation of evolution equations with stochastic perturbations serves a large variety of areas of applicability, among which mathematical finance as well. Pardoux and Peng (see, e.g., [11]) are dealing with systems of quasilinear backward parabolic stochastic partial differential equations driven by a stochastic Itô integral, and they provide a probabilistic representation for its unique classical solution via a system of backward "doubly stochastic" differential equations (BDSDE for short). Conversely, the solution of the latter is completely determined by the solution of the former (see the proof of Theorem 3.1).

It is well known the applicability of backward SDEs (BSDEs) in mathematical finance, for instance in the analysis of dynamic risk measures, as in Barrieu and El Karoui ([1]), in contingent claim valuation problems with constraints or in the theory of recursive utilities (see El Karoui et all [4]), or in term structure problems (as it is mentioned in a series of lecture by Josef Teichmann). Nonlinear SPDEs have applications in modelling of interest rates, in stochastic control with partial information (as it is specified in Lions and Souganidis [9]) etc. Other applications of SPDEs (including finance) may be found in Da Prato and Tubaro ([3]).

In Buckdahn and Ma ([2]) the authors consider a system of nonlinear SPDEs driven by Fisk-Stratonovich integrals with the diffusion term independent of the gradient of the solution, for which they prove, under weak conditions on the coefficients, the existence (and the uniqueness in a latter paper) of the so called stochastic viscosity solution, introduced by Lions and Souganidis for a general class of SPDEs in [9]. The approach is based on the previous work of Lions and Souganidis, by transforming the SPDE in a PDE with random coefficients, via a Doss-Sussman type transformation, which can be solved pathwisely. They use a perturbation method by considering the stochastic flow associated with the SDE (the so called stochastic characteristics, see also [9]) generated by the Stratonovich integral appearing in the SPDE. Similar techniques were used by Iftimie and Vârsan (see [5]) in the study of some evolution equations with stochastic perturbations of the same form as in [2], and where Doss-Sussman transformations given by Langevin's smooth approximations of Brownian motion were considered, and not the usual ones obtained by mollification of the Brownian motion or piecewise linear approximations.

In this paper we are dealing with the initial value problem associated to the nonlinear SPDE, considered in the classical sense

$$(1.1)\quad \begin{cases} \mathrm{d}u(t,x) & = \langle \nabla u(t,x), g_0(x)\rangle\, u(t,x)\mathrm{d}t + \sum_{i=1}^{m} \langle \nabla u(t,x), g_i(x)\rangle \circ \mathrm{d}W_i(t), \\ u(0,x) & = \varphi(x), t \in [0,T], x \in \mathbb{R}^n, \end{cases}$$

or equivalently

$$(1.2)\quad u(t,x) = \varphi(x) + \int_0^t \langle \nabla u(s,x), g_0(x)\, u(s,x)\rangle \mathrm{d}s + \sum_{i=1}^{m} \int_0^t \langle \nabla u(s,x), g_i(x)\rangle \circ \mathrm{d}W_i(s),$$

where the stochastic integral is understood in the Fisk-Stratonovich sense.

A main assumption is the commuting property of the vector fields $g_i, i = 0, \ldots, m$ with respect to the usual Lie bracket (see Assumption (**A**.4)), which is also known as a compatibility condition ([2], Remark 3.3) concerning the mentioned vector fields. This leads us to a gradient representation for the stochastic flow associated with the stochastic differential equation obtained by means of the (stochastic) system of characteristics defined by (1.1) (which is defined in analogy to the deterministic PDEs) and the corresponding fundamental solution $\psi(t,x)$ of the same SPDE. $\psi(t,x)$ will be described as the composition between the fundamental solution of deterministic nonlinear Hamilton-Jacobi equations (see Lemma 3.4 below)and the fundamental solution of a reduced SPDE (see equation (3.8)). We cannot expect that the property of unicity holds for this SPDE, and this is due to the strong nonlinearity nature of the problem. Notice that the drift function is not Lipschitz with respect to $(u, \nabla u)$, as it was the case in [11] and [2]. An example is also provided, for a model of Navier-Stokes equations with stochastic perturbations for which constant vector fields, both in the drift and diffusion part, are used in order to derive the existence of a global classical solution.

We are next interested in computing expectations of functionals involving the solution of some SDE, which is naturally related with the SDE obtained by writing the system of characteristics associated to system (1.1). This is accomplished by considering an appropriate conditional expectation, which satisfies a parameterized backward parabolic equation.

2 Preliminaries

Let $\{W(t), t \geqslant 0\}$ be a m-dimensional Wiener process on a complete filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\}$, where the filtration $\{\mathcal{F}_t\}$ stands for the augmentation under P of the natural filtration $\{\mathcal{F}_t^W\}$ generated by the Brownian motion. T is a fixed time horizon. We shall make use of the following assumptions

(A1) the vector fields $g_1, \dots, g_m$ belong to $\mathcal{C}_b^2(\mathbb{R}^n; \mathbb{R}^n)$; $g_0 \in \mathcal{C}_b^1(\mathbb{R}^n; \mathbb{R}^n)$ and is bounded.

(A2) the initial condition $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$ and admitts bounded first order partial derivatives.

(A3) $\rho := TMK < 1$, where $M := \sup\{|\nabla\varphi(x)|, x \in \mathbb{R}^n\}$ and $K := \sup\{|g_0(x)|, x \in \mathbb{R}^n\}$.

Throughout this paper we shall use the notations $\langle , \rangle$ for the inner product and ∇h for the gradient with respect to x of some function $h(t, x)$.

If $Y(t)$ and $X(t)$ are continuous one-dimensional semimartingales, the Fisk-Stratonovich integral of $Y(t)$ with respect to $X(t)$ is defined as

$$(2.1) \quad \int_0^t Y(s) \circ \mathrm{d}X(s) := \int_0^t Y(s)\mathrm{d}X(s) + \frac{1}{2}\langle Y, X\rangle_t,$$

where the stochastic integral entering the right hand side is the usual Itô integral and $\langle Y, X\rangle_t$ stands for the quadratic variation of the processes$(Y(t))$ and $(X(t))$. If $Y(t)$ is d-dimensional, we can still define the integral $\int_0^t Y(s) \circ \mathrm{d}X(s) := (\int_0^t Y_i(s) \circ \mathrm{d}X(s))_{1\leqslant i\leqslant d}$. We state the Itô's formula involving the Fisk-Stratonovich integral (see, e.g., [7], Problem 3.14, page 156 or [12], Theorem 34, page 82).

Proposition 1. *Let $Y(t)$ be a d-dimensional continuous semimartingale and $f : \mathbb{R}^d \to \mathbb{R}^k$ a vector function with the components belonging to $\mathcal{C}^3(\mathbb{R}^d)$. Then*

$$(2.2) \quad f(Y(t)) = f(Y(0)) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(Y(s)) \circ \mathrm{d}Y_i(s).$$

We shall also need the following result

Lemma 1. *Let $X(t), Y(t)$ be continuous semimartingales with decompositions $X(t) = X(0) + A(t) + \int_0^t M(s)\mathrm{d}W(s)$ and $Y(t) = X(0) + B(t) + \int_0^t N(s)\mathrm{d}W(s)$, where $A(t), B(t)$ are adapted, continuous processes with bounded variation and the processes defined by the stochastic integrals are (local) martingales (this decomposition holds for any continuous semimartingale, since the filtration $(\mathcal{F}_t)$ stands for the completion of the natural filtration generated by W, see [12], Theorem 43, Chapter IV). Then*

$$\int_0^t X(s) \circ \mathrm{d}\left(\int_0^s Y(r) \circ \mathrm{d}W(r)\right) = \int_0^t X(s)Y(s) \circ \mathrm{d}W(s). \tag{2.3}$$

Proof. The first term of the left hand side of the formula can be written as

$$\begin{aligned}
\int_0^t X(s) \circ \mathrm{d}\left(\int_0^s Y(r) \circ \mathrm{d}W(r)\right) &= \int_0^t X(s) \circ \mathrm{d}\left(\int_0^s Y(r)\mathrm{d}W(r) + \frac{1}{2}\int_0^s N(r)\mathrm{d}r\right)\\
&= \int_0^t X(s)\mathrm{d}\left(\int_0^s Y(r)\mathrm{d}W(r) + \frac{1}{2}\int_0^s N(r)\mathrm{d}r\right) + \frac{1}{2}\int_0^t M(s)Y(s)ds\\
&= \int_0^t X(s)Y(s)\mathrm{d}W(s) + \frac{1}{2}\int_0^t X(s)N(s)ds + \frac{1}{2}\int_0^t M(s)Y(s)ds\\
&= \int_0^t X(s)Y(s)\mathrm{d}W(s) + \frac{1}{2}\langle XY, W\rangle_t,
\end{aligned}$$

where the integration by parts formula (for semimartingales) was also used. □

The corresponding system of characteristics (see, e.g., [8], Chapter 6) is given by

$$\begin{cases}
\mathrm{d}\widehat{x}(t;\lambda) &= -\widehat{u}(t;\lambda)g_0(\widehat{x}(t;\lambda))\mathrm{d}t + \sum_{i=1}^m(-g_i)(\widehat{x}(t;\lambda)) \circ \mathrm{d}W_i(t);\\
\widehat{x}(0,\lambda) &= \lambda;\\
\mathrm{d}\widehat{u}(t,\lambda) &= 0, \widehat{u}(0,\lambda) = \varphi(\lambda); \lambda \in \mathbb{R}^n,
\end{cases} \tag{2.4}$$

Remark 1. *Notice that the integrals $\int_0^t(-g_i)(\widehat{x}(s;\lambda)) \circ \mathrm{d}W_i(s)$ and $-\int_0^t g_i(\widehat{x}(s;\lambda)) \circ \mathrm{d}W_i(s)$ are not equal.*

We deduce $\widehat{u}(t,\lambda) = \varphi(\lambda)$ and $\widehat{x}$ is the solution of the SDEs

$$\begin{aligned}
\widehat{x}(t;\lambda) &= \lambda - \varphi(\lambda)\int_0^t g_0(\widehat{x}(s;\lambda))\mathrm{d}s + \sum_{i=1}^m \int_0^t (-g_i)(\widehat{x}(s;\lambda)) \circ \mathrm{d}W_i(s)\\
&= \lambda - \int_0^t \left[\varphi(\lambda)g_0(\widehat{x}(s;\lambda)) - \frac{1}{2}\nabla g_i(\widehat{x}(s;\lambda))g_i(\widehat{x}(s;\lambda))\right]\mathrm{d}s\\
&\quad - \sum_{i=1}^m \int_0^t g_i(\widehat{x}(s;\lambda))\mathrm{d}W_i(s).
\end{aligned} \tag{2.5}$$

According to the formula (2.1) the (local) martingale part of $\int_0^t (-g_i)(\widehat{x}(s;\lambda)) \circ \mathrm{d}W_i(s)$ is given by - $\int_0^t g_i(\widehat{x}(s;\lambda))\mathrm{d}W_i(s)$, which is also the (local) martingale part of the process $\widehat{x}(t;\lambda)$ (see (2.4)). Hence, by virtue of Itô's Lemma, the martingale part of $(-g_i)(\widehat{x}(t;\lambda))$ is $\int_0^t \nabla g_i(\widehat{x}(s;\lambda))g_i(\widehat{x}(s;\lambda))\mathrm{d}W_i(s)$ and it implies that

$$\langle(-g_i^j)(\widehat{x}(\cdot;\lambda)), W_i(\cdot)\rangle_t = \int_0^t (\nabla g_i(\widehat{x}(s;\lambda))g_i(\widehat{x}(s;\lambda)))^j \,\mathrm{d}s,$$

for $j = 1,\dots,n$.

The assumptions imposed on the coefficients $g_i, i = 0,\dots,m$ ensure the existence of a unique solution $\widehat{x}_\varphi(t;\lambda)$ of the system (2.5). Under the same assumptions, the vector fields $g_i, i = 0, 1,\dots,m$ are complete, i.e. they generate globally defined flows $G_i(t,x) = G_i(t)(x)$, satisfying

$$\frac{\partial G_i}{\partial t}(t,x) = g_i(G_i(t,x)), \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^n;\ G_i(0,x) = x.$$

It is well known that for each t, $G_i(t)(\cdot)$ is a diffeomorphism, the map $(t,x) \in \mathbb{R}\times\mathbb{R}^n \mapsto G_i(t,x)$ is smooth and $G_i(t_1+t_2,x) = G_i(t_1)(G_i(t_2,x))$. Last property implies that $(G_i(t))^{-1}(\cdot) = G_i(-t)(\cdot) := H_i(t)(\cdot)$. We define $G(p)(x)$, $p = (t_1,\dots,t_m) \in \mathbb{R}^m, x \in \mathbb{R}^n$ as the composition of the flows associated to $g_1,\dots,g_m$, i.e.

(2.6) $G(p)(x) = G(p,x) := G_1(t_1) \circ \dots G_m(t_m)(x)$.

We assume from now on that the vector fields $g_0,\dots,g_m$ commute, i.e. the Lie bracket

(A4) $[g_i, g_j](x) := \nabla g_i(x)g_j(x) - \nabla g_j(x)g_i(x) = 0,$

and this means $G_i(t_i)\circ G_j(t_j) = G_j(t_j)\circ G_i(t_i)$, for $0 \leqslant i,j \leqslant m$. As a consequence, $G(p,x)$ is the solution of the gradient system defined by the original vector fields, i.e.

$$\frac{\partial G}{\partial t_i}(p,x) = g_i(G(p,x)).$$

Set also $H(p,x) := G(-p,x)$, for $p = (t_1,\dots,t_m)$.

3 Gradient representation of stochastic flow and construction of a solution of nonlinear SPDE

Next lemma provides a gradient representation for the stochastic flow $\widehat{x}_\varphi(t;\lambda)$.

Lemma 2. *The stochastic flow generated by the solution of the SDEs (2.5) can be represented as*

(3.1) $\widehat{x}_\varphi(t;\lambda) = G(-W(t)) \circ G_0(-t\varphi(\lambda))(\lambda) = H(W(t)) \circ H_0(t\varphi(\lambda))(\lambda)$.

Proof. Set $v(t,y) := G(-y) \circ G_0(-t\varphi(\lambda))(\lambda)$. It is obvious that $v \in \mathcal{C}^{1,3}(\mathbb{R} \times \mathbb{R}^m; \mathbb{R}^n)$ and a slightly modified version of Proposition 1 leads us to

$$\begin{aligned} v(t,W(t)) &= \lambda + \int_0^t \frac{\partial v}{\partial t}(s,W(s))\mathrm{d}s + \sum_{i=1}^m \int_0^t \frac{\partial v}{\partial y_i}(s,W(s)) \circ \mathrm{d}W_i(s) \\ &= \lambda - \varphi(\lambda) \int_0^t g_0(v(s,W(s)))\mathrm{d}s + \sum_{i=1}^m \int_0^t (-g_i)(v(s,W(s))) \circ \mathrm{d}W_i(s). \end{aligned}$$

The result follows by uniqueness of solutions of SDEs. □

The next step consists in finding the inverse mapping of the diffeomorphism $\lambda \to \widehat{x}_\varphi(t;\lambda)$, i.e. we solve the equation

(3.2) $\widehat{x}_\varphi(t;\lambda) = x$

with respect to the unknown λ. Taking into account the formula (3.1) and the properties of flows G_i (which are preserved by G), this is equivalent with

$$G_0(-t\varphi(\lambda))(\lambda) = G(W(t))(x) := z(t,x)$$

Consider first the equation $G_0(-t\varphi(\lambda))(\lambda) = z$, for arbitrary $t \in [0,T]$ and $z \in \mathbb{R}^n$, which can be rewritten as

(3.3) $G_0(t\varphi(\lambda))(z) = \lambda$.

Set $V(t,z,\lambda) := G_0(t\varphi(\lambda))(z)$.

Lemma 3. *The equation (3.3) admitts a unique solution given by a (deterministic) smooth mapping $\widehat{\psi}(t,z) \in \mathcal{C}^{1,1}([0,T] \times \mathbb{R}^n)$, such that*

$$|\widehat{\psi}(t,z) - z| \leqslant \frac{TK}{1-\rho}|\varphi(z)|.$$

In addition, $\widehat{\psi}(t,z)$ is the unique solution of the Hamilton-Jacobi equation

$$(3.4)\quad \begin{cases} \frac{\partial \widehat{\psi}}{\partial t}(t,z) &= \nabla\widehat{\psi}(t,z)\, g_0(z)\, \varphi(\widehat{\psi}(t,z)), \\ \widehat{\psi}(0,z) &= z. \end{cases}$$

Proof. Notice that the mapping $\lambda \in \mathbb{R}^n \mapsto V(t,z,\lambda)$ is a contractive mapping, uniformly with respect to $(t,z) \in [0,T] \times \mathbb{R}^n$, since

(3.5) $|\nabla_\lambda V(t,z,\lambda)| = |g_0(V(t,z,\lambda))||t\nabla\varphi(\lambda)| \leqslant \rho.$

The sequence $(\lambda_k)(t,z)$ defined by

$$\lambda_0(t,z) = z, \ \ \lambda_{k+1}(t,z) = V(t,z,\lambda_k(t,z))$$

satisfies

$$|\lambda_{k+1}(t,z) - \lambda_k(t,z)| \leqslant \rho^k |\lambda_1(t,z) - \lambda_0(t,z)|$$

and

$$|\lambda_1(t,z) - \lambda_0(t,z)| = |V(t,z,z) - z| \leqslant TK|\varphi(z)|,$$

and a standard procedure leads us to the first part of the lemma. Furthermore, using the properties of flows we get

$$\widehat{\psi}(0,z) = V(0,z,\widehat{\psi}(0,z)) = G_0(0,z) = z$$

and

$$V(t, G_0(-t\varphi(\lambda),\lambda),\lambda) = G_0(t\varphi(\lambda), G_0(-t\varphi(\lambda),\lambda)) = \lambda.$$

A straight differentiation with respect to t leads us to

$$\frac{\partial V}{\partial t}(t, G_0(-t\varphi(\lambda),\lambda),\lambda) - \nabla_z V(t, G_0(-t\varphi(\lambda),\lambda),\lambda) g_0(G_0(-t\varphi(\lambda),\lambda))\,\varphi(\lambda) = 0,$$

and in particular, for $\lambda = \widehat{\psi}(t,z)$, it yields

(3.6) $\dfrac{\partial V}{\partial t}(t,z,\widehat{\psi}(t,z)) - \nabla_z V(t,z,\widehat{\psi}(t,z)) g_0(z)\,\varphi(\widehat{\psi}(t,z)) = 0.$

On the other hand, differentiation with respect to t, λ in the equality $V(t,z,\widehat{\psi}(t,z)) = \widehat{\psi}(t,z)$ yields

$$\frac{\partial\widehat{\psi}}{\partial t}(t,z) = \frac{\partial V}{\partial t}(t,z,\widehat{\psi}(t,z)) + \nabla_\lambda V(t,z,\widehat{\psi}(t,z)) \frac{\partial\widehat{\psi}}{\partial t}(t,z)$$

and

$$\nabla\widehat{\psi}(t,z) = \nabla_z V(t,z,\widehat{\psi}(t,z)) + \nabla_\lambda V(t,z,\widehat{\psi}(t,z)) \nabla\widehat{\psi}(t,z).$$

Taking into account the estimate (3.5), it is easy to see that the matrix $I_n - \nabla_\lambda V(t,z,\widehat{\psi}(t,z))$ is invertible (here I_n stands for the $(n\times n)$-identity matrix) and it holds

$$\frac{\partial\widehat{\psi}}{\partial t}(t,z) = \Big[I_n - \nabla_\lambda V(t,z,\widehat{\psi}(t,z))\Big]^{-1}\frac{\partial V}{\partial t}(t,z,\widehat{\psi}(t,z)),$$

and

$$\nabla\widehat{\psi}(t,z) = \Big[I_n - \nabla_\lambda V(t,z,\widehat{\psi}(t,z))\Big]^{-1}\nabla_z V(t,z,\widehat{\psi}(t,z)).$$

Finally, combining the last two formulas with the equation (3.6) we obtain the PDE (3.4) satisfied by $\widehat{\psi}(t,z)$. $\square$

Next result is straightforward.

Corrolary 1. *The flow equation (3.2) allows a unique solution $\lambda = \psi(t,x)$, which can be represented as $\psi(t,x) := \widehat{\psi}(t,z(t,x))$, where recall that $z(t,x) = G(W(t)(x)$. Moreover, the mapping $\psi(t,x)$ is smooth with respect to (t,x) and is $(\mathcal{F}_t)$-adapted, for fixed x.*

Notice now that the composition of flows $G(p,x)$ is the solution of the following Hamilton-Jacobi equations

$$\frac{\partial G}{\partial t_i}(p,x) = \nabla G(p,x)g_i(x),\ G(0,x) = x. \tag{3.7}$$

For notationally convenience, let us prove this formula only for $m=1$. Obviously $G_1(t,G_1(-t,x)) = x$ and differentiation with respect to t yields

$$\frac{\partial G_1}{\partial t}(t,G_1(-t,x)) - \nabla G_1(t,G_1(-t,x))g_1(G_1(t,x)) = 0.$$

Replacing x with $G_1(t,x)$ we get the desired result. Since $z(t,x) = G(W(t),x)$, by virtue of Proposition 1 and formula (3.7) we obtain

$$\begin{aligned} z(t,x) &= x + \sum_{i=1}^m \int_0^t \frac{\partial G}{\partial t_i}(W(s),x)\circ \mathrm{d}W_i(s) \\ &= x + \sum_{i=1}^m \int_0^t \nabla G(W(s),x)g_i(x)\circ \mathrm{d}W_i(s) \\ &= x + \sum_{i=1}^m \int_0^t \nabla z(s,x)g_i(x)\circ \mathrm{d}W_i(s). \end{aligned} \tag{3.8}$$

Recall that g_0 commutes with $g_i, i = 1, \ldots, m$, which implies

$$G_0(t_0, z(t,x)) = z(t, G_0(t_0, x)).$$

Differentiation with respect to t_0 yields

$$g_0(G_0(t_0, z(t,x))) = \nabla z(t, G_0(t,x)) g_0(G_0(t_0,x)),$$

and replacing x by $G_0(-t_0, x)$ we get

(3.9) $g_0(z(t,x)) = \nabla z(t,x) g_0(x).$

We are now in position to state the main result of this section.

Theorem 1. *Assume* **(A.1)-(A.4)** *and set* $u(t,x) := \varphi(\psi(t,x))$. *Then* $u(t,x)$ *is a classical solution of the nonlinear SPDE* (1.1).

Proof. The stochastic rule of derivation stated in proposition 1 applied to $u(t,x) = \varphi(\widehat{\psi}(t, z(t,x)))$ reads

$$\begin{aligned} du(t,x) = \langle \nabla\varphi(\widehat{\psi}(t, z(t,x))), \frac{\partial \widehat{\psi}}{\partial t}(t, z(t,x))\rangle dt \\ + (\nabla\varphi(\widehat{\psi}(t, z(t,x))))^T \nabla\widehat{\psi}(t, z(t,x)) \circ dz(t,x). \end{aligned}$$

Taking into account the system of PDEs (3.4) satisfied by $\widehat{\psi}(t,z)$ and the formula (3.9), notice that the first term from the right hand side is equal to

$$\langle \nabla\varphi(\widehat{\psi}(t, z(t,x))), \nabla_z\widehat{\psi}(t, z(t,x)) g_0(z(t,x))\rangle \; \varphi(\widehat{\psi}(t, z(t,x))) = \langle \nabla u(t,x), g_0(x)\rangle \; u(t,x),$$

while by virtue of Lemma 1 and formula (3.8), the second term from the r.h.s. is rewritten as

$$\begin{aligned} \langle (\nabla\varphi(\widehat{\psi}(t, z(t,x))))^T \nabla\widehat{\psi}(t, z(t,x)) \nabla z(t,x), g_i(x)\rangle \circ dW_i(t) \\ = \langle \nabla u(t,x), g_i(x)\rangle \circ dW_i(t). \end{aligned}$$

The proof is complete. □

Remark 2. *The random smooth vector function* $\psi(t,x)$ *is a fundamental solution of SPDE* (1.1), *beeing constructed via n linearly independent solutions. It is obtained as the composition between the deterministic smooth mapping* $\widehat{\psi}(t,x)$ *(which verifies the Hamilton-Jacobi* *equations (3.4)) and $z(t,x)$, the fundamental solution of the reduced SPDE (3.8). It fulfills the nonlinear SPDE*

$$(3.10)\quad \begin{cases} \mathrm{d}\psi(t,x) & = \langle \nabla\psi(t,x), g_0(x)\rangle\ \varphi(\psi(t,x))\mathrm{d}t + \sum_{i=1}^m \langle \nabla\psi(t,x), g_i(x)\rangle \circ \mathrm{d}W_i(t), \\ \psi(0,x) & = x, t\in[0,T], x\in\mathbb{R}^n. \end{cases}$$

Remark 3. *The assumption* **(A.4)** *may be relaxed in the sense that it is sufficient to assume that the vector fields $g_1,\dots,g_m$ are in involution over $\mathbb{R}$, i.e.*

$$[g_i, g_j](x) = \sum_{k=1}^m \alpha_k g_k(x), \forall x \in \mathbb{R}^n,$$

with the scalars α_k depending on g_i, g_j. In this case a global gradient representation is valid, of the form

$$\nabla_p G(p,x) = (g_1(G(p,x)),\dots,g_m(G(p,x)))\,A(p),$$

where $A(p)$ is a nonsingular $(m\times m)$-matrix, for every $p\in\mathbb{R}^m$, and does not depend on the origin x. It yields the existence of the smooth vector fields $q_j(p)$, $j=1,\dots,m$ such that $\nabla_p G(p,x)q_j(p) = g_j(G(p,x))$, which implies $\nabla_p H(p,x)q_j(p) = -\partial_x H(p,x)g_j(x)$. Consider the stochastic differential system

$$y(t) = \lambda + \sum_{j=1}^m \int_0^t (-g_j)(y(s))\circ dW_j(s),$$

which is obtained by taking only the "diffusion part" of equation (2.4). When solving the auxiliary SDE

$$p(t) = -\sum_{j=1}^m \int_0^t q_j(p(s))\circ dW_j(s),$$

notice that the diffusion fields are not Lipschitz and do not have linear growth. Define a C_0^∞ function $\rho(p)$ which is equal to 1 on the closed ball $\{p\in\mathbb{R}^m \mid |p|\leqslant M\}$, where M is an arbitrary positive number. Set $\tilde{q}_j(p) := \rho(p)q_j(p)$. The SDE $p(t) = -\sum_{j=1}^m \int_0^t \tilde{q}_j(p(s))\circ dW_j(s)$ satisfies the conditions of existence and uniqueness of the solution and let $\tilde{p}(t)$ be it's solution. Define now the stopping time $\tau := \inf\{t\in[0,T] \mid |\tilde{p}(t)|\geqslant M\}$. It follows that the stopped process $\widehat{p}(t) := \tilde{p}(t\wedge\tau)$ takes values in B_M and satisfies

$$\widehat{p}(t) = -\sum_{j=1}^m \int_0^{t\wedge\tau} q_j(\widehat{p}(s))\circ dW_j(s).$$

If we assume that g_0 commutes with each g_j, $j = 1, \dots, m$, it is easy to check that the gradient representation for the stochastic flow $\widehat{x}_\varphi(t;\lambda)$ is given by $\widehat{x}_\varphi(t;\lambda) = G(-\widehat{p}(t)) \circ G_0(-t\varphi(\lambda))(\lambda)$, for $t \in [0,\tau]$. The results stated in lemma 3 and theorem 1 remain valid, but the differential equations appearing there are satisfied only for $t \leqslant \tau$ and we define now $\psi(t,x) = \widehat{\psi}(t, G(\widehat{p}(t), x))$, for $t \in [0,\tau]$.

4 An application: pathwise solutions of Navier Stokes equations with stochastic perturbations

Consider the following system of SPDEs

$$(4.1)\quad \begin{cases} du_i(t,x) &= \left[\frac{1}{2}\Delta u_i(t,x) + \langle \nabla u_i(t,x), u(t,x)\rangle\right] dt \\ & \quad + \sum_{k=1}^n \frac{\partial u_i}{\partial x_k}(t,x) dW_k(t), t \in [0,T], \\ u_i(0,x) &= \varphi_i(x), x \in \mathbb{R}^n, i = 1, \dots, n. \end{cases}$$

Here $(W(t))$ is a standard n-dimensional Brownian motion defined on a complete filtered probabilty space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P\}$, $\varphi_i \in \mathcal{C}^2(\mathbb{R}^n)$ with bounded first order derivatives and the stochastic integral is the usual Itô integral. We are looking for smooth solutions $u(t,x)$ which are $(\mathcal{F}_t)$ adapted for fixed x. Differentiation with respect to x_l yields

$$\frac{\partial u_i}{\partial x_l}(t,x) = \frac{\partial \varphi_i}{\partial x_l}(x) + \sum_{k=1}^n \int_0^t \left[\frac{1}{2}\frac{\partial^3 u_i}{\partial x_l \partial x_k^2}(s,x) + \frac{\partial^2 u_i}{\partial x_l \partial x_k}(s,x) u_k(s,x) \right.$$
$$\left. + \frac{\partial u_i}{\partial x_k}(s,x)\frac{\partial u_k}{\partial x_l}(s,x)\right] ds + \sum_{k=1}^n \int_0^t \frac{\partial^2 u_i}{\partial x_l \partial x_k}(s,x) dW_k(s),$$

where the derivatives with respect to x_l have to be understood in the L^2 sense, and since the mapping $u(t,\cdot)$ is smooth, they coincide with the classical ones. We deduce

$$\langle \frac{\partial u_i}{\partial x_l}(\cdot,x), W_l(\cdot)\rangle_t = \int_0^t \frac{\partial^2 u_i}{\partial x_l^2}(s,x) ds.$$

Hence, it is easy to see, via formula (2.1), that the system (4.1) can be rewritten as

$$(4.2)\quad \begin{cases} du_i(t,x) &= \langle \nabla u_i(t,x), \sum_{k=1}^n u_k(t,x) e_k\rangle dt \\ & \quad + \sum_{k=1}^n \langle \nabla u_i(t,x), e_k\rangle \circ dW_k(t), \\ u_i(0,x) &= \varphi_i(x), \end{cases}$$

where the system $\{e_1, \dots, e_n\}$ stands for the canonical basis of $\mathbb{R}^n$. Associate the stochastic system of characteristics

$$(4.3)\quad \begin{cases} d\widehat{x}(t;\lambda) & = -\sum_{k=1}^{n} \widehat{u}_k(t;\lambda)e_k dt - \sum_{k=1}^{n} e_k \circ dW_i(t),\ \widehat{x}(0,\lambda) = \lambda \in \mathbb{R}^n; \\ d\widehat{u}_i(t,\lambda) & = 0,\ t \in [0,T], \widehat{u}_i(0,\lambda) = \varphi_i(\lambda), \end{cases}$$

admitting the solutions $\widehat{u}_i(t,\lambda) = \varphi_i(\lambda)$ and $\widehat{x}(t;\lambda) = \widehat{x}_\varphi(t;\lambda) = \lambda - t\varphi(\lambda) - W(t)$. Assume that $TK = \rho < 1$, where $K := \sup\{|\nabla\varphi_i(\lambda|; \lambda \in \mathbb{R}^n, i = 1, \dots, n\}$. Hence, the equations $\widehat{x}_\varphi(t;\lambda) = x$ have a unique solution given by $\lambda = \psi(t,x) = \widehat{\psi}(t, x + W(t))$, with $\widehat{\psi}(t,z)$ satisfying the equation $z + t\varphi(\widehat{\psi}(t,z)) = \widehat{\psi}(t,z)$.

$\widehat{\psi}(t,z)$ is the solution of the Hamilton-Jacobi equation

$$\begin{cases} \frac{\partial\widehat{\psi}}{\partial t}(t,z) & = \nabla\widehat{\psi}(t,z)\varphi(\widehat{\psi}(t,z)), \\ \widehat{\psi}(0,z) & = z. \end{cases}$$

These remarks and computations lead us to

Theorem 2. *Under the assumptions from above, $u(t,x) := \varphi(\psi(t,x))$ is a solution of the Navier-Stokes equations* (4.2).

We ommit giving the proof since it follows the same ideas with those in Theorem 1.

5 A filtering problem for SDEs associated with parameterized backward parabolic equations

In the setting of Section 3, we consider the (slightly modified) stochastic system of characteristics (2.5) (associated to SPDE (1.1)), i.e, we replace g_i by $-g_i$ and by an abuse of notation we denote again its solution by $\widehat{x}_\varphi(t,\lambda)$. The goal of this section is to compute the expectation $E(h(\widehat{x}_\varphi(T;t,x)))$, which involves the non-Markovian solution of the SDE

$$(5.1)\quad \widehat{x}(s) = x + \varphi(\psi(t,x)) \int_t^s g_0(\widehat{x}(r))dr + \sum_{i=1}^{m} \int_t^s g_i(\widehat{x}(r)) \circ dW_i(r),\ s \in [t,T].$$

Recall that $\psi(t,x)$ was obtained as the solution of the flow equation $\widehat{x}_\varphi(t,\lambda) = x$, with respect to the unknown λ. Using a constants variation type formula, by replacing the parameter λ with the random vector function $\psi(t,x)$ in the SDE (2.5) we are lead to the equation (5.1), whose solution is described by the stochastic flow $\widehat{x}_\varphi(s;t,x), t \leqslant s \leqslant T$.

These type of expectations are usually computed via the backward Kolmogorov equation, fact which is no longer possible if we take into account the non-Markovian nature of the process which is involved. We achieve our goal by obtaining a nice formula for the conditional expectation $E[h(\widehat{x}_\varphi(T;t,x))|\psi(t,x)]$. Here $h \in \mathcal{C}^2(\mathbb{R}^n)$ and has bounded first order partial derivatives.

It is easily seen that the gradient representation of the stochastic flow $\widehat{x}_\varphi(T;t,x)$ is given by

(5.2) $\widehat{x}_\varphi(T;t,x) = G(W(T) - W(t)) \circ G_0((T-t)\varphi(\psi(t,x)))(x)$.

Set $v(t,x) := E\left[h(\widehat{x}_\varphi(T;t,x))|\psi(t,x)\right]$ and $y_\varphi(s;t,x,\lambda) := G(W(s) - W(t)) \circ G_0((s-t)\varphi(\lambda))(x)$, for $t \leqslant s \leqslant T$. Since $\psi(t,x) = \widehat{\psi}(t, G(-W(t),x))$ and $\widehat{\psi}(t,z)$ is deterministic (recall Lemma 3), it follows that the random variables $\psi(t,x)$ and $y_\varphi(T;t,x,\lambda)$ are independent. Clearly $\widehat{x}_\varphi(T;t,x) = y_\varphi(T;t,x,\psi(t,x))$. Therefore, the Independence Lemma (see [13], Lemma 2.3.4, page 73) leads us to the representation

$$v(t,x) = E\left[h(y_\varphi(T;t,x,\lambda))\right]\Big|_{\lambda=\psi(t,x)}.$$

Define $u(t,x;\lambda) := E\left[h(y_\varphi(T;t,x,\lambda))\right]$. Obviously, $y_\varphi(s;t,x,\lambda)$ is the solution of the SDE

$$y(s) = x + \varphi(\lambda)\int_t^s g_0(y(r))\mathrm{d}r + \sum_{i=1}^m \int_t^s g_i(y(r)) \circ \mathrm{d}W_i(r), \ s \in [t,T].$$

and $u(t,x;\lambda)$ satifies the Kolmogorov backward parabolic equation

$$(5.3) \quad \begin{cases} \frac{\partial u}{\partial t}(t,x;\lambda) & +\langle \nabla u(t,x;\lambda), g(x,\lambda)\rangle + \frac{1}{2}\sum_{i=1}^m \langle D^2 u(t,x;\lambda) g_i(x), g_i(x)\rangle = 0, \\ u(T,x) & = h(x), t \in [0,T], \end{cases}$$

where $g(x,\lambda) := g_0(x)\varphi(\lambda) + \frac{1}{2}\sum_{i=1}^m \nabla g_i(x) g_i(x)$.

The analysis from above can be summarized in the next statement

Theorem 3. *Let the assumptions* **(A.1)**, **(A.4)**. *Then the conditional expectation* $v(t,x) = E[h(\widehat{x}_\varphi(T;t,x))|\psi(t,x)]$ *can be represented as* $v(t,x) = u(t,x;\lambda)|_{\lambda=\psi(t,x)}$, *where* $u(t,x;\lambda)$ *is the solution of the backward parabolic equation* (5.3). *In addition, the expectation* $E(h(\widehat{x}_\varphi(T;t,x)))$ *can be computed as*

$$E(h(\widehat{x}_\varphi(T;t,x))) = E(v(t,x)).$$

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III. Functionals of SDE with jumps associated with nonlinear parabolic equations

1 Problems

Let $\{\widehat{x}_\varphi(t;\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$ be the stochastic flow generated by the following SDE with jumps

$$(1.1)\quad \begin{cases} d_t\widehat{x} = [\sum_{i=1}^{d} \varphi_j(\lambda) f_j(\widehat{x}(t-))]dt + \varphi_0(\lambda) f_0(\widehat{x}(t-))\delta y(t) + g(\widehat{x}(t-)) \circ dw(t), \\ \widehat{x}(0) = \lambda \in \mathbb{R}^n,\ t \in [0,T],\ y(0) = 0,\ \delta y(t) = y(t) - y(t-),\ \widehat{x}(t-) = \lim_{s \uparrow t} \widehat{x}(s), \end{cases}$$

where $\{y(t) \in [-\gamma,\gamma] : t \in [0,T]\}$ is a piecewise constant scalar process defined on a probability space $\{\Omega_2, \mathcal{F}_2, P_2\}$ satisfying $y(t,\omega_2) = y(\theta_i(\omega_2),\omega_2)$, $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$, $i = 0,1,\ldots,N-1$. Here $0 = \theta_0 < \theta_1 < \cdots < \theta_N = T$ is a partition such that $(y_i(\omega_2) := y(\theta_i(\omega_2),\omega_2), \theta_i(\omega_2))$ are $\mathcal{F}_2$-measurable. In addition, $\{(w(t), y(t)) : t \in [0,T]\}$ are independent processes on the filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}^t\}, P\}$ where $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, $\mathcal{F}^t = \mathcal{F}_1^t \times \mathcal{F}_2$, $P = P_1 \otimes P_2$ and $\{w(t) \in \mathbb{R} : t \in [0,T]\}$ is a scalar Wiener process over a complete filtered probability space $\{\Omega_1, \mathcal{F}_1 \supset \{\mathcal{F}_1^t\}, P_1\}$. The vector fields $\{g, f_0, \ldots, f_d\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n)$ and smooth scalar functions $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_d) \subset (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$ fulfil the following hypotheses,

(1.2) $\{g, f_0, f_1, \ldots, f_d\}$ mutually commute using Lie bracket,

(1.3) $\gamma V_0 K_0 = \rho_0 \in [0, \dfrac{1}{d+1})$, $TV_iK_i = \rho_i \in [0, \dfrac{1}{d+1})$, $i = 1,\ldots,d$,
where $\{|y(t)| \leqslant \gamma : t \in [0,T]\}$, $V_j = \sup\{|\partial_x\varphi_j(x)| : x \in \mathbb{R}^n\}$, $K_j = \sup\{|f_j(x)| : x \in \mathbb{R}^n\}$, $j = 0,1,\ldots,d$.

Problem (R1). (a) Under the hypotheses (1.2), (1.3), a unique $\mathcal{F}^t := \{\mathcal{F}_1^t \times \mathcal{F}_2\}$ adapted solution $\lambda = \psi(t,x)$ will exist such that

(1.4) $\widehat{x}_\varphi(t;\lambda) = x$, $t \in [0,T]$, $\psi(0,x) = x \in \mathbb{R}^n$,

$$(1.5)\quad \begin{cases} \{\psi(t,x) : t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)), x \in \mathbb{R}^n\} \text{ is a continuous mapping for each } \omega \in \Omega_2, \\ \text{satisfying a nonlinear SPDE, } i = 0,1,\ldots,N-1; \end{cases}$$

(b) Describe the evolution of a functional $u(t,x) = h(\psi(t,x))$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, $t \in [0,T]$, $x \in \mathbb{R}^n$, including $u_j(t,x) := \varphi_j(\psi(t,x))$, $j \in \{0,1,\ldots,d\}$.

Problem (R2). Using the unique solution $\{\lambda = \psi(t,x)\}$ of (R1), describe the evolution of the conditioned mean values

(1.6) $v_i(t,x) := E_1\{h(z_\psi(T;t,x)) \mid \psi(t,x)\}$, $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$, $x \in \mathbb{R}^n$,

for each $\omega_2 \in \Omega_2$, $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$, $i = 0,1,\ldots,N-1$, where $\{z_\psi(s;t,x) : s \in [t,T]\}$ is the unique solution of SDE with jumps

$$(1.7)\quad \begin{cases} d_s z = [\sum_{i=1}^{d} \varphi_i(\psi(t,x)) f_i(z(s-))]ds + \varphi_0(\psi(t,x)) f_0(z(s-))\delta y(s) + g(z(s-)) \circ dw(s), \\ z(t) = x,\ s \in [t,T]. \end{cases}$$

Recall that $\{w(t), y(t) : t \in [0,T]\}$ are independent processes on the complete filtered probability space $\{\Omega, \mathcal{F} \supset \{\mathcal{F}^t\}, P\}$, where $\omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, $\mathcal{F}^t = \{\mathcal{F}_1^t \times \mathcal{F}_2\}$ and $P = P_1 \otimes P_2$.

2 Solutions (hints)

R1 (a): Using (1.2), write the integral representation

(2.1) $\widehat{x}_\varphi(t,\lambda) = G(w(t)) \circ F_0(\tau_0(t;\lambda)) \circ F_1(\tau_1(t;\lambda)) \circ \cdots \circ F_d(\tau_d(t;\lambda))[\lambda]$,

where $G(\tau)[z]$ and $F_i(\sigma)[z]$ are the global flows generated by the complete vector fields g and f_i respectively. Here we use the notations

(2.2) $\tau_0(t;\lambda) = \varphi_0(\lambda) y(t)$, $\tau_i(t;\lambda) = \varphi_i(\lambda) t$, $t \in [0,T]$, $\lambda \in \mathbb{R}^n$, $1 \leqslant i \leqslant d$,

and integral representation (2.1) help us to replace $\widehat{x}_\varphi(t;\lambda) = x$ by the following integral equations

(2.3) $\lambda = V(t,x;\lambda) := F(-\tau(t;\lambda))[G(-w(t))[x]]$, $\tau = (\tau_0, \tau_1, \ldots, \tau_d)$,

where $F(\sigma)[z] = F_0(\sigma_0) \circ F_1(\sigma_1) \circ \cdots \circ F_d(\sigma_d)[z]$, $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_d)$. Applying (1.3), we compute

(2.4) $|\partial_\lambda V(t,x;\lambda)| \leqslant \sum_{i=0}^{d} \rho_j = \rho \in [0,1)$, for any $x, \lambda \in \mathbb{R}^n$, $t \in [0,T]$, which allow us to use Banach fixed point theorem for solving integral equations (2.3). In this respect, the unique solution of (2.3) will be found as a composition

$$(2.5)\ \psi(t,x)=\widehat{\psi}(t,z_0(t,x)),$$

where $z_0(t,x)=G(-w(t))[x]$, $t\in[0,T]$, $x\in\mathbb{R}^n$, is a continuous and $\mathcal{F}_1^t$-adapted process. Here, the piecewise continuous $\{\widehat{\psi}(t,z):t\in[0,T]\}$ and $\mathcal{F}_2$-measurable mapping is the unique solution of the integral equations

$$(2.6)\ \lambda=\widehat{V}(t,z;\lambda):=\widehat{F}(-\widehat{\tau}(t;\lambda))\circ F_0(-\tau_0(t;\lambda))[z],\ t\in[0,T],\ z\in\mathbb{R}^n,\ \widehat{\tau}=(\tau_1,\dots,\tau_n),$$

where $\widehat{F}(\widehat{\sigma})=F_1(\sigma_1)\circ\cdots\circ F_d(\sigma_d)$, $\widehat{\sigma}=(\sigma_1,\dots,\sigma_d)$. As far as $\{\lambda=\widehat{\psi}(t,z):t\in[0,T]\}$ is a piecewise continuous mapping, notice that it satisfies integral equations with jumps (2.6) as follows,

$$(2.7)\ \begin{cases}\widehat{\psi}(t,z)=\widehat{V}(t,z;\widehat{\psi}(t-,z)),\ t\in[0,T],\ \widehat{\psi}(0,z)=z\in\mathbb{R}^n,\\ \widehat{\psi}(\theta_i,z)=\widehat{V}(\theta_i,z;\widehat{\psi}(\theta_i-,z))=F_0[-\varphi_0(\widehat{\psi}(\theta_i-,z))\delta y(\theta_i)](\widehat{\psi}(\theta_i-,z)),\ 0\leqslant i\leqslant N-1,\end{cases}$$

where $\{\lambda=\widehat{\psi}(\theta_i-,z)\}$ is the unique solution of equations $\lambda=\widehat{V}(\theta_i-,z;\lambda)$. In addition, the corresponding approximating sequence $\{\lambda_k(t,z)\}_{k\geqslant 0}$ is constructed with the following properties

$$(2.8)\ \lambda_0(t,z)=z,\ \lambda_{k+1}(t,z)=\widehat{V}(t,z;\lambda_k(t-,z)),\ k\geqslant 0,\ t\in[0,T],\ z\in\mathbb{R}^n,$$

$$(2.9)\ \begin{cases}\widehat{\psi}(t,z)=\lim\limits_{k\to\infty}\lambda_k(t,z),\quad \widehat{\psi}(t-,z)=\lim\limits_{k\to\infty}\lambda_k(t-,z),\\ |\lambda_{k+1}(t,z)-\lambda_k(t,z)|\leqslant\rho^k|\lambda_1(t-,z)-\lambda_0(t-,z)|,\ k\geqslant 0,\ t\in[0,T].\end{cases}$$

Notice that,

$$(2.10)\ \begin{cases}\lambda_1(t-,z)=\widehat{V}(t-,z;z)=z-\sum\limits_{j=0}^{d}\int_0^1 f_j(F(-\theta\tau(t-;z))[z])\tau_j(t-;z)d\theta,\\ |\lambda_1(t-,z)-\lambda_0(t-,z)|=|\widehat{V}(t-,z;z)-z|\leqslant R(\gamma,T,z),\ t\in[0,T],\ z\in\mathbb{R}^n,\end{cases}$$

where $R(\gamma,T,z)=\gamma|\varphi_0(z)|K_0+T\sum\limits_{j=1}^{d}|\varphi_j(z)|K_j$. Using (2.10) into (2.9), we obtain

$$(2.11)\ |\widehat{\psi}(t,z)-z|\leqslant\frac{1}{1-\rho}R(\gamma,T,z),\ t\in[0,T],\ z\in\mathbb{R}^n.$$

Here $\{\widehat{\psi}(t,z) : t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)), z \in \mathbb{R}^n\}$ is a first order continuously differentiable mapping satisfying a quasilinear (H-J) equations with jumps

$$(2.12)\quad \begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)[\sum_{j=1}^{d} \varphi_j(\widehat{\psi}(t,z))f_j(z)] = 0,\ t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)) \\ \widehat{\psi}(\theta_i, z) = F_0[-\varphi_0(\widehat{\psi}(\theta_i-, z))\delta y(\theta_i)](\widehat{\psi}(\theta_i-, z)), \\ \widehat{\psi}(0,z) = z \in \mathbb{R}^n,\ i = 0, 1, \ldots, N-1. \end{cases}$$

The evolution of functionals $u(t,x) = h(\psi(t,x))$, $t \in [0,T]$, $x \in \mathbb{R}^n$, will be found considering $\psi(t,x) = \widehat{\psi}(t, z_0(t,x))$ (see (2.5)) and $\{\widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ is the unique solution of (2.6) which fulfils (H-J) equations with jumps (2.12). As a consequence, the evolution of $\psi(t,x) = \widehat{\psi}(t, z_0(t,x))$ can be obtained by applying the standard rule of stochastic derivation on each interval $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$ associated with $\mathcal{F}_1^t$-adapted and continuous process $z_0(t,x) = G(-w(t))[x]$. It leads us to a piecewise continuous and $\mathcal{F}^t = \{\mathcal{F}_1^t \times \mathcal{F}_2\}$-adapted process $\{\psi(t,x) : t \in [0,T]\}$, $x \in \mathbb{R}^n$, satisfying the following parabolic SPDE

$$(2.13)\quad \begin{cases} 0 = d_t\psi(t,x) + \partial_z\widehat{\psi}(t, z_0(t,x))[\sum_{j=1}^{d} \varphi_j(\psi(t,x))f_j(z_0(t,x))]dt \\ \qquad + [\partial_x\psi(t,x)g(x)]\widehat{\circ}dw(t),\ t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)),\ 0 \leqslant i \leqslant N-1,\ \omega_2 \in \Omega_2, \\ \psi(\theta_i, x) := \widehat{\psi}(\theta_i, z_0(\theta_i, x)) = F_0[-\varphi_0(\psi(\theta_i-, x))\delta y(\theta_i)](\psi(\theta_i-, x)),\ 1 \leqslant i \leqslant N-1, \\ \psi(0,x) = x \in \mathbb{R}^n (\text{ see (2.12)}). \end{cases}$$

Recall that the Fisk-Stratonovich integral "$\widehat{\circ}$" in (2.13) is computed using Ito integral "$\cdot$" as follows,

$$h(t,x)\widehat{\circ}dw(t) = h(t,x) \cdot dw(t) - \frac{1}{2}\partial_x h(t,x)g(x)dt.$$

R1 (b): The evolution of a functional $u(t,x) = h(\psi(t,x))$, $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$, $x \in \mathbb{R}^n$ is governed by a SPDE induced from the fundamental equations (2.13) as follows

$$(2.14)\quad \begin{cases} 0 = d_t u(t,x) + \partial_z\widehat{u}(t, z_0(t,x))[\sum_{k=1}^{d} u_k(t,x)f_k(z_0(t,x))]dt + \langle\partial_x u(t,x), g(x)\rangle\widehat{\circ}dw(t), \\ \quad t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)),\ \widehat{u}(t,z) := h(\widehat{\psi}(t,z)),\ u_k(t,x) := \varphi_k(\psi(t,x)),\ 1 \leqslant k \leqslant d, \\ \quad \text{for } i = 0, 1, \ldots, N-1, \text{ where } u(0,x) = h(x) \text{ and} \\ \quad u(\theta_i, x) = u(\theta_i-, x) + \int_0^1 \langle\partial_z h(\psi(\theta_i-, x) + \sigma\delta\psi(\theta_i, x)), \delta\psi(\theta_i, x)\rangle d\sigma. \end{cases}$$

Remark 2.1. Notice that due to the commutation property (1.2), we may and do replace the second term in (2.13) and (2.14) using

$$(2.15)\quad \begin{cases} \partial_z\widehat{\psi}(t,z_0(t,x))f_j(z_0(t,x)) = \partial_z\widehat{\psi}(t,z_0(t,x))[\partial_x z_0(t,x)][\partial_x z_0(t,x)]^{-1}f_j(z_0(t,x)) \\ \qquad = \partial_x\psi(t,x)f_j(x),\ j=1,\dots,d \\ \partial_z\widehat{u}(t,z_0(t,x))f_k(z_0(t,x)) = \partial_z\widehat{u}(t,z_0(t,x))[\partial_x z_0(t,x)][\partial_x z_0(t,x)]^{-1}f_k(z_0(t,x)) \\ \qquad = \langle \partial_x u(t,x), f_k(x)\rangle,\ k=1,\dots,d. \end{cases}$$

Inserting (2.15) into (2.13) and (2.14), we get the complete nonlinear SPDE satisfied by $\mathcal{F}^t$-adapted process $\psi(t,x)\in\mathbb{R}^n$ and $u(t,x)\in\mathbb{R}$, $t\in[0,T]$, $x\in\mathbb{R}^n$.

R2: Using the unique solution $\lambda=\psi(t,x)$, $t\in[\theta_i(\omega_2),\theta_{i+1}(\omega_2))$, $x\in\mathbb{R}^n$, found in (R1), we compute the evolution of the functionals

$$(2.16)\quad v_i(t,x) = E_1\{h(z_\psi(T;t,x)) \mid \psi(t,x)\},\ t\in[\theta_i(\omega_2),\theta_{i+1}(\omega_2)), x\in\mathbb{R}^n,\ 0\leqslant i\leqslant N-1,$$

for each $\omega_2\in\Omega_2$. Here E_1 stands for expectation with respect to P_1 and $\{z_\psi(s;t,x): s\in[t,T]\}$ satisfies SDE with jumps (1.6) and (1.7). We get the following integral representation (see (1.2)),

$$(2.17)\quad z_\psi(T;t,x) = G(w(T)-w(t))\circ F_0(\varphi_0(\psi(t,x))[y(T)-y(t)])\circ \widehat{F}(\widehat{\varphi}(\psi(t,x))(T-t))[x],$$

where $\widehat{F}(\widehat{\sigma})[x] = F_1(\sigma_1)\circ\cdots\circ F_d(\sigma_d)[x]$, $\widehat{\sigma}=(\sigma_1,\dots,\sigma_d)$ and $\widehat{\varphi}=(\varphi_1,\dots,\varphi_d)$. Notice that $z_\psi(T;t,x)$ in (2.17) and $h(z_\psi(T;t,x))$ in (2.16) are continuous mappings of the following independent random vectors $z_1 := w(T)-w(t)$ (see z_1 is independent of $\mathcal{F}^t$) and $z_2 := \psi(t,x)\in\mathbb{R}^n$ (see z_2 is $\mathcal{F}^t$-adapted). It suggests to compute the conditioned mean values in (2.16) using a parameterized functional $u_i(t,x;\lambda)$ given by

$$(2.18)\quad u_i(t,x;\lambda) = E_1 h(z_\lambda(T;t,x)),\ t\in[\theta_i(\omega_2),\theta_{i+1}(\omega_2)),\ x\in\mathbb{R}^n,\ \lambda\in\mathbb{R}^n,$$

where $z_\lambda(T;t,x)$ is obtained from (2.17) by replacing the random vector $z_2=\psi(t,x)$ with $\lambda\in\mathbb{R}^n$. Using (2.18), we write (2.16) as follows,

$$(2.19)\quad v_i(t,x) = u_i(t,x;\psi(t,x)),\ t\in[\theta_i(\omega_2),\theta_{i+1}(\omega_2)),\ x\in\mathbb{R}^n,$$

for each $\omega_2\in\Omega_2$ and $i=0,1,\dots,N-1$. In addition, $\{u_i(t,x;\lambda): t\in[\theta_i(\omega_2),\theta_{i+1}(\omega_2)), x\in\mathbb{R}^n\}$ fulfils a backward parabolic equation (Kolmogorov's equation) and for $i=N-1$ we get

$$(2.20)\quad \begin{cases} u_{N-1}(T,x;\lambda) = h(x),\ x \in \mathbb{R}^n, \\ \partial_t u_{N-1}(t,x;\lambda) + L_\lambda(u_{N-1})(t,x;\lambda) = 0,\ t \in [\theta_{N-1}(\omega_2), T),\ x \in \mathbb{R}^n. \end{cases}$$

In general, $u_i(t,x;\lambda)$ satisfies a similar Kolmogorov equation

$$(2.21)\quad \begin{cases} \partial_t u_i(t,x;\lambda) + L_\lambda(u_i)(t,x;\lambda) = 0,\ t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)),\ x \in \mathbb{R}^n, \\ u_i(\theta_{i+1}-, x;\lambda) := E_1 h(z_\lambda(T;\theta_{i+1}-, x)) \\ \text{where } z_\lambda(T;\theta_{i+1}-,x) = F_0(\delta y(\theta_{i+1})\varphi_0(\lambda))[z_\lambda(T;\theta_{i+1},x)],\ i = 0,1,\dots,N-1. \end{cases}$$

The corresponding parabolic operator $(\partial_t + L_\lambda)$ acting on each interval $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$ is associated with the following SDE

$$(2.22)\quad d_t z = [\sum_{j=1}^{d} \varphi_j(\lambda) f_j(z)]dt + g(z) \circ dw(t).$$

This lead us to a standard L_λ given by

$$(2.23)\quad L_\lambda(u)(x) = \langle \partial_x u(x), \sum_{j=1}^{d} \varphi_j(\lambda) f_j(x)\rangle + \frac{1}{2}\langle [\partial_x \langle \partial_x u(x), g(x)\rangle], g(x)\rangle.$$

Remark 2.2. In the case we assume

$$(2.24)\quad \{\varphi_1,\dots,\varphi_d\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n),\ \varphi_0 \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$$

is fulfilled, then the hypothesis (1.2) of (R1) and (R2) can be replaced by

$$(2.25)\quad \begin{cases} g \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n) \text{ commutes with } \{f_0, f_1, \dots, f_d\} \subset (\mathcal{C}_b \cap \mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n;\mathbb{R}^n) \\ \quad \text{and } f_0 \text{ commutes with } \{f_1, f_2, \dots, f_d\} \text{ using Lie bracket.} \end{cases}$$

Both problems (R1) and (R2) get a positive answer under the new conditions (2.24), (2.25) and a detailed analysis of this issue will be given in the next section.

3 Solutions for (R1) and (R2) under relaxed conditions (2.24), (2.25)

With the same notations as in section 1, consider SDE with jumps (1.1) and its stochastic flow $\{\widehat{x}_\varphi(t,\lambda) : t \in [0,T], \lambda \in \mathbb{R}^n\}$.

3.1 Problem $(\hat{R}1)$

(a) Under the hypotheses (2.24), (2.25) and $\gamma, T > 0$ are sufficiently small, a unique $\mathcal{F}^t = \mathcal{F}_1^t \times \mathcal{F}_2$-adapted solution $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ will exist such that

$$(3.1) \quad \begin{cases} \widehat{x}_\varphi(t;\lambda) = x,\ t \in [0,T],\ \psi(0,x) = x \in \mathbb{R}^n, \\ \{\psi(t,x) : t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)), x \in \mathbb{R}^n\} \text{ is a continuous mapping} \\ \qquad\qquad \text{fulfilling a SPDE for each } i = 0,1,\ldots,N-1,\ \omega_2 \in \Omega_2. \end{cases}$$

(b) Describe the evolution of a functional $u(t,x) = h(\psi(t,x))$, $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$, $t \in [0,T]$, $x \in \mathbb{R}^n$, using SPDE with jumps (see (2.14) and Remark 2.1).

3.2 Problem $(\hat{R}2)$

Under the conditions (2.24) and (2.25), describe the evolution of a functional (conditioned mean value) $v_i(t,x) = E_1\{h(z_\psi(T;t,x)) \mid \psi(t,x)\}$, $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$, $x \in \mathbb{R}^n$, for each $i \in \{0,1,\ldots,N-1\}$ and $\omega_2 \in \Omega_2$, where $h \in (C_b^1 \cap C^2)(\mathbb{R}^n)$ and $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ is found in $(\hat{R}1)$. Here $\{z_\psi(s;t,x) : s \in [t,T]\}$ satisfies the following SDE with jumps

$$(3.2) \quad \begin{cases} d_s z = [\sum_{i=1}^{d} \varphi_i(\psi(t,x)) f_i(z(s-))]ds + \varphi_0(\psi(t,x)) f_0(z(s-))\delta y(s) \\ \qquad + g(z(s-)) \circ dw(s),\ s \in [t,T], \\ z(t) = x,. \end{cases}$$

3.3 Solution for $(\hat{R}1)$ (hints)

(a) Consider the global flows $G(\tau)[z]$, $F_0(\sigma)[z]$ and $F(t;\lambda)[z]$, $\tau, \sigma, t \in \mathbb{R}$, $z \in \mathbb{R}^n$, generated by the complete vector fields g, f_0 and $\{f_\lambda(x) = \sum_{j=1}^{d} \varphi_j(\lambda) f_j(x) : x \in \mathbb{R}^n\}$ respectively, for each $\lambda \in \mathbb{R}^n$. Using (2.25), we write the first integral representation

$$(3.3) \quad \widehat{x}_\varphi(t;\lambda) = G(w(t)) \circ F_0(y(t)\varphi_0(\lambda)) \circ F(t;\lambda)[\lambda],\ t \in [0,T], x \in \mathbb{R}^n,$$

and this help us to replace $\widehat{x}_\varphi(t;\lambda) = x \in \mathbb{R}^n$ by the following integral equations

$$(3.4) \quad \lambda = V(t,x;\lambda) := F_0(-\varphi_0(\lambda)y(t)) \circ F(-t;\lambda)[G(-w(t))[x]],\ t \in [0,T],\ x \in \mathbb{R}^n.$$

A unique solution $\lambda = \psi(t,x)$ of (3.4) will be found as a composition

(3.5) $\psi(t,x) = \widehat{\psi}(t, z_0(t,x))$, $z_0(t,x) := G(-w(t))[x]$,

where $\{\lambda = \widehat{\psi}(t,z) : t \in [0,T], z \in \mathbb{R}^n\}$ is the unique solution of the following integral equations

(3.6) $\lambda = \widehat{V}(t,z;\lambda) := F_0(-\varphi_0(\lambda)y(t)) \circ F(-t;\lambda)[z]$, $t \in [0,T]$, $z \in \mathbb{R}^n$.

Here $\{\widehat{V}(t,z;\lambda) : t \in [0,T]\}$ fulfils ODE with jumps

$$(3.7) \quad \begin{cases} d_t\widehat{V} = -\{\sum_{i=1}^{d} \varphi_i(\lambda) f_i(\widehat{V}(t-,z;\lambda))dt + \varphi_0(\lambda) f_0(\widehat{V}(t-,z;\lambda))\delta y(t)\}, \\ \widehat{V}(0,z;\lambda) = z \in \mathbb{R}^n,\ \delta y(t) = y(t) - y(t-). \end{cases}$$

Notice that f_0 commutes with $\{f_1, \ldots, f_d\}$ (see (2.25)) and it implies that $F_0(\sigma)$ commutes with $F(t;\lambda)$, $\sigma, t \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$. This allows us to write

(3.8) $\widehat{V}(t,z;\lambda) = F_0(-\varphi_0(\lambda)y(t)) \circ F(-t;\lambda)[z] = F(-t;\lambda) \circ F_0(-\varphi_0(\lambda)y(t))[z]$

and to get $\{\widehat{V}(t,z;\lambda) : \lambda \in \mathbb{R}^n\}$ as a Lipschitz continuous mapping uniformly with respect to $t \in [0,T]$ and $z \in \mathbb{R}^n$ satisfying

(3.9) $|\widehat{V}(t,z;\lambda'') - \widehat{V}(t,z;\lambda')| \leqslant \rho|\lambda'' - \lambda'|$, $\lambda', \lambda'' \in \mathbb{R}^n$,

where $\rho \in [0,1)$ is a constant, provided $\gamma, T > 0$ are sufficiently small. In this respect, by a direct computation we get

$$\begin{aligned} \widehat{V}(t,z;\lambda'') - \widehat{V}(t,z;\lambda') &= F_0(-\varphi_0(\lambda')y(t))[F(-t;\lambda'')(z) - F(-t;\lambda')(z)] \\ &\quad + [F_0(-\varphi_0(\lambda'')y(t)) - F_0(-\varphi_0(\lambda')y(t))](F(-t;\lambda'')(z)) = T_1 + T_2. \end{aligned}$$

Rewrite the two right hand side terms as follows,

$$T_1 = z(t,\lambda'') - z(t,\lambda') + \int_0^{-\varphi_0(\lambda')y(t)} f_0(F_0(\sigma)[z(t,\lambda'') - z(t,\lambda')])d\sigma$$

where $z(t,\lambda) = F(-t,\lambda)[z]$, and

$$T_2 = \{\int_0^1 f_0(F_0(-[\varphi_0(\lambda')y(t) + \theta[\varphi_0(\lambda'') - \varphi_0(\lambda')]y(t)])(z(t,\lambda'')))d\theta\}[\varphi_0(\lambda'') - \varphi_0(\lambda')]y(t).$$

Notice that $|T_1| \leqslant \rho/2$ and $|T_2| \leqslant \rho/2$ for any $t \in [0,T]$, $|y(t)| \leqslant \gamma$ if $T, \gamma > 0$ are sufficiently small. A direct consequence of (3.9) is that Banach fixed point theorem can be applied to integral equations (3.6) using the following convergent sequence

$$(3.10)\quad \lambda_0(t,z) = z,\ \lambda_{k+1}(t,z) = \widehat{V}(t,z;\lambda_k(t-,z)),\ \lambda_{k+1}(t-,z) = \widehat{V}(t-,z;\lambda_k(t-,z))$$

for any $k \geqslant 0$, $t \in [0,T]$, $z \in \mathbb{R}^n$ such that

$$(3.11)\quad \widehat{\psi}(t,z) = \lim_{k\to\infty} \lambda_k(t,z),\ \widehat{\psi}(t,z) = \widehat{V}(t,z;\widehat{\psi}(t-,z)),\ \widehat{\psi}(t-,z) = \widehat{V}(t-,z;\widehat{\psi}(t-,z)).$$

Notice that

$$(3.12)\quad \begin{cases} |\lambda_{k+1}(t,z) - \lambda_k(t,z)| \leqslant \rho^k |\lambda_1(t-,z) - z|,\ k \geqslant 1, \\ \lambda_{k+1}(t,z) - z = \sum_{j=0}^{k} [\lambda_{j+1}(t,z) - \lambda_j(t,z)],\ k \geqslant 0 \end{cases}$$

and

$$(3.13)\quad |\widehat{\psi}(t,z) - z| \leqslant [\sum_{k=0}^{\infty} \rho^k] |\lambda_1(t-,z) - z| = \frac{1}{1-\rho} |\lambda_1(t-,z) - z|,\ t \in [0,T].$$

Compute $\lambda_1(t,z) := \widehat{V}(t,z;z) = H(\theta y(t), \theta t; z)|_{\theta=1}$, where

$$H(\theta y(t), \theta t; z) = F_0(-\theta \varphi_0(z) y(t)) \circ \widehat{F}(-\theta t; z)[z],\ \theta \in [0,1].$$

We get

$$(3.14)\quad \lambda_1(t,z) = z - \int_0^1 [\varphi_0(z) y(t) f_0(H(\theta y(t), \theta t; z)) + \sum_{i=1}^{d} \varphi_i(z) f_i(H(\theta y(t), \theta t; z))] d\theta,$$

for any $t \in [0,T]$, $z \in \mathbb{R}^n$. The following estimates are valid

$$(3.15)\quad \begin{cases} |\lambda_1(t,z) - z| \leqslant R(\gamma,T,z),\ |\lambda_1(t-,z) - z| \leqslant R(\gamma,T,z), \\ |\widehat{\psi}(t,z) - z| \leqslant \frac{1}{1-\rho} R(\gamma,T,z),\ |\widehat{\psi}(t-,z) - z| \leqslant \frac{1}{1-\rho} R(\gamma,T,z),\ t \in [0,T] \end{cases}$$

where $R(\gamma,T,z) = \gamma|\varphi_0(z)|K_0 + T \sum_{i=1}^{d} |\varphi_i(z)| K_i$ and $K_j = \sup\{|f_j(x)|, x \in \mathbb{R}^n\}$, $0 \leqslant j \leqslant d$. Define $\psi(t,z) = \widehat{\psi}(t, z_0(t,x))$, $z_0(t,z) = G(-w(t))[z]$, $t \in [0,T]$, $x \in \mathbb{R}^n$, and using (H-J) equations (see (2.12)) fulfiled by $\{\widehat{\psi}(t,z)\}$,

$$(3.16)\quad \begin{cases} \partial_t\widehat{\psi}(t,z) + \partial_z\widehat{\psi}(t,z)[\sum_{j=1}^{d}\varphi_j(\widehat{\psi}(t,z))f_j(z)] = 0,\ t\in[\theta_i,\theta_{i+1}), \\ \widehat{\psi}(\theta_i,z) = F_0[-\varphi_0(\widehat{\psi}(\theta_i-,z))\delta y(\theta_i)](\widehat{\psi}(\theta_i-,z)), \\ \widehat{\psi}(0,z) = z\in\mathbb{R}^n,\ 0\leqslant i\leqslant N-1, \end{cases}$$

we get SPDE satisfied by $\{\psi(t,x) : t\in[\theta_i,\theta_{i+1}), x\in\mathbb{R}^n\}$ (see (2.3), (2.15)). More precisely, applying the stochastic rule of derivation associated with $\{\widehat{\psi}(t,z)\}$ and $z = z_0(t,x)$,

$$(3.17)\quad \begin{cases} d_t\psi(t,x) + \partial_x\psi(t,x)[\sum_{j=1}^{d}\varphi_j(\psi(t,x))f_j(x)]dt + [\partial_x\psi(t,x)g(x)]\widehat{\circ}dw(t) = 0, \\ \psi(\theta_i,x) = F_0[-\varphi_0(\psi(\theta_i-,x))\delta y(\theta_i)](\psi(\theta_i-,x)),\ t\in[\theta_i,\theta_{i+1}),\ 0\leqslant i\leqslant N-1, \end{cases}$$

where $\psi(0,x) = \psi(0-,x) = x\in\mathbb{R}^n$ and $h(t,x)\widehat{\circ}dw(t) := -\frac{1}{2}\partial_x h(t,x)g(x)dt + h(t,x)\cdot dw(t)$.

(b) The evolution of a functional $u(t,x) = h(\psi(t,x))$, $t\in[\theta_i,\theta_{i+1})$, $h\in(\mathcal{C}_b^1\cap\mathcal{C}^2)(\mathbb{R}^n)$ is determined by the fundamental equations (3.17) as follows

$$(3.18)\quad \begin{cases} 0 = d_t u(t,x) + \partial_z\widehat{u}(t,z_0(t,x))[\sum_{k=1}^{d}\varphi_k(\psi(t,x))f_k(z_0(t,x))]dt \\ \qquad + \langle\partial_x u(t,x), g(x)\rangle\widehat{\circ}dw(t),\ t\in[\theta_i,\theta_{i+1}), \\ u(\theta_i,x) = u(\theta_i-,x) + \int_0^1\langle\partial_\lambda h(\psi(\theta_i-,x)+\sigma\delta\psi(\theta_i,x)), \delta\psi(\theta_i,x)\rangle d\sigma, \\ u(0,x) = h(x),\ i = 0,1,\ldots,N-1 \end{cases}$$

where $\widehat{u}(t,z) = h(\widehat{\psi}(t,z))$, $z_0(t,x) = G(-w(t))[x]$. Notice that g commutes with $\{f_1,\ldots,f_d\}$ and the second term in (3.18) becomes (see (2.15) of Remark 2.1)

$$(3.19)\quad \partial_z\widehat{u}(t,z_0(t,x))f_k(z_0(t,x)) = \langle\partial_x u(t,x), f_k(x)\rangle,\ t\in[0,T],\ x\in\mathbb{R}^n,\ 1\leqslant k\leqslant d.$$

Using (3.19) into (3.18) we get the corresponding SPDE satisfied by $\{u(t,x)\}$,

$$(3.20)\quad \begin{cases} 0 = d_t u(t,x) + \langle\partial_x u(t,x), \sum_{k=1}^{d}\varphi_k(\psi(t,x))f_k(x)\rangle dt \\ \qquad + \langle\partial_x u(t,x), g(x)\rangle\widehat{\circ}dw(t),\ t\in[\theta_i,\theta_{i+1}), \\ u(\theta_i,x) = u(\theta_i-,x) + \int_0^1\langle\partial_\lambda h(\psi(\theta_i-,x))\rangle d\sigma, \\ u(0,x) = h(x),\ i = 0,1,\ldots,N-1 \end{cases}$$

3.4 Solution for ($\hat{\mathrm{R}}2$) (hints)

With the same notations as in §3.3 (see ($\hat{\mathrm{R}}2$)), consider the unique $\mathcal{F}^t$-adapted solution $\{\lambda = \psi(t,x) : t \in [0,T], x \in \mathbb{R}^n\}$ found in ($\hat{\mathrm{R}}1$) (see (3.17)) and define $\{z_\psi(s;t,x) : s \in [t,T]\}$ such that (3.2) is satisfied. Assuming (2.24), (2.25), we are able to describe the evolution of a functional

$$(3.21)\quad v_i(t,x) := E_1\{h(z_\psi(T;t,x)) \mid \psi(t,x)\},\ t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)),\ x \in \mathbb{R}^n,\ \omega_2 \in \Omega_2,$$

where "E_1" stands for expectation with respect to P_1 and $h \in (\mathcal{C}_b^1 \cap \mathcal{C}^2)(\mathbb{R}^n)$. Under (2.24) and (2.25) we may and do represent $\{z_\psi(T;t,x) : t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)), x \in \mathbb{R}^n\}$ as follows

$$(3.22)\quad z_\psi(T;t,x) = G(w(T) - w(t)) \circ F_0(\varphi_0(\psi(t,x))[y(T) - y(t)]) \circ \widehat{F}(T-t;\psi(t,x))[x],$$

where $\widehat{F}(\sigma;\lambda)[x]$, $\sigma \in \mathbb{R}$ is the global flow generated by ODE

$$(3.23)\quad \frac{dz}{d\sigma} = \sum_{i=1}^{d} \varphi_i(\lambda) f_i(z) := \widehat{f}(\sigma;\lambda),\ z(0) = x \in \mathbb{R}^n.$$

The integral representation (3.22) help us to see that $z_\psi(T;t,x)$ is a continuous mapping of two independent random vectors $z_1 = w(T) - w(t) \in \mathbb{R}$ which is independent of $\{\mathcal{F}^t\}$ and $z_2 = \psi(t,x) \in \mathbb{R}^n$ which is $\mathcal{F}^t$-adapted. As a direct consequence of (3.22), we compute $v_i(t,x)$ in (3.21) as follows

$$(3.24)\quad v_i(t,x) = u_i(t,x;\psi(t,x)),\ t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)),\ x \in \mathbb{R}^n,$$

where $u_i(t,x;\lambda) := E_1 h(z_\lambda(T;t,x))$, $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$, $x \in \mathbb{R}^n$. Here $\{z_\lambda(s;t,x) : s \in [t,T]\}$ satisfies SDE with jumps

$$(3.25)\quad \begin{cases} d_s z_\lambda = [\sum_{i=1}^{d} \varphi_i(\lambda) f_i(z_\lambda(s-))]ds + \varphi_0(\lambda) f_0(z_\lambda(s-))\delta y(s) + g(z_\lambda(s-)) \circ dw(s), \\ z_\lambda(t) = x,\ s \in [t,T]. \end{cases}$$

Using (3.22) for the unique solution of (3.25) we write

$$(3.26)\quad z_\lambda(T;t,x) = F_0(\varphi_0(\lambda)[y(T) - y(t)]) \circ \widehat{F}(T-t;\lambda) \circ G(w(T) - w(t))[x],$$

for any $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^n$. Notice that

$$\{u_i(t,x;\lambda) = E_1 h(z_\lambda(T;t,x)) : t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)), x \in \mathbb{R}^n\}$$

is a smooth mapping for each $\omega_2 \in \Omega_2$, $\lambda \in \mathbb{R}^n$, $i = 0, 1, \ldots, N-1$, and the corresponding backward parabolic equation (Kolmogorov's equation) can be deduced using a standard procedure. In this respect, for a fixed $t \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2))$ and $\delta > 0$ sufficiently small, rewrite $u_i(t,x;\lambda)$ as follows

$$(3.27)\quad u_i(t,x;\lambda) = E_1 h(z_\lambda(T; t+\delta, \widehat{z}_\lambda(t+\delta; t,x))),\ t, t+\delta \in [\theta_i(\omega_2), \theta_{i+1}(\omega_2)),$$

where $\{z_\lambda(s;t,x) : s \in [t, t+\delta]\}$ fulfils SDE

$$(3.28)\quad \begin{cases} d_s \widehat{z}_\lambda = [\sum_{i=1}^{d} \varphi_i(\lambda) f_i(\widehat{z}_\lambda(s))]ds + g(\widehat{z}_\lambda(s)) \circ dw(s), \\ \widehat{z}_\lambda(t) = x,\ s \in [t, t+\delta]. \end{cases}$$

Notice that $\{w(T) - w(t+\delta)\}$ and $\{\widehat{z}_\lambda(t+\delta; t, x)\}$ are independent random vectors and computation of the mean value in (3.27) can be done using the conditioned mean value with respect to $\{\widehat{z}_\lambda(t+\delta; t,x)\}$. We obtain

$$(3.29)\quad \begin{cases} u_i(t,x;\lambda) = E_1[E_1\{h(z_\lambda(T; t+\delta, \widehat{z}_\lambda(t+\delta; t,x))) \mid \widehat{z}_\lambda(t+\delta; t,x)\}] \\ \qquad = E_1 u_i(t+\delta, \widehat{z}_\lambda(t+\delta; t,x); \lambda). \end{cases}$$

Combining (3.29) with $\{u_i(t+\delta, z; \lambda) : z \in \mathbb{R}^n\}$ is second order continuously differentiable of $z \in \mathbb{R}^n$ we may and do the standard rule of stochastic derivation which lead us to

$$(3.30)\quad \begin{cases} u_i(t,x;\lambda) = u_i(t+\delta, x; \lambda) + E_1 \int_t^{t+\delta} \langle \partial_x u_i(t+\delta, \widehat{z}\lambda(s); \lambda), f_\lambda(\widehat{z}_\lambda(s))\rangle ds \\ \qquad + E_1 \int_t^{t+\delta} \langle \partial_x u_i(t+\delta, \widehat{z}_\lambda(s); \lambda), g(\widehat{z}_\lambda(s))\rangle \circ dw(s). \end{cases}$$

Here $f_\lambda(z) := \sum_{i=1}^{d} \varphi_i(\lambda) f_i(z)$ and the Fisk-Stratonovich integral "$\circ$" is computed by

$$(3.31)\quad \varphi(\widehat{z}_\lambda(s)) \circ dw(s) = \frac{1}{2}\langle \partial_x \varphi(\widehat{z}_\lambda(s)), g(\widehat{z}_\lambda(s))\rangle ds + \varphi(\widehat{z}_\lambda(s)) \cdot dw(s)$$

using Ito stochastic integral "$\cdot$". A direct computation allows us to write

$$(3.32)\quad \partial_t^+ u_i(t,x;\lambda) := \lim_{\delta \downarrow 0} \frac{u_i(t+\delta, x; \lambda) - u_i(t,x;\lambda)}{\delta} = -L_\lambda(u_i(t,\cdot;\lambda))(x),$$

where $L_\lambda(v)(x) = \langle \partial_x v(x), f_\lambda(x)\rangle + \frac{1}{2}\langle \partial_x[\langle \partial_x v(x), g(x)\rangle], g(x)\rangle$ for any $v \in \mathcal{C}^2(\mathbb{R}^n)$. In addition, using $u_{N-1} := E_1 h(z_\lambda(T; t, x))$, for $t \in [\theta_{N-1}(\omega_2), T)$, we get the final condition

$$(3.33)\quad u_{N-1}(T, x; \lambda) = \lim_{t \uparrow T} E_1 h(z_\lambda(T; t, x)) = h(x),\ x \in \mathbb{R}^n.$$

These remarks allows us to conclude that $\{u_i(t, x; \lambda) : t \in (\theta_i(\omega_2), \theta_{i+1}(\omega_2)), x \in \mathbb{R}^n\}$ is a smooth function (see $\partial_t u_i(t, x; \lambda)$, $\partial_{x_k} u_i(t, x; \lambda)$ and $\partial^2_{x_k x_j} u_i(t, x; \lambda)$ are continuous functions) satisfying the following parabolic equations

$$(3.34)\quad \partial_t u_i(t, x; \lambda) + L_\lambda(u_i(t, \cdot; \lambda))(x) = 0,\ t \in (\theta_i(\omega_2), \theta_{i+1}(\omega_2)),\ x \in \mathbb{R}^n,\ 0 \leqslant i \leqslant N-1,$$

and final conditions

$$(3.35)\quad u_{N-1}(T, x; \lambda) = h(x),\ x \in \mathbb{R}^n.$$

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