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FLORIN FELIX NICHITA AND CĂLIN POPESCU

Abstract

We associate with each (co)algebra a bicomplex by encoding the (co)multiplication in one of the differentials, and the (co)unit in the other. Akin to the (co)bar construction, the construction enables us to encode a morphism entwining an algebra with a coalgebra as a bicomplex morphism entwining the associated bicomplexes.

Next, we consider a related simplicial construction which associates a simplicial module with each (co)algebra. In this case, an entwining structure yields a simplicial map entwining the associated constructions, on one hand, and a chain map entwining the chain complexes on the constructions, on the other. Furthermore, the components of the simplicial entwining combine to yield another chain map between the chain complexes on the entwined products of constructions. The two chain maps thus obtained turn out to be compatible (they commute) up to chain homotopy with any pair of natural chain transformations for the Eilenberg-Zilber theorem which are the respective identities in dimension zero.

Key Words: (co)algebra, entwining structure, (co)bar construction, simplicial module, bicomlex.
2000 Mathematics Subject Classification: Primary 16W30, Secondary 18G30, 18G35.

1 Introduction

Attemps to construct a unifying theory for algebra and coalgebra structures led to Yang-Baxter operators derived from (co)algebra structures [9]. Applications of these constructions occur in Quantum Group Theory [6], [10], Knot Theory [8], Theoretical Physics [3] etc.

In this paper, we consider another approach. Thus, we associate with any algebra structure a bicomplex by encoding the multiplication in one of the differentials, and the unit in the other. Dually, we associate with any coalgebra structure a bicomplex by encoding the comultiplication in one of the differentials, and the counit in the other. Akin to the (co)bar construction, the construction enables us to encode a morphism entwining an algebra with a coalgebra as a bicomplex morphism entwining the associated bicomplexes.

Next, we consider a related simplicial construction which associates a simplicial module with each (co)algebra. In this case, an entwining structure yields a simplicial map entwining the associated constructions, on one hand, and a chain map entwining the chain complexes on the constructions, on the other. Furthermore, the components of the simplicial entwining combine to yield another chain map between the chain complexes on the entwined products of constructions. The two chain maps thus obtained turn out to be compatible (they commute) up to chain homotopy with any pair of natural chain transformations for the Eilenberg-Zilber theorem which are the respective identities in dimension zero.

The constructions considered in this paper easily lend themselves to extensions to modules over (co)algebras, (differential) graded objects and the like. Part of the ingredients have been considered in various setups [1], [2], [4]. Details on Hopf algebras and entwining structures can be found in [3], [5], [11].

2 Conventions, Notation, Main Results

Fix a commutative ring K once and for all. Unless otherwise stated, throughout this paper, modules and tensor products are over K. We also use the following standard conventions on identity morphisms and tensor powers: For any module X, the identity morphism on X is also denoted by X,

$$X^{\otimes n} = \begin{cases} 0, & \text{for integer } n < 0, \\ K, & \text{for } n = 0, \\ \underbrace{X \otimes \cdots \otimes X}_{n}, & \text{for integer } n > 0, \end{cases}$$

and if $f : X \to Y$ is a morphism of modules, then $f^{\otimes n} : X^{\otimes n} \to Y^{\otimes n}$ is given by

$$f^{\otimes n} = \begin{cases} 0, & \text{for } n < 0, \\ K, & \text{for } n = 0, \\ \underbrace{f \otimes \cdots \otimes f}_{n}, & \text{for } n > 0. \end{cases}$$

Given an algebra (A, μ, η) with associative multiplication $\mu : A^{\otimes 2} \to A$ and unit $\eta : K \to A$, define inductively

$$\partial'_n : A^{\otimes n} \to A^{\otimes(n-1)} \quad \text{and} \quad \partial''_n : A^{\otimes n} \to A^{\otimes(n+1)},$$

by

$$\partial'_n = \begin{cases} 0, & \text{for } n < 2, \\ -A \otimes \partial'_{n-1} + \mu \otimes A^{\otimes(n-2)}, & \text{for } n \geq 2, \end{cases} \tag{1}$$

and

$$\partial''_n = \begin{cases} 0, & \text{for } n < 0, \\ -\partial''_{n-1} \otimes A + A^{\otimes n} \otimes \eta, & \text{for } n \geq 0, \end{cases} \tag{2}$$

respectively. It turns out that

$$\partial'_{n-1}\partial'_n = 0, \quad \partial''_n\partial''_{n-1} = 0, \quad \text{and} \quad \partial''_{n-1}\partial'_n = \partial'_{n+1}\partial''_n. \tag{3}$$

We may therefore associate with A a bicomplex $\hat{A} = \left(\hat{A}_{p,q}, d'_{p,q}, d''_{p,q}\right)$ by setting

$$\hat{A}_{p,q} = A^{\otimes(p-q)}, \quad d'_{p,q} = \partial'_{p-q} \quad \text{and} \quad d''_{p,q} = (-1)^p \partial''_{p-q}.$$

The rows (respectively, columns) of this bicomplex are shifted copies of one another. In addition, each row is eventually acyclic: for each q,

$$h_{p,q} = \eta \otimes A^{\otimes(p-q)} : \hat{A}_{p,q} = A^{\otimes(p-q)} \to \hat{A}_{p+1,q} = A^{\otimes(p+1-q)}$$

is a contracting homotopy in the range $p \geq q$. The construction is easily made functorial: With every algebra morphism $f : A \to A'$ associate the bicomplex morphism $\hat{f} : \hat{A} \to \hat{A}'$ given by

$$\hat{f}_{p,q} : \hat{A}_{p,q} \to \hat{A}'_{p,q}, \quad \hat{f}_{p,q} = f^{\otimes(p-q)}.$$

The fact that $\hat{f}$ commutes with the respective differentials is a routine verification.

Dually, given a coalgebra (C, Δ, ϵ) with coassociative comultiplication $\Delta : C \to C^{\otimes 2}$ and counit $\epsilon : C \to K$, define inductively

$$\nabla'_n : C^{\otimes n} \to C^{\otimes(n-1)} \quad \text{and} \quad \nabla''_n : C^{\otimes n} \to C^{\otimes(n+1)},$$

by

$$\nabla'_n = \begin{cases} 0, & \text{for } n < 1, \\ -C \otimes \nabla'_{n-1} + \epsilon \otimes C^{\otimes(n-1)}, & \text{for } n \geq 1, \end{cases} \tag{1'}$$

and

$$\nabla''_n = \begin{cases} 0, & \text{for } n < 1, \\ -\nabla''_{n-1} \otimes C + C^{\otimes(n-1)} \otimes \Delta, & \text{for } n \geq 1, \end{cases} \tag{2'}$$

respectively. As expected,

$$\nabla'_{n-1}\nabla'_n = 0, \quad \nabla''_n\nabla''_{n-1} = 0, \quad \text{and} \quad \nabla''_{n-1}\nabla'_n = \nabla'_{n+1}\nabla''_n. \tag{3'}$$

We may therefore associate with C a bicomplex $\hat{C} = \left(\hat{C}_{r,s}, d'_{r,s}, d''_{r,s}\right)$ by setting

$$\hat{C}_{r,s} = C^{\otimes(r-s)}, \quad d'_{r,s} = \nabla'_{r-s} \quad \text{and} \quad d''_{r,s} = (-1)^r \nabla''_{r-s}.$$

The rows (respectively, columns) of this bicomplex are also shifted copies of one another. In this case, each column is eventually acyclic: for each r,

$$h_{r,s} = C^{\otimes(r-s-1)} \otimes \epsilon : \hat{C}_{r,s} = C^{\otimes(r-s)} \to \hat{C}_{r,s+1} = C^{\otimes(r-s-1)}$$

is a contracting homotopy in the range $s \leq r$. Again, the construction is easily made functorial: With every coalgebra morphism $f : C \to C'$ associate the bicomplex morphism $\hat{f} : \hat{C} \to \hat{C}'$ given by

$$\hat{f}_{r,s} : \hat{C}_{r,s} \to \hat{C'}_{r,s}, \quad \hat{f}_{r,s} = f^{\otimes(r-s)}.$$

The fact that $\hat{f}$ commutes with the respective differentials is also a routine verification.

Next, consider an algebra (A, μ, η) entwined with a coalgebra (C, Δ, ϵ) *via* $\psi : C \otimes A \to A \otimes C$; that is, ψ is a module morphism satisfying the following four conditions:

(a) $\psi(C \otimes \mu) = (\mu \otimes C)(A \otimes \psi)(\psi \otimes A)$;

(b) $\psi(C \otimes \eta) = \eta \otimes C$;

(a′) $(A \otimes \Delta)\psi = (\psi \otimes C)(C \otimes \psi)(\Delta \otimes A)$; and

(b′) $(A \otimes \epsilon)\psi = \epsilon \otimes A$.

The entwining morphism ψ gives raise to a doubly indexed collection of module morphisms

$$\psi_{n,m} : C^{\otimes n} \otimes A^{\otimes m} \to A^{\otimes m} \otimes C^{\otimes n},$$

entwining the tensor powers of A and C. The $\psi_{n,m}$ are defined inductively as follows: If either index is negative, set $\psi_{n,m} = 0$; set further $\psi_{n,0} = C^{\otimes n}$, $\psi_{0,m} = A^{\otimes m}$, $\psi_{1,1} = \psi$ and define

$$\psi_{n,m} = (\psi_{0,m-1} \otimes \psi_{1,1} \otimes \psi_{n-1,0})(\psi_{1,m-1} \otimes \psi_{0,1} \otimes \psi_{n-1,0})(\psi_{1,0} \otimes \psi_{n-1,m} \otimes \psi_{0,0}),$$

for $m \geq 1$ and $n \geq 1$; in a slightly less cumbersome notation,

$$\psi_{n,m} = \left(A^{\otimes(m-1)} \otimes \psi \otimes C^{\otimes(n-1)}\right)\left(\psi_{1,m-1} \otimes A \otimes C^{\otimes(n-1)}\right)(C \otimes \psi_{n-1,m}).$$

Recall now the bicomplexes $\hat{A} = \left(\hat{A}_{p,q}, d'_{p,q}, d''_{p,q}\right)$ and $\hat{C} = \left(\hat{C}_{r,s}, d'_{r,s}, d''_{r,s}\right)$ associated with A and C, respectively, and consider the tensor product bicomplexes

$$\hat{C}\hat{\otimes}\hat{A} = \left(\left(\hat{C} \otimes \hat{A}\right)_{m,n}, \delta'_{m,n}, \delta''_{m,n}\right) \quad \text{and} \quad \hat{A}\hat{\otimes}\hat{C} = \left(\left(\hat{A} \otimes \hat{C}\right)_{m,n}, \delta'_{m,n}, \delta''_{m,n}\right).$$

Explicitly,

$$\left(\hat{C} \otimes \hat{A}\right)_{m,n} = \coprod_{r+p=m,\, s+q=n} \hat{C}_{r,s} \otimes \hat{A}_{p,q},$$

and the differentials are given by

$$\delta'_{m,n} = \sum_{r+p=m,\, s+q=n} \left(d'_{r,s} \otimes \hat{A}_{p,q} + (-1)^r \hat{C}_{r,s} \otimes d'_{p,q}\right),$$

$$\delta''_{m,n} = \sum_{r+p=m,\, s+q=n} \left((-1)^p d''_{r,s} \otimes \hat{A}_{p,q} + (-1)^{r+s} \hat{C}_{r,s} \otimes d''_{p,q}\right);$$

similarly,

$$\left(\hat{A}\otimes\hat{C}\right)_{m,n} = \coprod_{p+r=m,\, q+s=n} \hat{A}_{p,q}\otimes\hat{C}_{r,s}\,,$$

and the differentials are given by

$$\delta'_{m,n} = \sum_{p+r=m,\, q+s=n} \left(d'_{p,q}\otimes\hat{C}_{r,s} + (-1)^p\hat{A}_{p,q}\otimes d'_{r,s}\right),$$

$$\delta''_{m,n} = \sum_{p+r=m,\, q+s=n} \left((-1)^r d''_{p,q}\otimes\hat{C}_{r,s} + (-1)^{p+q}\hat{A}_{p,q}\otimes d''_{r,s}\right).$$

The $\psi_{n,m}$ defined above fit together to yield an overall entwining morphism

$$\hat{\psi} : \hat{C}\otimes\hat{A}\to\hat{A}\otimes\hat{C}$$

of bigraded modules whose components

$$\hat{\psi}_{m,n} : \left(\hat{C}\otimes\hat{A}\right)_{m,n}\to\left(\hat{A}\otimes\hat{C}\right)_{m,n}$$

are given by

$$\hat{\psi}_{m,n} = \sum_{r+p=m,\, s+q=n} (-1)^{rp+sq}\,\hat{\psi}_{r,s,p,q}\,,$$

where

$$\hat{\psi}_{r,s,p,q} = \psi_{r-s,p-q} : \quad \hat{C}_{r,s}\otimes\hat{A}_{p,q} = C^{\otimes(r-s)}\otimes A^{\otimes(p-q)}\to \hat{A}_{p,q}\otimes\hat{C}_{r,s} = A^{\otimes(p-q)}\otimes C^{\otimes(r-s)}.$$

We are now in a position to state our first result.

Theorem A. *$\hat{\psi}$ is a morphism of bicomplexes:*

$$\delta'_{m,n}\hat{\psi}_{m,n} = \hat{\psi}_{m-1,n}\delta'_{m,n} \quad \textit{and} \quad \delta''_{m,n}\hat{\psi}_{m,n} = \hat{\psi}_{m,n-1}\delta''_{m,n}.$$

We now move in a slightly different direction and consider a related simplicial construction. For standard definitions, notation and results, we refer to [7]. Written explicitly, the ∂'_n, ∂''_n, ∇'_n and ∇''_n have the form

$$\partial'_n = \sum_{i\in\mathbb{Z}}(-1)^i\partial'_{n,i}, \qquad \partial''_n = \sum_{i\in\mathbb{Z}}(-1)^{n+i}\partial''_{n,i},$$
$$\nabla'_n = \sum_{i\in\mathbb{Z}}(-1)^i\nabla'_{n,i} \quad \text{and} \quad \nabla''_n = \sum_{i\in\mathbb{Z}}(-1)^{n+i-1}\nabla''_{n,i},$$

where

$$\begin{aligned} \partial'_{n,i} &: A^{\otimes n}\to A^{\otimes(n-1)}, & \partial'_{n,i} &= A^{\otimes i}\otimes\mu\otimes A^{\otimes(n-i-2)},\\ \partial''_{n,i} &: A^{\otimes n}\to A^{\otimes(n+1)}, & \partial''_{n,i} &= A^{\otimes i}\otimes\eta\otimes A^{\otimes(n-i)},\\ \nabla'_{n,i} &: C^{\otimes n}\to C^{\otimes(n-1)}, & \nabla'_{n,i} &= C^{\otimes i}\otimes\epsilon\otimes C^{\otimes(n-i-1)}, \quad \text{and}\\ \nabla''_{n,i} &: C^{\otimes n}\to C^{\otimes(n+1)}, & \nabla''_{n,i} &= C^{\otimes i}\otimes\Delta\otimes C^{\otimes(n-i-1)}. \end{aligned}$$

It turns out that we may associate with A a simplicial module SA, by setting $S_nA = A^{\otimes(n+2)}$, $n = 0, 1, \cdots$, with face operators $d_{n,i} = \partial'_{n+2,i}$, $i = 0, \cdots, n$, and degeneracy operators $s_{n,i} = \partial''_{n,i}$, $i = 0, \cdots, n$. Similarly, we may associate with C a simplicial module SC, by setting $S_nC = C^{\otimes(n+1)}$, $n = 0, 1, \cdots$, with face operators $d_{n,i} = \nabla'_{n+1,i}$, $i = 0, \cdots, n$, and degeneracy operators $s_{n,i} = \nabla''_{n+1,i}$, $i = 0, \cdots, n$. The standard identities

$$\begin{aligned}
d_{n-1,i}d_{n,j} &= d_{n-1,j-1}d_{n,i}\,, && \text{for } i < j, \\
s_{n,i}s_{n-1,j} &= s_{n,j+1}s_{n-1,i}\,, && \text{for } i \leq j, \text{ and} \\
d_{n+1,i}s_{n,j} &= s_{n-1,j-1}d_{n,i}\,, && \text{for } i < j, \\
&= \text{identity}, && \text{for } i = j \text{ or } i = j+1, \text{ and} \\
&= s_{n-1,j}d_{n,i-1}\,, && \text{for } i > j+1,
\end{aligned}$$

are readily checked in both cases. As expected, if A and C are entwined *via* ψ, then the $\psi_{n+1,n+2}$, $n = 0, 1, \cdots$, are the components of a simplicial map $SC \times SA \to SA \times SC$ between the product simplicial modules (recall that face operators and degeneracies in the product are defined componentwise). Consider further the associated (positive) chain comlexes $(\boldsymbol{K}SA, \partial')$, $\boldsymbol{K}_nSA = S_nA$, and $(\boldsymbol{K}SC, \nabla')$, $\boldsymbol{K}_nSC = S_nC$, along with $(\boldsymbol{K}(SC \times SA), \delta)$ and $(\boldsymbol{K}(SA \times SC), \delta)$, where

$$\boldsymbol{K}_n(SC \times SA) = S_nC \otimes S_nA, \quad \delta_n = \sum_{i=0}^{n}(-1)^i\nabla'_{n+1,i} \otimes \partial'_{n+2,i}\,, \quad \text{and}$$

$$\boldsymbol{K}_n(SA \times SC) = S_nA \otimes S_nC, \quad \delta_n = \sum_{i=0}^{n}(-1)^i\partial'_{n+2,i} \otimes \nabla'_{n+1,i}\,.$$

By the Eilenberg-Zilber theorem, there are natural chain equivalences

$$\boldsymbol{K}(SC \times SA) \rightleftarrows \boldsymbol{K}SC \otimes \boldsymbol{K}SA \quad \text{and} \quad \boldsymbol{K}(SA \times SC) \rightleftarrows \boldsymbol{K}SA \otimes \boldsymbol{K}SC.$$

The standard natural chain transformations which do the job are Alexander-Whitney's (rightwards) and Eilenberg-Zilber's (leftwards); both are the identity in dimension zero. Now, if A and C are entwined *via* ψ, then the doubly indexed ψ's combine together to yield two morphisms of graded modules:

$$\tilde{\psi} : \boldsymbol{K}(SC \times SA) \to \boldsymbol{K}(SA \times SC), \quad \tilde{\psi}_n = \psi_{n+1,n+2}\,, \quad \text{and}$$

$$\check{\psi} : \boldsymbol{K}SC \otimes \boldsymbol{K}SA \to \boldsymbol{K}SA \otimes \boldsymbol{K}SC, \quad \check{\psi}_n = \sum_{q+p=n}(-1)^{qp}\psi_{q+1,p+2}\,.$$

Theorem B. *Both $\tilde{\psi}$ and $\check{\psi}$ are chain transformations compatible up to chain homotopy with any pair of natural chain transformations for the Eilenberg-Zilber theorem,*

$$\begin{aligned}
\theta^{C,A} &: \boldsymbol{K}(SC \times SA) \rightleftarrows \boldsymbol{K}SC \otimes \boldsymbol{K}SA : \zeta^{C,A} \quad \textit{and} \\
\theta^{A,C} &: \boldsymbol{K}(SA \times SC) \rightleftarrows \boldsymbol{K}SA \otimes \boldsymbol{K}SC : \zeta^{A,C},
\end{aligned}$$

which are the respective identities in dimension zero: $\check{\psi}\theta^{C,A} \simeq \theta^{A,C}\tilde{\psi}$, *so* $\zeta^{A,C}\check{\psi} \simeq \tilde{\psi}\zeta^{C,A}$ *as well.*

The remainder of the paper is devoted to a detailed proof of Theorems A and B. Related facts are also considered.

3 Proofs, Formulae, Lemmas

The Morphisms ∂'_n, ∂''_n, ∇'_n and ∇''_n

In the previous section we stated without proof relations (3) and (3′). Our purpose here is to fill in the details.

We begin with an alternative description of the ∂'_n, ∂''_n, ∇'_n and ∇''_n.

1. Lemma.

(a) $\partial'_n = \partial'_{n-1} \otimes A + (-1)^n A^{\otimes(n-2)} \otimes \mu$;

(b) $\partial''_n = A \otimes \partial''_{n-1} + (-1)^n \eta \otimes A^{\otimes n}$;

(a′) $\nabla'_n = \nabla'_{n-1} \otimes C + (-1)^{n-1} C^{\otimes(n-1)} \otimes \epsilon$; *and*

(b′) $\nabla''_n = C \otimes \nabla''_{n-1} + (-1)^{n-1} \Delta \otimes C^{\otimes(n-1)}$.

Proof. With reference to the definitions, all relations follow by induction on n. As an example, we prove **(a)**. Clearly, the statement holds for $n < 2$. For $n \geq 2$, use (1) and the induction hypothesis to get

$$\begin{aligned}
\partial'_n &= -A \otimes \partial'_{n-1} + \mu \otimes A^{\otimes(n-2)} \\
&= -A \otimes \left(\partial'_{n-2} \otimes A + (-1)^{n-1} A^{\otimes(n-3)} \otimes \mu\right) + \mu \otimes A^{\otimes(n-2)} \\
&= -A \otimes \partial'_{n-2} \otimes A + (-1)^n A^{\otimes(n-2)} \otimes \mu + \mu \otimes A^{\otimes(n-2)} \\
&= \left(-A \otimes \partial'_{n-2} + \mu \otimes A^{\otimes(n-3)}\right) \otimes A + (-1)^n A^{\otimes(n-2)} \otimes \mu \\
&= \partial'_{n-1} \otimes A + (-1)^n A^{\otimes(n-2)} \otimes \mu.
\end{aligned}$$

Remark. Lemma 1 actually provides equivalent definitions for the ∂'_n, ∂''_n, ∇'_n and ∇''_n. For instance, had we defined $\partial'_n : A^{\otimes n} \to A^{\otimes(n-1)}$ by

$$\partial'_n = \begin{cases} 0, & \text{for } n < 2, \\ \partial'_{n-1} \otimes A + (-1)^n A^{\otimes(n-2)} \otimes \mu, & \text{for } n \geq 2, \end{cases}$$

then (1) would have followed in the same way.

Relations (3) and (3′) follow by induction on n from the lemma below.

2. Lemma.

(a) $\partial'_{n-1}\partial'_n = A \otimes \partial'_{n-2}\partial'_{n-1}$;

(b) $\partial''_n\partial''_{n-1} = \partial''_{n-1}\partial''_{n-2} \otimes A$;

(c) $\partial'_{n+1}\partial''_n - \partial''_{n-1}\partial'_n = -A \otimes \left(\partial'_n\partial''_{n-1} - \partial''_{n-2}\partial'_{n-1}\right)$;

(a′) $\nabla'_{n-1}\nabla'_n = C \otimes \nabla'_{n-2}\nabla'_{n-1}$;

(b′) $\nabla''_n\nabla''_{n-1} = \nabla''_{n-1}\nabla''_{n-2} \otimes C$; *and*

(c′) $\nabla'_{n+1}\nabla''_n - \nabla''_{n-1}\nabla'_n = -C \otimes \left(\nabla'_n\nabla''_{n-1} - \nabla''_{n-2}\nabla'_{n-1}\right)$.

Proof. We prove the first three statements; the last three are established in the same way.

(a) Use (1) to write both ∂''s in the left-hand member and get

$$\begin{aligned}\partial'_{n-1}\partial'_n =& A \otimes \partial'_{n-2}\partial'_{n-1} - \mu \otimes \partial'_{n-2} - \\ & \left(\mu \otimes A^{\otimes(n-3)}\right)\left(A \otimes \partial'_{n-1}\right) + \mu(\mu \otimes A) \otimes A^{\otimes(n-3)}.\end{aligned}$$

Finally, use (1) again to write ∂'_{n-1} in the third summand above along with associativity of multiplication, $\mu(A \otimes \mu) = \mu(\mu \otimes A)$, to obtain the desired result.

(b) Use (2) to write both ∂'''s in the left-hand member along with equality of the compositions $K \xrightarrow{\eta} A \xrightarrow{\cong} A \otimes K \xrightarrow{A\otimes\eta} A \otimes A$ and $K \xrightarrow{\cong} K \otimes K \xrightarrow{\eta\otimes\eta} A \otimes A$ to get

$$\partial''_n\partial''_{n-1} = \partial''_{n-1}\partial''_{n-2} \otimes A - \partial''_{n-1} \otimes \eta - \partial''_{n-2} \otimes A \otimes \eta + A^{\otimes(n-1)} \otimes \eta \otimes \eta.$$

Finally, use (2) again to write ∂''_{n-1} in the second summand above to obtain the desired result.

(c) Begin with the left-hand member: Use (1) to write the ∂''s, and Lemma 1(b) to write the ∂'''s, to get

$$\begin{aligned}\partial'_{n+1}\partial''_n =& - A \otimes \partial'_n\partial''_{n-1} + (-1)^{n+1}\eta \otimes \partial'_n + \\ & \left(\mu \otimes A^{\otimes(n-1)}\right)\left(A \otimes \partial''_{n-1}\right) + (-1)^n\mu(\eta \otimes A) \otimes A^{\otimes(n-1)},\end{aligned} \tag{4}$$

$$\begin{aligned}\partial''_{n-1}\partial'_n =& - A \otimes \partial''_{n-2}\partial'_{n-1} + \mu \otimes \partial''_{n-2} + \\ & (-1)^n\eta \otimes A \otimes \partial'_{n-1} + (-1)^{n-1}(\eta \otimes A)\mu \otimes A^{\otimes(n-2)}.\end{aligned} \tag{5}$$

In the right-hand member of (4), use again (1) to write ∂'_n in the second summand, and Lemma 1(b) to write ∂''_{n-1} in the third, along with $\mu(A \otimes \eta) = A = \mu(\eta \otimes A)$ to get

$$\begin{aligned}\partial'_{n+1}\partial''_n =& - A \otimes \partial'_n\partial''_{n-1} + (-1)^n\eta \otimes A \otimes \partial'_{n-1} + \\ & (-1)^{n+1}\eta \otimes \mu \otimes A^{\otimes(n-2)} + \mu \otimes \partial''_{n-2}.\end{aligned} \tag{6}$$

Finally, since the compositions $A \otimes A \xrightarrow{\cong} K \otimes A \otimes A \xrightarrow{\eta \otimes \mu} A \otimes A$ and $A \otimes A \xrightarrow{\mu} A \xrightarrow{\cong} K \otimes A \xrightarrow{\eta \otimes A} A \otimes A$ are equal, subtraction of (5) from (6) yields the desired result.

Remark. In any tensor product in Lemma 2, factors can be swapped. The above proof works through *mutatis mutandis.*

The Morphisms $\psi_{n,m}$

As stated in the introduction, the definition of the $\psi_{n,m}$ may seem far-fetched and unwieldy. Our purpose here is to make it more tractable by describing an alternative, but equivalent approach.

To begin with, let $\psi_{n,m} = 0$, if either index is negative, $\psi_{n,0} = C^{\otimes n}$, and $\psi_{0,m} = A^{\otimes m}$. Set

$$\psi_{1,m} = \left(A^{\otimes(m-1)} \otimes \psi\right)(\psi_{1,m-1} \otimes A), \tag{7}$$

for $m > 0$, and check by induction on m that

$$\psi_{1,m} = (A \otimes \psi_{1,m-1})\left(\psi \otimes A^{\otimes(m-1)}\right), \tag{8}$$

for all $m > 0$ (or *vice versa*). Finally, set

$$\psi_{n,m} = \left(\psi_{1,m} \otimes C^{\otimes(n-1)}\right)(C \otimes \psi_{n-1,m}), \tag{9}$$

for all $n > 1$ and all m, and check by induction on (n, m) that

$$\psi_{n,m} = (\psi_{n-1,m} \otimes C)\left(C^{\otimes(n-1)} \otimes \psi_{1,m}\right), \tag{10}$$

for all n and all m (or *vice versa*).

We could equally well have started by first filling in the rows in the first quadrant. More precisely, define $\psi_{n,m}$ as before if either index is non-positive. Now, set

$$\psi_{n,1} = \left(\psi \otimes C^{\otimes(n-1)}\right)(C \otimes \psi_{n-1,1}), \tag{7$'$}$$

for $n > 0$, and check by induction on n that

$$\psi_{n,1} = (\psi_{n-1,1} \otimes C)\left(C^{\otimes(n-1)} \otimes \psi\right), \tag{8$'$}$$

for $n > 0$ (or *vice versa*). Finally, set

$$\psi_{n,m} = \left(A^{\otimes(m-1)} \otimes \psi_{n,1}\right)(\psi_{n,m-1} \otimes A), \tag{9$'$}$$

for all n and all $m > 1$, and check by induction on (n,m) that

$$\psi_{n,m} = (A \otimes \psi_{n,m-1}) \left(\psi_{n,1} \otimes A^{\otimes(m-1)}\right), \qquad (10')$$

for all n and all m (or *vice versa*).

The two definitions actually agree, as can be checked by a rather lengthy induction on (n,m). Altogether, they combine to yield

$$\begin{aligned}\psi_{n,m} &= \left(A^{\otimes(m-1)} \otimes \psi \otimes C^{\otimes(n-1)}\right) \left(\psi_{1,m-1} \otimes A \otimes C^{\otimes(n-1)}\right) (C \otimes \psi_{n-1,m}) \\ &= (\psi_{n-1,m} \otimes C) \left(C^{\otimes(n-1)} \otimes A \otimes \psi_{1,m-1}\right) \left(C^{\otimes(n-1)} \otimes \psi \otimes A^{\otimes(m-1)}\right),\end{aligned}$$

for all n and all m.

Our next task is to show that the $\psi_{n,m}$ commute with the differentials — Lemma 3 below , and with face operators and degeneracies Lemma 4 in the sequel.

3. Lemma.

(a) $(\partial'_m \otimes C^{\otimes n}) \psi_{n,m} = \psi_{n,m-1} (C^{\otimes n} \otimes \partial'_m)$;

(b) $(\partial''_m \otimes C^{\otimes n}) \psi_{n,m} = \psi_{n,m+1} (C^{\otimes n} \otimes \partial''_m)$;

(a′) $(A^{\otimes m} \otimes \nabla'_n) \psi_{n,m} = \psi_{n-1,m} (\nabla'_n \otimes A^{\otimes m})$; *and*

(b′) $(A^{\otimes m} \otimes \nabla''_n) \psi_{n,m} = \psi_{n+1,m} (\nabla''_n \otimes A^{\otimes m})$.

Consequently,

(â) $\left(d'_{p,q} \otimes \hat{C}_{r,s}\right) \hat{\psi}_{r,s,p,q} = \hat{\psi}_{r,s,p-1,q} \left(\hat{C}_{r,s} \otimes d'_{p,q}\right)$;

(b̂) $\left(d''_{p,q} \otimes \hat{C}_{r,s}\right) \hat{\psi}_{r,s,p,q} = \hat{\psi}_{r,s,p,q-1} \left(\hat{C}_{r,s} \otimes d''_{p,q}\right)$;

(â′) $\left(\hat{A}_{p,q} \otimes d'_{r,s}\right) \hat{\psi}_{r,s,p,q} = \hat{\psi}_{r-1,s,p,q} \left(d'_{r,s} \otimes \hat{A}_{p,q}\right)$; *and*

(b̂′) $\left(\hat{A}_{p,q} \otimes d''_{r,s}\right) \hat{\psi}_{r,s,p,q} = \hat{\psi}_{r,s-1,p,q} \left(d''_{r,s} \otimes \hat{A}_{p,q}\right)$.

Proof. Only the first two statements will be proved. The next two are proved *mutatis mutandis*, and the last four are obvious rephrasings of the first four. The proofs are by induction on (n,m).

(a) The statement holds trivially if either $n < 0$ or $m < 2$, and is easily checked for any pair of the form $(0,m)$. Consider first a pair (n,m) with $n > 1$

and transform the left-hand member successively as follows:

$$\begin{aligned}
&\left(\partial'_m \otimes C^{\otimes n}\right) \psi_{n,m} \\
&= \left(\partial'_m \otimes C^{\otimes n}\right) \left(\psi_{n-1,m} \otimes C\right) \left(C^{\otimes(n-1)} \otimes \psi_{1,m}\right) \quad \text{(by (10))} \\
&= \left(\left(\partial'_m \otimes C^{\otimes(n-1)}\right) \psi_{n-1,m} \otimes C\right) \left(C^{\otimes(n-1)} \otimes \psi_{1,m}\right) \\
&= \left(\psi_{n-1,m-1} \left(C^{\otimes(n-1)} \otimes \partial'_m\right) \otimes C\right) \left(C^{\otimes(n-1)} \otimes \psi_{1,m}\right) \\
&\quad \text{(by the induction hypothesis)} \\
&= \left(\psi_{n-1,m-1} \otimes C\right) \left(C^{\otimes(n-1)} \otimes \partial'_m \otimes C\right) \left(C^{\otimes(n-1)} \otimes \psi_{1,m}\right) \\
&= \left(\psi_{n-1,m-1} \otimes C\right) \left(C^{\otimes(n-1)} \otimes \left(\partial'_m \otimes C\right) \psi_{1,m}\right) \\
&= \left(\psi_{n-1,m-1} \otimes C\right) \left(C^{\otimes(n-1)} \otimes \psi_{1,m-1} \left(C \otimes \partial'_m\right)\right) \\
&\quad \text{(by the induction hypothesis)} \\
&= \left(\psi_{n-1,m-1} \otimes C\right) \left(C^{\otimes(n-1)} \otimes \psi_{1,m-1}\right) \left(C^{\otimes n} \otimes \partial'_m\right) \\
&= \psi_{n,m-1} \left(C^{\otimes n} \otimes \partial'_m\right). \quad \text{(by (10))}
\end{aligned}$$

The case $(1, m)$ is dealt with by induction on m. As remarked at the beginning of the proof, the statement holds trivially for $m < 2$. So, let $m \geq 2$ and use (1) and (8) to get

$$\begin{aligned}
\left(\partial'_m \otimes C\right) \psi_{1,m} = &- \left(A \otimes \partial'_{m-1} \otimes C\right) \left(A \otimes \psi_{1,m-1}\right) \left(\psi \otimes A^{\otimes(m-1)}\right) + \\
&\left(\mu \otimes A^{\otimes(m-2)} \otimes C\right) \left(A \otimes \psi_{1,m-1}\right) \left(\psi \otimes A^{\otimes(m-1)}\right).
\end{aligned} \tag{11}$$

Now, deal with each summand in (11) separately. The sign left aside, the first summand transforms successively as follows:

$$\begin{aligned}
&\left(A \otimes \partial'_{m-1} \otimes C\right) \left(A \otimes \psi_{1,m-1}\right) \left(\psi \otimes A^{\otimes(m-1)}\right) \\
&= \left(A \otimes \left(\partial'_{m-1} \otimes C\right) \psi_{1,m-1}\right) \left(\psi \otimes A^{\otimes(m-1)}\right) \\
&= \left(A \otimes \psi_{1,m-2} \left(C \otimes \partial'_{m-1}\right)\right) \left(\psi \otimes A^{\otimes(m-1)}\right) \\
&\quad \text{(by the induction hypothesis)} \\
&= \left(A \otimes \psi_{1,m-2}\right) \left(A \otimes C \otimes \partial'_{m-1}\right) \left(\psi \otimes A^{\otimes(m-1)}\right) \\
&= \left(A \otimes \psi_{1,m-2}\right) \left(\psi \otimes \partial'_{m-1}\right) \\
&= \left(A \otimes \psi_{1,m-2}\right) \left(\psi \otimes A^{\otimes(m-2)}\right) \left(C \otimes A \otimes \partial'_{m-1}\right) \\
&= \psi_{1,m-1} \left(C \otimes A \otimes \partial'_{m-1}\right). \quad \text{(by (8))}
\end{aligned}$$

Next, transform the second summand in (11) as follows:

$$\begin{aligned}
&\left(\mu \otimes A^{\otimes(m-2)} \otimes C\right)(A \otimes \psi_{1,m-1})\left(\psi \otimes A^{\otimes(m-1)}\right)\\
&= \left(\mu \otimes A^{\otimes(m-2)} \otimes C\right)\\
&\quad \left(A \otimes (A \otimes \psi_{1,m-2})\left(\psi \otimes A^{\otimes(m-2)}\right)\right)\left(\psi \otimes A^{\otimes(m-1)}\right) \quad \text{(by (8))}\\
&= \left(\mu \otimes A^{\otimes(m-2)} \otimes C\right)(A \otimes A \otimes \psi_{1,m-2})\left(A \otimes \psi \otimes A^{\otimes(m-2)}\right)\\
&\quad \left(\psi \otimes A^{\otimes(m-1)}\right)\\
&= (\mu \otimes \psi_{1,m-2})\left(A \otimes \psi \otimes A^{\otimes(m-2)}\right)\left(\psi \otimes A^{\otimes(m-1)}\right)\\
&= (A \otimes \psi_{1,m-2})\left(\mu \otimes C \otimes A^{\otimes(m-2)}\right)\left(A \otimes \psi \otimes A^{\otimes(m-2)}\right)\\
&\quad \left(\psi \otimes A^{\otimes(m-1)}\right)\\
&= (A \otimes \psi_{1,m-2})\left((\mu \otimes C)(A \otimes \psi)(\psi \otimes A) \otimes A^{\otimes(m-2)}\right)\\
&= (A \otimes \psi_{1,m-2})\left(\psi(C \otimes \mu) \otimes A^{\otimes(m-2)}\right)\\
&\quad \text{(by condition (a) in the definition of } \psi)\\
&= (A \otimes \psi_{1,m-2})\left(\psi \otimes A^{\otimes(m-2)}\right)\left(C \otimes \mu \otimes A^{\otimes(m-2)}\right)\\
&= \psi_{1,m-1}\left(C \otimes \mu \otimes A^{\otimes(m-2)}\right). \quad \text{(by (8))}
\end{aligned}$$

Finally, plug the outcome of both calculations into (11) to get the desired result by (1):

$$\begin{aligned}
(\partial'_m \otimes C)\,\psi_{1,m} &= \psi_{1,m-1}\left(-C \otimes A \otimes \partial'_{m-1} + C \otimes \mu \otimes A^{\otimes(m-2)}\right)\\
&= \psi_{1,m-1}\,(C \otimes \partial'_m)\,.
\end{aligned}$$

(b) The statement holds trivially if either index is negative, and is easily checked for any pair of the form $(0, m)$. For a pair (n, m) with $n > 1$, use (10) to proceed inductively as in the first step of the proof of **(a)**. For a pair of the form $(1, m)$ with $m > 0$, use (2) and (7) to get

$$\begin{aligned}
(\partial''_m \otimes C)\,\psi_{1,m} = &-\left(\partial''_{m-1} \otimes A \otimes C\right)\left(A^{\otimes(m-1)} \otimes \psi\right)(\psi_{1,m-1} \otimes A) +\\
&\left(A^{\otimes m} \otimes \eta \otimes C\right)\psi_{1,m}.
\end{aligned} \tag{12}$$

The sign left aside, transform the first summand in (12) as follows:

$$\begin{aligned}
&\left(\partial''_{m-1}\otimes A\otimes C\right)\left(A^{\otimes(m-1)}\otimes\psi\right)\left(\psi_{1,m-1}\otimes A\right)\\
&=\left(\partial''_{m-1}\otimes\psi\right)\left(\psi_{1,m-1}\otimes A\right)\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(\partial''_{m-1}\otimes C\otimes A\right)\left(\psi_{1,m-1}\otimes A\right)\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(\left(\partial''_{m-1}\otimes C\right)\psi_{1,m-1}\otimes A\right)\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(\psi_{1,m}\left(\partial''_{m-1}\otimes C\right)\otimes A\right)\quad\text{(by the induction hypothesis)}\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(\psi_{1,m}\otimes A\right)\left(C\otimes\partial''_{m-1}\otimes A\right)\\
&=\psi_{1,m+1}\left(C\otimes\partial''_{m-1}\otimes A\right).\quad\text{(by (7))}
\end{aligned}$$

Next, transform the second summand in (12) as follows:

$$\begin{aligned}
&\left(A^{\otimes m}\otimes\eta\otimes C\right)\psi_{1,m}\\
&=\left(A^{\otimes m}\otimes\psi\left(C\otimes\eta\right)\right)\psi_{1,m}\quad\text{(by condition (b) in the definition of }\psi)\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(A^{\otimes m}\otimes C\otimes\eta\right)\psi_{1,m}\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(\psi_{1,m}\otimes\eta\right)\\
&=\left(A^{\otimes m}\otimes\psi\right)\left(\psi_{1,m}\otimes A\right)\left(C\otimes A^{\otimes m}\otimes\eta\right)\\
&=\psi_{1,m+1}\left(C\otimes A^{\otimes m}\otimes\eta\right).\quad\text{(by (7))}
\end{aligned}$$

Finally, plug the outcome of both calculations into (12) to get the desired result by (2):

$$\begin{aligned}
\left(\partial''_m\otimes C\right)\psi_{1,m}&=\psi_{1,m+1}\left(-C\otimes\partial''_{m-1}\otimes A+C\otimes A^{\otimes m}\otimes\eta\right)\\
&=\psi_{1,m+1}\left(C\otimes\partial''_m\right).
\end{aligned}$$

Remark. Part **(a)** of Lemma 3 can equally well be proved by using Lemma 1(**a**), (7) and (9) instead of (1), (8) and (10), respectively. Similarly, part **(b)** of Lemma 3 can also be established by resorting to Lemma 1(**b**), (8) and (9) instead of (2), (7) and (10), respectively.

4. Lemma.

(a) $\left(\partial'_{m,i}\otimes C^{\otimes n}\right)\psi_{n,m}=\psi_{n,m-1}\left(C^{\otimes n}\otimes\partial'_{m,i}\right)$;

(b) $\left(\partial''_{m,i}\otimes C^{\otimes n}\right)\psi_{n,m}=\psi_{n,m+1}\left(C^{\otimes n}\otimes\partial''_{m,i}\right)$;

(a′) $\left(A^{\otimes m}\otimes\nabla'_{n,i}\right)\psi_{n,m}=\psi_{n-1,m}\left(\nabla'_{n,i}\otimes A^{\otimes m}\right)$; *and*

(b′) $\left(A^{\otimes m}\otimes\nabla''_{n,i}\right)\psi_{n,m}=\psi_{n+1,m}\left(\nabla''_{n,i}\otimes A^{\otimes m}\right)$.

The proof is similar to that of Lemma 3 and hence omitted. Recalling that face operators and degeneracies in the product are defined componentwise, Lemma 4 shows that the $\psi_{n+1,n+2}$, $n = 0, 1, \cdots$, are the components of a simplicial map $SC \times SA \to SA \times SC$ between the product simplicial modules, so $\tilde{\psi}$ defined at the end of Section 2 is a chain transformation.

We now proceed to prove the theorems.

Proof of the Theorem A. Both relations are proved along the same lines, so only one of them will be dealt with in full detail. As an example, let us show that $\delta''_{m,n}\hat{\psi}_{m,n} = \hat{\psi}_{m,n-1}\delta''_{m,n}$. To this end, recall the definitions and use parts ($\hat{\mathrm{b}}$) and ($\hat{\mathrm{b}}'$) of Lemma 3 to rewrite the left-hand member successively as follows:

$$\begin{aligned}\delta''_{m,n}\hat{\psi}_{m,n} &= \sum_{p+r=m,\ q+s=n} (-1)^{(p+1)r+qs}\left(d''_{p,q}\otimes \hat{C}_{r,s}\right)\hat{\psi}_{r,s,p,q} + \\ &\sum_{p+r=m,\ q+s=n} (-1)^{p(r+1)+q(s+1)}\left(\hat{A}_{p,q}\otimes d''_{r,s}\right)\hat{\psi}_{r,s,p,q} \\ &= \sum_{p+r=m,\ q+s=n} (-1)^{(p+1)r+qs}\hat{\psi}_{r,s,p,q-1}\left(\hat{C}_{r,s}\otimes d''_{p,q}\right) + \\ &\sum_{p+r=m,\ q+s=n} (-1)^{p(r+1)+q(s+1)}\hat{\psi}_{r,s-1,p,q}\left(d''_{r,s}\otimes \hat{A}_{p,q}\right).\end{aligned}$$

With reference again to the definitions, write the right-hand member explicitly to get

$$\begin{aligned}\hat{\psi}_{m,n-1}\delta''_{m,n} &= \sum_{p+r=m,\ q+s=n} (-1)^{p(r+1)+q(s-1)}\hat{\psi}_{r,s-1,p,q}\left(d''_{r,s}\otimes \hat{A}_{p,q}\right) + \\ &\sum_{p+r=m,\ q+s=n} (-1)^{(p+1)r+qs}\hat{\psi}_{r,s,p,q-1}\left(\hat{C}_{r,s}\otimes d''_{p,q}\right).\end{aligned}$$

Comparison of the two yields the desired result.

Proof of the Theorem B. We already noticed after Lemma 4 that $\tilde{\psi}$ is a chain transformation. To show that $\check{\psi}$ is a chain transformation, recall the definitions and proceed as in the proof of Theorem A we omit the details.

To prove the second part of the theorem, recall [7] that for any simplicial modules X and Y there exists a natural chain transformation $\boldsymbol{K}(X \times Y) \to \boldsymbol{K}X \otimes \boldsymbol{K}Y$ which is the identity in dimension zero, and any two such are chain homotopic *via* a natural chain homotopy. Similarly, there exists a natural chain transformation $\boldsymbol{K}X \otimes \boldsymbol{K}Y \to \boldsymbol{K}(X \times Y)$ which is the identity in dimension zero, and any two such are chain homotopic *via* a chain homotopy natural in X and Y. Finally, there is a natural chain homotopy between any two natural chain transformations $\boldsymbol{K}(X\times Y) \rightrightarrows \boldsymbol{K}(X\times Y)$ which are the identity in dimension zero.

The standard chain transformations for the Eilenberg-Zilber chain equivalence theorem are Alexander-Whitney's, $AW^{X,Y}$, and Eilenberg-Zilber's, $EZ^{X,Y}$:

$$AW^{X,Y} : \boldsymbol{K}(X \times Y) \rightleftarrows \boldsymbol{K}X \otimes \boldsymbol{K}Y : EZ^{X,Y};$$

both are the identity in dimension zero. Explicit formulae for $AW^{X,Y}$ and $EZ^{X,Y}$ can be found in [7]. Back to our case, it suffices to show that

$$\check{\psi}\, AW^{SC,SA} = AW'^{SA,SC}\, \tilde{\psi},$$

where $AW'^{SA,SC} \simeq AW^{SA,SC}$ is an Alexander-Whitney-like natural chain transformation which is the identity in dimension zero. To this end, recall the definitions of $\check{\psi}$ and $\tilde{\psi}$, write $AW^{SC,SA}$ explicitly,

$$AW_n^{SC,SA} = \sum_{q+p=n} \underbrace{\nabla'_{q+2,q+1} \cdots \nabla'_{n+1,n}}_{p} \otimes \underbrace{\partial'_{p+3,0} \cdots \partial'_{n+2,0}}_{q},$$

and use parts **(a)** and **(a′)** of Lemma 4 to get

$$\begin{aligned}\check{\psi}_n AW_n^{SC,SA} &= \sum_{p+q=n} (-1)^{pq} \left(\partial'_{p+3,0} \cdots \partial'_{n+2,0} \otimes \nabla'_{q+2,q+1} \cdots \nabla'_{n+1,n}\right) \psi_{n,n} \\ &= AW_n'^{SA,SC} \tilde{\psi}_n ,\end{aligned}$$

where

$$AW_n'^{SA,SC} = \sum_{p+q=n} (-1)^{pq}\, \partial'_{p+3,0} \cdots \partial'_{n+2,0} \otimes \nabla'_{q+2,q+1} \cdots \nabla'_{n+1,n} .$$

Compare

$$AW_n^{SA,SC} = \sum_{p+q=n} \partial'_{p+3,p+1} \cdots \partial'_{n+2,n} \otimes \nabla'_{q+2,0} \cdots \nabla'_{n+1,0} ;$$

to notice that in each summand in AW', the first tensor factor is (simplicially) a back face, and the second a front face, while in each summand in AW, the first tensor factor is a front face, and the second a back face. Back to $AW'^{SA,SC}$, it is defined by face operators, so it is natural. It reduces to the identity in dimension zero. And it is a chain map by the standard identities for face operators — the lengthy verifications are omitted.

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