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An update on the maximal function

Ilie Valuşescu

Abstract

New results about the maximal functions are presented. Using choice sequence techniques, a directe proof for the form of the maximal function is obtained. Some properties of the maximal function are analysed, and some connections with the linear systems theory are found. It is shown that in the contraction case the spectral factors are restrictions of the maximal function. Also, the behaviour of discrete linear systems is studied with the maximal functions of the main operator T and T^*, respectively, which become the observability and the controllability operators of the system.

AMS Classification: 47A20, 47A30.

Keywords: Contractions, unitary dilation, choice sequences, maximal function, spectral factors, linear systems.

1 Preliminaries

The maximal function arised in the context of factorization of operator valued analytic functions. Firstly, in the bounded case ([18], Prop.V.4.2), where the obtained maximal outer function is a contractive one, and then in the general setting of a semispectral measure [16], when the maximal outer function is not a bounded one, but an L^2-bounded analytic function. Trying to illustrate the maximal function in the particular case of the semispectral measure attached to a contraction on a Hilbert space, a concrete form for the maximal function was obtained [20], which latter proved to have interesting properties and applications, especially in the linear systems theory.

The maximal function of a semispectral measure was an usefull tool in obtaining the linear Wiener filter for prediction in the generalized infinite dimensional case, and in solving various prediction problems (see e.g. [17]). The maximal function of a contraction, besides the intrinsic properties, proves to play a more or less explicit role in the study of linear systems having as the main operator a contraction. Actually, the maximal function of T and T^* generate the observability and controllability operators, and can be usefull in the study of the behaviour of linear systems.

In the present paper some properties and applications of the maximal function will be presented, mainly following [21]-[23].

As usually, if $\mathcal{H}$ is a complex separable Hilbert space, and $\mathcal{L}(\mathcal{H})$-the C^*-algebra of the linear bounded operators on $\mathcal{H}$, as usually, for a contraction $T \in \mathcal{L}(\mathcal{H})$, $(\|Th\| \leq \|h\|)$ the defect operators $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ are attached. Also, the corresponding defect spaces will be $\mathcal{D}_T = \overline{D_T\mathcal{H}}$ and $\mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$.

For two suitable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, a *choice sequence* [9] is a finite or infinite sequence $\{G_n\}$ of contractions such that $G_1 : \mathcal{H}_1 \to \mathcal{H}_2$, and for $k \geq 2$, $G_n : \mathcal{D}_{G_{n-1}} \to \mathcal{D}_{G^*_{n-1}}$. Of course, if $\{G_n\}_{n=1}^{N}$ is a finite choice sequence, then for any contraction $G_{N+1} : \mathcal{D}_{G_N} \to \mathcal{D}_{G^*_N}$, $\{G_n\}_{n=1}^{N+1}$ is a choice sequence, too.

By an $\mathcal{L}(\mathcal{H})$-valued *semispectral measure* on the unit torus $\mathbb{T}$ we mean a map $\sigma \to F(\sigma)$ from the family $\mathcal{B}(\mathbb{T})$ of Borel subsets σ of $\mathbb{T}$ into $\mathcal{L}(\mathcal{H})$, such that for any $h \in \mathcal{H}$ the map $\sigma \to \langle F(\sigma)h, h\rangle_{\mathcal{H}}$ is a positive Radon measure on $\mathbb{T}$.

An $\mathcal{L}(\mathcal{H})$-valued *semispectral measure* E is *spectral*, if $E(\mathbb{T}) = I_{\mathcal{H}}$ and $E(\sigma_1 \cap \sigma_2) = E(\sigma_1) \cap E(\sigma_2)$. To each $\mathcal{L}(\mathcal{K})$-valued spectral measure E on $\mathbb{T}$, an $\mathcal{L}(\mathcal{H})$-valued semispectral measure can be attached, taking for any bounded operator $V \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ the map

$$F(\sigma) = V^*E(\sigma)V.$$

The converse is assured by the following Naimark spectral dilation theorem.

THEOREM 1.1. *Let F be an $\mathcal{L}(\mathcal{H})$-valued semispectral measure on $\mathbb{T}$. There exists a Hilbert space $\mathcal{K}$ and an $\mathcal{L}(\mathcal{K})$-valued spectral measure E on $\mathbb{T}$ such that*

$$F(\sigma) = V^*E(\sigma)V \qquad (\sigma \in \mathcal{B}(\mathbb{T})). \tag{1.1}$$

The triplet $[\mathcal{K}, V, E]$ which verifies (1.1) *is called a spectral dilation of F. The spectral dilation is minimal if*

$$\mathcal{K} = \bigcap_{\sigma \in \mathcal{B}(\mathbb{T})} E(\sigma)V\mathcal{H}. \tag{1.2}$$

By the representation of the C^*-algebra $C(\mathbb{T})$ into $\mathcal{L}(\mathcal{K})$, which maps the function 1 into the identity operator $I_{\mathcal{K}}$, we have the unitary operator $U \in \mathcal{L}(\mathcal{K})$ corresponding to the $\mathcal{L}(\mathcal{K})$-valued spectral measure E, defined by

$$U = \int_0^{2\pi} e^{it} dE(t). \tag{1.3}$$

Also, if we take for any $n \in \mathbb{Z}$

$$R(n) = V^* U^{*n} V, \tag{1.4}$$

then $R(n)$ is an $\mathcal{L}(\mathcal{H})$-valued positive definite map on the group $\mathbb{Z}$ such that

$$R(n) = \int_0^{2\pi} \mathrm{e}^{-int} \mathrm{d}F(t). \tag{1.5}$$

In [11], or [6], a structure of the Naimark dilation can be found in terms of the choice sequences attached to the Fourier coefficients of the semispectral measure. In the proof was used the following result [4] on row-contractions, i..e., n-tuples of the form

$$T^{(n)} = (T_1, \ldots, T_n) : \bigoplus_{k=1}^{n} \mathcal{H}_k = \mathcal{H} \to \bigoplus_{k=1}^{n} \mathcal{H}'_k = \mathcal{H}',$$

which are contractions.

THEOREM 1.2. *$T^{(n)}$ is a contraction if and only if $G_1 = T_1$ is a contraction, and for $k \geq 2$,*

$$T_k = D_{G_1^*} \cdots D_{G_{k-1}^*},$$

where $G_k : \mathcal{H}_k \to \mathcal{D}_{G_{k-1}^*}$ are contractions.

The correspondence between $T^{(n)}$ and $\{G_k\}_{k=1}^n$ is one-to-one, and the identification of the defect spaces of $T^{(n)}$ can be explicitly given by the following unitary operators:

$$\alpha_n : \mathcal{D}_{T^{(n)}} \to \mathcal{D}_{G_1} \oplus \mathcal{D}_{G_2} \oplus \cdots \oplus \mathcal{D}_{G_n}$$

and

$$\beta_n : \mathcal{D}_{T^{(n)*}} \to \mathcal{D}_{G_n^*},$$

where

$$\alpha_n = \begin{bmatrix} D_{G_1} & -G_1^* G_2 & \cdots & -G_1^* D_{G_2^*} \cdots D_{G_{n-1}^*} G_n \\ 0 & D_{G_2} & \cdots & -G_2^* D_{G_3^*} \cdots D_{G_{n-1}^*} G_n \\ \vdots & & \ddots & \cdots \\ 0 & \cdots & \cdots & D_{G_n} \end{bmatrix} \tag{1.6}$$

and

$$\beta_n D_{T^{(n)*}} = D_{G_n^*} \cdots D_{G_1^*}. \tag{1.7}$$

This kind of row-contractions was denoted by

$$L = L(\{G_k\}_{k=1}^n) = \{G_1, D_{G_1^*}G_2, \ldots, D_{G_1^*}\cdots D_{G_{n-1}^*}G_n). \tag{1.8}$$

Using the duality for a formula depending on the parameters G_n

$$\widetilde{\text{form}}(\{G_k\}_{k=1}^n) = \text{form}(\{G_k^*\}_{k=1}^n)^*, \tag{1.9}$$

the column-contractions are defined by

$$\widetilde{L} = \widetilde{L}(\{G_k\}_{k=1}^n) = \{G_1, G_2D_{G_1}, \ldots, G_nD_{G_{n-1}}\cdots D_{G_1})^t, \tag{1.10}$$

where"t" is used for matrix transpose.

The results are extended to row-contractions of infinite length

$$L(\{G_k\}_{k=1}^\infty) = \{G_1, D_{G_1^*}G_2, D_{G_1^*}D_{G_2^*}G_3, \ldots), \tag{1.11}$$

and the unitary operators between the defect spaces are identified as

$$\alpha(L) : \mathcal{D}_L \to \bigoplus_{n=1}^{\infty} \mathcal{D}_{G_n} = \mathcal{D}(L)$$

and

$$\beta(L) : \mathcal{D}_{L^*} \to \mathcal{D}_*(L).$$

Following [6], the structure of the minimal Naimark dilation of an $\mathcal{L}(\mathcal{H})$-valued semispectral measure is given by $\{\mathcal{K}, W\}$ where $\mathcal{K}$ is a Hilbert space containing $\mathcal{H}$, and W is a unitary operator in $\mathcal{L}(\mathcal{K})$ such that

$$S_n = P_{\mathcal{H}}^{\mathcal{K}} W^n | \mathcal{H} \qquad (n \in \mathbb{Z}), \tag{1.12}$$

where $\{S_n\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of the semispectral measure F, and $P_{\mathcal{H}}^{\mathcal{K}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. If $\{G_n\}_{n=1}^\infty$ is the choice sequence associated with $\{S_n\}$, then the dilation space $\mathcal{K}$ has the form

$$\mathcal{K} = \cdots \oplus \mathcal{D}_*(L) \oplus \oplus \mathcal{D}_*(L) \oplus \mathcal{H} \oplus \bigoplus_{n=1}^{\infty} \mathcal{D}_{G_n}. \tag{1.13}$$

2 The maximal function

Let $\mathcal{H}$ and $\mathcal{H}'$ be separable Hilbert spaces. An operator valued *analytic function* on $\mathbb{D}$ is a function $\Theta(\lambda)$ which admits a Taylor expansion of the form

$$\Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n \qquad (\lambda \in \mathbb{D}), \tag{2.1}$$

where the coefficients $\Theta_n \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$. The series is supposed to be convergent in norm,weakly, or strongly, which amounts to the same for the power series. Following the notations from [18] we shall denote such a function by the triplet $\{\mathcal{H}, \mathcal{H}', \Theta(\lambda)\}$.

An operator valued analytic function $\{\mathcal{H}, \mathcal{H}', \Theta(\lambda)\}$ is a *bounded* function, if there exists a positive constant M such that for each $\lambda \in \mathbb{D}$

$$\|\Theta(\lambda)\| \leq M. \tag{2.2}$$

A lot of the nice properties from the scalar case was recovered for the bounded operator valued analytic functions. To be mentioned the very usefull property of almost everywhere (a.e.) nontangential limits on the boundary, such that for each bounded operator valued analytic function $\{\mathcal{H}, \mathcal{H}', \Theta(\lambda)\}$ on $\mathbb{D}$, there exists the bounded operator valued function $\{\mathcal{H}, \mathcal{H}', \Theta(e^{it})\}$ a.e. on $\mathbb{T}$. (For detailles see [18]).

For prediction purposes, the class of bounded analytic functions was enlarged to the class of so called L^2-bounded functions [16]. An *L^2-bounded analytic function* on $\mathbb{D}$ is an operator valued analytic function $\{\mathcal{H}, \mathcal{H}', \Theta(\lambda)\}$ with the property that there exists a positive constant M such that

$$\sup_{0\leq r<1} \frac{1}{2\pi} \int_0^{2\pi} \left\|\Theta(re^{it})h\right\|^2 dt \leq M \left\|h\right\|^2 \qquad (h \in \mathcal{H}), \tag{2.3}$$

or equivalently

$$\sum_{n=0}^{\infty} \left\|\Theta_n h\right\|^2 \leq M \left\|h\right\|^2 \qquad (h \in \mathcal{H}). \tag{2.4}$$

Unfortunately for L^2-bounded functions the nontangential convergence is no longer valid, but this class of function was able to permit the extension to the infinite dimensional prediction theory.

The factorization theorems play a crucial role in prediction problems, and the factorization for operator valued semispectral measures [16] permited the extension fom the multivariate (matrix) case to the infinite dimensional one. In [16] was proved the following generalized Lowdenslager–Sz.-Nagy–Foias factorization theorem.

THEOREM 2.1. *Let F be an $\mathcal{L}(\mathcal{H})$-valued semispectral measure on $\mathbb{T}$, and $[\mathcal{K}, V, E]$ its minimal spectral dilation. There exists a unique L^2-bounded outer function $\{\mathcal{H}, \mathcal{F}_1, \Theta_1(\lambda)\}$ with the following properties:*

(i) $F_{\Theta_1} \leq F$,

(ii) for any L^2-bounded analytic function $\{\mathcal{H}, \mathcal{F}, \Theta(\lambda)\}$ such that $F_\Theta \leq F$, we have also $F_\Theta \leq F_{\Theta_1}$.

The properties (i) and (ii) determine the outer function $\Theta_1(\lambda)$ up to a left unitary constant factor.

The equality in (i) holds if and only if

$$\bigcap_{n\geq 0} U^n \mathcal{K}_+ = \{0\}, \tag{2.5}$$

where U is the unitary operator corresponding to the spectral measure E, and

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} U^n V \mathcal{H}, \tag{2.6}$$

The unique L^2-bounded outer function $\{\mathcal{H}, \mathcal{F}_1, \Theta_1(\lambda)\}$, obtained by the previous factorization theorem, is called the *maximal function* of the $\mathcal{L}(\mathcal{H})$-valued semispectral measure F.

In the particular case of a contraction T from $\mathcal{L}(\mathcal{H})$, a semispectral measure F_T is attached taking

$$F_T(\sigma) = V^* E(\sigma) V, \tag{2.7}$$

where E is the spectral measure corresponding to the unitary dilation U of T on the Hilbert space $\mathcal{K}$.

In [20] was proved the following structure theorem for the maximal outer function of a contraction.

PROPOSITION 2.2. *The maximal function of a contraction $T \in \mathcal{L}(\mathcal{H})$ coincides with $\{\mathcal{H}, \mathcal{D}_{T^*}, \Theta_1(\lambda)\}$, where*

$$\Theta_1(\lambda) = D_{T^*}(I - \lambda T^*)^{-1}, \qquad (\lambda \in \mathbb{D}). \tag{2.8}$$

Proof. The first proof [20] made a unitary correspondence between the coefficients of the obtained maximal outer function and the coefficients of $\Theta_1(\lambda)$ given by (2.8). Here, using choice sequences technique, another proof is given, similar with the proof given in [6] for the generalized Lowdenslager–Sz.-Nagy–Foias factorization theorem (Theorem 2.1).

If we consider the Fourier coefficients S_n of the semispectral measure F_T given by the positive definite kernel

$$S_n = \begin{cases} T^n & , n > 0 \\ I & , n = 0 \\ T^{*|n|} & , n < 0, \end{cases} \tag{2.9}$$

which corresponds to the choice sequence $\{G_n\}_{n=1}^{\infty}$, where $G_1 = T$, and $G_k = 0$ for $k \geq 2$, then the Naimark dilation on $\mathcal{K}$, where

$$\mathcal{K} = \cdots \oplus \mathcal{D}_{T^*} \oplus \mathcal{D}_{T^*} \oplus \mathcal{H} \oplus \mathcal{D}_T \oplus \mathcal{D}_T \cdots ,$$

is given by the Schäffer matrix form of Sz.-Nagy dilation for a contraction

$$W = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & I & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & D_{T^*} & T & 0 & \cdots \\ \cdots & 0 & 0 & -T^* & D_T & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is convenient to take into consideration the unitary operator $\widetilde{W}^*$ from $\mathcal{L}(\widetilde{\mathcal{K}})$, where

(2.10) $$\widetilde{\mathcal{K}} = \cdots \oplus \mathcal{D}_*(\tilde{L}) \oplus \mathcal{H} \oplus \mathcal{D}_{G_1^*} \oplus \mathcal{D}_{G_2^*} \oplus \cdots ,$$

and $\widetilde{W}^* = W(\{G_n^*\}_{n=1}^{\infty})$. Defining the subspace

(2.11) $$\mathcal{K}_- = \bigvee_{n=0}^{\infty} \widetilde{W}^{*n}\mathcal{H} = \mathcal{H} \oplus \mathcal{D}_{G_1^*} \oplus \mathcal{D}_{G_2^*} \oplus \cdots ,$$

and the isometry

(2.12) $$W_- = \widetilde{W}^*|\mathcal{K}_-,$$

then the Wold decomposition is given by

(2.13) $$\mathcal{K}_- = \bigoplus_{n=0}^{\infty} \widetilde{W}^{*n}\mathcal{G}_- \oplus \mathcal{R}_-,$$

where $\mathcal{G}_- = \mathcal{K}_- \ominus W_-\mathcal{K}_-$, and $\mathcal{R}_- = \bigcap_{n=0}^{\infty} \widetilde{W}^{*n}\mathcal{K}_-$. Therefore

(2.14) $$\mathcal{G}_- = \widetilde{W}^*(\cdots \oplus 0 \oplus \mathcal{D}_*(\tilde{L}) \oplus 0_{\mathcal{H}} \oplus \cdots) = \widetilde{W}^*\mathcal{G},$$

and it follows [6] that the form of the maximal outer function is given by

(2.15) $$\Theta_1(\lambda) = P_{\mathcal{G}}^{\widetilde{\mathcal{K}}}\widetilde{W}(I - \lambda\widetilde{W})^{-1}|\mathcal{H} \qquad (\lambda \in \mathbb{D}).$$

Taking into account the fact that $G_1 = T$ and $G = 0$ for $k \geq 2$, and the natural identification between $\mathcal{G}$ and $\mathcal{D}_{T^*}$, it follows that the maximal function of T has the form (2.8), namely

$$\Theta_1(\lambda) = D_{T^*}(I - \lambda T^*)^{-1}, \qquad (\lambda \in \mathbb{D}).$$

□

Analogously can be proved that the maximal function $\{\mathcal{H}, \mathcal{D}_T, \Theta_2(\lambda)\}$ of the adjoint contraction T^* has the form

$$\Theta_2(\lambda) = D_T(I - \lambda T)^{-1}, \qquad (\lambda \in \mathbb{D}). \tag{2.16}$$

Between the maximal function $\Theta_1(\lambda)$, and the characteristic function $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$ of the contraction T,

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T] | \mathcal{D}_T, \qquad (\lambda \in \mathbb{D}), \tag{2.17}$$

taking into account (2.8), there exists the obvious relation

$$\Theta_T(\lambda) = [-T + \lambda \Theta_1(\lambda) D_T] | \mathcal{D}_T, \qquad (\lambda \in \mathbb{D}), \tag{2.18}$$

or written into the matrix form

$$\Theta_T(\lambda) = \begin{bmatrix} I & \lambda \Theta_1(\lambda) \end{bmatrix} \begin{bmatrix} -T \\ D_T \end{bmatrix}, \tag{2.19}$$

as a factorization into an analytic part and an isometry.

In [20] was found another relation between these two analytic functions, namely

$$\Theta_T(\lambda) D_T = \Theta_1(\lambda)(\lambda I - T), \qquad (\lambda \in \mathbb{D}), \tag{2.20}$$

which can be also written as

$$\begin{bmatrix} \Theta_T(\lambda) & \Theta_1(\lambda) \end{bmatrix} \begin{bmatrix} D_T \\ T - \lambda I \end{bmatrix} = 0. \tag{2.21}$$

In the particular case when T is a strict contraction ($\|T\| < 1$), the relation (2.20) can be written as

$$\Theta_T(\lambda) = \Theta_1(\lambda)(\lambda I - T) D_T^{-1}, \qquad (\lambda \in \mathbb{D}). \tag{2.22}$$

Taking the characteristic function $\{\mathcal{D}_{T^*}, \mathcal{D}_T, \Theta_{T^*}(\lambda)\}$ of T^*,

$$\Theta_{T^*}(\lambda) = [-T^* + \lambda D_T(I - \lambda T)^{-1} D_{T^*}] | \mathcal{D}_{T^*}, \qquad (\lambda \in \mathbb{D}), \tag{2.23}$$

an analogous calculus leads to the connexion between the maximal function of T^* and its characteristic function $\Theta_{T^*}(\lambda)$ given by

$$\Theta_{T^*}(\lambda)D_{T^*} = \Theta_2(\lambda)(\lambda I - T^*), \qquad (\lambda \in \mathbb{D}). \tag{2.24}$$

In some investigations (see e.g. [13]), generalizing the $C_{0.}$ and $C_{.0}$ case, an operator T on $\mathcal{H}$ is called *stable* if $T^n \longrightarrow 0$, and $*$-*stable* if $T^{*n} \longrightarrow 0$, strongly as $n \longrightarrow 0$. Actually, in the linear systems theory, a system with the main operator T is called stable, if it is stable and $*$-stable.

PROPOSITION 2.3. *If T is a $*$-stable contraction, then its maximal function $\{\mathcal{H}, \mathcal{D}_{T^*}, \Theta_1(\lambda)\}$ is bounded, and the attached operator Θ_1 defined from $\mathcal{H}$ into $H^2(\mathcal{D}_{T^*})$ by*

$$(\Theta_1 h)(\lambda) = \Theta_1(\lambda)h \tag{2.25}$$

is an isometry. Moreover, the Sz.-Nagy–Foias functional model [18] reduces to a functional representation given by $\Theta_1(\lambda)$. Namely, the imbedding of $\mathcal{H}$ is given by

$$\boldsymbol{H} = \{u \in H^2(\mathcal{D}_{T^*}) | u(\lambda) = \Theta_1(\lambda)h,\ h \in \mathcal{H}\}, \tag{2.26}$$

and the contraction T is represented by

$$\boldsymbol{T}^* u(\lambda) = \frac{1}{\lambda}[\Theta_1(\lambda)h - \Theta_1(0)h]. \tag{2.27}$$

Proof. The fact that for a stable contraction T the imbedding of $\mathcal{H}$ into the Sz.-Nagy–Foias functional model $\mathbf{H}$ is given by (2.26) was proved in [20], and is based on the fact that the functional model (see [18], Ch.VI) is obtained by a unitary imbedding Φ of the dilation space $\mathcal{K}$ into a functional space. If T is $*$-stable, then $\mathcal{K} = M(\mathcal{L}_*)$, where $\mathcal{L}_* = U\mathcal{L}^*$, and $\Phi = \Phi^{\mathcal{D}_{T^*}}$-the Fourier representation on $H^2(\mathcal{D}_{T^*})$.

For any contraction T and $h \in \mathcal{H}$ we have

$$\sum_{k=0}^{n} \left\|D_{T^*}T^{*k}h\right\|^2 = \sum_{k=0}^{n} \langle D_{T^*}^2 T^{*k}h, T^{*k}h\rangle = \sum_{k=0}^{n}(\left\|T^{*k}h\right\|^2 - \left\|T^{*k+1}h\right\|^2) =$$
$$= \sum_{k=0}^{n} \left\|T^{*k}h\right\|^2 - \sum_{k=1}^{n+1} \left\|T^{*k}h\right\|^2 = \|h\|^2 - \left\|T^{*k+1}h\right\|^2.$$

Since T is $*$-stable, the previous relation becomes

$$\sum_{n=0}^{\infty} \|D_{T^*}T^{*n}h\|^2 = \|h\|^2,$$

and taking into account that

$$\Theta_1(\lambda) = \sum_{n=0}^{\infty} D_{T^*} T^{*n} \lambda^n,$$

it follows that the attached operator $\Theta_1 : \mathcal{H} \to H^2(\mathcal{D}_{T^*})$ is an isometry. $\square$

A dual representation can be found in the stable case for T^*.

Other relations between the maximal function and the characteristic function will be given in the context of linear systems, in the next sections.

As a remark, even for the characteristic function we have the dual formula $\Theta_{T^*}(\lambda) = \Theta_T(\overline{\lambda})^*$, between the maximal functions there exists no such a duality relation except the case when T is a normal operator. Moreover, the characteristic function is a contractive analytic function, while the maximal function is generally not a bounded one, but an L^2-bounded analytic function.

A particular remark is the fact that, if the semispectral measure F is Harnack equivalent with the Lebesgue measure on $\mathbb{T}$, i.e., there exists a positive constant c such that

$$c \cdot \mathrm{d}t \leq \mathrm{d}F(t) \leq c^{-1} \cdot \mathrm{d}t,$$

then the attached maximal function is bounded, and has a bounded inverse. This is the case when a linear Wiener filter for the prediction of operatorial processes, in infinite dimensional setting, can be obtained (see e.g., [17]).

Some conditions for the boundedness of the maximal function of a contraction can be found in [20]. Such a way, the maximal function is bounded if and only if the corresponding semispectral measure is boundedly dominated by the Lebesgue measure on $\mathbb{T}$. Also, a contraction T has the semispectral measure F_T of the form $\mathrm{d}F_T(t) = \Theta(\mathrm{e}^{it})^*\Theta(\mathrm{e}^{it})$, with $\Theta(\lambda)$ a bounded analytic function, if and only if $T \in C_{.0}$ and the spectral radious $\rho(t) < 1$.

3 Spectral factors

Applying Theorem 2.1 for the $\mathcal{L}(\mathcal{H})$-valued semispectral measure of the form

$$\mathrm{d}F(t) = N(t)^2 \mathrm{d}t, \tag{3.1}$$

where, for $0 \leq t \leq 2\pi$, $N(t)$ is a function whose values are self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and which is measurable (strongly, or weakly, which amounts to the same for separable $\mathcal{H}$), with the property that $0 \leq N(t) \leq I$, then the following Sz.-Nagy–Foias factorization theorem is obtained (see [18], Prop.V.4.2).

THEOREM 3.1. *There exists a contractive outer function* $\{\mathcal{H}, \mathcal{F}_1, \Theta_1(\lambda)\}$ *with the following properties:*

(i) $\Theta_1(e^{it})^*\Theta_1(e^{it}) \leq N(t)^2$ *a.e. ;*

(ii) for every other contractive analytic function $\{\mathcal{H}, \mathcal{F}, \Theta(\lambda)\}$ *such that* $\Theta(e^{it})^*\Theta(e^{it}) \leq N(t)^2$ *a.e., we have also*

$$\Theta(e^{it})^*\Theta(e^{it}) \leq \Theta_1(e^{it})^*\Theta_1(e^{it}) \quad a.e.$$

Moreover, these properties determine the outer function $\Theta_1(\lambda)$ *up to a constant unitary factor from the left. In order that equality holds in (i) a.e., it is necessary and sufficient that the condition*

$$\bigcap_{n\geq 0} e^{int}\overline{NH^2(\mathcal{H})} = \{0\} \tag{3.2}$$

be satisfied, where the self-adjoint operator N *on* $L^2(\mathcal{H})$ *is defined by the equality* $(Nv)(t) = N(t)v(t)$.

In [6] for any contractive analytic function the measurable functions $N_L(t)^2 = I - \Theta(e^{it})^*\Theta(e^{it})$, and $N_R(t)^2 = I - \Theta(e^{it})\Theta(e^{it})^*$ was considered, and the corresponding contractive maximal outer functions was obtained, using Theorem 2.1. These particular maximal functions was called [6] the *left spectral factor* and the *right spectral factor*, respectively.

PROPOSITION 3.2. *Let* $\{\mathcal{H}_1, \mathcal{H}_2, \Theta(\lambda)\}$ *be a contractive analytic function on* $\mathbb{D}$*. The spectral factors have the form*

$$\Theta_L(\lambda) = P_{\mathcal{G}}^{\tilde{\mathcal{K}}}(I - \lambda\tilde{A})^{-1}\tilde{C} \qquad (\lambda \in \mathbb{D}), \tag{3.3}$$

and

$$\Theta_R(\lambda) = P_{\mathcal{H}}^{\mathcal{K}}C(I - \lambda A)^{-1}|\tilde{\mathcal{G}} \qquad (\lambda \in \mathbb{D}), \tag{3.4}$$

where $\tilde{\mathcal{K}}$ *has the form (2.10),* $\mathcal{G}$ *is given by (2.14),* A, B, C*, and the duals* $\tilde{A}, \tilde{B}, \tilde{C}$*, are obtained by an appropriate choice sequence (Schur parameters) associated to* $\Theta(\lambda)$.

A complete proof can be found in [6].

The spectral factors, as particular cases of maximal functions, play an interesting role in the study of linear systems.

Let $\mathcal{H}, \mathcal{U}, \mathcal{Y}$ be separable Hilbert spaces and $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. A linear system $\sigma = (A, B, C, D; \mathcal{H}, \mathcal{U}, \mathcal{Y})$ of the form

$$\begin{cases} h_{n+1} = & Ah_n + Bu_n, \quad (n \geq 0) \\ y_n = & Ch_n + Du_n, \end{cases} \tag{3.5}$$

where $\{h_n\} \subset \mathcal{H}$, $\{u_n\} \subset \mathcal{U}$, $\{y_n\} \subset \mathcal{Y}$, is called a *discrete-time system.*

Usually the spaces $\mathcal{H}$, $\mathcal{U}$, $\mathcal{Y}$ are called, respectively, the *state* space, the *input* space, and the *output* space, and the operators A, B, C and D are called, respectively, the *main* operator, the *control* operator, the *observation* operator, and the *feedthrough* operator of the system σ.

Let us define the bloc operator matrix (*colligation*) $S : \mathcal{H} \oplus \mathcal{U} \to \mathcal{H} \oplus \mathcal{Y}$ by

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}. \tag{3.6}$$

Then (3.5) can be written into a matrix form as follows

$$\begin{bmatrix} h_{n+1} \\ y_n \end{bmatrix} = S \begin{bmatrix} h_n \\ u_n \end{bmatrix}.$$

The system σ will be called: *passive*, *isometric*, *co-isometric*, *conservative* if S is, respectively, a contraction, an isometry, a co-isometry, a unitary operator.

The operator valued function $\Theta_\sigma(\lambda) : \mathcal{U} \to \mathcal{Y}$, $(\lambda \in \mathbb{D})$, attached to a system σ by

$$\Theta_\sigma(\lambda) = D + \lambda C(I_{\mathcal{H}} - \lambda A)^{-1} B \qquad (\lambda \in \mathbb{D}), \tag{3.7}$$

is the *transfer function* (or frequency response function) of the system.

The name is justified by the fact that $\Theta_\sigma(\lambda)$ make a connection between the transformed input and transformed output of the system. Indeed, application of the Z-transform

$$\hat{h}(\lambda) = \sum_{n \geq 0} h_n \lambda^n$$

to the system (3.5) and elimination of the state variable leads to

$$\hat{y}(\lambda) = \Theta_\sigma(\lambda) \hat{u}(\lambda) \qquad (\lambda \in \mathbb{D}). \tag{3.8}$$

The transfer function is the basic connection between state-space and frequency-domain in the linear system theory.

The references of the linear systems are very large, I mention here only a few of them [7, 8, 14, 3, 15, 1]. In the following only discrete-time linear systems will be considered.

If σ is a passive system, then $\Theta_\sigma(\lambda)$ is a contractive holomorphic function on $\mathbb{D}$, i.e. $\Theta_\sigma(\lambda)$ belongs to the Schur class $S(\mathcal{U}, \mathcal{Y})$.

For a system σ, the following subspaces of $\mathcal{H}$ are considered:

$$\mathcal{C} = \bigvee_{n\geq 0} A^n B\mathcal{U} \qquad \text{(the } \textit{controllable} \text{ space)} \tag{3.9}$$

and

$$\mathcal{O} = \bigvee_{n\geq 0} A^{*n} C^* \mathcal{Y} \qquad \text{(the } \textit{observable} \text{ space).} \tag{3.10}$$

Generally we have

$$\mathcal{H} = (\mathcal{C} \bigvee \mathcal{O}) \oplus (\mathcal{C}^{\perp} \cap \mathcal{O}^{\perp})$$

The system σ is called *controllable* if $\mathcal{C} = \mathcal{H}$, *observable* if $\mathcal{O} = \mathcal{H}$, and *minimal* if σ is both observable and controllable. The system σ is *simple* if $\mathcal{C} \bigvee \mathcal{O} = \mathcal{H}$.

From(3.9) it follows that $(\mathcal{C})^{\perp} = \bigcap_{n=0}^{\infty} \ker(B^* A^{*n})$, and from (3.10) we have $(\mathcal{O})^{\perp} = \bigcap_{n=0}^{\infty} \ker(CA^n)$. Hence the following characterizations occur: the system σ is, respecively, controllable iff $\bigcap_{n=0}^{\infty} \ker(B^* A^{*n}) = \{0\}$, observable iff $\bigcap_{n=0}^{\infty} \ker(CA^n) = \{0\}$, and simple iff $(\bigcap_{n=0}^{\infty} \ker(B^* A^{*n})) \cap (\bigcap_{n=0}^{\infty} \ker(CA^n)) = \{0\}$.

An explicit form for the spectral factors attached to $\Theta_\sigma(\lambda)$ was found in [12], using the maximal unilateral shift contained in the completely non-unitary contraction A, as follows.

PROPOSITION 3.3. *The spectral factors of* $\Theta_\sigma(\lambda)$ *have the form*

$$\Theta_R(\lambda) = P_{\Omega}^{\mathcal{H}}(I - \lambda A)^{-1} B \qquad (\lambda \in \mathbb{D}), \tag{3.11}$$

and

$$\Theta_L(\lambda) = C(I - \lambda A)^{-1}|\Omega_* \qquad (\lambda \in \mathbb{D}), \tag{3.12}$$

where Ω, *and* Ω_* *are the generating subspaces of the maximal unilateral shift contained in* A, *and* A^*, *respectively.*

4 Some applications

In this section, some applications of the maximal function in the linear systems theory are given, and some characterization in terms of the maximal function are obtained. Firstly we will analyse the particular case of a system given by the rotation operator of T, and then how the results can be generalized, using the maximal function, is presented.

Let us consider the system $\mathcal{J}$ given by the following unitary operator (the rotation operator of T, or Julia operator)

$$J(T) = R_T = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}. \tag{4.1}$$

In this particular case, the controllable and the observable subspaces of $\mathcal{H}$ will be, respectively,

$$\mathcal{C} = \bigvee_{n=0}^{\infty} T^n D_{T^*} \mathcal{D}_{T^*}, \qquad \mathcal{O} = \bigvee_{n=0}^{\infty} T^{*n} D_T \mathcal{D}_T, \tag{4.2}$$

and the corresponding orthogonals in the state space $\mathcal{H}$ will be

$$\mathcal{C}^\perp = \bigcap_{n=0}^{\infty} \ker(D_{T^*} T^{*n}) = \bigcap_{n} \ker D_{T^{*n}} = \{h \in \mathcal{H};\ \|T^{*n}h\| = \|h\|\}, \tag{4.3}$$

$$\mathcal{O}^\perp = \bigcap_{n=0}^{\infty} \ker(D_T T^n) = \bigcap_{n} \ker D_{T^n} = \{h \in \mathcal{H};\ \|T^n h\| = \|h\|\}. \tag{4.4}$$

Thus $T|\mathcal{O}^\perp$ and $T^*|\mathcal{C}^\perp$ are isometric operators and

$$\mathcal{C}^\perp \cap \mathcal{O}^\perp = \{h \in \mathcal{H};\ \|T^n h\| = \|h\| = \|T^{*n}h\|\} = \mathcal{H}_0, \tag{4.5}$$

where $\mathcal{H}_0$ is the subspace of the unitary part from the canonical decomposition [18] of the contraction $T = T_0 \oplus T_1 = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix}$ into its unitary part and the completely non-unitary (c.n.u.) part on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

Also, let us remark that the transfer function of $\mathcal{J}$ is just the characteristic function of T^*

$$\Theta_{\mathcal{J}}(\lambda) = -T^* + \lambda D_T (I - \lambda T)^{-1} D_{T^*} = \Theta_{T^*}(\lambda).$$

If we consider the system $\mathcal{J}^*$ given by the unitary bloc matrix $J(T^*) = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix}$, then the transfer function of $\mathcal{J}^*$ will be given by the analytic function $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$, the characteristic function of T.

Obviously $R_{T^*} = R_T^*$, and the corresponding linear systems $\mathcal{J}$ and $\mathcal{J}^*$ are dual, namely, if $\mathcal{J}$ is observable, then $\mathcal{J}^*$ is controllable, and conversely.

Proposition 4.1. *The system $\mathcal{J}$ given by $J(T)$ is conservative and simple, if and only if the main operator T is a completely non-unitary contraction. Moreover, if $T\mathcal{O}^{\perp} = \mathcal{O}^{\perp}$, then $\mathcal{J}$ is observable, and if $T^*\mathcal{C}^{\perp} = \mathcal{C}^{\perp}$, then $\mathcal{J}$ is a controllable system.*

Proof. Obviously that the system $\mathcal{J}$ is conservative, being governed by the unitary bloc matrix operator $J(T)$. If $\mathcal{J}$ is simple, then $\mathcal{H} = \mathcal{C} \bigvee \mathcal{O}$, and $\mathcal{C}^{\perp} \cap \mathcal{O}^{\perp} = \{h \in \mathcal{H};\ \|T^n h\| = \|h\| = \|T^{*n}h\|\} = \mathcal{H}_0 = \{0\}$, i.e., the subspace of the unitary part of T is the null space, and T is a completely non-unitary contraction.

Conversely, if T is c.n.u., then $\mathcal{H}_0 = \{0\}$, and $\mathcal{H} = \mathcal{C} \bigvee \mathcal{O}$, i.e., the system $\mathcal{J}$ is simple.

It is obvious that $\mathcal{O}^{\perp}$ is invariant to T, and T restricted to $\mathcal{O}^{\perp}$ is an isometry. If $T\mathcal{O}^{\perp} = \mathcal{O}^{\perp}$, then $\mathcal{O}^{\perp}$ reduces T to a unitary operator. Since $\mathcal{J}$ is simple, i.e., T is c.n.u., it follows that $\mathcal{O}^{\perp} = \{0\}$, or $\mathcal{O} = \mathcal{H}$, and $\mathcal{J}$ is observable.

Analogously, if $T^*\mathcal{C}^{\perp} = \mathcal{C}^{\perp}$, then $\mathcal{C}^{\perp}$ reduces T^* to a unitary operator, which implies that $\mathcal{C}^{\perp} = \{0\}$, i.e., the system $\mathcal{J}$ is controllable. □

From prediction point of view, we are interested in the cases when the maximal function attached to the distribution of a process is not a null function, to obtain the best linear predictor and the prediction-error in terms of the coefficients of the maximal function. In the case of discrete linear systems with the main operator a contraction, the fact that the maximal functions $\Theta_1(\lambda)$ or $\Theta_2(\lambda)$ of the main operator, or its adjoint, are the null functions, give some informations about the corresponding structure.

Proposition 4.2. *Let $\mathcal{J}$ be the conservative system given by the Julia operator $J(T)$ of a completely non-unitary contraction T.*

1). If the maximal function $\{\mathcal{H}, \mathcal{D}_{T^*}, \Theta_1(\lambda)\}$ of the main operator T is the null function, then the system $\mathcal{J}$ is observable.

2). If the maximal function $\{\mathcal{H}, \mathcal{D}_T, \Theta_2(\lambda)\}$ of T^* is the null function, then the system $\mathcal{J}$ is controllable.

Proof. If the maximal function of the contraction T

$$\Theta_1(\lambda) = D_{T^*}(I - T^*)^{-1} = \sum_{k=0}^{\infty} C_k \lambda^k = \sum_{k=0}^{\infty} D_{T^*} T^{*k} \lambda^k \equiv \{0\},$$

then the coefficients $C_k = 0$, i.e., $D_{T^*} T^{*n} h = 0$ for any $h \in \mathcal{H}$, and by (4.3) it follows that $\mathcal{C}^{\perp} = \mathcal{H}$, i.e., $\mathcal{C} = \{0\}$. By the previous Proposition, the system $\mathcal{J}$ is simple, thus it follows that $\mathcal{O} = \mathcal{H}$, and the system $\mathcal{J}$ is observable.

Analogous, if $\{\mathcal{H}, \mathcal{D}_T, \Theta_2(\lambda) \equiv \{0\}\}$, then by (4.4) we have $\mathcal{O}^{\perp} = \mathcal{H}$, and it follows that $\mathcal{C} = \mathcal{H}$, i.e., the system $\mathcal{J}$ is controllable. □

Corollary 4.3. *If the main operator T of the system $\mathcal{J}$ is an isometric (co-isometric) operator, then $\mathcal{J}$ is controllable (observable). If T is unitary, then the system $\mathcal{J}$ is minimal.*

A stronger characterization for the controllability (observability) of the system $\mathcal{J}$ can be done with the maximal functions of the main operator T as follows.

Proposition 4.4. *The discrete linear system $\mathcal{J}$ is controllable if and only if the operator Θ_1 from $\mathcal{H}$ into $H^2(\mathcal{D}_{T^*})$ attached to the maximal function $\{\mathcal{H}, \mathcal{D}_{T^*}, \Theta_1\}$ by (2.25) is one to one.*

Proof. If the system $\mathcal{J}$ is controllable, then $\mathcal{C}_{\mathcal{J}} = \mathcal{H}$, where $\mathcal{C}_{\mathcal{J}}$ is given by (4.2), or equivalently, $\mathcal{C}_{\mathcal{J}}^{\perp} = \bigcap \ker(D_{T^*}T^{*n}) = \{0\}$. That is, $D_{T^*}T^{*n}h = 0$ for any $n \geq 0$ if and only if $h = 0$. Taking into account that

$$\Theta_1(\lambda)h = D_{T^*}h + D_{T^*}T^*\lambda h + D_{T^*}T^{*2}\lambda^2 h + \cdots$$

it follows that $\ker \Theta_1 = 0$.

Conversely, if $\ker \Theta_1 = 0$, then $\Theta_1 h = 0$ if and only if $h = 0$, i.e., $D_{T^*}T^{*n}h = 0$ for any $n \geq 0$ if and only if $h = 0$, which implies that $\mathcal{C}_{\mathcal{J}}^{\perp} = \{0\}$, or equivalently $\mathcal{C}_{\mathcal{J}} = \mathcal{H}$, and the system is controllable. □

Analogously, for the system $\mathcal{J}^*$ can be proved the following

Proposition 4.5. *The discrete linear system $\mathcal{J}^*$ is controllable if and only if the operator Θ_2 from $\mathcal{H}$ into $H^2(\mathcal{D}_T)$ attached to the maximal function $\{\mathcal{H}, \mathcal{D}_T, \Theta_2\}$ by*

$$(\Theta_2 h)(\lambda) = \Theta_2(\lambda)h$$

is one to one.

Therefore the operators Θ_1 and Θ_2 corresponding to the maximal functions $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ of T and T^*, respectively, contain the information about the structure of the corresponding systems.

Actually, following [13], the observability, and controllability properties of a system can be introduced with some observable, and controllable operators, which represent the maximal function of the main operator of the system, as follows. Let us introduce the *observability operator* O_σ from $\mathcal{H}$ into $H^2(\mathcal{Y})$ defined by

$$O_\sigma h = C(I - \lambda A)^{-1}h \qquad (\lambda \in \mathbb{D}, h \in \mathcal{H}), \tag{4.6}$$

and the *controllability operator* $\Omega_\sigma : \mathcal{H} \to \mathcal{U}$, defined by

$$\Omega_\sigma h = B^*(I - \lambda A^*)^{-1}h \qquad (\lambda \in \mathbb{D}, h \in \mathcal{H}). \tag{4.7}$$

Now we introduce the observability, and the controllability of a system, as follows. The system σ is *observable* if and only if the observability operator is one to one, and σ is *controllable* if and only if the controllability operator is one to one. These definitions are equivalent with the classical one, given with the observability, and controlabillity subspaces of the system. Indeed, if the observability operator O_σ is one to one, then $O_\sigma h = 0$ if and only if $h = 0$. By (4.6), this is equivalent with $CA^n h = 0$ for all $n \geq 0$, if and only if $h = 0$. That is

$$\bigcap_{n\geq 0} \ker(CA^n) = \{0\},$$

which it is equivalent with the fact that the linear span of $A^{*n}C^*\mathcal{Y}$ is the wholle space $\mathcal{H}$, i.e., $\mathcal{O}_\sigma = \mathcal{H}$.

If Ω_σ is one to one, then it follows that $B^*A^{*n}h = 0$ for all $n \geq 0$ if and only if $h = o$, which it is equivalent with

$$\bigcap_{n\geq 0} \ker(B^*A^{*n}) = \{0\},$$

that is $\mathcal{C}_\sigma = \bigvee_{n\geq 1} A^n B\mathcal{U} = \mathcal{H}$.

Let us consider the linear systems

$$\mathcal{J} = (T, D_{T^*}, D_T, -T^*; \mathcal{H}, \mathcal{D}_{T^*}, \mathcal{D}_T),$$

and

$$\mathcal{J}^* = (T^*, D_T, D_{T^*}, -T; \mathcal{H}, \mathcal{D}_T, \mathcal{D}_{T^*})$$

generated, respectively, by the rotation operators $R_T = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix}$, and $R_{T^*} = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix}$.

COROLLARY 4.6. *For the system $\mathcal{J}$ given by the rotation operator R_T, the controllability operator is given by the maximal function $\{\mathcal{H}, \mathcal{D}_{T^*}, \Theta_1(\lambda)\}$ of the main operator T by*

$$\Omega_{\mathcal{J}} = \Theta_1, \tag{4.8}$$

where $\Theta_1 : \mathcal{H} \longrightarrow H^2(\mathcal{D}_{T^*})$ is given by

$$(\Theta_1 h)(\lambda) = \Theta_1(\lambda)h \qquad (\lambda \in \mathbb{D}), \tag{4.9}$$

and the observability operator is

$$O_{\mathcal{J}} = \Theta_2, \tag{4.10}$$

where $\{\mathcal{H}, \mathcal{D}_T, \Theta_2(\lambda)\}$ is the maximal function of T^*.

Remark that the controllability (observability) operator of $\mathcal{J}$ become the observability (controllability) operator of $\mathcal{J}^*$, and conversely.

Many other remarkable tools in systems theory can be formulated with the maximal function. Such a way, the *observability gramian*

$$G_\sigma^O = O_\sigma^* O_\sigma,$$

and the *controllability gramian*

$$G_\sigma^C = \Omega_\sigma^* \Omega_\sigma,$$

which in the rotation case $\mathcal{J}$ become

$$G_{\mathcal{J}}^C = \Theta_1^* \Theta_1 \qquad \text{and} \qquad G_{\mathcal{J}}^O = \Theta_2^* \Theta_2.$$

Remark that a system is observable if and only if the observability gramian is strictly positive, and controllable if and only if the controllability gramian is strictly positive, respectively.

For a general study of linear systems, we are interested in bloc matrices operators of the form $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. By definition, if S is, respectively, contraction, isometry, co-isometry, unitary, then the corresponding system is, respectively, passive, isometric, co-isometric, conservative. In the following, some explicit relations show the utility of the maximal function in the investigation of passive systems, but the study can be generalized for other type of systems, too. For the study of passive systems, the following theorem ([5], Theorem 1.3), which gives a characterization of such a bloc matrix S to be a contraction, is very helpfull.

THEOREM 4.7. *The formula*

$$X = -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1} \tag{4.11}$$

establishes a one-to-one correspondence between all operators $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{K}_2)$ *such that* $S = \begin{bmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & X \end{bmatrix}$ *is a contraction, and all* $\Gamma \in \mathcal{L}(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$. *Moreover,* $\mathcal{D}_S$ *can be identified with* $\mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_\Gamma$, *and* $\mathcal{D}_{S^*}$ *can be identified with* $\mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma^*}$.

Therefore, in our case, $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, is a contraction if and only if there exist the uniquelly determined contractions $\Gamma_1 \in \mathcal{L}(\mathcal{U}, \mathcal{D}_{A^*})$, $\Gamma_2 \in \mathcal{L}(\mathcal{D}_A, \mathcal{Y})$ and $\Gamma \in \mathcal{L}(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$, such that

$$(4.12) \qquad S = \begin{bmatrix} A & D_{A^*}\Gamma_1 \\ \Gamma_2 D_A & -\Gamma_2 A^*\Gamma_1 + D_{\Gamma_2^*}\Gamma D_{\Gamma_1} \end{bmatrix}.$$

If for the passive system governed by the contractive bloc matrix S, having the main operator the contraction A, we construct the conservative system governed by the unitary bloc matrix $R_A = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}$, then S can be obviously written in the form

$$(4.13) \qquad S = \begin{bmatrix} I & 0 \\ 0 & \Gamma_2 \end{bmatrix} \underbrace{\begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}}_{R_A} \begin{bmatrix} I & 0 \\ 0 & \Gamma_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D_{\Gamma_2^*}\Gamma D_{\Gamma_1} \end{bmatrix},$$

or using Corollary 3.5 from [13], S can be factorized as follows

$$S = \begin{bmatrix} I & 0 & 0 \\ 0 & \Gamma_2 & D_{\Gamma_2^*} \end{bmatrix} \begin{bmatrix} A & D_{A^*} & 0 \\ D_A & -A^* & 0 \\ 0 & 0 & \Gamma \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma_1 \\ 0 & D_{\Gamma_1} \end{bmatrix}.$$

These facts suggest that, having a characterization of conservative systems with the maximal functions, then the study can be extended to the passive systems, too.

Using (4.13), the following relation between the controllable and observable subspaces of a passive system, and the the corresponding conservative attached system can be proved.

PROPOSITION 4.8. *Let $\sigma = (A, B, C, D; \mathcal{H}, \mathcal{U}, \mathcal{Y})$ be a passive system corresponding to the bloc matrix S, and $\mathcal{J} = (A, D_{A^*}, D_A, -A^*; \mathcal{H}, \mathcal{D}_A, \mathcal{D}_{A^*})$ the conservative system corresponding to R_A, where A is the main operator of σ. The observable and the controllable subspaces of σ are contained , respectively, into the observable and the controllable subspaces of $\mathcal{J}$.*

Proof. The system σ is passive, hence the bloc matrix S is contraction, and if we use the structure given by (4.12), then the proof is a straightforward verification, starting from the definition of the corresponding subspaces. Let $\mathcal{C}_\sigma$ and $\mathcal{C}_\mathcal{J}$ be the controllable subspaces of σ and $\mathcal{J}$, respectively, and $\mathcal{O}_\sigma$, $\mathcal{O}_\mathcal{J}$ the corresponding observable subspaces. Then, taking into account that $\Gamma_1 \in \mathcal{L}(\mathcal{U}, \mathcal{D}_{A^*})$ and $\Gamma_2^* \in \mathcal{L}(\mathcal{Y}, \mathcal{D}_A)$, we have

$$\mathcal{C}_\sigma = \bigvee_{n=0}^{\infty} A^n D_{A^*}\Gamma_1 \mathcal{Y} \subseteq \bigvee_{n=0}^{\infty} A^n D_{A^*}\mathcal{D}_{A^*} = \mathcal{C}_\mathcal{J}$$

and

$$\mathcal{O}_\sigma = \bigvee_{n=0}^{\infty} A^{*n}(\Gamma_2 D_A)^* \mathcal{Y} = \bigvee_{n=0}^{\infty} A^{*n} D_A \Gamma_2^* \mathcal{Y} \subseteq \bigvee_{n=0}^{\infty} A^{*n} D_A \mathcal{D}_A = \mathcal{O}_{\mathcal{J}}.$$

Therefore $\mathcal{C}_\sigma \subseteq \mathcal{C}_{\mathcal{J}}$ and $\mathcal{O}_\sigma \subseteq \mathcal{O}_{\mathcal{J}}$. □

From these inclusions it results the following

COROLLARY 4.9. *If the passive system σ is controllable, observable, minimal, or simple, then the attaced conservative system $\mathcal{J}$ becomes, respectively, controllable, observable, minimal, or simple.*

Finally, let us recall some more results and connections with the maximal function. One of them is that between the maximal function and the characteristic function there exist the following relation, usefull in solving interpolation problems (see [13], Cap.IX, Theorem 6.4): if T is $*$-stable, then the operator W_T from the space of the minimal isometric dilation $\mathcal{K} = \mathcal{H} \oplus H^2(\mathcal{D}_T)$ into $H^2(\mathcal{D}_{T^*})$ defined by

$$W_T(h \oplus f) = \Theta_1 h + \Theta_T f,$$

or into matricial form

$$W_T = \begin{bmatrix} \Theta_1 & \Theta_T \end{bmatrix} : \mathcal{H} \oplus H^2(\mathcal{D}_T) \to H^2(\mathcal{D}_{T^*})$$

is unitary.

As a remark, the stable and $*$-stable systems, having a contraction as the main operator, can be analysed also in a functional form, taking account by Proposition 2.3, into a functional model generated by the maximal function of the main operator.

Usefull in the study of the linear systems can be the fact that the defect functions of the transfer function of $\mathcal{J}$ and $\mathcal{J}^*$ generate positive definite kernels as follows

PROPOSITION 4.10. *Let $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$ be the characteristic function of the contraction T. There exist the following relations*

$$\frac{I - \Theta_T(\lambda)\Theta_T(\mu)^*}{1 - \lambda\overline{\mu}} = \Theta_1(\lambda)\Theta_1(\mu)^* \tag{4.14}$$

and

$$\frac{I - \Theta_T(\mu)^*\Theta_T(\lambda)}{1 - \lambda\overline{\mu}} = \Theta_2(\overline{\mu})\Theta_2(\overline{\lambda})^*, \tag{4.15}$$

where $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ are the maximal functions of T and T^*, respectively.

Proof. It is known ([18], Chap.VI, (1.4)) that the defect function of the characteristic function $\{\mathcal{D}_T, \mathcal{D}_{T^*}, \Theta_T(\lambda)\}$ is obtained by

$$\langle f, f\rangle - \langle \Theta_T(\lambda)f, \Theta_T(\mu)f\rangle = (1-\lambda\overline{\mu})\left\langle (I-\lambda T^*)^{-1}D_Tf, (I-\mu T^*)^{-1}D_Tf\right\rangle$$

that is

$$\Delta^2_{\Theta_T}(\lambda,\mu) = I - \Theta_T(\mu)^*\Theta_T(\lambda) = (1-\lambda\overline{\mu})D_T(I-\overline{\mu}T)^{-1}(I-\lambda T^*)^{-1}D_T,$$

and taking into account by (2.16) it follows that

$$\frac{I-\Theta_T(\mu)^*\Theta_T(\lambda)}{1-\lambda\overline{\mu}} = \Theta_2(\overline{\mu})\Theta_2(\overline{\lambda})^*.$$

An analogous calculus leads to

$$\Delta^2_{\Theta_{T^*}}(\lambda,\mu) = I - \Theta_T(\lambda)\Theta_T(\mu)^* = (1-\lambda\overline{\mu})D_{T^*}(I-\lambda T^*)^{-1}(I-\overline{\mu}T)^{-1}D_{T^*},$$

and by (2.8) it follows that

$$\frac{I-\Theta_T(\lambda)\Theta_T(\mu)^*}{1-\lambda\overline{\mu}} = \Theta_1(\lambda)\Theta_1(\mu)^*$$

□

A similar result can be obtained for the characteristic function $\Theta_{T^*}(\lambda)$, which is the transfer function of the system $\mathcal{J}$ given by the rotation operator R_T.

Proposition 4.11. *Let $\{\mathcal{D}_{T^*}, \mathcal{D}_T, \Theta_{T^*}(\lambda)\}$ be the characteristic function of the contraction T^*. There exist the following relations*

$$\frac{I-\Theta_{T^*}(\lambda)\Theta_{T^*}(\mu)^*}{1-\lambda\overline{\mu}} = \Theta_2(\lambda)\Theta_2(\mu)^* \tag{4.16}$$

and

$$\frac{I-\Theta_{T^*}(\mu)^*\Theta_{T^*}(\lambda)}{1-\lambda\overline{\mu}} = \Theta_1(\overline{\mu})\Theta_1(\overline{\lambda})^*, \tag{4.17}$$

where $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$ are the maximal functions of T and T^*, respectively.

From the above results can be seen that in the contraction case the spectral factors, which are particular cases of maximal functions, can be obtained from (4.14) and (4.15) for $\mathcal{J}^*$, and from (4.16) and (4.17) for the system $\mathcal{J}$ given by the rotation of T.

Actually, for a system generated by a unitary bloc matrix S given by (4.12), a generalized result can be obtained.

Proposition 4.12. *Let $\sigma = (A, B, C, D; \mathcal{H}, \mathcal{U}, \mathcal{Y})$ be a system given by the unitary bloc matrix operator*

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix},$$

and $\Theta_\sigma(\lambda)$ the transfer function of σ. Then we have the following positive definite kernels on $\mathbb{D}^2$

$$\frac{I - \Theta_\sigma(\mu)^* \Theta_\sigma(\lambda)}{1 - \lambda\overline{\mu}} = B^*(I - \overline{\mu}A^*)^{-1}(I - \lambda A)^{-1}B \qquad (\lambda, \mu \in \mathbb{D}), \tag{4.18}$$

and

$$\frac{I - \Theta_\sigma(\lambda) \Theta_\sigma(\mu)^*}{1 - \lambda\overline{\mu}} = C(I - \lambda A)^{-1}(I - \overline{\mu}A^*)^{-1}C^* \qquad (\lambda, \mu \in \mathbb{D}). \tag{4.19}$$

Proof. Since S is a unitary operator, it follows from $S^*S = I$ and $SS^* = I$ that

$C^*C = I - A^*A$, $B^*B = I - D^*D$, $D^*C = -B^*A$, $BD^* = -AC^*$.

Taking into account the previous relations we have

$$\begin{aligned} I - \Theta_\sigma(\mu)^*\Theta_\sigma(\lambda) &= I - [D^* + \overline{\mu}B^*(I - \overline{\mu}A^*)^{-1}C^*][D + \lambda C(I - \lambda A)^{-1}B] = \\ &= I - D^*D - \lambda D^*C(I - \lambda A)^{-1}B - \overline{\mu}B^*(I - \overline{\mu}A^*)^{-1}C^*D - \lambda\overline{\mu}B^*(I - \overline{\mu}A^*)^{-1}C^*C \\ &\cdot(I - \lambda A)^{-1}B = B^*B + \lambda B^*A(I - \lambda A) - 1B + \overline{\mu}B^*(I - \overline{\mu}A^*)^{-1}A^*B - \\ &- \lambda\overline{\mu}B^*(I - \overline{\mu}A^*)^{-1}C^*C(I - \lambda A)^{-1}B = B^*(I - \overline{\mu}A^*)^{-1}[(I - \overline{\mu}A^*)(I - \lambda A) + \\ &+ \lambda(I - \overline{\mu}A^*)A + \overline{\mu}A^*(I - \lambda A) - \lambda\overline{\mu}(I - A^*A)](I - \lambda A)^{-1}B = \\ &= (1 - \lambda\overline{\mu})B^*(I - \overline{\mu}A^*)^{-1}(I - \lambda A)^{-1}B, \end{aligned}$$

and (4.18) is verified.

An analogous calculus leads to (4.19), and the proof is finished. □

Using the previous Proposition, the general form of the spectral factors can be derived, but in this paper we are mainly interested in the connection between the spectral factors and the maximal functions $\Theta_1(\lambda)$ and $\Theta_2(\lambda)$. Let us remark that in this case the subspaces Ω and Ω_* become

$$\Omega = \mathcal{O}^\perp \ominus T\mathcal{O}^\perp \qquad \text{and} \qquad \Omega_* = \mathcal{C}^\perp \ominus T^*\mathcal{C}^\perp. \tag{4.20}$$

The following result gives another characterization for the behaviour of the rotation systems, in terms of the maximal functions.

Proposition 4.13. *The conservative simple system $\mathcal{J}$ attached to the rotation of a completely non-unitary contraction T is*

1) observable if and only if $\Theta_1(\lambda)|\Omega = 0$,

2) controllable if and only if $\Theta_2(\lambda)|\Omega_* = 0$,

where $\Theta_1(\lambda)$, and $\Theta_2(\lambda)$, are the maximal functions of T, and T^*, respectively.

Proof. If $\mathcal{J}$ is observable, then $\mathcal{O} = \mathcal{H}$, i.e., $\mathcal{O}^\perp = \{0\}$, and by (4.20) it follows that $\Omega = \{0\}$, and $\Theta_1(\lambda)|\Omega = 0$.

Conversely, if $\Theta_1(\lambda)|\Omega = 0$, then for any $g \in \mathcal{D}_{T^*}$, and $\omega \in \Omega$, we have

$$0 = \langle \Theta_1(\lambda)\omega, g\rangle = \langle \omega, \Theta_1(\lambda)^* g\rangle = \langle \omega, P_\Omega \Theta_1(\lambda)^* g\rangle ,$$

that is

$$0 = P_\Omega \Theta_1(\lambda)^* g = P_\Omega [D_{T^*}(I - \lambda T^*)^{-1}]^* g =$$
$$= P_\Omega (I - \overline{\lambda} T)^{-1} D_{T^*} g = P_\Omega \sum_{n=0}^{\infty} \overline{\lambda}^n T^n D_{T^*} g.$$

It follows that for any $n \geq 0$, and $g \in \mathcal{D}_{T^*}$, we have $P_\Omega T^n D_{T^*} g = 0$, and by (4.2) it results that

$$P_\Omega \bigvee_{n=0}^{\infty} T^n D_{T^*} \mathcal{D}_{T^*} = P_\Omega \mathcal{C} = \{0\},$$

i.e., $\Omega \subset \mathcal{C}^\perp$. But $\Omega \subset \mathcal{O}^\perp$, and it results that $\Omega \subset \mathcal{C}^\perp \cap \mathcal{O}^\perp = \mathcal{H}_0 = \{0\}$, where $\mathcal{H}_0$ is the space of unitary part of the c.n.u. contraction T. Therefore $\Omega = \{0\}$, and by (4.20) we have $\mathcal{O}^\perp \ominus T\mathcal{O}^\perp = \{0\}$, or equivalent, $T\mathcal{O}^\perp = \mathcal{O}^\perp$. By (4.4) $T|\mathcal{O}^\perp$ is an isometry, therefore $\mathcal{O}^\perp$ reduces T to a unitary operator. T being c.n.u., it results $\mathcal{O}^\perp = \{0\}$, and it follows that $\mathcal{O} = \mathcal{H}$, i.e. $\mathcal{J}$ is observable.

If $\mathcal{J}$ is controllable, then $\mathcal{C} = \mathcal{H}$, i.e., $\mathcal{C}^\perp = \{0\}$, which implies $\Omega_* = \{0\}$, and $\Theta_2(\lambda)|\Omega_* = 0$.

Conversely, if $\Theta_2(\lambda)|\Omega_* = 0$, then for any $v \in \Omega_*$, and any $g \in \mathcal{D}_T$ we have

$$\langle \Theta_2(\lambda)v, g\rangle = \langle v, \Theta_2(\lambda)^* g\rangle = \langle v, P_{\Omega_*}\Theta_2(\lambda)^* g\rangle = 0,$$

that is

$$P_{\Omega_*}\Theta_2(\lambda)^* g = P_{\Omega_*}[D_T(I - \lambda)^{-1}]^* g =$$
$$= P_{\Omega_*}(I - \overline{\lambda} T^*)^{-1} D_T g = P_{\Omega_*} \sum_{n=0}^{\infty} \overline{\lambda}^n T^{*n} D_T g = 0.$$

It follows that for any $n \geq 0$, and $g \in \mathcal{D}_T$ we have $P_{\Omega_*} T^{*n} D_T g = 0$, and by (4.2) it results that

$$P_{\Omega_*} \bigvee_{n=0}^{\infty} T^{*n} D_T \mathcal{D}_T = P_{\Omega_*} \mathcal{O} = \{0\},$$

i.e., $\Omega_* \subset \mathcal{O}^\perp$. Since $\Omega_* \subset \mathcal{C}^\perp$, it follows that $\Omega_* \subset \mathcal{O}^\perp \cap \mathcal{C}^\perp = \mathcal{H}_0 = \{0\}$, i.e., $\Omega_* = \{0\}$, which implies $T^*\mathcal{C}^\perp = \mathcal{C}^\perp$, and the fact that $\mathcal{C}^\perp$ reduces T^* to a unitary operator. But T is completely non-unitary, so we have $\mathcal{C}^\perp = \{0\}$, or equivalent, $\mathcal{C} = \mathcal{H}$, and the system $\mathcal{J}$ is controllable. □

By duality of $\mathcal{J}$ and $\mathcal{J}^*$ we also have

PROPOSITION 4.14. *If T is a completely non-unitary contraction, then the conservative simple system $\mathcal{J}^*$ given by the rotation of T^* is controllable if and only if $\Theta_1(\lambda)|\Omega = 0$, and observable if and only if $\Theta_2(\lambda)|\Omega_* = 0$, where $\Theta_1(\lambda)$, and $\Theta_2(\lambda)$, are the maximal functions of T, and T^*, respectively.*

From the previous results, it follows the following corollary for the conservative simple systems generated by the rotation of completely non-unitary contractions.

COROLLARY 4.15. *The conservative simple systems $\mathcal{J}$ and $\mathcal{J}^*$, having the main operator a completely non-unitary contraction, are minimal if and only if the corresponding spectral factors are zero functions, or equivalent, if and only if $\Theta_1(\lambda)|\Omega = 0$ and $\Theta_2(\lambda)|\Omega_* = 0$, where $\Theta_1(\lambda)$, and $\Theta_2(\lambda)$, are the maximal functions of T, and T^*, respectively.*

For composing of systems, the Redheffer cascading systems theory, or Redheffer product, is very helpfull. In [13] a short introduction into the theory of Redheffer products with applications to the structure of matrix contractions and linear systems can be found. Such a way, if $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ generates a system, $R_A = \begin{bmatrix} A & D_{A^*} \\ D_A & -A^* \end{bmatrix}$ is the rotation of A, and $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is the identity matrix for the Redheffer products, then, knowing the investigation with the maximal functions on the rotation operator conservative system, consequences for passive systems can derive also using the following form for the Theorem 4.7.

THEOREM 4.16. *The matrix*

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{U} \to \mathcal{H}' \oplus \mathcal{Y}$$

is a contraction if and only if it can be represented as a Redheffer product of the form

$$S = \left(J(R'_{X^*} \circ \begin{bmatrix} 0 & \Gamma \\ 0 & 0 \end{bmatrix} \circ R_Y)\right) \circ R_A,$$

where A, X, Y and Γ are contractions acting between appropriate spaces.

To be mentioned also that the rotation R'_A is obtained by the rotation of A^* composed to the identity operator $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ as follows

$$R'_A = \begin{bmatrix} -A & D_{A^*} \\ D_A & A^* \end{bmatrix} = JR_{A^*}J.$$

A complete proof can be found in [13], Cap.XIV, Corollary 2.8.

Another explicit relation betveen the maximal function and the characteristic function of a contraction was obtained in [19], where using Redheffer products, the composing of the system generated by $JR_{T^*} = \begin{bmatrix} D_{T^*} & -T \\ T^* & D_T \end{bmatrix}$ with the extended feedback system $\{I, \lambda I\}$ is expressed as the extended feedback system of the maximal function $\Theta_1(\lambda)$ and the characteristic function $\Theta_T(\lambda)$. Also an extended form for the maximal function $\Theta_1(\lambda)$ to $\Theta_1(X)$, where $X \in \mathcal{L}(\mathcal{H})$, is used to obtain a generalized form of the characteristic function.

All previous facts and the strong relations between the maximal functions and transfer functions show up that the maximal functions are used implicitly in the study of the linear systems, and can become an explicit tools for investigation.

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