One- and two-level additive methods for variational and quasi-variational inequalities of the second kind

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Abstract

We introduce and analyze some one- and two-level additive Schwarz methods for variational and quasi-variational inequalities of the second kind. The methods are introduced as subspace correction algorithms for problems in a reflexive Banach space. We prove that these methods are globally convergent and give, under some assumptions, error estimates. If we utilize the finite element spaces the introduced algorithms are in fact one- and two-level Schwarz methods. In this case we prove that the assumptions we made for the general convergence result hold, and are able to write the convergence rate depending on the overlapping and mesh parameters. We get that our methods have an optimal convergence rate, i.e. their converge is the same as in the case of linear equations. In this way, we prove that the two-level introduced methods are very efficient for this type of problems because their convergence is almost independent of the mesh and overlapping parameters.

Keywords: domain decomposition methods, subspace correction methods, variational inequalities, multilevel methods
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1 Introduction

Schwarz methods are widely used for linear problems because they provide robust and efficient solution methods. However, their generalization to non–linear problems is not straightforward, in particular the estimate of the convergence rate for the multilevel methods is far from being trivial. The convergence of the projected Gauss–Seidel relaxation (or successive coordinate minimization) for variational inequalities of the second kind in $\mathbf{R}^d$ has been proved in [9]. There, the non-differentiable term has been decomposed as a sum of terms, each of them depending only on one vector component. Such a localizing decomposition can be obtained, for instance, if the continuous problem is discretized by finite elements and the non-differentiable term is approximated by numerical integration. The projected Gauss-Seidel method is a particular case of a Schwarz method in which the domain is decomposed into the interior of the supports of the nodal basis functions. Consequently, the above representation of the non-differentiable term can be viewed as a decomposition in concordance with the domain decomposition. A straightforward generalization of the convergence proof in [9] to more general decompositions can be obtained using this idea, but it fails if, in order to get a faster convergence of the non-linear Schwarz method, a two-level or multilevel method is considered. This is due to the fact that on the coarser levels the nonlinearities are not decoupled. A remedy can be found in adapting minimization techniques for the construction and analysis of multigrid and domain decomposition methods, see [11]–[13]. In [4], two-level multiplicative Schwarz methods have been proposed for variational and quasi-variational inequalities of the second kind. The convergence rates of these methods are almost independent of the mesh and overlapping parameters. On the other hand, it is well-known that the additive methods are the best on parallel machines even if their convergence is a little more slow. In this paper, we prove that the additive variants of the methods in [4] are also globally convergent and have an optimal convergence rate.

The paper is organized as follows. Section 2 is devoted to a general framework in a reflexive Banach space. We introduce here an assumption on the convex set and the correction subspaces, which will be necessary in the convergence proof of the algorithms. Mainly, this hypothesis refers to the decomposition of the elements in the convex set, and introduces a constant C_0 which will play an important role in the writing of the convergence rate. In Section 3, we introduce a subspace correction algorithm for variational inequalities of the second kind, and prove that, under the above assumption, it is globally convergent. We also estimate its convergence rate. In Section 4, we introduce two subspace correction algorithms for the quasi-variational inequalities. As in the previous section, we prove their convergence and estimate the convergence rate, using the assumption introduced in Section 2 and asking that the non-differentiable term to satisfy a certain property. Using this property, we also show that the quasi-variational inequality has an unique solution, and the convergence condition of the algorithms and the existence and uniqueness condition of the solution are of the same type. Section 5 is devoted to the one- and two-level methods. If we associate the correction subspaces to a domain decomposition, the abstract algorithms introduced in Sections 3 and 4 are Schwarz methods. We show that the assumption introduced in Section 2 holds for general enough convex sets and we explicitly write the constant C_0 depending on the mesh and domain decomposition parameters. In this way, we get that the convergence rates of the one- and two-level methods for the variational and quasi-variational inequalities of the second kind are similar with the convergence rates obtained for equations, ie., we get an optimal convergence. In the case of the two-level methods, the convergence rate is almost independent of the mesh and domain decomposition parameters.

2 General framework

Let V be a reflexive Banach space and $V_1, \cdots, V_m$ be some closed subspaces of V. Also, let $K \subset V$ be a non empty closed convex set for which we make the following

ASSUMPTION 2.1. *There exists a constant $C_0 > 0$ such that for any $w, v \in K$ there exist $v_i \in V_i$, $i = 1, \cdots, m$, which satisfy*

$$w + v_i \in K \text{ for } i = 1, \cdots, m, \tag{2.1}$$

$$v - w = \sum_{i=1}^{m} v_i, \text{ and} \tag{2.2}$$

$$\sum_{i=1}^{m} ||v_i|| \leq C_0 ||v - w||. \tag{2.3}$$

We consider a Gâteaux differentiable functional $F : V \to \mathbf{R}$, and assume that there exist two real numbers $p,\ q > 1$ such that for any real number $M > 0$ there exist two constants $\alpha_M,\ \beta_M > 0$ for which

$$\alpha_M \|v - u\|^p \leq \langle F'(v) - F'(u), v - u \rangle, \text{ and} \tag{2.4}$$

$$\|F'(v) - F'(u)\|_{V'} \leq \beta_M \|v - u\|^{q-1}, \tag{2.5}$$

for any $u, v \in V$ with $\|u\|, \|v\| \leq M$. Above, we have denoted by F' the Gâteaux derivative of F, and we have marked that the constants α_M and β_M may depend on M. It is evident that if (2.4) and (2.5) hold, then for any $u, v \in V$, $\|u\|, \|v\| \leq M$, we have

$$\alpha_M \|v - u\|^p \leq \langle F'(v) - F'(u), v - u \rangle \leq \beta_M \|v - u\|^q. \tag{2.6}$$

Following the way in [10], we can prove that for any $u, v \in V$, $\|u\|, \|v\| \leq M$, we have

$$\begin{aligned} \langle F'(u), v-u\rangle + \tfrac{\alpha_M}{p}\|v-u\|^p \leq F(v) - F(u) \leq \\ \langle F'(u), v-u\rangle + \tfrac{\beta_M}{q}\|v-u\|^q. \end{aligned} \tag{2.7}$$

We point out that since F is Gâteaux differentiable and satisfies (2.4), F is a convex functional (see Proposition 5.5 in [8], pag. 25). Also, we can prove that $q \leq 2 \leq p$.

3 Subspace correction algorithm for variational inequalities of the second kind

Let $\varphi : V \to \mathbf{R}$ be a convex lower semicontinuous functional and we assume that $F + \varphi$ is coercive in the sense that

$$\frac{F(v)+\varphi(v)}{\|v\|} \to \infty, \text{ as } \|v\| \to \infty, \ v \in K, \tag{3.1}$$

if K is not bounded. In addition to the hypotheses of Assumption 2.1, we suppose that

$$\sum_{i=1}^{m} \varphi(w+v_i) \leq (m-1)\varphi(w) + \varphi(v) \tag{3.2}$$

for any $v, w \in K$ and $v_i \in V_i$, $i = 1, \ldots, m$, which satisfy Assumption 2.1.

Now, we consider the problem

$$u \in K : \langle F'(u), v-u\rangle + \varphi(v) - \varphi(u) \geq 0, \text{ for any } v \in K. \tag{3.3}$$

It is well known (see [14], Theorem 8.5, page 251, for instance) that the above problem has a unique solution, and it is also the unique solution of the minimization problem

$$u \in K : F(u) + \varphi(u) \leq F(v) + \varphi(v), \text{ for any } v \in K. \tag{3.4}$$

From (2.7) we see that, for a given $M > 0$ such that the solution u of (3.3) satisfies $\|u\| \leq M$, we have

$$\begin{aligned} &\tfrac{\alpha_M}{p}\|v-u\|^p \leq F(v) - F(u) + \varphi(v) - \varphi(u), \\ &\text{for any } v \in K, \ \|v\| \leq M. \end{aligned} \tag{3.5}$$

The proposed algorithm corresponding to the subspaces $V_1, \cdots, V_m$ and the convex set K is written as follows

Algorithm 3.1. *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we simultaneously solve the inequalities*

$$
\begin{aligned}
& w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K: \\
(3.6)\quad & \langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle + \varphi(u^n + v_i) - \varphi(u^n + w_i^{n+1}) \geq 0, \\
& \text{for any } v_i \in V_i,\ u^n + v_i \in K,
\end{aligned}
$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \frac{r}{m}\sum_{i=1}^{m} w_i^{n+1}$, where $0 < r \leq 1$ is a fixed constant.

This algorithm does not assume a decomposition of the convex set K depending on the subspaces V_i. Evidently, problem (3.6) has an unique solution, and it is equivalent with

$$
\begin{aligned}
& w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K: \\
(3.7)\quad & F(u^n + w_i^{n+1}) + \varphi(u^n + w_i^{n+1}) \leq F(u^n + v_i) + \varphi(u^n + v_i), \\
& \text{for any } v_i \in V_i,\ u^n + v_i \in K.
\end{aligned}
$$

We have the following general convergence result.

Theorem 3.1. *Let V be a reflexive Banach, $V_1, \cdots, V_m$ some closed subspaces of V, and K a non empty closed convex subset of V which satisfies Assumption 2.1. Also, we assume that F is Gâteaux differentiable and satisfies (2.4) and (2.5), the functional φ is convex and lower semicontinuous, satisfies (3.2), and $F + \varphi$ is coercive if K is not bounded. If u is the solution of problem (3.3) and u^n, $n \geq 0$, are its approximations obtained from Algorithm 3.1, then there exists an $M > 0$ such that $\max(\|u\|, \max_{n\geq 0}\|u^n\|, \max_{n\geq 0, 1\leq i\leq m}\|u^n + w_i^{n+1}\|) \leq M$ and we have the following error estimations:*

(i) if $p = q = 2$ we have

$$
(3.8)\qquad
\begin{aligned}
& F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \\
& \left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right],
\end{aligned}
$$

$$
(3.9)\qquad \|u^n - u\|^2 \leq \tfrac{2}{\alpha_M}\left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right].
$$

(ii) if $p > q$ we have

$$
(3.10)\qquad
\begin{aligned}
& F(u^n) + \varphi(u^n) - F(u) - \varphi(u) \leq \\
& \frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}},
\end{aligned}
$$

$$(3.11)\qquad \|u-u^n\|^p \le \frac{p}{\alpha_M}\frac{F(u^0)+\varphi(u^0)-F(u)-\varphi(u)}{\left[1+nC_2(F(u^0)+\varphi(u^0)-F(u)-\varphi(u))^{\frac{p-q}{q-1}}\right]^{\frac{q-1}{p-q}}}.$$

The constants C_1 and C_2 are given in (3.26) and (3.30), respectively.

Proof. This proof is similar with that given for the minimization of non-quadratic functionals in [3]. Except the part concerning the functional φ, which will be detailed, we will point out here only the main steps of the proof. Also, the proof uses the same techniques of that in the multiplicative case in [4] or [6].

In view of the convexity of F, we get

$$F(u^{n+1}) = F(u^n + \frac{r}{m}\sum_{i=1}^{m} w_i^{n+1}) = F((1-r)u^n + \sum_{i=1}^{m}\frac{r}{m}(u^n + w_i^{n+1})) \le$$
$$(1-r)F(u^n) + \frac{r}{m}\sum_{i=1}^{m} F(u^n + w_i^{n+1})$$

A similar result can be obtained for φ, ie., we have

$$(3.12)\qquad \begin{aligned} F(u^{n+1}) &\le (1-r)F(u^n) + \frac{r}{m}\sum_{i=1}^{m} F(u^n + w_i^{n+1}) \\ \varphi(u^{n+1}) &\le (1-r)\varphi(u^n) + \frac{r}{m}\sum_{i=1}^{m} \varphi(u^n + w_i^{n+1}) \end{aligned}$$

Using equation (3.7) and these inequalities, we get

$$F(u^{n+1}) + \varphi(u^{n+1}) \le F(u^n) + \varphi(u^n)$$

Therefore, using (3.4), for any $n \ge 0$ and $i = 1, \cdots, m$, we get

$$(3.13)\qquad \begin{aligned} &F(u) + \varphi(u) \le F(u^n + w_i^{n+1}) + \varphi(u^n + w_i^{n+1}) \le \\ &F(u^n) + \varphi(u^n) \le F(u^0) + \varphi(u^0), \end{aligned}$$

Consequently, from the coerciveness of $F + \varphi$ if K is not bounded, we get that there exists $M > 0$, such that

$$(3.14)\qquad \begin{aligned} &\|u\| \le M,\ \|u^n\| \le M,\ \|u^n + w_i^{n+1}\| \le M \\ &\text{for any } n \ge 0,\ i = 1, \cdots, m. \end{aligned}$$

From (3.6) and (2.7), for any $n \ge 0$ and $i = 1, \cdots, m$, we have

$$(3.15)\qquad \begin{aligned} &F(u^n) - F(u^n + w_i^{n+1}) + \varphi(u^n) - \varphi(u^n + w_i^{n+1}) \ge \\ &\tfrac{\alpha_M}{p}\|w_i^{n+1}\|^p, \end{aligned}$$

and, in view of (3.5), we have

$$F(u^n + w_i^{n+1}) - F(u) + \varphi(u^n + w_i^{n+1}) - \varphi(u) \geq \frac{\alpha_M}{p}||u^n + w_i^{n+1} - u||^p, \tag{3.16}$$

for any $n \geq 0$ and $i = 1, \cdots, m$, and

$$F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) \geq \frac{\alpha_M}{p}||u^{n+1} - u||^p, \tag{3.17}$$

for any $n \geq 0$. Writing

$$\overline{u}^{n+1} = u^n + \sum_{i=1}^{m} w_i^{n+1}, \tag{3.18}$$

from the convexity of F, we get

$$F(u^{n+1}) \leq (1 - \frac{r}{m})F(u^n) + \frac{r}{m}F(\overline{u}^{n+1}) \tag{3.19}$$

Applying Assumption 2.1 for $w = u^n$ and $v = u$, we get a decomposition $u_1^n, \cdots, u_m^n$ of $u - u^n$. From (2.1), we can replace v_i by u_i^n in (3.6), and from (3.19) and (2.7), we obtain

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \tfrac{r}{m}\tfrac{\alpha_M}{p}||u - \overline{u}^{n+1}||^p \leq \\
&(1 - \tfrac{r}{m})\left[F(u^n) - F(u)\right] + \tfrac{r}{m}\left[F(\overline{u}^{n+1}) - F(u) + \tfrac{\alpha_M}{p}||u - \overline{u}^{n+1}||^p\right] + \\
&\varphi(u^{n+1}) - \varphi(u) \leq \\
&(1 - \tfrac{r}{m})[F(u^n) - F(u)] + \tfrac{r}{m}\langle F'(\overline{u}^{n+1}), \overline{u}^{n+1} - u\rangle + \varphi(u^{n+1}) - \varphi(u) \leq \\
&(1 - \frac{r}{m})[F(u^n) - F(u)] + \frac{r}{m}\sum_{i=1}^{m}\langle F'(u^n + w_i^{n+1}) - F'(\overline{u}^{n+1}), u_i^n - w_i^{n+1}\rangle + \\
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + u_i^n) - \varphi(u^n + w_i^{n+1})] + \varphi(u^{n+1}) - \varphi(u)
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \tfrac{r}{m}\tfrac{\alpha_M}{p}||u - \overline{u}^{n+1}||^p \leq \\
&(1 - \frac{r}{m})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] + \\
&\frac{r}{m}\sum_{i=1}^{m}\langle F'(u^n + w_i^{n+1}) - F'(\overline{u}^{n+1}), u_i^n - w_i^{n+1}\rangle + \\
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + u_i^n) - \varphi(u^n + w_i^{n+1})] + \\
&\tfrac{r}{m}[\varphi(u^n) - \varphi(u)] + \varphi(u^{n+1}) - \varphi(u^n)
\end{aligned} \tag{3.20}$$

As in [3], using (2.5) and (2.3), we get

$$\sum_{i=1}^{m}\langle F'(u^n+w_i^{n+1})-F'(\overline{u}^{n+1}),u_i^n-w_i^{n+1}\rangle \le$$
$$\beta_M\left(\sum_{i=1}^{m}||w_i^{n+1}||\right)^{q-1}\sum_{i=1}^{m}||u_i^n-w_i^{n+1}|| \le$$
$$\beta_M m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}\sum_{i=1}^{m}(||u_i^n||+||w_i^{n+1}||) \le$$
$$\beta_M m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}\left(C_0||u-u^n||+\sum_{i=1}^{m}||w_i^{n+1}||\right) \le$$
$$\beta_M m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}\cdot$$
$$\left(C_0||u-\bar{u}^{n+1}||+(1+C_0)\sum_{i=1}^{m}||w_i^{n+1}||\right) \le$$
$$\beta_M C_0 m^{\frac{(p-1)(q-1)}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q-1}{p}}||u-\bar{u}^{n+1}||+$$
$$\beta_M(1+C_0)m^{\frac{(p-1)q}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q}{p}}$$

But, for any $\varepsilon>0$ $r>1$ and $x,y\ge 0$, we have $x^{\frac{1}{r}}y\le \varepsilon x+\frac{1}{\varepsilon^{\frac{1}{r-1}}}y^{\frac{r}{r-1}}$. Consequently, we get

$$\begin{aligned}&\sum_{i=1}^{m}\langle F'(u^n+w_i^{n+1})-F'(\overline{u}^{n+1}),u_i^n-w_i^{n+1}\rangle \le\\ &\beta_M(1+C_0)m^{\frac{(p-1)q}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q}{p}}+\\ &\beta_M C_0\frac{m^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q-1}{p-1}}+\\ &\beta_M C_0\varepsilon m^{\frac{(p-1)(q-1)}{p}}||u-\bar{u}^{n+1}||^p\end{aligned}\tag{3.21}$$

for any $\varepsilon>0$.

Also, using (3.12) and (3.2), we get

$$\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n+u_i^n)-\varphi(u^n+w_i^{n+1})]+$$
$$\frac{r}{m}[\varphi(u^n)-\varphi(u)]+\varphi(u^{n+1})-\varphi(u^n)\leq$$
$$\frac{r}{m}\left[\sum_{i=1}^{m}\varphi(u^n+u_i^n)-(m-1)\varphi(u^n)-\varphi(u)\right]\leq 0$$

Consequently, from (3.20) and (3.21), we have

$$\begin{aligned}&F(u^{n+1})-F(u)+\varphi(u^{n+1})-\varphi(u)+\\&\frac{r}{m}\left[\frac{\alpha_M}{p}-\beta_M C_0\varepsilon m^{\frac{(p-1)(q-1)}{p}}\right]||u-\overline{u}^{n+1}||^p\leq\\&(1-\frac{r}{m})[F(u^n)-F(u)+\varphi(u^n)-\varphi(u)]+\\&\frac{r}{m}\beta_M\left[(1+C_0)m^{\frac{(p-1)q}{p}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q}{p}}+\right.\\&\left.C_0\frac{m^{\frac{(p-1)(q-1)}{p}}}{\varepsilon^{\frac{1}{p-1}}}\left(\sum_{i=1}^{m}||w_i^{n+1}||^p\right)^{\frac{q-1}{p-1}}\right]\end{aligned}\tag{3.22}$$

for any $\varepsilon>0$.

Now, in view of (3.12) and (3.15), we get

$$F(u^{n+1})\leq(1-r)F(u^n)+\frac{r}{m}\sum_{i=1}^{m}F(u^n+w_i^{n+1})\leq$$
$$F(u^n)-\frac{r}{m}\frac{\alpha_M}{p}\sum_{i=1}^{m}||w_i^{n+1}||^p+\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n)-\varphi(u^n+w_i^{n+1})]$$

Consequently, we have

$$\begin{aligned}&\frac{r}{m}\frac{\alpha_M}{p}\sum_{i=1}^{m}||w_i^{n+1}||^p\leq F(u^n)-F(u^{n+1})+\varphi(u^n)-\varphi(u^{n+1})+\\&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n)-\varphi(u^n+w_i^{n+1})]-\varphi(u^n)+\varphi(u^{n+1})\end{aligned}\tag{3.23}$$

But, in view of (3.12), we have

$$\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n)-\varphi(u^n+w_i^{n+1})]-\varphi(u^n)+\varphi(u^{n+1})\leq 0,$$

and consequently,

$$\sum_{i=1}^{m} ||w_i^{n+1}||^p \leq \frac{m}{r}\frac{p}{\alpha_M}\left[F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})\right] \tag{3.24}$$

Finally, from (3.22) and (3.24), we get

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) + \\
&\frac{r}{m}\left[\frac{\alpha_M}{p} - \beta_M C_0 \varepsilon m^{\frac{(p-1)(q-1)}{p}}\right] ||u - \overline{u}^{n+1}||^p \leq \\
&(1 - \frac{r}{m})[F(u^n) - F(u) + \varphi(u^n) - \varphi(u)] + \\
&\frac{r}{m}\beta_M \left[\left(\frac{m}{r}\right)^{\frac{q}{p}} \frac{(1+C_0)m^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}}\left(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})\right)^{\frac{q}{p}} + \right.\\
&\left.\left(\frac{m}{r}\right)^{\frac{q-1}{p-1}} \frac{C_0 m^{\frac{(p-1)(q-1)}{p}}}{(\frac{\alpha_M}{p})^{\frac{q-1}{p-1}}\varepsilon^{\frac{1}{p-1}}}\left(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})\right)^{\frac{q-1}{p-1}}\right]
\end{aligned}$$

With

$$\varepsilon = \frac{\alpha_M}{p}\frac{1}{\beta_M C_0 m^{\frac{(p-1)(q-1)}{p}}},$$

the above equation becomes,

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u^{n+1}) - \varphi(u) \leq \\
&\frac{m-r}{r}[F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})] + \\
&\beta_M \left[\left(\frac{m}{r}\right)^{\frac{q}{p}} \frac{(1+C_0)m^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}}\cdot\right.\\
&\left(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})\right)^{\frac{q}{p}} + \\
&\left(\frac{m}{r}\right)^{\frac{q-1}{p-1}} \frac{\beta_M^{\frac{1}{p-1}} C_0^{\frac{p}{p-1}} m^{q-1}}{(\frac{\alpha_M}{p})^{\frac{q}{p-1}}}\cdot \\
&\left.\left(F(u^n) - F(u^{n+1}) + \varphi(u^n) - \varphi(u^{n+1})\right)^{\frac{q-1}{p-1}}\right]
\end{aligned} \tag{3.25}$$

Using (3.5), we see that error estimations in (3.9) and (3.11) can be obtained from (3.8) and (3.10), respectively.

Now, if $p = q = 2$, from the above equation, we easily get equation (3.8), where

$$C_1 = \frac{m}{r}\left[1 - \frac{r}{m} + (1+C_0)m\frac{\beta_M}{\frac{\alpha_M}{2}} + C_0^2 m \left(\frac{\beta_M}{\frac{\alpha_M}{2}}\right)^2\right] \tag{3.26}$$

Finally, if $q < p$, from (3.13) and (3.25), we get

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) \le \\ &C_3 \left[F(u^n) + \varphi(u^n) - F(u^{n+1}) - \varphi(u^{n+1})\right]^{\frac{q-1}{p-1}}. \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} &C_3 = \frac{m-r}{r}[F(u^0) - F(u) + \varphi(u^0) - \varphi(u)]^{\frac{p-q}{p-1}} + \\ &\left(\frac{m}{r}\right)^{\frac{q}{p}} \frac{\beta_M(1+C_0)m^{\frac{(p-1)q}{p}}}{(\frac{\alpha_M}{p})^{\frac{q}{p}}} \cdot \\ &\left(F(u^0) - F(u) + \varphi(u^0) - \varphi(u)\right)^{\frac{p-q}{p(p-1)}} + \\ &\left(\frac{m}{r}\right)^{\frac{q-1}{p-1}} \frac{\beta_M^{\frac{p}{p-1}} C_0^{\frac{p}{p-1}} m^{q-1}}{(\frac{\alpha_M}{p})^{\frac{q}{p-1}}} \end{aligned} \tag{3.28}$$

Now, from (3.27), we get

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) + \frac{1}{C_3^{\frac{p-1}{q-1}}}[F(u^{n+1}) + \varphi(u^{n+1}) - \\ &F(u) - \varphi(u)]^{\frac{p-1}{q-1}} \le F(u^n) + \varphi(u^n) - F(u) - \varphi(u), \end{aligned}$$

and, like in [1] or [3], we have

$$\begin{aligned} &F(u^{n+1}) + \varphi(u^{n+1}) - F(u) - \varphi(u) \le \\ &\left[(n+1)C_2 + \left(F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right)^{\frac{q-p}{q-1}}\right]^{\frac{q-1}{q-p}}, \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} &C_2 = \\ &\frac{p-q}{(p-1)\left(F(u^0) + \varphi(u^0) - F(u) - \varphi(u)\right)^{\frac{p-q}{q-1}} + (q-1)C_3^{\frac{p-1}{q-1}}}. \end{aligned} \tag{3.30}$$

Equation (3.29) is another form of the first estimate in (3.10). □

4 Subspace correction algorithms for quasi-variational inequalities

Let $\varphi : V \times V \to \mathbf{R}$ be a functional such that, for any $u \in K$, $\varphi(u, \cdot) : K \to \mathbf{R}$ is convex and lower semicontinuous. We assume that $F + \varphi$ is coercive in the sense that

$$\frac{F(v) + \varphi(u, v)}{\|v\|} \to \infty, \text{ as } \|v\| \to \infty, \ v \in K, \text{ for any } u \in K \tag{4.1}$$

if K is not bounded.

In this section we assume that $p = q = 2$ in (2.4) and (2.5). Also, we assume that for any $M > 0$ there exists $c_M > 0$ such that

$$
\begin{aligned}
&|\varphi(v_1, w_2) + \varphi(v_2, w_1) - \varphi(v_1, w_1) - \varphi(v_2, w_2)| \leq \\
&c_M \|v_1 - v_2\| \|w_1 - w_2\|
\end{aligned} \tag{4.2}
$$

for any $v_1,\ v_2,\ w_1\ w_2 \in K$, $\|v_1\|,\ \|v_2\|,\ \|w_1\|\ \|w_2\| \leq M$. In addition to the hypotheses of Assumption 2.1, we suppose that

$$
\sum_{i=1}^{m} \varphi(u, w + v_i) \leq (m-1)\varphi(u, w) + \varphi(u, v) \tag{4.3}
$$

for any $u \in K$ and for any $v, w \in K$ and $v_i \in V_i$, $i = 1, \ldots, m$, which satisfy Assumption 2.1.

Now, we consider the quasi-variational inequality

$$
u \in K : \langle F'(u), v - u\rangle + \varphi(u, v) - \varphi(u, u) \geq 0, \text{ for any } v \in K. \tag{4.4}
$$

Since φ is convex in the second variable and F is differentiable and satisfies (2.4), problem (4.4) is equivalent with the minimization problem

$$
u \in K : F(u) + \varphi(u, u) \leq F(v) + \varphi(u, v), \text{ for any } v \in K. \tag{4.5}
$$

With a similar proof to that of Theorem 2.1 in [15], we can show that problem (4.4) has a unique solution if there exists a constant $\varkappa < 1$ such that $\frac{c_M}{\alpha_M} \leq \varkappa$, for any $M > 0$. In view of (2.7) we see that, for a given $M > 0$ such that the solution u of (4.4) satisfies $\|u\| \leq M$, we have

$$
\begin{aligned}
&\tfrac{\alpha_M}{2}\|v - u\|^2 \leq F(v) - F(u) + \varphi(u, v) - \varphi(u, u), \\
&\text{for any } v \in K,\ \|v\| \leq M.
\end{aligned} \tag{4.6}
$$

A first algorithm corresponding to the subspaces $V_1, \cdots, V_m$ and the convex set K is written as follows

ALGORITHM 4.1. *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \geq 0$, we simultaneously solve the inequalities*

$$
\begin{aligned}
&w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : \\
&\langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1}\rangle + \varphi(u^n + w_i^{n+1}, u^n + v_i) - \\
&\varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1}) \geq 0, \text{ for any } v_i \in V_i,\ u^n + v_i \in K,
\end{aligned} \tag{4.7}
$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \dfrac{r}{m}\displaystyle\sum_{i=1}^{m} w_i^{n+1}$, where $0 < r \leq 1$ is a fixed constant.

Evidently, problem (4.7) is equivalent with the finding of $w_i^{n+1} \in V_i$, $u^n + w_i^{n+1} \in K$, which satisfies

$$
\begin{aligned}
&w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K :\ F(u^n + w_i^{n+1})+ \\
&\varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1}) \le F(u^n + v_i)+ \\
&\varphi(u^n + w_i^{n+1}, u^n + v_i),\ \text{for any } v_i \in V_i,\ u^n + v_i \in K.
\end{aligned}
\tag{4.8}
$$

for any $v_i \in V_i$, $u^n + v_i \in K$.

A simplified variant of Algorithm 4.1 can be written as

ALGORITHM 4.2. *We start the algorithm with an arbitrary $u^0 \in K$. At iteration $n+1$, having $u^n \in K$, $n \ge 0$, we solve the inequalities*

$$
\begin{aligned}
&w_i^{n+1} \in V_i,\ u^n + w_i^{n+1} \in K : \\
&\langle F'(u^n + w_i^{n+1}), v_i - w_i^{n+1} \rangle + \varphi(u^n, u^n + v_i) - \\
&\varphi(u^n, u^n + w_i^{n+1}) \ge 0, \textit{for any } v_i \in V_i,\ u^n + v_i \in K,
\end{aligned}
\tag{4.9}
$$

for $i = 1, \cdots, m$, and then we update $u^{n+1} = u^n + \frac{r}{m}\sum_{i=1}^{m} w_i^{n+1}$, where $0 < r \le 1$ is a fixed constant.

We can apply Theorem 8.5 in [14], page 251, to prove that problem (4.9) has an unique solution. Also, like for problem (4.4), with a similar proof to that of Theorem 2.1 in [15], we can prove that problem (4.7) has a unique solution if there exists a constant $\varkappa < 1$ such that $\frac{c_M}{\alpha_M} \le \varkappa$, for any $M > 0$.

The following theorem proves that if c_M is small enough, then Algorithms 4.1 and 4.2 are convergent.

Theorem 4.1. *Let V be a reflexive Banach, $V_1, \cdots, V_m$ some closed subspaces of V, and K a non empty closed convex subset of V which satisfies Assumption 2.1. Also, we assume that F is Gâteaux differentiable and satisfies (2.4) and (2.5) with $p = q = 2$, and the functional φ is convex, lower semicontinuous in the second variable, satisfies (4.2), (4.3) and coerciveness condition (4.1), if K is not bounded. Then, if u is the solution of problem (4.4), u^n, $n \ge 0$, are its approximations obtained from one of Algorithms 4.1 or 4.2, and*

$$
\frac{c_M}{\alpha_M} \le \chi_M \tag{4.10}
$$

for any $M > 0$, where χ_M is the smallest positive solution of equation (4.27), then there exists an $M > 0$ such that $\max(\|u\|, \max_{n\ge 0} \|u^n\|, \max_{n\ge 0, 1\le i\le m} \|u^n + w_i^{n+1}\|) \le M$ and we have the following error estimations

$$
\begin{aligned}
&F(u^n) + \varphi(u, u^n) - F(u) - \varphi(u,u) \le \\
&\left(\tfrac{C_1}{C_1+1}\right)^n \left[F(u^0) + \varphi(u, u^0) - F(u) - \varphi(u,u)\right],
\end{aligned}
\tag{4.11}
$$

$$(4.12)\qquad \|u^n-u\|^2 \le \frac{2}{\alpha_M}\left(\frac{C_1}{C_1+1}\right)^n \left[F(u^0)+\varphi(u^0)-F(u)-\varphi(u)\right].$$

The constant C_1 is given in (4.24) in which ε_1, ε_2 and ε_3 are given in (4.26).

Proof. As in the proof of Theorem 3.1, we will omit here the details which are similar with those in [3], [4] or [6]. Also, u and v in equations (2.4), (2.5), and v_i, w_i, $i=1,2$ in (4.2), will be replaced only by the solution of problem (4.4) or its approximations obtained from Algorithms 4.1 or 4.2. Consequently, we are interested in the boundedness of the sequences u^n and $u^n+w_i^{n+1}$, $n\ge 0$, $i=1,\dots,m$. To this end, we take

$$(4.13)\qquad M=\max(\|u\|, \max_{0\le n\le k}\|u^n\|, \max_{0\le n\le k, 1\le i\le m}\|u^n+w_i^{n+1}\|),$$

for a given $k\ge 0$ and prove (4.11) for $n=1,\dots,k$. As we shall see in the following, estimation (4.11) also holds if we replace u^n by $u^n+w_i^{n+1}$, $i=1,\cdots,m$. Consequently, in view of the coerciveness condition (4.1),

$$M\le \max(\|u\|, \sup\{\|v\| : F(v)+\varphi(u,v)\le F(u^0)+\varphi(u,u^0)\})<\infty,$$

where u is the solution of problem (4.4). Therefore, we may conclude that there exists a real constant $M>0$ such that

$$(4.14)\qquad \begin{array}{l}\|u\|\le M,\ \|u^n\|\le M,\ \|u^n+w_i^{n+1}\|\le M\\ \text{for any } n\ge 0,\ i=1,\cdots,m.\end{array}$$

In the following, for a fixed n, we take M given in (4.13). Since the proof of the theorem is almost the same for both algorithms, we prove the theorem only for Algorithm 4.1.

In view of (4.6), we notice that (4.12) can be obtained from (4.11).

Now, we prove (4.11). From (4.7) and (2.7), we get that, for any $n\ge 0$ and $i=1,\cdots,m$,

$$(4.15)\qquad \begin{array}{l}F(u^n)-F(u^n+w_i^{n+1})+\varphi(u^n+w_i^{n+1},u^n)-\\ \varphi(u^n+w_i^{n+1},u^n+w_i^{n+1})\ge \frac{\alpha_M}{2}||w_i^{n+1}||^2,\end{array}$$

Also, in view of (4.6), we get

$$(4.16)\qquad F(u^{n+\frac{i}{m}})-F(u)+\varphi(u,u^{n+\frac{i}{m}})-\varphi(u,u)\ge \frac{\alpha_M}{2}\|u^{n+\frac{i}{m}}-u\|^2$$

for $n\ge 0$ and $i=1,\cdots,m$. Applying Assumption 2.1 for $w=u^n$ and $v=u$, we get a decomposition $u_1^n,\cdots,u_m^n$ of $u-u^n$. With $\bar u^{n+1}$ defined in (3.18), from (2.1), we can replace v_i by u_i^n in (4.7), and in view of (2.7) and the convexity of F, we obtain

$$F(u^{n+1}) - F(u) + \varphi(u, u^{n+1}) - \varphi(u, u) + \tfrac{r}{m}\tfrac{\alpha_M}{2}||u - \overline{u}^{n+1}||^2 \leq$$
$$(1 - \tfrac{r}{m})\left[F(u^n) - F(u)\right] + \tfrac{r}{m}\left[F(\overline{u}^{n+1}) - F(u) + \tfrac{\alpha_M}{2}||u - \overline{u}^{n+1}||^2\right] +$$
$$\varphi(u, u^{n+1}) - \varphi(u, u) \leq$$
$$(1 - \tfrac{r}{m})[F(u^n) - F(u)] + \tfrac{r}{m}\langle F'(\overline{u}^{n+1}), \overline{u}^{n+1} - u\rangle + \varphi(u, u^{n+1}) - \varphi(u, u) \leq$$
$$(1 - \frac{r}{m})[F(u^n) - F(u)] + \frac{r}{m}\sum_{i=1}^{m}\langle F'(u^n + w_i^{n+1}) - F'(\overline{u}^{n+1}), u_i^n - w_i^{n+1}\rangle +$$
$$\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] +$$
$$\varphi(u, u^{n+1}) - \varphi(u, u)$$

Consequently, we have

$$\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u, u^{n+1}) - \varphi(u, u) + \\
&\frac{r}{m}\frac{\alpha_M}{2}||u - \overline{u}^{n+1}||^2 \leq \\
&(1 - \frac{r}{m})[F(u^n) - F(u) + \varphi(u, u^n) - \varphi(u, u)] + \\
&\frac{r}{m}\sum_{i=1}^{m}\langle F'(u^n + w_i^{n+1}) - F'(\overline{u}^{n+1}), u_i^n - w_i^{n+1}\rangle + \\
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] + \\
&\tfrac{r}{m}[\varphi(u, u^n) - \varphi(u, u)] + \varphi(u, u^{n+1}) - \varphi(u, u^n)
\end{aligned} \tag{4.17}$$

Using (2.5) and (2.3) for $p = q = 2$, and the Hölder inequality, similarly with (3.21), we get

$$\begin{aligned}
&\sum_{i=1}^{m}\langle F'(u^n + w_i^{n+1}) - F'(\overline{u}^{n+1}), u_i^n - w_i^{n+1}\rangle \leq \\
&\beta_M m\left[1 + C_0(1 + \frac{1}{2\varepsilon_1})\right]\sum_{i=1}^{m}||w_i^{n+1}||^2 + \\
&\beta_M C_0\tfrac{\varepsilon_1}{2}||u - \bar{u}^{n+1}||^2
\end{aligned} \tag{4.18}$$

for any $\varepsilon_1 > 0$. Similarly with (3.12), from the convexity of φ in the second variable, we get

$$\varphi(u, u^{n+1}) \leq (1 - r)\varphi(u, u^n) + \frac{r}{m}\sum_{i=1}^{m}\varphi(u, u^n + w_i^{n+1})$$

Using this equation, in view of (4.3), (4.2) and (2.3), we have

$$
\begin{aligned}
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})]+\\
&\frac{r}{m}[\varphi(u, u^n) - \varphi(u, u)] + \varphi(u, u^{n+1}) - \varphi(u, u^n) \leq\\
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})]+\\
&+\frac{r}{m}\sum_{i=1}^{m}\varphi(u, u^n + w_i^{n+1}) - \frac{r}{m}[(m-1)\varphi(u, u^n) + \varphi(u, u)] \leq\\
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})]+\\
&+\frac{r}{m}\sum_{i=1}^{m}[\varphi(u, u^n + w_i^{n+1}) - \varphi(u, u^n + u_i^n)] \leq\\
&\frac{r}{m}c_M\sum_{i=1}^{m}\|u^n + w_i^{n+1} - u\|\|w_i^{n+1} - u_i^n\| \leq\\
&\frac{r}{m}c_M\left[\|\bar{u}^{n+1} - u\| + \sum_{i=1}^{m}\|w_i^{n+1}\|\right]\sum_{i=1}^{m}\left(\|w_i^{n+1}\| + \|u_i^n\|\right) \leq\\
&\frac{r}{m}c_M\left[\|\bar{u}^{n+1} - u\| + \sum_{i=1}^{m}\|w_i^{n+1}\|\right]\cdot\\
&\left[C_0\|\bar{u}^{n+1} - u\| + (1 + C_0)\sum_{i=1}^{m}\|w_i^{n+1}\|\right]
\end{aligned}
$$

or

$$
\begin{aligned}
&\frac{r}{m}\sum_{i=1}^{m}[\varphi(u^n + w_i^{n+1}, u^n + u_i^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})]+\\
&\frac{r}{m}[\varphi(u, u^n) - \varphi(u, u)] + \varphi(u, u^{n+1}) - \varphi(u, u^n) \leq\\
&\frac{r}{m}c_M[C_0 + (1 + 2C_0)\frac{\varepsilon_2}{2}]\|\bar{u}^{n+1} - u\|^2+\\
&rc_M[1 + C_0 + \frac{1 + 2C_0}{2\varepsilon_2}]\sum_{i=1}^{m}\|w_i^{n+1}\|^2
\end{aligned}
\tag{4.19}
$$

for any $\varepsilon_2 > 0$. Consequently, from (4.17)–(4.19), we have

$$
\begin{aligned}
&F(u^{n+1}) - F(u) + \varphi(u, u^{n+1}) - \varphi(u, u) + \\
&\left\{\frac{\alpha_M}{2} - \beta_M C_0 \frac{\varepsilon_1}{2} - c_M[C_0 + (1+2C_0)\frac{\varepsilon_2}{2}]\right\} ||u - \overline{u}^{n+1}||^2 \leq \\
&\frac{m-r}{r}[F(u^n) - F(u^{n+1}) + \varphi(u, u^n) - \varphi(u, u^{n+1})] + \\
&\left\{\beta_M m \left[1 + C_0(1 + \frac{1}{2\varepsilon_1})\right] + c_M m \left[1 + C_0 + \frac{1+2C_0}{2\varepsilon_2}\right]\right\} \cdot \\
&\sum_{i=1}^m ||w_i^{n+1}||^2
\end{aligned} \tag{4.20}
$$

for any ε_1, $\varepsilon_2 > 0$.

In view of (3.12) and (4.15), we have

$$
\begin{aligned}
&F(u^{n+1}) \leq (1-r)F(u^n) + \frac{r}{m}\sum_{i=1}^m F(u^n + w_i^{n+1}) \leq F(u^n) - \\
&\frac{r}{m}\frac{\alpha_M}{2}\sum_{i=1}^m ||w_i^{n+1}||^2 + \frac{r}{m}\sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\frac{r}{m}\frac{\alpha_M}{2}\sum_{i=1}^m ||w_i^{n+1}||^2 \leq F(u^n) - F(u^{n+1}) + \\
&\varphi(u, u^n) - \varphi(u, u^{n+1}) + \frac{r}{m}\sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n) - \\
&\varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] - \varphi(u, u^n) + \varphi(u, u^{n+1})
\end{aligned} \tag{4.21}
$$

Similarly with (4.19), we have

$$
\begin{aligned}
&\frac{r}{m}\sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] - \\
&\varphi(u, u^n) + \varphi(u, u^{n+1}) \leq \\
&\frac{r}{m}\sum_{i=1}^m [\varphi(u^n + w_i^{n+1}, u^n) - \varphi(u^n + w_i^{n+1}, u^n + w_i^{n+1})] + \\
&\frac{r}{m}\sum_{i=1}^m [\varphi(u, u^n + w_i^{n+1}) - \varphi(u, u^n)] \leq \\
&\frac{r}{m} c_M \sum_{i=1}^m ||u^n + w_i^{n+1} - u|| ||w_i^{n+1}|| \leq \\
&\frac{r}{m} c_M \left(\sum_{i=1}^m ||w_i^{n+1}|| + ||\bar{u}^{n+1} - u||\right) \sum_{i=1}^m ||w_i^{n+1}|| \leq \\
&\frac{r}{m} c_M (1 + \frac{1}{2\varepsilon_3}) m \sum_{i=1}^m ||w_i^{n+1}||^2 + \frac{r}{m} c_M \frac{\varepsilon_3}{2} ||\overline{u}^{n+1} - u||^2
\end{aligned} \tag{4.22}
$$

for any $\varepsilon_3 > 0$. In view of (4.21) and (4.22), we get

$$\begin{aligned}&\left[\frac{\alpha_M}{2} - c_M(1+\frac{1}{2\varepsilon_3})m\right]\sum_{i=1}^{m}||w_i^{n+1}||^2 \leq \\ &\frac{m}{r}\left[F(u^n) - F(u^{n+1}) + \varphi(u,u^n) - \varphi(u,u^{n+1})\right] + \\ &c_M\frac{\varepsilon_3}{2}||\overline{u}^{n+1} - u||^2\end{aligned} \tag{4.23}$$

for any $\varepsilon_3 > 0$. If we write

$$\begin{aligned}&C_1 = \tfrac{m-r}{r} + C_4\tfrac{m}{r}\\ &C_2 = \tfrac{\alpha_M}{2} - c_M(1+\tfrac{1}{2\varepsilon_3})m\\ &C_3 = \tfrac{\alpha_M}{2} - \beta_M C_0\tfrac{\varepsilon_1}{2} - c_M\left(C_0 + \tfrac{1+2C_0}{2}\varepsilon_2\right) - c_M\tfrac{\varepsilon_3}{2}C_4\\ &C_4 = \frac{m}{C_2}\left[\beta_M\left(1 + C_0(1+\frac{1}{2\varepsilon_1})\right) + \right.\\ &\left. c_M\left(1 + C_0 + \tfrac{1+2C_0}{2\varepsilon_2}\right)\right]\end{aligned} \tag{4.24}$$

then, from (4.20) and (4.23), on the condition $C_2 > 0$, we get

$$\begin{aligned}&F(u^{n+1}) - F(u) + \varphi(u,u^{n+1}) - \varphi(u,u) + C_3||u - \overline{u}^{n+1}||^2 \leq \\ &C_1\left[F(u^n) - F(u^{n+1}) + \varphi(u,u^n) - \varphi(u,u^{n+1})\right]\end{aligned} \tag{4.25}$$

Now, if $C_3 \geq 0$, then (4.11) can be obtained from (4.25).

We can easily see that C_3, as a function of ε_1, ε_2, and ε_3, reaches its maximum value for

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{c_M m}{\frac{\alpha_M}{2} - c_M m}, \tag{4.26}$$

and this is

$$\begin{aligned}&C_{3max} = \tfrac{\alpha_M}{2} - c_M C_0 - [\beta_M C_0 + c_M(1+2C_0)]\tfrac{c_M m}{\frac{\alpha_M}{2} - c_M m}\\ &-(1+C_0)(\beta_M + c_M)\frac{c_M^2 m^2}{(\frac{\alpha_M}{2} - c_M m)^2}.\end{aligned}$$

Condition $C_{3max} \geq 0$ is satisfied if

$$(\frac{1}{2} - C_0\frac{c_M}{\alpha_M})\frac{\alpha_M}{\beta_M} \geq (1+3C_0)\frac{\frac{c_M}{\alpha_M}m}{\frac{1}{2} - \frac{c_M}{\alpha_M}m} + 2(1+C_0)\frac{(\frac{c_M}{\alpha_M})^2 m^2}{(\frac{1}{2} - \frac{c_M}{\alpha_M}m)^2}$$

We see that equation

$$(\frac{1}{2} - C_0\chi_M)\frac{\alpha_M}{\beta_M} = (1+3C_0)\frac{\chi_M m}{\frac{1}{2} - \chi_M m} + 2(1+C_0)\frac{(\chi_M)^2 m^2}{(\frac{1}{2} - \chi_M m)^2} \tag{4.27}$$

has a solution $\chi_M \in (0, \frac{1}{2C_0})$, and if it is the smallest one and we take $\frac{c_M}{\alpha_M} \le \chi_M$, then $C_{3max} \ge 0$.

The value of C_2 for ε_3 in (4.26) is

$$C_{2\max} = \frac{1}{2}\left(\frac{\alpha_M}{2} - c_M m\right).$$

Since we can always take $C_0 \ge m$, the above solution χ_M of equation (4.27) satisfies $\chi_M < \frac{1}{2m}$, and therefore, we get $C_{2\max} > 0$ for any $\frac{c_M}{\alpha_M} \le \chi_M$.

□

5 Convergence rates for the one- and two-level methods

Algorithms 3.1, 4.1 and 4.2 can be viewed as additive Schwarz methods in a subspace correction variant if we use the Sobolev spaces. The convergence rates given in Theorems 3.1 and 4.1 depend on the functionals F and φ, the number m of the subspaces and the constant C_0 introduced in Assumption 2.1. The number of subspaces can be associated with the number of colors needed to mark the subdomains such that the subdomains with the same color do not intersect with each other. Since this number of colors depends on the dimension of the Euclidean space where the domain lies, we can conclude that our convergence rates essentially depend on the constant C_0. We shall see in this section that, if we utilize the finite element spaces, Assumption 2.1 as well as conditions (3.2) and (4.3) hold for closed convex sets K satisfying a general enough property. Also, we are able to explicitly write the dependence of C_0 on the domain decomposition and mesh parameters. Therefore, from Theorems 3.1 and 4.1, we can conclude that the one- and two-level methods globally converge for variational and quasi-variational inequalities of the second kind if conditions (2.4) and (2.5) on F, and condition (4.2) on φ, in the case of quasi-variational inequalities, hold. Moreover, from the dependence of C_0 on the mesh and domain decomposition parameters, the convergence rate is optimal, ie. is similar with that in the case of linear equations, for instance. The convergence rate of the two-level method depends very weakly on the mesh and domain decomposition parameters, and, for some particular choices, it is even independent of them.

The convergence of these two methods for the minimization of non-quadratic functionals has been studied in [3]. It is proved there that Assumption 2.1 holds for the spaces and the convex sets we use in this paper. Consequently, we will focus especially on the conditions (3.2) and (4.3) in the proofs .

5.1 One-level methods

We consider a simplicial regular mesh partition $\mathcal{T}_h$ of mesh size h (see [7], p. 124, for instance) over the domain $\Omega \subset \mathbf{R}^d$. We assume that the domain Ω is decomposed as

$$\Omega = \bigcup_{i=1}^{m} \Omega_i \tag{5.1}$$

and that $\mathcal{T}_h$ supplies a mesh partition for each subdomain Ω_i, $i = 1, \ldots, m$. In addition, we suppose that the overlapping parameter of this decomposition is δ.

We associate to the decomposition (5.1), a unity partition $\{\theta_i\}_{1\le i\le m}$, with $\theta^i \in C^0(\bar\Omega)$, $\theta^i|_\tau \in P_1(\tau)$ for any $\tau \in \mathcal{T}_h$, $i = 1, \cdots, m$,

$$0 \le \theta^i \le 1 \text{ on } \Omega, \text{ supp } \theta_i \subset \overline{\Omega}_i \text{ and } \sum_{i=1}^{m} \theta_i = 1 \tag{5.2}$$

which satisfies

$$|\partial_{x_k}\theta^i| \le C/\delta, \text{ a.e. in } \Omega, \text{ for any } k = 1, \ldots, d \tag{5.3}$$

As in (5.3), we denote in the following by C a generic constant which does not depend on either the mesh or the domain decomposition.

We consider the piecewise linear finite element space

$$V_h = \{v \in C^0(\bar\Omega) : v|_\tau \in P_1(\tau), \ \tau \in \mathcal{T}_h, \ v = 0 \text{ on } \partial\Omega\}, \tag{5.4}$$

and also, for $i = 1, \ldots, m$, let

$$V_h^i = \{v \in V_h : v = 0 \text{ in } \Omega\backslash\Omega_i\} \tag{5.5}$$

be the subspaces of V_h corresponding to the domain decomposition $\Omega_1, \ldots, \Omega_m$. The spaces V_h and V_h^i, $i = 1, \ldots, m$, are considered as subspaces of $W^{1,s}$, for some fixed $1 \le s \le \infty$. We denote by $\|\cdot\|_{0,s}$ the norm in L^s, and by $\|\cdot\|_{1,s}$ and $|\cdot|_{1,s}$ the norm and seminorm in $W^{1,s}$, respectively.

The convex set K_h is defined as a subset of V_h satisfying the following property.

Property 5.1. *If $v, w \in K_h$, and if $\theta \in C^0(\bar\Omega)$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \le \theta \le 1$, then $L_h(\theta v + (1-\theta)w) \in K_h$.*

Above, we have denoted by L_h the P_1-Lagrangian interpolation operator which uses the function values at the nodes of the mesh $\mathcal{T}_h$.

In the case of the variational inequalities of the second kind, we assume that the functional φ is of the form

$$\varphi(v) = \sum_{\kappa\in\mathcal{N}_h} s_\kappa(h)\phi(v(x_\kappa)) = \sum_{k\in\mathcal{N}_h} s_\kappa(h)\phi_\kappa(v) \tag{5.6}$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is a continuous and convex function, $\mathcal{N}_h$ is the set of nodes of the mesh partition $\mathcal{T}_h$, and $s_\kappa(h) \geq 0$, $\kappa \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. For the ease of notation, we have written $\phi_\kappa(v) = \phi(v(x_\kappa))$. We see that ϕ_κ, $\kappa \in \mathcal{N}_h$, can be viewed as some functionals $\phi_\kappa : V_h \to \mathbf{R}$ which satisfy

$$\phi_\kappa(L_h(\theta v + (1-\theta)w)) \leq \theta(x_\kappa)\phi_\kappa(v) + (1-\theta(x_\kappa))\phi_\kappa(w) \tag{5.7}$$

for any $v, w \in K_h$, and any function $\theta : \bar{\Omega} \to \mathbf{R}$ which satisfy $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \leq \theta \leq 1$.

For the quasi-variational inequalities, we assume that the functional φ is of the form

$$\varphi(u,v) = \sum_{\kappa\in\mathcal{N}_h} s_\kappa(h)\phi(u, v(x_\kappa)) = \sum_{k\in\mathcal{N}_h} s_\kappa(h)\phi_\kappa(u,v) \tag{5.8}$$

where $\phi : V_h \times \mathbf{R} \to \mathbf{R}$ is continuous, and, as above, $s_\kappa(h) \geq 0$, $\kappa \in \mathcal{N}_h$, are some non-negative real numbers which may depend on the mesh size h. Also, we assume that $\varphi(u,\cdot) : \mathbf{R} \to \mathbf{R}$ is convex for any $u \in V_h$, and, for the ease of notation, we have written $\phi_\kappa(u,v) = \phi(u, v(x_\kappa))$. We see that ϕ_κ, $\kappa \in \mathcal{N}_h$, can be viewed as some functionals $\phi_\kappa : V_h \times V_h \to \mathbf{R}$ which satisfy

$$\phi_\kappa(u, L_h(\theta v + (1-\theta)w)) \leq \theta(x_\kappa)\phi_\kappa(u,v) + (1-\theta(x_\kappa))\phi_\kappa(u,w) \tag{5.9}$$

for any $u \in V_h$, $v, w \in K_h$, and any function $\theta : \bar{\Omega} \to \mathbf{R}$ with the properties $\theta \in C^0(\bar{\Omega})$, $\theta|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, and $0 \leq \theta \leq 1$. The functionals $\varphi(u)$ and $\varphi(u,v)$ defined in (5.6) and (5.8), respectively, can be viewed as numerical approximations of some functionals defined on V_h.

We can conclude from the following proposition that the error estimations in Theorems 3.1 and 4.1 hold for the one-level multiplicative Schwarz method applied to the solution of variational and quasi-variational inequalities of the second kind.

Proposition 5.1. *Assumption* 2.1 *holds for the piecewise linear finite element spaces, $V = V_h$ and $V_i = V_h^i$, $i = 1, \ldots, m$, for any convex set $K = K_h \subset V_h$ having Property* 5.1. *Also, conditions (3.2) and (4.3) for functionals φ of the form (5.6) and (5.8), respectively, are satisfied. The constant in (2.3) can be written as*

$$C_0 = Cm\left(1 + \frac{1}{\delta}\right), \tag{5.10}$$

where C is independent of the mesh and domain decomposition parameters.

Proof. If the convex set K_h has Property 5.1, we can prove that Assumption 2.1 holds with

$$v_i = L_h\left(\theta^i(v-w)\right), \quad i=1,\dots,m, \tag{5.11}$$

and the constant C_0 in (5.10).

To prove that condition (3.2) holds for a functional φ of the form (5.6), it is sufficient to show that

$$\sum_{i=1}^{m} \phi_\kappa(w+v_i) \le (m-1)\phi_\kappa(w) + \phi_\kappa(v) \tag{5.12}$$

for the $v_i \in V_h^i$, $i=1,\dots,m$, we have defined in (5.11). In view of (5.7), we have

$$\phi_\kappa(w+v_i) = \phi_\kappa(w+L_h(\theta^i(v-w)) = \phi_\kappa(L_h(\theta^i v + (1-\theta^i)w) \le \theta^i(x_\kappa)\phi_\kappa(v) + (1-\theta^i(x_\kappa))\phi_\kappa(w)$$

and therefore, (5.12) holds.

To prove that condition (4.3) holds for a functional φ of the form (5.8), it sufficient to prove that

$$\sum_{i=1}^{m} \varphi_\kappa(u, w+v_i) \le (m-1)\varphi_\kappa(u,w) + \varphi_\kappa(u,v) \tag{5.13}$$

for any $u \in V_h$ and the $v_i \in V_h^i$, $i=1,\dots,m$, we have defined in (5.11). Since $\phi_\kappa(u,v)$ satisfies (5.9) which is similar with (5.7), the proof of (5.13) is similar with that of (5.12). □

5.2 Two-level methods

We consider two simplicial mesh partitions $\mathcal{T}_h$ and $\mathcal{T}_H$ of the domain $\Omega \subset \mathbf{R}^d$ of mesh sizes h and H, respectively. The mesh $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, and we assume that both the families, of fine and coarse meshes, are regular. Mesh sizes h and H are supposed to approach zero and we shall consider a family of mesh pairs (h, H).

As in the previous section, we consider an overlapping decomposition (5.1), the mesh partition $\mathcal{T}_h$ of Ω supplying a mesh partition for each Ω_i, $1 \le i \le m$. Also, we assume that the overlapping size is δ. In addition, we suppose that there exists a constant C, independent of both meshes, such that the diameter of the connected components of each Ω_i are less than CH. We point out that the domain Ω may be different from

$$\Omega_0 = \bigcup_{\tau \in \mathcal{T}_H} \tau, \tag{5.14}$$

but we assume that if a node of $\mathcal{T}_H$ lies on $\partial\Omega_0$ then it also lies on $\partial\Omega$, and there exists a constant C, independent of both meshes, such that

$$\operatorname{dist}(x, \Omega_0) \leq CH \tag{5.15}$$

for any node x of $\mathcal{T}_h$.

Now, besides the spaces V_h and V_h^i, $i = 1, \ldots, m$ defined in (5.4) and (5.5), we introduce the continuous, piecewise linear finite element space corresponding to the H-level,

$$V_H^0 = \left\{ v \in C^0(\bar{\Omega}_0) : v|_\tau \in P_1(\tau),\ \tau \in \mathcal{T}_H,\ v = 0 \text{ on } \partial\Omega_0 \right\}, \tag{5.16}$$

where the functions v are extended with zero in $\Omega \backslash \Omega_0$. The convex set $K_h \subset V_h$ is defined as a subset of V_h having Property 5.1.

The two-level Schwarz methods are also obtained from Algorithms 3.1, 4.1 and 4.2 in which we take $V = V_h$, $K = K_h$, and the subspaces $V_0 = V_H^0$, $V_1 = V_h^1$, $V_2 = V_h^2, \ldots, V_m = V_h^m$. As in the previous section, the spaces V_h, V_H^0, V_h^1, $V_h^2, \ldots, V_h^m$, are considered as subspaces of $W^{1,s}$ for $1 \leq s \leq \infty$. We note that, this time, the decomposition of the domain Ω contains m overlapping subdomains, but we utilize $m+1$ subspaces of V, $V_0, V_1, \ldots, V_m$, in Algorithms 3.1, 4.1 and 4.2. Naturally, if we prove that Assumption 2.1, written for $m + 1$ subspaces, is satisfied for the previous choice of the convex set K and the subspaces $V_0, V_1, \ldots, V_m$ of V, we can conclude that these algorithms converge if we prove in addition that the functionals φ of the form (5.6) or (5.8), satisfy (3.2) or (4.3), respectively. To this end, we consider the operator $I_H : V_h \to V_H^0$, which has been introduced in [2] and has the following properties (see Lemma 4.3 in [2]) for any $v \in V_h$:

$$\|I_H v - v\|_{0,s} \leq CHC_{d,s}(H, h)|v|_{1,s} \tag{5.17}$$

and

$$\|I_H v\|_{0,s} \leq C\|v\|_{0,s} \text{ and } |I_H v|_{1,s} \leq CC_{d,s}(H, h)|v|_{1,s}, \tag{5.18}$$

where

$$C_{d,s}(H, h) = \begin{cases} 1 & \text{if } d = s = 1 \text{ or} \\ & 1 \leq d < s \leq \infty \\ \left(\ln \frac{H}{h} + 1\right)^{\frac{d-1}{d}} & \text{if } 1 < d = s < \infty \\ \left(\frac{H}{h}\right)^{\frac{d-s}{s}} & \text{if } 1 \leq s < d < \infty, \end{cases} \tag{5.19}$$

Moreover, for any $x \in \Omega$, we have

$$
\begin{aligned}
&0 \le I_H v(x) \le v(x) \text{ if } v(x) > 0,\\
&0 \ge I_H v(x) \ge v(x) \text{ if } v(x) < 0,\\
&\text{and } I_H v = 0 \text{ on } \tau \in \mathcal{T}_H \text{ if there exists a } x \in \tau \text{ such that } v(x) = 0
\end{aligned}
$$

for any $v \in V_h$. Consequently, writing

$$
\theta_v(x) = \begin{cases} \frac{I_H v(x)}{v(x)} & \text{if } v(x) \ne 0 \\ 0 & \text{if } v(x) = 0, \end{cases}
$$

then $\theta_v \in C^0(\bar{\Omega})$, $\theta_v|_\tau \in C^1(\tau)$ for any $\tau \in \mathcal{T}_h$, $0 \le \theta_v \le 1$, and

$$
I_H v = \theta_v v \tag{5.20}
$$

for any $v \in V_h$.

Now, we can prove the following proposition which, in particular, shows that the constant C_0 in Assumption 2.1 is independent of the mesh and domain decomposition parameters if H/δ and H/h are kept constant when $h \to 0$.

Proposition 5.2. *Assumption* 2.1 *is satisfied for the piecewise linear finite element spaces* $V = V_h$ *and* $V_0 = V_H^0$, $V_i = V_h^i$, *and* $i = 1, \ldots, m$, *defined in (5.4), (5.5), and (5.16), respectively, any convex set* $K = K_h$ *with Property 5.1. Also, conditions (3.2) and (4.3) for functionals* φ *of the form (5.6) and (5.8), respectively, are satisfied. The constant in (2.3) of Assumption 2.1 can be taken of the form*

$$
C_0 = C(m+1)\left(1 + \frac{H}{\delta}\right) C_{d,s}(H,h), \tag{5.21}
$$

where C *is independent of the mesh and domain decomposition parameters, and* $C_{d,s}(H,h)$ *is given in* (5.19).

Proof. By means of I_H and the functions θ^i, $i = 1, \ldots, m$, with properties (5.2) and (5.3), we define

$$
v_0 = I_H(v - w). \tag{5.22}
$$

and

$$
v_i = L_h(\theta^i(v - w - v_0)), \tag{5.23}
$$

for $i = 1, \ldots, m$. Using properties (5.17) and (5.18) of the operator I_H, we can prove that $v_0, v_1, \ldots, v_m$, defined in (5.22) and (5.23), satisfy Assumption 2.1 with the constant C_0 given in (5.21).

To prove that condition (3.2) holds for a functional φ of the form (5.6), it is sufficient to prove that

$$\sum_{i=0}^{m} \phi_\kappa(w+v_i) \le m\phi_\kappa(w) + \phi_\kappa(v) \tag{5.24}$$

for the $v_i \in V_i$, $i = 0, \dots, m$, we have defined in (5.22) and (5.23). Using (5.20) and Property 5.1, we get that $v - v_0 \in K_h$. Like in the proof of Proposition 5.1, where this time we consider $v - v_0$ in the place of v, we get

$$\phi_\kappa(w+v_0) + \sum_{i=1}^{m} \phi_\kappa(w+v_i) \le \phi_\kappa(w+v_0) + (m-1)\phi_\kappa(w) + \phi_\kappa(v-v_0),$$

and, in view of (5.20) and (5.7), we have

$$\begin{aligned}&\phi_\kappa(w+v_0) + \phi_\kappa(v-v_0) = \phi_\kappa(w+\theta_{v-w}(v-w)) + \phi_\kappa(v-\theta_{v-w}(v-w)) \le \\ &(1-\theta_{v-w}(x_\kappa))\phi_\kappa(w) + \theta_{v-w}(x_\kappa)\phi_\kappa(v) + \\ &(1-\theta_{v-w}(x_\kappa))\phi_\kappa(v) + \theta_{v-w}(x_\kappa)\phi_\kappa(w) = \phi_\kappa(w) + \phi_\kappa(v)\end{aligned}$$

Equation (5.24) follows from the last two equations.

To prove that condition (4.3) holds for a functional φ of the form (5.8), it is sufficient to prove that

$$\sum_{i=0}^{m} \phi_\kappa(u, w+v_i) \le m\phi_\kappa(u,w) + \phi_\kappa(u,v) \tag{5.25}$$

for $v_i \in V_i$, $i = 0, \dots, m$, we have defined in (5.22) and (5.23). Like in Proposition 5.1, since $\phi_\kappa(u,v)$ satisfies (5.9) which is similar with (5.7), the proof of (5.25) is similar with that of (5.24). □

Remark 5.1. In this Section 5, we have assumed that the functional φ is of the form (5.6) or (5.8) in the case of variational or quasi-variational inequalities, respectively. We notice that the proofs of Propositions 5.1 and 5.2 also hold if we replace the functional $\varphi(u,v)$ of form (5.8) with

$$\varphi(u,v) = \sum_{\kappa\in\mathcal{N}_h} s_\kappa(h)\phi(u(x_\kappa), v(x_\kappa)) = \sum_{k\in\mathcal{N}_h} s_\kappa(h)\phi_\kappa(u,v) \tag{5.26}$$

where $s_\kappa(h) \ge 0$, and $\phi : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ is continuous and convex in the second variable. We have denoted above $\phi_\kappa(u,v) = \phi(u(x_\kappa), v(x_\kappa))$, $\kappa \in \mathcal{N}_h$. In general, (5.6), (5.8) or (5.26) represent numerical approximations of some integrals. Concerning to condition (4.2) imposed on φ of the form (5.8) or (5.26), in the case of quasi-variational inequalities, we have to check it for each particular problem we solve.

The results of this section have referred to problems in $W^{1,s}$ with Dirichlet boundary conditions. We point out that similar results can be obtained for problems in $(W^{1,s})^d$ or problems with mixed boundary conditions.

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