On generalized quasiregular mappings

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Abstract: We study classes of continuous, open, discrete mappings satisfying some modular inequalities and we show that this thing ensures important geometric properties, extending partially known results from the theory of quasiregular mappings, like Liouville, Picard, Montel theorems or equicontinuity results. Most of the theorems are obtained in dimension $n = 2$.

Keywords: mappings satisfying generalized modular inequalities.

AMS 2000 Subject Classification: 30C65.

1 Introduction.

If $D \subset \mathbf{R}^n$ is a domain, we say that a map $f : D \to \mathbf{R}^n$ is of finite distortion if $f \in W^{1,1}_{loc}(D, \mathbf{R}^n)$, $J_f \in L^1_{loc}(D)$ and there exists $K : D \to [0, \infty]$ measurable and finite a.e. so that $|f'(x)|^n \leq K(x) \cdot J_f(x)$ a.e. If $f \in W^{1,n}_{loc}(D, \mathbf{R}^n)$ and $K \in L^\infty(D)$, we obtain the known class of quasiregular mappings. If the homeomorphism $f : D \to D'$ between two domains from $\mathbf{R}^n$ is quasiregular, we say that f is quasiconformal. For more information about the theory of quasiregular mappings, we send the reader to [22,23], [30-32].

If $x = (x_1, ..., x_n) \in \mathbf{R}^n$, we set $|x| = (\sum\limits_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ and if $A \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$, $detA \neq 0$, $p > 0$, we set $|A| = \sup\limits_{|h|=1} |A(h)|$, $l(A) = \inf\limits_{|h|=1} |A(h)|$, $H(A) = |A|/l(A)$, $K_{0,p}(A) = |A|^p/|detA|$, $K_{I,p}(A) = |detA|/l(A)^p$, and we put $K_0(A) = K_{0,n}(A)$, $K_I(A) = K_{I,n}(A)$. If $D \subset \mathbf{R}^n$ is a domain, $f : D \to \mathbf{R}^n$ is a.e. differentiable on D and $J_f(x) \neq 0$ a.e. on D, we can define a.e. the mappings $K_{0,p}(f) : D \to [0, \infty]$ by $K_{0,p}(f)(x) = K_{0,p}(f'(x))$ a.e. in D and $K_{I,p}(f) : D \to [0, \infty]$ by $K_{I,p}(f)(x) = K_{I,p}(f'(x))$ a.e. in D and we set $K_0(f) = K_{0,n}(f)$, $K_I(f) = K_{I,n}(f)$. If $D \subset \mathbf{R}^n$ is a domain, we define the map $H_{I,p}(f) : D \to \mathbf{R}$, $H_{I,p}(f)(x) = K_{I,p}(f'(x))$ if f is differentiable in x and $J_f(x) \neq 0$, $H_{I,p}(f)(x) = 0$ otherwise.

If Γ is a path family from $\mathbf{R}^n$, we set $F(\Gamma) = \{\rho : \mathbf{R}^n \to [0, \infty]$ Borel maps$| \int\limits_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$ and for $p > 1$ we have the usual p-modulus $M_p(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{\mathbf{R}^n} \rho^p(x)dx$. If $\omega : D \to [0, \infty]$ is measurable and finite a.e., $\omega > 0$ a.e., we define the weight p-modulus of weight ω by $M^p_\omega(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{\mathbf{R}^n} \omega(x)\rho^p(x)dx$.

A quasiregular mapping is open and discrete and the known modular inequality of Poleckii says that if $f : D \to \mathbf{R}^n$ is quasiregular and $K_I(f) \leq K$, then $M_n(f(\Gamma)) \leq K M_n(\Gamma)$ for every path family Γ from D. This modular inequality is the key for proving most of the important geometric properties of quasiregular mappings. If f is a map of finite distortion and either $K_I(f) \in BMO(D)$ or $exp(\mathcal{A} \circ K_0(f)) \in L^1_{loc}(D)$ for some Orlicz function $\mathcal{A}$, then, using some weight modular inequalities, in [5-6], [12-15], [18-21], [24-25] are established a lot of geometric properties in this classes of functions. If $f : D \to D'$ is a ring homeomorphism, or if $f : D \to \mathbf{R}^n$

is a map of finite length distortion, recent results concerning equicontinuity and boundary extension are established in [16-17], [19], [26-29]. In [7] and [8] we give further extensions of this type using a generalized Poleckii's modular inequality of type "$M_n(f(\Gamma)) \leq M^n_{K_I(f)}(\Gamma)$ for every path family Γ from D" which is established in the new introduced class of mappings.

On the other side, in [9] and Chapter 12 from [19] are established for some classes of homeomorphisms $f : D \to D^{'}$ (called homeomorphisms with finite mean dilatations), which are not necessarily quasiregular, modular inequalities of type "$M_q(f(\Gamma)) \leq \gamma(M_p(\Gamma))$ for every path family Γ from D, some $1 < q < p$ and a continuous increasing function $\gamma : [0.\infty) \to [0, \infty)$ with $\lim\limits_{t\to 0} \gamma(t) = 0$, namely $\gamma(t) = Kt^{q/p}$ for $t \geq 0$". This thing raises the question if for such mappings (or even for more general mappings) are valid some of the basic properties of quasiregular mappings. Anyway, for $p = q = n$ we remain in the class of quasiconformal mappings since in [4] it is proved that if $f : D \to D^{'}$ is a homeomorphism between two domains from $\mathbf{R}^n$ so that there exists a continuous, increasing function $\gamma : [0, \infty) \to [0, \infty)$ with $\lim\limits_{t\to 0} \gamma(t) = 0$ and so that $M_n(f(\Gamma)) \leq \gamma(M_n(\Gamma))$ for every path family Γ from D, then it results that f is quasiconformal.

We shall show that the answer is positive, especially ini the case $n = 2$. We show that if $D \subset \mathbf{R}^n$ is a domain, $n - 1 < q < p \leq n$ and $f : D \to \mathbf{R}^n$ is continuous, open, discrete so that there exists $\gamma : [0, \infty) \to [0, \infty)$ continuous, increasing with $\lim\limits_{t\to 0} \gamma(t) = 0$ and so that $M_q(f(\Gamma)) \leq \gamma(M_p(\Gamma))$ for every path family Γ from $\mathbf{R}^n$, then a lot of the classical results from the geometric theory of quasiregular mappings remain valid in this class of mappings.

In fact we establish more general results and we show that for some mappings satisfying modular inequalities, which are not necessary quasiregular and which are not a priori in some of the above mentioned classes of mappings studied in [5-9], [11-21], [24-29], we can prove equicontinuity, eliminability results, Picard, Montel theorems and we give estimates of the modulus of continuity.

We denote by μ_n the Lebesgue measure from $\mathbf{R}^n$, by V_n the volume of the unit ball from $\mathbf{R}^n$ and by ω_{n-1} the area of the unit sphere from $\mathbf{R}^n$. If $a, b \in \overline{\mathbf{R}}^n$, we denote by $q(a, b)$ the chordal distance between a and b and $q(a, b) = |a - b|/(1 + |a|^2)^{\frac{1}{2}} \cdot (1 + |b|^2)^{\frac{1}{2}}$ if $a, b \in \mathbf{R}^n$, $q(a, \infty) = 1/(1 + |a|^2)^{\frac{1}{2}}$ if $a \in \mathbf{R}^n$ and if $A \subset \overline{\mathbf{R}}^n$, we set $q(A)$ the diameter of A considering the chordal metric on $\overline{\mathbf{R}}^n$.

If E, F are Hausdorff spaces and $f : E \to F$ is a map, we say that f is open if f-carries open sets into open sets and we say that f is discrete if $f^{-1}(y)$ is discrete or empty for every $y \in F$. If $p : [0, 1] \to F$ is a path and $x \in E$ is so that $f(x) = p(0)$, we say that $q : [0, 1] \to E$ is a lifting of p from x if q is a path, $q(0) = x$, $f \circ q = p$, and we say that $q : [0, a) \to E$ is a maximal lifting of p from the point $x \in E$ so that $f(x) = p(0)$ if $q(0) = x$, q is a path, $0 < a \leq 1$, $f \circ q = p|[0, a)$ and a is maximal with this property. If $D \subset \mathbf{R}^n$ is a domain, $f : D \to \mathbf{R}^n$ is continuous, open, discrete, $x \in D$, $p : [0, 1] \to f(D)$ is a path so that $f(x) = p(0)$, there exists always a maximal lifting of p from x.

If $D \subset \mathbf{R}^n$ is a domain, $b \in \partial D$ and $f : D \to \mathbf{R}^n$ is a map, we put $\mathbf{C}(f, b) = \{z \in \overline{\mathbf{R}}^n |$ there exists $b_p \in D, b_p \to b$ so that $f(b_p) \to z\}$ and if $A \subset D$, $y \in \mathbf{R}^n$, we put $N(y, f, A) = Card f^{-1}(y) \bigcap A$ and $N(f, A) = \sup\limits_{y\in\mathbf{R}^n} N(y, f, A)$. Also, if $x \in D$, we set $L(x, f) = \limsup\limits_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|}$.

If $D \subset \mathbf{R}^n$ is a domain, $E, F \subset \overline{D}$, we denote by $\Delta(E, F, D) = \{\gamma : [a, b] \to \overline{D}$ path $|\gamma(a) \in E, \gamma(b) \in F$ and $\gamma((a, b)) \subset D\}$ and if $x \in \overline{D}$, $0 < a < b$, we denote by $\Gamma_{x,a,b,D} = \Delta(\overline{B}(x, a) \bigcap D, S(x, b) \bigcap D, (B(x, b) \setminus \overline{B}(x, a)) \bigcap D)$ and we set $\Gamma_{x,a,b} = \Gamma_{x,a,b,\mathbf{R}^n}$. If $D \subset \mathbf{R}^n$ is a domain and $f : D \to \mathbf{R}$ is a map so that $f \in L^1_{loc}(D)$, we set $\fint_A f(x)dx = \int_A f(x)dx/\mu_n(A)$ for every $A \subset D$ bounded.

If $p > 1$, we denote by $W^{1,p}_{loc}(D, \mathbf{R}^m)$ the Sobolev space of all functions $f : D \to \mathbf{R}^m$ which are locally in L^p together with their first order weak derivatives. We say that f is ACL if f is continuous and for every cube $Q \subset D$ with the sides parallel to coordinate axes and every face S of Q it results that $f|P_S^{-1}(y) \bigcap Q : P_S^{-1}(y) \bigcap Q \to \mathbf{R}^m$ is absolutely continuous for a.e. $y \in S$, where $P_S : \mathbf{R}^n \to S$ is the projection on S. An ACL map has a.e. partial derivatives and if $p \geq 1$ we say that f is ACL^p if f is ACL and the partial derivatives are locally in L^p. We see from Prop. 1.2, page 66 from [23] that if $p \geq 1$ and $f \in C(D, \mathbf{R}^m)$, then f is ACL^p if and only if $f \in W^{1,p}_{loc}(D, \mathbf{R}^m)$.

If $D \subset \mathbf{R}^n$ is a domain, $f : D \to \mathbf{R}^n$ is continuous, open, discrete, $x \in D$ and $r > 0$ is so that $\overline{B}(x, r) \subset D$, we set $L(x, f, r) = \sup_{|y-x|=r} |f(y) - f(x)|$, $l(x, f, r) = \inf_{|y-x|=r} |f(y) - f(x)|$, $H(x, f) = \limsup_{r\to 0} \frac{L(x,f,r)}{l(x,f,r)}$, and we know that if f is differentiable in x and $J_f(x) \neq 0$,, then $H(x, f) = H(f^{`}(x)) = \frac{|f^{`}(x)|}{l(f^{`}(x))}$. It is known that a homeomorphism $f : D \to D^{`}$ between two domains from $\mathbf{R}^n$ is quasiconformal if and only if there exists $H \geq 1$ so that $H(x, f) \leq H$ for every $x \in D$.

If X, Y are metric spaces and W is a family of mappings $f : X \to Y$, we say that the family W is equicontinuous at a point $x \in D$ if for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ so that $d(f(y), f(x)) \leq \epsilon$ if $d(x, y) \leq \delta_\epsilon$ for every $f \in W$, and we say that the family W is equicontinuous if it is equicontinuous at every point $x \in X$. If $D \subset \mathbf{R}^n$ is a domain and W is a family of mappings $f : D \to \mathbf{R}^n$, we say that the family W is bounded if for every $K \subset D$ compact there exists $M(K) > 0$ so that $|f(z)| \leq M(K)$ for every $z \in K$ and every $f \in W$.

We say that $\Phi : [0, \infty) \to [0, \infty)$ is a Young function if there exists $\varphi : [0, \infty) \to [0, \infty)$ continuous, increasing so that there exists $K > 0$ so that $\varphi(2t) \leq K\varphi(t)$ for every $t \geq 0$, $\Phi(t) = \int_0^t \varphi(s)ds$ for $t \geq 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

Letting $B_\Phi = \{p > 0| \frac{\Phi(t)}{t^p}$ is increasing$\}$ and $C_\Phi = \{p > 0| \frac{\Phi(t)}{t^p}$ is deacreasing$\}$, we see that $B_\Phi \neq \phi$, $C_\Phi \neq \phi$ and if $p(\Phi) = \sup B_\Phi$, $q(\Phi) = \inf C_\Phi$, then $1 \leq p(\Phi) \leq q(\Phi) < \infty$. We also have that $\Phi(1)\lambda^{p(\Phi)} \leq \Phi(\lambda) \leq \Phi(1)\lambda^{q(\Phi)}$ if $\lambda \geq 1$, $\Phi(1)\lambda^{q(\Phi)} \leq \Phi(\lambda) \leq \Phi(1)\lambda^{p(\Phi)}$ if $\lambda < 1$, and there exists $C > 0$ so that $\Phi(\lambda) \leq C\lambda^{q(\Phi)}$ for every $\lambda \geq 0$ and $\Phi(\lambda t) \leq \Phi(t) max\{\lambda^{p(\Phi)}, \lambda^{q(\Phi)}\}$ for $\lambda, t > 0$.

2 The M_N modulus.

We present the modulus used in this paper in the general setting of metric measure spaces, although we shall use it only on euclidian spaces.

We say that (X, μ, d) is a metric measure space if the Borel sets are measurable, μ is a regular measure and d is a metric on X. If $\gamma : [a, b] \to X$ is a path and $\Delta = (a = t_0 < t_1 <, ..., < t_n = b) \in \mathcal{D}([a, b])$, we set $V_\Delta(\gamma) = \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))$ and if there exists $M > 0$ such that $V_\Delta(\gamma) \leq M$ for every $\Delta \in \mathcal{D}([a, b])$, we say that γ is rectifiable and we put $l(\gamma) = \sup_{\Delta\in\mathcal{D}([a,b])} V_\Delta(\gamma)$. If $\gamma : [a.b] \to X$ is rectifiable, we set $s_\gamma(t) = l(\gamma|[a, t])$ for $t \in [a, b]$ and we define a reparametrisation $\gamma^\circ : [0, l(\gamma)] \to X$ of the path γ given by $\gamma(t) = \gamma^\circ(s_\gamma(t))$ for $t \in [a, b]$.

If $\gamma : [a,b] \to X$ is rectifiable and $\rho : X \to [0,\infty]$ is a Borel map, we set $\int\limits_{\gamma} \rho ds = \int\limits_{0}^{l(\gamma)} \rho(\gamma^{\circ}(t))dt$ and if $\gamma : [a,b] \to X$ is locally rectifiable, we set $\int\limits_{\gamma} \rho ds = \sup \int\limits_{\alpha} \rho ds$, where the supremum is taken over all closed subpaths α for γ. We set $\mathcal{M}(X) = \{u : X \to [0,\infty] | u \text{ is measurable}\}$ and we set $\mathcal{A}(X)$ the set of all path families Γ from X. If $\Gamma \in \mathcal{A}(X)$, we set $F(\Gamma) = \{\rho : X \to [0,\infty]$ Borel maps $| \int\limits_{\gamma} \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$.

We say that $M : \mathcal{A}(X) \to [0,\infty]$ is a modulus on X if

a) $M(\Phi) = 0$.

b) If $\Gamma_1 \subset \Gamma_2$, $\Gamma_1, \Gamma_2 \in \mathcal{A}(X)$, then $M(\Gamma_1) \leq M(\Gamma_2)$.

c) If $\Gamma_1, ..., \Gamma_n, ...,$ are from $\mathcal{A}(X)$, then $M(\bigcup\limits_{n=1}^{\infty} \Gamma_n) \leq \sum\limits_{n=1}^{\infty} M(\Gamma_n)$.

Theorem 1. Let (X,μ,d) be a metric measure space, $p : X \to [0,\infty]$ measurable and finite a.e., $\omega : X \to [0,\infty]$ measurable and finite a.e., $\omega > 0$ a.e., $\Phi : [0,\infty) \to [0,\infty)$ a homeomorphism, $\Psi : [0,\infty) \times [1,\infty) \to [0,\infty)$ a Borel map such that all the maps $\Psi_s : [0,\infty) \to [0,\infty)$ given by $\Psi_s(t) = \Psi(t,s)$ for $t \geq 0$, $s \geq 1$ are homeomorphisms for every fixed $s \geq 1$ and let $N : \mathcal{M}(X) \to [0,\infty]$ be defined by $N(u) = \int\limits_X \omega(x)\psi_{p(x)}(\Phi(|u|(x)))dx$ for $u \in \mathcal{M}(X)$. Let $M_N : \mathcal{A}(X) \to [0,\infty]$ be given by $M_N(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} N(\rho)$ for every $\Gamma \in \mathcal{A}(X)$. Then M_N is a modulus on X.

Proof: Using Lusin's theorem, we can find a Borel function $q : X \to [0,\infty]$ so that $q = p$ a.e. in X. We have to show that if $\Gamma_1, ..., \Gamma_n, ...,$ are from $\mathcal{A}(X)$ and $\Gamma = \bigcup\limits_{n=1}^{\infty} \Gamma_n$, then $M_N(\Gamma) \leq \sum\limits_{n=1}^{\infty} M_N(\Gamma_n)$. We can suppose that $\sum\limits_{n=1}^{\infty} M_N(\Gamma_n) < \infty$ and let $\epsilon > 0$. We can find $\rho_n \in F(\Gamma_n)$ so that $N(\rho_n) \leq M_N(\Gamma_n) + \frac{\epsilon}{2^{n+1}}$ for every $n \in N$. Let $\rho : X \to [0,\infty]$ be given by $\rho(x) = (\Psi_{q(x)} \circ \Phi)^{-1}(\sum\limits_{n=1}^{\infty} (\Psi_{q(x)} \circ \Phi)(\rho_n(x)))$ for $x \in X$. Then ρ is a Borel map and let $i \in N$ be fixed. We see that $(\Psi_{q(x)} \circ \Phi)(\rho(x)) = (\Psi_{q(x)} \circ \Phi)(\Psi_{q(x)} \circ \Phi)^{-1}(\sum\limits_{n=1}^{\infty} (\Psi_{q(x)} \circ \Phi)(\rho_n(x))) = \sum\limits_{n=1}^{\infty} (\Psi_{q(x)} \circ \Phi)(\rho_n(x)) \geq (\Psi_{q(x)} \circ \Phi)(\rho_i(x))$ for every $x \in X$. Since the maps $\Psi_{q(x)}$ and Φ are increasing for every $x \in X$, we see that $\rho(x) \geq \rho_i(x)$ for every $x \in X$.

We proved that $\rho \geq \rho_i$ for $i \in N$ and this shows that $\rho \in F(\Gamma)$. Then $M_N(\Gamma) \leq N(\rho) = \int\limits_X \omega(x)(\Psi_{p(x)} \circ \Phi)(\rho(x))dx = \int\limits_X \omega(x)(\Psi_{q(x)} \circ \Phi)(\rho(x))dx = \int\limits_X \omega(x)(\Psi_{q(x)} \circ \Phi)(\Psi_{q(x)} \circ \Phi)^{-1}(\sum\limits_{n=1}^{\infty} (\Psi_{q(x)} \circ \Phi)(\rho_n(x)))dx = \int\limits_X \omega(x) \sum\limits_{n=1}^{\infty} (\Psi_{q(x)} \circ \Phi)(\rho_n(x))dx = \sum\limits_{n=1}^{\infty} \int\limits_X \omega(x)(\Psi_{q(x)} \circ \Phi)(\rho_n(x))dx = \sum\limits_{n=1}^{\infty} \int\limits_X \omega(x)\ (\Psi_{p(x)} \circ \Phi)(\rho_n(x))dx = \sum\limits_{n=1}^{\infty} N(\rho_n) \leq \sum\limits_{n=1}^{\infty} (M_N(\Gamma_n) + \frac{\epsilon}{2^{n+1}}) =$
$= \sum\limits_{n=1}^{\infty} M_N(\Gamma_n) + \epsilon.$

Letting $\epsilon \to 0$, we find that $M_N(\Gamma) \leq \sum\limits_{n=1}^{\infty} M_N(\Gamma_n)$.

If $X = \mathbf{R}^n$ and $\omega = 1$, $\Psi(t,s) = t$, $\Phi(t) = t^p$ for $t \geq 0$, $s \geq 1$ and some $p > 1$, we obtain the usual p modulus M_p given by $M_p(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{\mathbf{R}^n} \rho^p(x)dx$ for every $\Gamma \in \mathcal{A}(X)$. If $X = \mathbf{R}^n$, $\Psi(t,s) = t$, $\Phi(t) = t^p$ for $t \geq 0$, $s \geq 1$ and some $p > 1$, we obtain the usual p modulus of weight

ω, $M^p_\omega(\Gamma) = \inf_{\rho\in F(\Gamma)} \int_{\mathbf{R}^n} \omega(x)\rho^p(x)dx$ for $\Gamma \in \mathcal{A}(X)$, which is essentially due to Cabiria Andreian (see for instance [2]).

If $\omega = 1$, $\Phi(t) = t$, $\Psi(t,s) = t^s$ for $t \geq 0$, $s \geq 1$, then $N(u) = \int_{\mathbf{R}^n} |u(x)|^{p(x)}dx$ for $u \in \mathcal{M}(\mathbf{R}^n)$ and for the modulus M_N given by $M_N(\Gamma) = \inf_{\rho\in F(\Gamma)} \int_{\mathbf{R}^n} |\rho(x)|^{p(x)}dx$ for $\Gamma \in \mathcal{A}(\mathbf{R}^n)$ we can use a result from [10] to prove a Fuglede type theorem.

An important particular case we shall have in mind is obtained for $\omega = 1$, $\Psi(t,s) = t^s$ for $t \geq 0$, $s \geq 1$, for which $N(u) = \int_{\mathbf{R}^n} \Phi(|u|(x))^{p(x)}dx$ for $u \in \mathcal{M}(\mathbf{R}^n)$ and for which we have the modulus M_N given by $M_N(\Gamma) = \inf_{\rho\in F(\Gamma)} \int_{\mathbf{R}^n} \Phi(\rho(x))^{p(x)}dx$ for $\Gamma \in \mathcal{A}(\mathbf{R}^n)$.

3 Estimates of the modulus M_N.

From now on the space X will be a domain D from $\mathbf{R}^n$, μ will be the Lebesgue measure from $\mathbf{R}^n$ and d the euclidian metric on D. We shall work with a particular operator $N : \mathcal{M}(D) \to [0,\infty]$ of type $N(u) = \int_D \Psi_{p(x)}(\Phi(|u|(x)))dx$ for $u \in \mathcal{M}(D)$. The map $\omega : D \to [0,\infty]$ is measurable and finite a.e., $\omega > 0$ a.e., $p : D \to [1,\infty]$ is measurable and finite a.e., $\Phi : [0,\infty) \to [0,\infty)$ is a Young function and $\Psi : [0,\infty) \times [1,\infty) \to [0,\infty)$ is a Borel map so that all the maps $\Psi_c : [0,\infty) \to [0,\infty)$ given by $\Psi_c(t) = \Psi(t,c)$ for $t \geq 0$, $c \geq 1$ are Young functions for every fixed $c \geq 1$ and there exists $q,r,s > 0$ so that $q \leq \Psi_{p(x)}(1)$, $1 \leq r \leq p(\Psi_{p(x)}) \leq q(\Psi_{p(x)}) \leq s < \infty$ for every $x \in D$. We shall always keep in mind that the operator N is defined by the functions ω, p, Φ, Ψ and the constants q,r,s. We also denote by $g : [0,\infty) \to [0,\infty)$ the function given by $g(t) = t^{s/r}$ if $0 \leq t < 1$, $g(t) = t$ for $t \geq 1$.

Theorem 2. Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $x \in \overline{D}$, $0 < a < b$, $N : \mathcal{M}(D) \to [0,\infty]$ given by $N(u) = \int_D \Psi_{p(z)}(\Phi(|u|(z)))dz$ for $u \in \mathcal{M}(D)$, let $C,m > 0$ so that $\Phi(\lambda t) \leq C\lambda^m\Phi(t)$ for every $\lambda, t > 0$, let $B_k = B(x, be^{-k})$ for $k \geq 0$ and let $A = \sum_{k=0}^{\infty} \int_{B_k\cap D} \omega(z)\Psi_{p(z)}(\Phi(e^{k+1}/b(k+1)))dz$. Then there exists $\alpha \in \{r,s\}$ so that $M_N(\Gamma_{x,a,b,D}) \leq AC^\alpha/(\ln\ln(be/a))^{m\alpha}$ and $\alpha = s$ if $1 \leq C/(\ln\ln(be/a))^m$ and $\alpha = r$ if $C/(\ln\ln(be/a))^m < 1$.

Proof: Let $t_k = be^{-k}$, $A_k = B_k \setminus B_{k+1}$ for $k \geq 0$ and let $l \in N$ be so that $t_{l+1} \leq a < t_l$. Then $A_k \subset B_k$, $k+1 \leq \ln(be/|z-x|)$ for $z \in B_k$, $be^{-(k+1)} \leq |z-x|$ for $z \in A_k$, $k \geq 0$, hence $1/(|z-x|\ln(be/|z-x|)) \leq e^{k+1}/b(k+1)$ for $z \in A_k$, $k \geq 0$. Let $\rho : \mathbf{R}^n \to [0,\infty)$ be defined by $\rho(z) = (1/\ln\ln(be/a))(1/(|z-x|\ln(be/|z-x|)))$ for $z \in B(x,b) \setminus \overline{B}(x,a)$, $\rho(z) = 0$ otherwise. We see that $\rho X_D \in F(\Gamma_{x,a,b,D})$ and let $\alpha(z) = q(\Psi_{p(z)})$ if $1 \leq C/(\ln\ln(be/a))^m$, $\alpha(z) = p(\Psi_{p(z)})$ if $C/(\ln\ln(be/a))^m < 1$ for $z \in D$. Then

$$M_N(\Gamma_{x,a,b,D}) \leq N(\rho X_D) = \int_D \omega(z)\Psi_{p(z)}(\Phi(\rho(z)))dz \leq$$

$$\int_{(B(x,b)\setminus\overline{B}(x,a))\cap D} \omega(z)\Psi_{p(z)}(\Phi((1/\ln\ln(be/a)))(1/(|z-x|\ln(be/|z-x|)))dz \leq$$

$$\sum_{k=0}^{l} \int_{A_k\cap D} \omega(z)\Psi_{p(z)}((C/(\ln\ln(be/a))^m)\Phi(1/(|z-x|\ln(be/|z-x|)))dz \leq$$

$$\sum_{k=0}^{l} \int_{B_k\cap D} \omega(z)(C/(\ln\ln(be/a))^m)^{\alpha(z)}\Psi_{p(z)}(\Phi(e^{k+1}/b(k+1)))dz \leq AC^\alpha/(\ln\ln(be/a))^{m\alpha}.$$

The theorem is now proved.

Remark 1. Suppose that in the preceding theorem $\Psi(t,c) = t^c$ for $t \geq 0$, $c \geq 1$. Then $p(\Psi_c) = q(\Psi_c) = c$ for $c \geq 1$ and in this case we see that $A = \sum_{k=0}^{\infty} \int_{B_k\cap D} \omega(z)\Phi^{p(z)}(e^{k+1}/b(k+1))dz$, $N(u) = \int_D \omega(z)\Phi(|u|(z))^{p(z)}dz$ for $u \in \mathcal{M}(D)$ and we have the inequality $M_N(\Gamma_{x,a,b,D}) \leq AC^\alpha/(\ln\ln(be/a))^{m\alpha}$ with $\alpha \in \{r,s\}$ and $1 \leq r \leq p(z) \leq s < \infty$ for $z \in D$.

If in addition $p(z) = p$ for $z \in D$, then $A = \sum_{k=0}^{\infty} \int_{B_k\cap D} \omega(z)\Phi^p(e^{k+1}/b(k+1))dz$, $N(u) = \int_D \omega(z)\Phi^p(|u|(z))dz$ for $u \in \mathcal{M}(D)$ and $M_N(\Gamma_{x,a,b,D}) \leq AC^p/(\ln\ln(be/a))^{mp}$.

An important particular case is obtained when $p(z) = 1$ for $z \in D$, $\Psi(t,c) = t$, $\Phi(t) = t^p$ for $t \geq 0$, $c \geq 1$. Then $N(u) = \int_D \omega(z)|u|(z)^p dz$ if $u \in \mathcal{M}(D)$, $A = \sum_{k=0}^{\infty} \int_{B_k\cap D} \omega(z)(e^{k+1}/b(k+1))^p dz$ and we have the inequality $M_N(\Gamma_{x,a,b,D}) \leq A/(\ln\ln(be/a))^p$.

We obtain in this way some modular estimates from [18] and [7].

Theorem 3. Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $x \in \overline{D}$, $0 < a < b$, $m > 1$, $\alpha, l > 0$, $\omega : D \to [0,\infty]$ measurable and finite a.e., $\omega > 0$ a.e. so that $\int_{B(x,\delta)\cap D} \omega(z)dz \leq M(b)\delta^l(\ln(be/\delta))^\alpha$ for $0 < \delta < b$ and suppose that either $m < l$, or $l = m$ and $0 \leq \alpha < m-1$, and let $C(b) = M(b)b^{l-m}e^m \sum_{k=0}^{\infty} e^{k(m-l)}\frac{1}{(k+1)^{m-\alpha}}$. Then $C(b) < \infty$ and if $N : \mathcal{M}(D) \to [0,\infty]$ is defined by $N(u) = \int_D \omega(z)|u|^m(z)dz$ for $u \in \mathcal{M}(D)$, we have that $M_N(\Gamma_{x,a,b,D}) \leq C(b)/(\ln\ln(be/a))^m$.

Proof: We see that $\int_{B(x,be^{-k})} \omega(z)dz \leq M(b)b^l e^{-kl}(\ln(be/be^{-k}))^\alpha \leq M(b)b^l e^{-kl}(k+1)^\alpha$ for every $k \geq 0$. Then $A = \sum_{k=0}^{\infty} \int_{B(x,be^{-k})\cap D} \omega(z)(e^{k+1}/b(k+1))^m dz \leq M(b)b^{l-m}e^m \sum_{k=0}^{\infty} e^{k(m-l)}\frac{1}{(k+1)^{m-\alpha}} < \infty$ if $m < l$ or if $m = l$ and $0 \leq \alpha < m-1$. Taking $C(b) = A$, we see that $M_N(\Gamma_{x,a,b,D}) \leq C(b)/(\ln\ln(be/a))^m$.

Another important case is obtained for $\omega = 1$, $\Psi(t,c) = t^c$, $\Phi(t) = t$ for $t \geq 0$, $c \geq 1$.

Corollary 1. Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $x \in \overline{D}$, $0 < a < b < e$, $p : D \to [1,\infty]$ measurable and finite a.e. so that there exists $r, s \geq 1$ so that $1 \leq r \leq p(z) \leq s \leq n$ for every $z \in D$ and let $C(b) = V_n e^s b^{n-s} \sum_{k=0}^{\infty} e^{-k(n-s)}\frac{1}{(k+1)^s}$. Then $C(b) < \infty$ and if $N : \mathcal{M}(D) \to [0,\infty]$ is defined by $N(u) = \int_D |u(z)|^{p(z)}dz$ for $u \in \mathcal{M}(D)$, it results that $M_N(\Gamma_{x,a,b,D}) \leq C(b)/(\ln\ln(be/a))^\alpha$, where $\alpha \in \{r,s\}$, and we can take $\alpha = s$ if $b \leq e^{e-1}a$ and we can take $\alpha = r$ if $e^{e-1}a < b$.

Proof: We see that $e^{k+1}/b(k+1) = \frac{e}{b}\frac{e^k}{(k+1)} \geq 1$ for $k \geq 0$, hence $A = \sum_{k=0}^{\infty} \int_{B(x,be^{-k})\cap D} (e^{k+1}/b(k+1))^{p(z)}dz \leq \sum_{k=0}^{\infty} \int_{B(x,be^{-k})\cap D} (e^{k+1}/b(k+1))^s dz \leq V_n e^s b^{n-s} \sum_{k=0}^{\infty} e^{-k(n-s)}\frac{1}{(k+1)^s} < \infty$. We take now $C(b) = A$ and we apply Theorem 2.

Remark 2. We showed in the preceding theorems that $M_N(\Gamma_{x,a,b,D}) \leq C(b)/(\ln\ln(be/a))^m$ for some fixed $m \geq 1$, some fixed $b > 0$ and every $0 < a < b$. This implies the important fact that $\lim_{a\to 0} M_N(\Gamma_{x,a,b,D}) = 0$. In particular, $\lim_{a\to 0} M_p(\Gamma_{x,a,b,D}) = 0$ for $b > 0$ fixed and $p \le n$ and it can be shown that $M_p(\Gamma_{x,a,b,D}) \nrightarrow 0$ if $a \to 0$ and $p > n$.

Following the ideas from [19], Chapter 6, we can find some other cases when $\lim_{a\to 0} M^p_\omega(\Gamma_{x,a,b,D}) = 0$, for some weight ω and $p \le n$.

Theorem 4. Let $n \ge 2$, $D \subset \mathbf{R}^n$ a domain, $x \in \overline{D}$, $0 < b$, $\omega : D \to [0,\infty]$ measurable and finite a.e., $\eta : (0,b) \to [0,\infty]$ measurable and finite a.e. so that $\int_0^b \eta(t)dt = \infty$ and $I(a) = \int_a^b \eta(t)dt < \infty$ for every $0 < a < b$, let $A_{x,a,b,D} = \{z \in D | a < |z-x| < b\}$ for $0 < a < b$ and suppose that $\lim_{a\to 0} \int_{A_{x,a,b,D}} \omega(z)\eta(|z-x|)^p dz/I(a)^p = 0$. Then $\lim_{a\to 0} M^p_\omega(\Gamma_{x,a,b,D}) = 0$.

Proof. Let $\beta : (0,b) \to [0,\infty]$ be a Borel map so that $\beta = \eta$ a.e. and let $0 < a < b$. Let $\rho_a : \mathbf{R}^n \to [0,\infty]$ be defined by $\rho_a(z) = \beta(|z-x|)/I(a)$ for $z \in A_{x,a,b,D}$, $\rho_a(z) = 0$ otherwise. Then $\rho_a \in F(\Gamma_{x,a,b,D})$ and $M^p_\omega(\Gamma_{x,a,b,D}) \le \int_{\mathbf{R}^n} \omega(z)\rho_a(z)^p dz = \int_{A_{x,a,b,D}} \omega(z)\eta(|z-x|)^p dz/I(a)^p \to 0$ if $a \to 0$.

As in [19], Chapter 6, we can take $\eta(t) = \frac{1}{t}$ for $t \in (0,b)$ and then $I(a) = \int_a^b \frac{dt}{t} = \ln(b/a)$ for $0 < a < b$, and we can take $\eta(t) = \frac{1}{t\ln t}$ for $t \in (0,b)$ and then $I(a) = \int_a^b \frac{dt}{t\ln t} = \ln\ln(b/a)$ for $0 < a < b$. If we let $\tilde{\omega} : \mathbf{R}^n \to [0,\infty]$, $\tilde{\omega}(z) = \omega(z)$ if $z \in D$, $\tilde{\omega}(z) = 0$ otherwise and we set $w_x(t) = \fint_{S(x,t)} \tilde{\omega}(z) d_{S(x,t)}$ for $0 < t < b$, then $J(a) = \int_{A_{x,a,b,D}} \omega(z)\eta(|z-x|)^p dz = \omega_{n-1} \int_a^b \omega_x(t)t^{n-1}\eta^p(t)dt$ for $a < t < b$.

We find now some lower bounds for the modulus $M_N(\Delta(E,F,D))$ in some particular cases.

Lemma 1. Let $\rho : \mathbf{R}^n \to [0,\infty]$ be a Borel map, let $D \subset \mathbf{R}^n$ be a bounded domain, let the functions p, Ψ, Φ and the constants q, r, s be as in the definition of the operator N and let $d > 0$ be so that $d \le \fint_D \rho(z)dz$. Then $N(p) \ge \min\{q, q2^{(r-s)/r}\Phi^s(d)g(\mu_n(D))\}$.

Proof: Let $M = \min\{1, \mu_n(D)^{(r-s)/r}\}$, $D_1 = \{x \in D | \Phi(\rho(x)) \ge 1\}$, $D_2 = \{x \in D | \Phi(\rho(x)) < 1\}$. Then $\Phi^r(d) \le \Phi^r(\fint_D \rho(z)dz) \le \fint_D \Phi^r(\rho(z))dz$, hence $\int_D \Phi^r(\rho(z))dz \ge \Phi^r(d)\mu_n(D)$.

Using Hölder's inequality, we have $\int_{D_2} \Phi^s(\rho(z))dz \ge (\int_{D_2} \Phi^r(\rho(z))dz)^{s/r}\mu_n(D)^{(r-s)/r}$. We see that $N(\rho) = \int_D \Psi_{p(z)}(\Phi(\rho(z)))dz = \int_{D_1} \Psi_{p(z)}(\Phi(\rho(z)))dz + \int_{D_2} \Psi_{p(z)}(\Phi(\rho(z)))dz \ge \int_{D_1} \psi_{p(z)}(1)$ $\Phi(\rho(z))^{p(\Psi_{p(z)})}dz + \int_{D_2} \Psi_{p(z)}(1)\Phi(\rho(z))^{q(\Psi_{p(z)})}dz \ge q(\int_{D_1} \Phi^r(\rho(z))dz + \int_{D_2} \Phi^s(\rho(z))dz)$.

It results that $N(\rho) \ge q$ if $\int_{D_1} \Phi^r(\rho(z))dz \ge 1$.

If $\int_{D_1} \Phi^r(\rho(z))dz < 1$, then $N(\rho) \ge q((\int_{D_1} \Phi^r(\rho(z))dz)^{s/r} + (\int_{D_2} \Phi^r(\rho(z))dz)^{s/r}\mu_n(D)^{(r-s)/r}) \ge qM((\int_{D_1} \Phi^r(\rho(z))dz)^{s/r} + (\int_{D_2} \Phi^r(\rho(z))dz)^{s/r}) \ge qM2^{(r-s)/r}(\int_{D_1} \Phi^r(\rho(z))dz + \int_{D_2} \Phi^r(\rho(z))dz)^{s/r} = qM2^{(r-s)/r}(\int_D \Phi^r(\rho(z))dz)^{s/r} \ge qM2^{(r-s)/r}\Phi^s(d)\mu_n(D)^{s/r} = q2^{(r-s)/r}\Phi^s(d)g(\mu_n(D))$.

Remark 3. If $\Psi(t,c) = t^c$ for $t \ge 0$, $c \ge 1$, then $\Psi_c(1) = 1$ for $c \ge 1$ and if also $p = 1$, then $N(\rho) = \int_D \Phi(\rho(z))dz \ge \Phi(d)\mu_n(D)$.

Theorem 5. Let $n \geq 2$, $x \in \mathbf{R}^n$, $0 < a < b$, $D = B(x,b) \setminus \overline{B}(x,a)$ and let $N : \mathcal{M}(D) \to [0,\infty]$, $N(u) = \int\limits_D \Psi_{p(z)}(\Phi(|u|(z)))dz$ for $u \in \mathcal{M}(D)$. Then $M_N(\Gamma_{x,a,b,D}) \geq \min\{q, q2^{(r-s)/r} \cdot\Phi^s(\frac{a^{n-1}\omega_{n-1}}{V_n(b^n-a^n)})g(V_n(b^n-a^n))\}$.

Proof: Let $Q = [0,\pi]^{n-2} \times [0,2\pi]$ and let $\theta : (0,\infty) \times Q \to \mathbf{R}^n$ be the polar coordinates. We know that if $t > 0$ and $f : S(0,t) \to \mathbf{R}$ is continuous, then $\int\limits_{S(0,t)} f(z)d_{S(0,t)} = \int\limits_Q f(\theta(t,y))J_\theta(t,y)dy$. We can suppose that $x = 0$ and let $\rho \in F(\Gamma_{x,a,b,D})$. We define $f : S(0,1) \to \mathbf{R}$ by $f(y) = \int\limits_a^b t^{n-1}\rho(ty)dt$ for $y \in S(0,1)$and let $\gamma_y : [0,1] \to \mathbf{R}^n$ be defined by $\gamma_y(t) = ty$ for $t \in [a,b]$ and $y \in S(0,1)$. Then $\gamma_y^\circ = \gamma_y$ for every $y \in S(0,1)$ and $1 \leq \int\limits_{\gamma_y} \rho ds = \int\limits_a^b \rho(ty)dt$, hence $a^{n-1} \leq \int\limits_a^b t^{n-1}\rho(ty)dt = f(y)$ for every $y \in S(0,1)$.

Integrating over $y \in S(0,1)$, we have $a^{n-1}\omega_{n-1} \leq \int\limits_{S(0,1)} f(y)dy = \int\limits_Q f(\theta(1,y))J_\theta(1,y)dy = \int\limits_Q(\int\limits_a^b t^{n-1}\rho(\theta(1,y)tdt)J_\theta(1,y)dy = \int\limits_Q\int\limits_a^b \rho(\theta(t,y))J_\theta(t,y)dtdy = \int\limits_D \rho(z)dz$, hence $\frac{a^{n-1}\omega_{n-1}}{V_n(b^n-a^n)} \leq$ $\leq \fint\limits_D \rho(z)dz$.

We use now the preceding theorem to see that $M_N(\Gamma_{x,a,b,D}) \geq \min\{q, q\cdot 2^{(r-s)/r}\Phi^s(\frac{a^{n-1}\omega_{n-1}}{V_n(b^n-a^n)})$ $g(\mu_n(D))\}$.

Remark 4. Using Remark 3, we see that if $\Psi(t,c) = t^c$ for $t \geq 0$, $c \geq 1$ and $p = 1$, then $M_N(\Gamma_{x,a,b}) \geq \Phi(\frac{a^{n-1}\omega_{n-1}}{V_n(b^n-a^n)})V_n(b^n-a^n)$.

Theorem 6. Let $n \geq 2$, H_1 and H_2 be parallel hiperplanes from $\mathbf{R}^n$ with $d(H_1,H_2) = h > 0$, $P : \mathbf{R}^n \to H_1$ be the projection on H_1, let $E_1 \subset H_1$ with $\mu_{n-1}(E_1) > 0$, $E_2 = H_2 \cap P^{-1}(E_1)$, let D be the set of all points from $P^{-1}(E_1)$ between the hiperplanes H_1 and H_2 and let $\Gamma = \Delta(E_1,E_2,D)$. Then, if $N : \mathcal{M}(D) \to [0,\infty]$ is given by $N(u) = \int\limits_D \Psi_{p(z)}(\Phi(|u|(z)))dz$ for $u \in \mathcal{M}(D)$, it results that $M_N(\Gamma) > 0$.

Proof: We can suppose that $E_1 \subset \mathbf{R}^{n-1}$, $E_2 = E_1 + he_n$, $D = \{x \in \mathbf{R}^n | 0 \leq x_n \leq h$ and $(x_1,...,x_{n-1}) \in E_1\}$. Let $\rho \in F(\Gamma)$ and let $\gamma_y : [0,h] \to D$, $\gamma_y(t) = \rho(y + te_n)$ for $t \in [0,h]$, $y \in E_1$. Then $\gamma_y \in \Gamma$ and $1 \leq \int\limits_{\gamma_y} \rho ds = \int\limits_o^h \rho(y+te_n)dt$ for $y \in E_1$. Integrating over $y \in E_1$ we obtain that $\mu_{n-1}(E_1) \leq \int\limits_{E_1}\int\limits_0^h \rho(y+te_n)dtdy = \int\limits_D \rho(z)dz$, hence $\frac{1}{h} \leq \fint\limits_D \rho(z)dz$.

Using Lemma 1 we see that $M_n(\Gamma) \geq \min\{q, q2^{(r-s)/r}\Phi^s(\frac{1}{h})g(\mu_{n-1}(E_1)\cdot h)\} > 0$.

Theorem 7. Let $n \geq 2$, E,F be disjoint sets from $\mathbf{R}^n$ so that E contains a ball $B_1 = B(x_1,r_1)$ and $\mu_n(F) > 0$, and let $\Gamma = \Delta(E,F,\mathbf{R}^n)$. Then $M_N(\Gamma) > 0$.

Proof: Since $\mu_n(F) > 0$, there exists $x_2 \in F \setminus E$ so that $\mu_n(F \cap B(x_2,r)) > 0$ for every $r > 0$. Let d be the line which joins x_1 and x_2, let $r > 0$ be such that $d(x_1,x_2) \geq 6r$, $3r \leq r_1$ and let $D = \{z \in \mathbf{R}^n | d(z,d) < r\}$. We can find hyperplanes H_1 and H_2 perpendicular on d so that $x_1 \in H_1$, $d(x_2,H_2) \leq r$ and $\mu_{n-1}(H_2 \cap F \cap D) > 0$. Let $E_1 = H_1 \cap D$, $E_2 = H_2 \cap D \cap F$. Then $E_1 \subset E$, $E_2 \subset F$, $\mu_{n-1}(E_2) > 0$ and let $\Gamma_1 = \Delta(E_1,E_2,D)$. We see from Theorem 6 that $M_N(\Gamma_1) > 0$ and since $\Gamma_1 \subset \Gamma$, we find that $M_N(\Gamma) \geq M_N(\Gamma_1) > 0$.

Theorem 8. Let E,F be disjoint sets from $\mathbf{R}^n$, $x \in \mathbf{R}^n$, $0 < a < b$, $D = B(x,b) \setminus \overline{B}(x,a)$

so that $S(x,t) \cap E \neq \phi$, $S(x,t) \cap F \neq \phi$ for $a < t < b$, let $\Gamma = \Delta(E,F,D)$ and let $N : \mathcal{M}(D) \to [0,\infty]$ be defined by $N(u) = \int_D \Psi_{p(z)}(\Phi(|u|(z)))dz$ for $u \in \mathcal{M}(D)$. Then, if $n = 2$, it results that $M_N(\Gamma) \geq C(q,r,s,a,b) > 0$ and if $n \geq 2$ and $M_N = M_p$, then $M_N(\Gamma) > C(n,p)(b^{n-p} - a^{n-p})$ if $n-1 < p$ and $p \neq n$ and $M_N(\Gamma) \geq C(n) \ln \frac{b}{a}$ if $p = n$.

Proof: We know from Theorem 10.12, page 31 from [30] that $M_N(\Gamma) \geq C(n) \ln \frac{b}{a}$ and we see from [3] that if $n-1 < p < n$ or if $p > n$, then $M_p(\Gamma) \geq C(n,p)(b^{n-p} - a^{n-p})$.

Suppose now that $n = 2$. We can suppose that $x = 0$ and let $\theta : [0,\pi] \times [0,2\pi] \to \mathbf{R}^2$ be the plane polar coordinates. Let $\rho \in F(\Gamma)$ and $a < t < b$. We can find a path $\gamma_t : [a_t, b_t] \to \mathbf{R}^2$ so that $Im\gamma_t \subset S(0,t)$, $\gamma_t \in \Gamma$, and $\gamma_t^\circ(u) = \theta(t, a_t + \frac{u}{t})$ for $u \in [0, t(b_t - a_t)]$. We have $1 \leq \int_{\gamma_t} \rho ds = \int_0^{l(\gamma_t)} \rho(\gamma_t^\circ(u))du = \int_0^{t(b_t-a_t)} \rho(\theta(t, a_t + \frac{u}{t}))du = t\int_{a_t}^{b_t} \rho(\theta(t,\varphi))d\varphi \leq \int_0^{2\pi} t\rho(\theta(t,\varphi))d\varphi$.

Integrating over $t \in (a,b)$ we obtain that $b - a \leq \int_a^b \int_0^{2\pi} \rho(\theta(t,\varphi))J_\theta(t,\varphi)dtd\varphi = \int_D \rho(z)dz$, hence $1/\pi(a+b) \leq ⨍_D \rho(z)dz$. Using Lemma 1 we see that $M_N(\Gamma) \geq \min\{q, q2^{(r-s)/r}\Phi^s(1/\pi(a+b))g(\pi(b^2-a^2))\}$.

Remark 5. If $\Psi(t,c) = t^c$ for $t \geq 0$, $c \geq 1$ and $p = 1$, then $M_N(\Gamma) \geq \pi(b^2 - a^2)\Phi(1/\pi(a+b))$.

4 Generalized quasiregular mappings.

Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain and $f : D \to \mathbf{R}^n$ be continuous, open, discrete. We say that f is a generalized quasiregular mapping if there exists M_1, M_2 modulus on D and $\gamma : [0,\infty) \to [0,\infty)$ with $\lim_{t \to 0} \gamma(t) = 0$ so that $M_1(f(\Gamma)) \leq \gamma(M_2(\Gamma))$ for every path family Γ from D.

We give now an example of a mapping satisfying a modular inequality:

Proposition 1. Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $1 < q < p$, $f \in ACL^q(D, \mathbf{R}^n)$, f a.e. differentiable so that there exists $K : D \to [0,\infty]$ measurable and finite a.e. so that $|f'(x)|^p \leq K(x)|J_f(x)|$ a.e., $K \in L^{q/(p-q)}(D)$, $N(f,D) < \infty$ and let $C = N(f,D)^{q/p}(\int_D K(x)^{q/(p-q)}dx)^{(p-q)/p}$. Then $M_q(\Gamma) \leq C(M_p(f(\Gamma)))^{q/p}$ for every path family Γ from D.

Proof: Let Γ be a path family from D and let $\rho' \in F(f(\Gamma))$. Let $\rho : \mathbf{R}^n \to [0,\infty]$, $\rho(x) = \rho'(f(x)) \cdot L(x,f)$ if $x \in D$, $\rho(x) = 0$ otherwise and let $\Gamma_0 = \{\gamma \in \Gamma | f \circ \gamma^\circ$ is absolutely continuous$\}$. Using Fuglede's theorem (see [30] Theorem 28.2, page 93), we have $M_q(\Gamma) = M_q(\Gamma_0)$ and from Theorem 3.3, page 93 from [30], we see that $\rho \in F(\Gamma_0)$. Using the change of variable formulae (3) from [11] and Hölder's inequality, we have

$$\int_D \rho^q(x)dx = \int_D \rho'^q(f(x))L(x,f)^q dx = \int_D \rho'^q(f(x))|f'(x)|^q dx \leq \int_D \rho'^q(f(x))K^{q/p}(x)|J_f(x)|^{q/p}dx \leq$$

$$\leq (\int_D \rho'^p(f(x))|J_f(x)|dx)^{q/p}(\int_D K(x)^{q/(p-q)}dx)^{(p-q)/p} \leq$$

$$\leq (\int_{\mathbf{R}^n} N(y,f,D)\rho'^p(y)dy)^{q/p}(\int_D K(x)^{q/(p-q)}dx)^{(p-q)/p} \leq C(\int_{\mathbf{R}^n} \rho'^p(y)dy)^{q/p}.$$

It results that $M_q(\Gamma) = M_q(\Gamma_0) \leq \int\limits_D \rho^q(x)dx \leq C(\int\limits_{\mathbf{R}^n} \rho^{`p}(y)dy)^{q/p}$ and hence that $M_q(\Gamma) \leq C(M_p(f(\Gamma)))^{q/p}$.

Proposition 2. Let $n \geq 2$, $1 < q < p$, $D, D^`$ domains from $\mathbf{R}^n$, $h : D^` \to D$ a homeomorphism, $f = h^{-1}$ so that $f \in ACL^q(D, D^`)$, $f$ is a.e. differentiable and $J_f(x) \neq 0$ a.e., in $D$, $H_{I,q}(h) \in L^{p/(p-q)}(D^`)$ and let $C = (\int\limits_{D^`} H_{I,q}(h)(y)^{p/(p-q)}dy)^{(p-q)/p}$. Then $M_q(h(\Gamma^`)) \leq C(M_p(\Gamma^`))^{q/p}$ for every path family $\Gamma^`$ from $D^`$.

Proof: Let $A = \{x \in D | f$ is differentiable in x and $J_f(x) \neq 0\}$ and $B = \{y \in D^` | h$ is differentiable in y and $J_h(y) \neq 0\}$. Then $f(A) \subset B$ and $\mu_n(CA) = 0$. We have, using the change of variable formulae (3) from [11] that

$$\int\limits_D K_{0,p}(f)(x)^{q/(p-q)}dx = \int\limits_A (|f^`(x)|^p/|J_f(x)|)^{q/(p-q)}dx = \int\limits_A |f^`(x)|^{pq/(p-q)}/|J_f(x)|^{q/(p-q)}dx =$$

$$= \int\limits_A (J_h(f(x))^{p/(p-q)}/l(h^`(f(x))^{pq/(p-q)}))|J_f(x)|dx \leq \int\limits_{f(A)} |J_h(y)|^{p/(p-q)}/l(h^`(y))^{pq/(p-q)}dy \leq$$

$$\leq \int\limits_B |J_h(y)|^{p/(p-q)}/l(h^`(y))^{pq/(p-q)}dy = \int\limits_B (|J_h(y)|/l(h^`(y))^q)^{p/(p-q)}dy =$$

$$= \int\limits_{D^`} H_{I,q}(h)(y)^{p/(p-q)}dy < \infty.$$

Let $\Gamma^`$ be a path family from $D^`$ and let $\Gamma = h(\Gamma^`)$. Then $\Gamma^` = f(\Gamma)$ and using Proposition 1, we see that $M_q(h(\Gamma^`)) = M_q(\Gamma) \leq CM_p(f(\Gamma))^{q/p} = CM_p(\Gamma^`)^{q/p}$.

Proposition 2 is closely related to Theorem 3 from [9] and the class $B(G)$ from [9].

We give now an example of a homeomorphism $h : (0,1)^n \to D^`$ which is not quasiconformal, but is a generalized quasiregular mapping, since it satisfies a modular inequality $M_q(h(\Gamma^`)) \leq CM_p(\Gamma^`)^{q/p}$ for every path family $\Gamma^`$ from $(0,1)^n$ and some $1 < q < p$. The example is from [9]. See also [19], page 240.

Example 1. Let $D = (0,1)^n$, $1 < q < p$, $0 < c < (p-q)/(pq-p)$ and let $h : D \to \mathbf{R}^n$ be defined by $h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, \frac{x_n^{1+c}}{1+c})$ for $x = (x_1, ..., x_n) \in D$. We see that h is a homeomorphism onto a domain $D^`$ from $\mathbf{R}^n$ and $h \in C^1(D, D^`)$, $J_h(x) = x_n^c \neq 0$, $l(h^`(x)) = x_n^c$, $|h^`(x)| = 1$ for every $x \in D$, hence $H(x,h) = |h^`(x)|/l(h^`(x)) = x_n^{-c} \to \infty$ if $x \to 0$ and hence h is not quasiconformal. We have $H_{I,q}(h)(x) = |J_h(x)|/l(h^`(x))^q = x_n^{c(1-q)}$ for $x \in D$ and let $C = (\int\limits_D H_{I,q}(h)(x)^{p/(p-q)}dx)^{(p-q)/p}$. Then $C = (\int\limits_0^1 x_n^{pc(1-q)/(p-q)}dx_n)^{(p-q)/p} = ((p-q)/(pc-pqc+p-q))^{(p-q)/p} < \infty$, and from Proposition 2 we see that $M_q(h(\Gamma^`)) \leq CM_p(\Gamma^`)^{q/p}$ for every path family $\Gamma^`$ from D.

5 Geometric properties of generalized quasiregular mappings.

We shall prove first some geometric properties of generalized quasiregular mappings in the general setting of the operator N in dimension $n \geq 2$.

Theorem 9. (Generalization of Schwarz's lemma and modulus of continuity).

Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $f : D \to \mathbf{R}^n$ continuous, open, discrete and bounded, let M be a modulus on D so that there exists $\varphi : (0,\infty) \to (0,\infty)$ continuous, increasing with $\lim_{t\to 0} \varphi(t) = 0$ and $M(\Gamma_{x,a,b}) \leq \varphi(1/\ln\ln(be/a))$ for every $x \in D$ and every $b > 0$ so that $\overline{B}(x,b) \subset D$ and let $\gamma : [0,\infty) \to [0,\infty)$ be continuous, increasing with $\lim_{t\to 0} \gamma(t) = 0$ and so that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D.

Then there exists $\delta > 0$ and $F : (0,\infty) \to (0,\infty)$ continuous, increasing with $\lim_{t\to 0} F(t) = 0$ and so that $l(x,f,a) \leq F(1/\ln\ln(be/a))$ for every $x \in D$ and every $0 < a < b\delta$ so that $\overline{B}(x,b) \subset D$.

Proof: Let $d = df(D)$ and let $h_4 : (0,d) \to (0,\infty)$ be defined by $h_4(t) = \frac{t^{n-1}\omega_{n-1}}{V_n(d^n - t^n)}$ for $t \in (0,d)$. Then $\lim_{t\to 0} h_4(t) = 0$ and $h_4 : (0,d) \to (0,\infty)$ is an increasing homeomorphism. There exists $\delta_0 \in (0,d)$ so that $h_4(\delta_0) = 1$ and $h_4(t) \leq 1$ if $t \in [0,\delta_0]$, $h_4(t) > 1$ if $t \in (\delta_0, d)$. Let $C_0 = q2^{(r-s)/r}\Phi^s(1)$. Let $h_1 : (0,\delta_0] \to \mathbf{R}$, $h_1(t) = C_0(h_4(t))^{sq(\Phi)} g(V_n(d^n - t^n))$ for $t \in (0,\delta_0]$, $h_2 : (\delta_0, d) \to \mathbf{R}$, $h_2(t) = C_0(h_4(t))^{sp(\Phi)} g(V_n(d^n - t^n))$ for $t \in (\delta_0, d)$ and let $h : (0,d) \to (0,\infty)$ be defined by $h(t) = h_1(t)$ for $t \in (0,\delta_0]$, $h(t) = h_2(t)$ for $t \in (\delta_0,d)$. Then h is continuous on $(0,d)$ and let $c \in (0,d)$ be so that $V_n(d^n - c^n) = 1$.

Suppose first that $0 < c < \delta_0$. Then $h_1(t) = C_0(t^{n-1}\omega_{n-1})^{sq(\Phi)}(1/(V_n(d^n - t^n))^{sq(\Phi)-1})$ if $t \in (0,c)$, $h_1(t) = C_0(t^{n-1}\omega_{n-1})^{sq(\Phi)}(1/(V_n(d^n - t^n))^{sq(\Phi)-s/r})$ if $t \in [c,\delta_0)$, and since $sq(\Phi) - 1 \geq 0$, $sq(\phi) - s/r \geq 0$, we see that h_1 is strictly increasing. Also, $h_2(t) = C_0(t^{n-1}\omega_{n-1})^{sp(\Phi)}(1/(V_n(d^n - t^n))^{sp(\Phi)-s/r})$ if $t \in (\delta_0, d)$, hence h_2 is strictly increasing and let $\alpha = \lim_{t\to d} h_2(t)$. Then $h : (0,d) \to (0,\alpha)$ is an increasing homeomorphism.

Suppose now that $\delta_0 < c$. Then $h_1(t) = C_0(t^{n-1}\omega_{n-1})^{sq(\Phi)}(1/(V_n(d^n - t^n))^{sq(\Phi)-1})$ if $0 < t \leq \delta_0$, hence h_1 is strictly increasing and $h_2(t) = C_0(t^{n-1}\omega_{n-1})^{sp(\Phi)}(1/(V_n(d^n - t^n))^{sp(\Phi)-1})$ if $\delta_0 < t \leq c$ and $h_2(t) = C_0(t^{n-1}\omega_{n-1})^{sp(\Phi)}(1/(V_n(d^n - t^n))^{sp(\Phi)-s/r}$ if $c < t < d$. Since $sp(\Phi) - 1 \geq 0$, $sp(\Phi) - s/r \geq 0$, we see that h_2 is strictly increasing and if $\alpha = \lim_{t\to d} h_2(t)$, we see that $h : (0,d) \to (0,\alpha)$ is an increasing homeomorphism. We proved that in both cases $h : (0,d) \to (0,\alpha)$ is an increasing homeomorphism.

Let $h_3 : (0,d) \to (0,\infty)$, $h_3(t) = q2^{(r-s)/r}\Phi^s(h_4(t))g(V_n(d^n - t^n))$ for $t \in (0,d)$. We see that $h_3(t) = q2^{(r-s)/r}\Phi^s(h_4(t))g(V^n(d^n - t^n)) \geq q2^{(r-s)/r}\Phi^s(1)g(V_n(d^n - t^n))\min\{h_4(t)^{sp(\Phi)}, h_4(t)^{sq(\Phi)}\} = h(t)$ for $t \in (0,d)$. Let $\beta = \min\{q, \alpha q2^{(r-s)/r}\}$ and let $\delta = \exp(1 - \exp(1/(\gamma \circ \varphi)^{-1}(\beta)))$. Let $x \in D$ and $b > 0$ be so that $\overline{B}(x,b) \subset D$ and $0 < a < b\delta$, and let U be the component of $f^{-1}(B(f(x), l(x,f,a)))$ containing x. We see that $U \subset B(x,a)$ and let $\Gamma^{‘} = \Gamma_{f(x),l(x,f,a),d}$. Let Γ be the family of all maximal lifting of some paths from $\Gamma^{‘}$ starting from some points from U. Then $\Gamma > \Gamma_{x,a,b}$, $\Gamma^{‘} > f(\Gamma)$ and from Theorem 5 we see that $\min\{q, q2^{(r-s)/r}h_3(l(x,f,a))\} \leq M_N(\Gamma^{‘}) \leq M_N(f(\Gamma)) \leq \gamma(M(\Gamma)) \leq \gamma(M(\Gamma_{x,a,b})) \leq \gamma(\varphi(1/\ln\ln(be/a))) < \beta \leq q$.

This implies that $q2^{(r-s)/r}h_3(l(x,f,a)) \leq \gamma(\varphi(1/\ln\ln(be/a))) < \beta \leq \alpha q2^{(r-s)/r}$, hence $h(l(x,f,a)) \leq h_3(l(x,f,a)) < q^{-1}2^{(s-r)/r}\gamma(\varphi(1/\ln\ln(be/a))) < \alpha$. Let $v : (0,\alpha) \to (0,d)$ be the inverse of $h : (0,d) \to (0,\alpha)$. Since $q^{-1}2^{(s-r)/r}\gamma(\varphi(1/\ln\ln(be/a))) \in Imh$, we can take $F = v(q^{-1}2^{(s-r)/r}\gamma \circ \varphi)$ and we see that F is continuous, increasing, $\lim_{t\to 0} F(t) = 0$ and $l(x,f,a,) \leq F(1/\ln\ln(be/a))$ if $\overline{B}(x,b) \subset D$ and $0 < a < b\delta$.

Remark 6. Suppose that in the preceding theorem we additionally have the relation "$L(x,f,a) \leq \alpha(l(x,f,a))$ for some continuous, increasing function $\alpha : (0,\infty) \to (0,\infty)$, every $x \in D$ and every $0 < a < b\delta$ so that $\overline{B}(x,b) \subset D$". Then, if $x \in D$ is so that $\overline{B}(x,b) \subset D$ and $0 < a < b\delta$, we see that $L(x,f,a) \leq \alpha(l(x,f,a)) \leq \alpha(F(1/\ln\ln(be/a)))$. We obtain in this case that if $x \in D$, $b > 0$ is so that $\overline{B}(x,b) \subset D$, then $|f(y) - f(x)| \leq \alpha(F(1/\ln\ln(be/|y-x|)))$ if $0 < |y - x| < b\delta$.

Theorem 10. (Generalization of Liouville's theorem) Let $n \geq 2$, $f : \mathbf{R}^n \to \mathbf{R}^n$ a function which is either constant, or continuous, open, discrete and bounded, let M be a modulus on $\mathbf{R}^n$ so that there exists $\varphi : (0,\infty) \to (0,\infty)$ continuous,, increasing with $\lim\limits_{t\to 0}\varphi(t) = 0$ and $M(\Gamma_{x,a,b}) \leq \varphi(1/\ln\ln(be/a))$ for every $x \in D$ and $b > 0$ so that $\overline{B}(x,b) \subset D$ and let $\gamma : [0,\infty) \to [0,\infty)$ be continuous, increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$ and so that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D. Then f is constant.

Proof: Suppose that f is not constant. Then f is continuous, open, discrete and bounded and let $x \in \mathbf{R}^n$ and $0 < a$ be fixed. Using the preceding theorem, there exists $\delta > 0$ and a continuous, increasing function $F : (0,\infty) \to (0,\infty)$ with $\lim\limits_{t\to 0} F(t) = 0$ and so that $l(x,f,a) \leq F(1/\ln\ln(be/a))$ if $0 < a < b\delta$. Letting $b = ma$ with $m > \delta^{-1}$, we see that $l(x,f,a) \leq F(1/\ln\ln m) \to 0$ if $m \to \infty$. It results that $l(x,f,a) = 0$ and this contradicts the fact that f is open, discrete. We proved that f is constant.

Definition. Let $D \subset \mathbf{R}^n$ a domain, M a modulus on D and $E \subset \overline{\mathbf{R}}^n$. We say that E is of zero M modulus (and we write $M(E) = 0$) if the M modulus of the family of all paths passing through some points from E is zero.

If $x \in D$, $b > 0$ is so that $\overline{B}(x,b) \subset D$ and $\lim\limits_{a\to 0} M(\Gamma_{x,a,b}) = 0$ (and such a condition holds in Theorem 2,3,4), then $M(\{x\}) = 0$. If $A \subset D$ is at most countable and $M(\{x\}) = 0$ for every $x \in A$, then $M(A) = 0$.

Theorem 11. (Generalization of Picard's theorem). Let $n \geq 2$, $E \subset \mathbf{R}^n$ closed, $f : \mathbf{R}^n \setminus E \to \mathbf{R}^n$ continuous, open, discrete, M a modulus on $\mathbf{R}^n$ so that $M(E \cup \{\infty\}) = 0$, let $\gamma : [0,\infty) \to [0,\infty)$ be so that $\gamma(0) = 0$ and $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from $\mathbf{R}^n$. Then $\mu_n(\overline{Cf(\mathbf{R}^n \setminus E)}) = \phi$.

Proof: Suppose that there exists $F \subset \overline{Cf(\mathbf{R}^n \setminus E)}$ so that $\mu_n(F) > 0$. Let $K \subset \mathbf{R}^n \setminus E$ be compact so that $Intf(K) \neq \phi$. Then $f(K) \cap \overline{Cf(\mathbf{R}^n \setminus E)} = \phi$ and let $\Gamma' = \Delta(f(K), \overline{Cf(\mathbf{R}^n \setminus E)}, \mathbf{R}^n)$ and Γ be the family of all maximal lifting of some paths from Γ' starting from some points from K. Then $\Gamma' > f(\Gamma)$ and since every path from Γ has at least a limit point in $E \cup \{\infty\}$, we see that $M(\Gamma) = 0$, and from Theorem 7 we see that $M_N(\Gamma') > 0$. We have that $0 < M_N(\Gamma') \leq M_N(f(\Gamma)) \leq \gamma(M(\Gamma)) = \gamma(0) = 0$, and we reached a contradiction. We therefore proved that $\mu_n(\overline{Cf(\mathbf{R}^n \setminus E)}) = 0$.

6 Stronger geometric properties of generalized quasiregular mappings.

As we can see in the classical case of quasiregular mappings,which satisfies a modular inequality of type $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ with $M_N = M = M_n$ and $\gamma(t) = Kt$ for $t \geq 0$, an essential step in proving some basic geometric properties in this class of mappings is the fact that $M_n(\Delta(E,F,B(x,b) \setminus \overline{B}(x,a))) \geq C_n \ln\frac{b}{a} = C(a,b,n) > 0$ for every $x \in \mathbf{R}^n$ and every $E, F \subset \mathbf{R}^n$ so that $S(x,t) \cap E \neq \phi$, $S(x,t) \cap F \neq \phi$ for every $a < t < b$.

We proved in Theorem 8 such a result in the case $n = 2$ for the operator $N : \mathcal{M}(D) \to [0,\infty]$ given by $N(u) = \int\limits_D \Psi_{p(x)}(\Phi(|u|(x))dx$ for $u \in \mathcal{M}(D)$ and in the case $n \geq 3$ for $M_N = M_p$ with $p > n - 1$, and so we can expect that also for some other generalized quasiregular mappings some similar results can hold.

Throughout this chapter the operator $N : \mathcal{M}(D) \to [0,\infty]$ will be given in the case $n = 2$ by the formulae $N(u) = \int\limits_D \Psi_{p(x)}(\Phi(|u|(x)))dx$ for $u \in \mathcal{M}(D)$, with the functions Ψ, Φ, p and the constants q, r, s as before, and if $n \geq 3$, then $M_N = M_p$ with $p > n-1$.

Theorem 12. (Generalization of Schwarz's lemma and modulus of continuity). Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, $f : D \to \mathbf{R}^n$ continuous, open, discrete and bounded, let M a modulus on D so that there exists $\varphi : (0,\infty) \to (0,\infty)$ continuous, increasing with $\lim\limits_{t\to 0} \varphi(t) = 0$ and $M(\Gamma_{x,a,b}) \leq \varphi(1/\ln\ln(be/a))$ for every $x \in D$ and every $b > 0$ so that $\overline{B}(x,b) \subset D$, let $\gamma : [0,\infty) \to [0,\infty)$ be continuous, increasing with $\lim\limits_{t\to 0} \gamma(t) = 0$ and suppose that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D. Then there exists $\delta > 0$ and $F : (0,\infty) \to (0,\infty)$ continuous, increasing with $\lim\limits_{t\to 0} F(t) = 0$ and so that $|f(y) - f(x)| \leq F(1/\ln\ln(be/|y-x|))$ for every $x \in D$ and every $0 < |y-x| < b\delta$ so that $\overline{B}(x,b) \subset D$, and we can take $\delta = \exp(1 - \exp(1/(\gamma\circ\varphi)^{-1}(q)))$ if $n = 2$ and $\delta = 1$ if $M_N = M_p$ with $p > n-1$.

Proof: Let $d = df(D)$, let $x \in D$ and $b > 0$ be so that $\overline{B}(x,b) \subset D$ and let $y \in B(x,b)$. Let $y_1 \in S(x,|y-x|)$ be so that $L(x,f,|y-x|) = |f(y_1) - f(x)|$, and let P be the point from the line determined by $f(x)$ and $f(y_1)$, opposite to $f(y_1)$ and so that $|P - f(x)| = d$. Let $E = f(\overline{B}(x,|y-x|))$, $F = CB(f(x),d)$ and let $\Gamma` = \Delta(E,F,\mathbf{R}^n)$. Let $\Gamma$ be the family of all maximal lifting of some paths from $\Gamma`$ starting from some points of $\overline{B}(x,|y-x|)$. Then $\Gamma` > f(\Gamma)$, $\Gamma > \Gamma_{x,|y-x|,b}$ and $S(P,t)\cap E \neq \phi$, $S(P,t)\cap F \neq \phi$ for $d < t < d + |f(y_1) - f(x)|$.

Suppose that $n = 2$. From Theorem 8 we have
$\min\{q, q2^{(r-s)/r}\Phi^s(1/\pi(2d+|f(y_1)-f(x)|))g(\pi(d+|f(y_1)-f(x)|)^2-d^2) \leq M_N(\Gamma`) \leq M_N(f(\Gamma)) \leq \gamma(M(\Gamma)) \leq \gamma(M(\Gamma_{x,|y-x|,b})) \leq \gamma(\varphi(1/\ln\ln(be/|y-x|))) < q$ if $|y-x| < b\delta$.

This implies that if $0 < |y-x| < b\delta$, then
$q2^{(r-s)/r}\Phi^s(1/\pi(2d+|f(y_1)-f(x)|))g(\pi|f(y_1)-f(x)|(2d+|f(y_1)-f(x)|)) \leq \gamma(\varphi(1/\ln\ln(be/|y-x|)))$ and hence $q2^{(r-s)/r}\Phi^s(1/3\pi d)g(\pi|f(y)-f(x)|^2) \leq \gamma(\varphi(1/\ln\ln(be/|y-x|)))$.

Letting $F = (\pi^{-1}g^{-1}(q^{-1}2^{(s-r)/r}\Phi^{-s}(1/3\pi d)\gamma\circ\varphi))^{\frac{1}{2}}$, we see that F is continuous, increasing, $\lim\limits_{t\to 0} F(t) = 0$ and $|f(y) - f(x)| \leq F(1/\ln\ln(be/|y-x|))$ if $0 < |y-x| < b\delta$ and $\overline{B}(x,b) \subset D$.

Suppose now that $M_N = M_n$. Then $M_N(\Gamma`) = M_n(\Gamma`) \geq C(n)\ln(\frac{d+|f(y_1)-f(x)|}{d}) \geq C(n)\ln(1+\frac{|f(y)-f(x)|}{d}) \geq \frac{C(n)}{2d}|f(y)-f(x)|$, hence $\frac{C(n)}{2d}|f(y)-f(x)| \leq M_N(\Gamma`) \leq M_N(f(\Gamma)) \leq \gamma(M(\Gamma)) \leq \gamma(M(\Gamma_{x,|y-x|,d})) \leq \gamma(\varphi(1/\ln\ln(be/|y-x|)))$ if $|y-x| < b$ and $\overline{B}(x,b) \subset D$.

Letting $F = \frac{2d}{C(n)}\gamma\circ\varphi$, we see that $F : (0,\infty) \to (0,\infty)$ is continuous, increasing, $\lim\limits_{t\to 0} F(t) = 0$ and $|f(y) - f(x)| \leq F(1/\ln\ln(be/|y-x|))$ if $x \in D$, $\overline{B}(x,b) \subset D$ and $0 < |y-x| < b$.

Suppose now that $M_N = M_p$, with $p > n-1$, $p \neq n$. Then $M_N(\Gamma`) = M_p(\Gamma`) \geq C(n,p)((d+|f(y_1)-f(x)|)^{n-p} - d^{n-p}) \geq C(n,p)((d+|f(y)-f(x)|)^{n-p} - d^{n-p}) \geq C(n,p)(n-p)|f(y)-f(x)|(2d)^{n-p-1}$. It results that
$C(n,p)(n-p)(2d)^{n-p-1}|f(y)-f(x)| \leq M_N(\Gamma`) \leq M_N(f(\Gamma)) \leq \gamma(M(\Gamma)) \leq \gamma(M(\Gamma_{x,|y-x|,b})) \leq \gamma(\varphi(1/\ln\ln(be/|y-x|)))$ if $|y-x| < b$ and $\overline{B}(x,b) \subset D$.

Letting $F = (1/(C(n,p)(n-p)(2d)^{n-p-1}))\gamma\circ\varphi$, we see that $F : (0,\infty) \to (0,\infty)$ is continuous, increasing, $\lim\limits_{t\to 0} F(t) = 0$ and $|f(y) - f(x)| \leq F(1/\ln\ln(be/|y-x|))$ if $x \in D$, $\overline{B}(x,b) \subset D$ and $0 < |y-x| < b$.

Theorem 13. (Generalization of Picard's theorem). Let $n \geq 2$, $E \subset \mathbf{R}^n$ closed, $f : \mathbf{R}^n \setminus E \to \mathbf{R}^n$ continuous, open, discrete, M a modulus on $\mathbf{R}^n$ so that $M(E \cup \{\infty\}) = 0$ and let $\gamma : [0,\infty) \to [0,\infty)$ be so that $\gamma(0) = 0$ and $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from $\mathbf{R}^n$. Then $\overline{Cf(\mathbf{R}^n \setminus E)}$ is totally disconnected.

Proof: Suppose that there exists $F \subset \overline{Cf(\mathbf{R}^n \setminus E)}$ compact, connected so that $Card F > 1$

and let $K \subset \mathbf{R}^n \setminus E$ be compact, connected with $Card f(K) > 1$ and $f(K) \cap F = \phi$. Let $\Gamma^{‘} = \Delta(f(K), F, \mathbf{R}^n)$ and let Γ be the family of all maximal lifting of some paths from $\Gamma^{‘}$ starting from some points from K. Then $\Gamma^{‘} > f(\Gamma)$, $M(E \cup \{\infty\}) = 0$ and since every path from Γ has at least a limit point in $E \cup \{\infty\}$, it results that $M(\Gamma) = 0$. Using Theorem 8, we see that $M_N(\Gamma^{‘}) > 0$, hence $0 < M_N(\Gamma^{‘}) \leq M_N(f(\Gamma)) \leq \gamma(M(\Gamma)) = \gamma(0) = 0$ and we reached a contradiction. We therefore proved that $\overline{Cf(\mathbf{R}^n \setminus E)}$ is totally disconnected.

The following equicontinuity result extends Corollary 2.7, page 66 from [23].

Theorem 14. Let $n \geq 2$, $D \subset \mathbf{R}^n$ a domain, M a modulus on D so that $\lim_{a \to 0} M(\Gamma_{x,a,b}) = 0$ for every $x \in D$ and every $b > 0$ so that $\overline{B}(x, b) \subset D$, let $\gamma : [0, \infty) \to [0, \infty)$ be increasing so that $\lim_{t \to 0} \gamma(t) = 0$, let W be a family of continuous, open, discrete mappings $f : D \to \mathbf{R}^n$ so that there exists $\delta > 0$ and sets $M_f \subset CImf$ so that $\overline{M}_f$ is compact, connected, $Card\overline{M}_f > 1$, $q(\overline{M}_f) \geq \delta$ for every $f \in W$ and suppose that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D and every $f \in W$. Then the family W is equicontinuous, and we take on D the euclidean metric and we take on $\overline{\mathbf{R}}^n$ the chordal metric.

Proof: Let $x \in D$ and $\epsilon > 0$ so that $\overline{B}(x, \epsilon) \subset D$. Suppose that the family W is not equicontinuous at x. Then there exists $\alpha > 0$, $r_m \to 0$ and $f_m \in W$ so that $q(f_m(\overline{B}(x, r_m))) > \alpha$ for every $m \in N$ and let $Q_m = f_m(\overline{B}(x, r_m))$ for $m \in N$. Since $Imf_m \cap M_{f_m} = \phi$ and Imf_m are open sets for every $m \in N$, we see that $Imf_m \cap \overline{M}_{f_m} = \phi$, hence $Q_m \cap \overline{M}_{f_m} = \phi$ for every $m \in N$. Let $\Gamma^{‘}_m = \Delta(Q_m, \overline{M}_{f_m}, \mathbf{R}^n)$ and let Γ_m be the family of all maximal lifting of some paths from $\Gamma^{‘}_m$ starting from some points of $\overline{B}(x, r_m)$ for $m \in N$. Since every path from Γ_m has at least a limit point outside $B(x, \epsilon)$, we see that $\Gamma_m > \Gamma_{x, r_m, \epsilon}$ and we also see that $\Gamma^{‘}_m > f(\Gamma_m)$ for every $m \in N$. Using Theorem 7.1, page 11 from [33], we can find compact, connected sets Q and Y so that $q(Q) \geq \alpha$, $q(Y) \geq \delta$ and so that $\lim_{m \to \infty} Q_m = Q$, $\lim_{m \to \infty} \overline{M}_{f_m} = Y$. We can therefore find $z \in \mathbf{R}^n$ and $0 < a < b$ such that $S(z, t) \cap Y \neq \phi$, $S(z, t) \cap Q \neq \phi$, for $a < t < b$. This implies that we can find $m_0 \in N$ so that $S(z, t) \cap Q_m \neq \phi$, $S(z, t) \cap \overline{M}_{f_m} \neq \phi$ for every $a < t < b$ and every $m \geq m_0$.

Using Theorem 8, we can find a constant C so that $M_N(\Gamma^{‘}_m) > C > 0$ for every $m \geq m_0$, and $C = C(q, r, s, a, b)$ if $n = 2$, $C = C(n, a, b)$ if $n \geq 3$ and $M_N = M_n$, $C = C(n, p, a, b)$ if $n \geq 3$ and $M_N = M_p$ with $p > n - 1$, $p \neq n$. It results that if $m \geq m_0$ we have $0 < C < M_N(\Gamma^{‘}_m) \leq M_N(f(\Gamma_m)) \leq \gamma(M(\Gamma_m)) \leq \gamma(M(\Gamma_{x, r_m, \epsilon})) \to 0$ if $m \to \infty$, and we reached a contradiction. It results that the family W is equicontinuous at x.

Theorem 15. (Generalization of Montel's theorem). Let $n \geq 2$, $D \subset \mathbf{R}^n$ be a domain, M a modulus on D so that $\lim_{a \to 0} M(\Gamma_{x,a,b}) = 0$ for every $x \in D$ and every $b > 0$ so that $\overline{B}(x, b) \subset D$, let $\gamma : [0, \infty) \to [0, \infty)$ be increasing so that $\lim_{t \to 0} \gamma(t) = 0$ and let W be a bounded family of continuous, open, discrete mappings $f : D \to \mathbf{R}^n$ so that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D and every $f \in W$. Then W is equicontinuous, and we take the euclidian metric on D and the chordal metric on $\overline{\mathbf{R}}^n$.

Remark 7. An important particular case of Theorem 14 is when $M_f = Y$ for every $f \in W$, i.e. when there exists a single set Y avoided by every map $f \in W$ and so that $\overline{Y}$ is compact, connected and $Card\overline{Y} > 1$.

Another important particular case of Theorem 14 is obtained when every map f from the family W is a homeomorphism, extending a known result from the theory of quasiconformal mappings from [30], Theorem 19.2, page 65.

Theorem 16. Let $n \geq 2$, $D \subset \mathbf{R}^n$ be a domain, M a modulus on D so that $\lim_{a \to 0} M(\Gamma_{x,a,b}) = 0$ for every $x \in D$ and every $b > 0$ so that $\overline{B}(x, b) \subset D$, let $\gamma : [0, \infty) \to [0, \infty)$ be increasing so that $\lim_{t\to 0}\gamma(t)=0$ and let W be a family of homeomorphisms $f: D \to D_f \subset \mathbf{R}^n$ so that there exists $\delta > 0$ so that for every $f \in W$ there exists points $a_f, b_f \notin Imf$ so that $q(a_f, b_f) \geq \delta$ and suppose that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D and every $f \in W$. Then W is equicontinuous, and we take the euclidian metric on D and we take the chordal metric on $\overline{\mathbf{R}}^n$.

Proof: Let $x \in D$ and $\epsilon > 0$ be fixed so that $\overline{B}(x,\epsilon) \subset D$. Let $f \in W$. Since $f|S(x,\epsilon): S(x,\epsilon) \to f(S(x,\epsilon))$ is a homeomorphism, we use Jordan's theorem to see that $\mathbf{R}^n \setminus f(S(x,\epsilon))$ has exactly two components A_f and B_f. If $f(B(x,\epsilon)) \cap A_f \neq \phi$, then $f(B(x,\epsilon)) = A_f$ and hence $a_f, b_f \in B_f$. It results that B_f is a domain so that $q(B_f) \geq \delta$ and $B_f \cap f(B(x,\epsilon)) = \phi$ for every $f \in W$. Let $W_\epsilon = \{f|B(x,\epsilon)| f \in W\}$. Using Theorem 14 we see that the family W_ϵ is equicontinuous at x, hence the family W is equicontinuous at x.

Remark 8. As in [30] Theorem 19.4, page 66, we can prove that we can replace in Theorem 16 the condition "there exists $\delta > 0$ such that fort every $f \in W$ there exists points $a_f, b_f \notin Imf$ so that $q(a_f, b_f) \geq \delta$" with one of the following conditions:

a) there exists $\delta > 0$ and $x_1, x_2 \in D$ such that for every $f \in W$ there exists a point $a_f \notin Imf$ with $q(a_f, f(x_i)) \geq \delta$, $i = 1, 2$.

b) there exists $\delta > 0$ and points x_1, x_2, x_3 so that $q(f(x_i), f(x_j)) \geq \delta$ for $i \neq j$, $i, j = 1, 2, 3$ and for every $f \in W$.

We also have the following eliminability result, which is a partial extension of Theorem 2.9, page 66 from [23].

Theorem 17. Let $n \geq 2$, $D \subset \mathbf{R}^n$ be a domain, $x \in D$, M a modulus on D so that $\lim_{a\to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \subset D$, let $\gamma: [0,\infty) \to [0,\infty)$ be increasing so that $\lim_{t\to 0}\gamma(t) = 0$ and let $E \subset D$ closed in D and nowhere disconnecting so that $x \in E$ and $M(E) = 0$. Let $f: D \setminus E \to \mathbf{R}^n$ be continuous, open, discrete so that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from $D \setminus E$ and suppose that there exists $r_x > 0$ and a set $Y \subset Cf(B(x,r_x) \setminus E)$ so that $\overline{Y}$ is compact, connected and $Card\overline{Y} > 1$. Then there exists $\lim_{z\to x} f(z) \in \overline{\mathbf{R}}^n$.

Proof: We can suppose that $\overline{B}(x,r_x) \subset D$. Suppose that $CardC(f,x) > 1$ and let $b_1, b_2 \in C(f,x)$, $b_1 \neq b_2$. Let $x_j, y_j \in B(x,r_x) \setminus E$ be so that $x_j \neq y_j$ for $j \in N$, $f(x_j) \to b_1$, $f(y_j) \to b_2$ and let $r_j \to 0$ be so that $x_j, y_j \in B(x,r_j)$, $0 < r_j < r_x$ for every $j \in N$. Since E is nowhere disconnecting, we can find a compact, connected set $C_j \subset B(x,r_j) \setminus E$ so that $x_j, y_j \in C_j$ for every $j \in N$. Let $\delta = |b_1 - b_2|$. We can suppose that $q(f(C_j)) \geq \lambda\delta$, $q(\overline{Y}) \geq \lambda\delta$ for every $j \in N$ and some $0 < \lambda < 1$. Since $f(B(x,r_x) \setminus E)$ is an open set and $Y \cap f(B(x,r_x) \setminus E) = \phi$, it results that $\overline{Y} \cap f(B(x,r_x) \setminus E) = \phi$ and hence $f(C_j) \cap \overline{Y} = \phi$ for every $j \in N$. Using Theorem 7.1, page 11 from [33], we can find a compact, connected set Q so that $f(C_j) \to Q$ and $q(Q) \geq \lambda\delta$. We can therefore find $z \in \mathbf{R}^n$ and $0 < a < b$ such that $S(z,t) \cap \overline{Y} \neq \phi$, $S(z,t) \cap Q \neq \phi$ for $a < t < b$, hence we can find $j_0 \in N$ so that $S(z,t) \cap f(C_j) \neq \phi$ for every $a < t < b$ and every $j \geq j_0$.

Let $\Gamma_j^{‘} = \Delta(f(C_j), \overline{Y}, \mathbf{R}^n)$ and let Γ_j be the family of all maximal lifting of some paths from $\Gamma_j^{‘}$ starting from some points from C_j for $j \in N$. Let $\Gamma_{1j} = \{\varphi \in \Gamma | \varphi$ has at least a limit point in $E\}$ and let $\Gamma_{2j} = \{\varphi \in \Gamma | \varphi$ has at least a limit point outside $B(x,r_x)\}$ for $j \in N$. We see that $M(\Gamma_{1j}) = 0$, that $\Gamma_j = \Gamma_{1j} \cup \Gamma_{2j}$, $\Gamma_{2j} > \Gamma_{x,r_j,r_x}$ and $\Gamma_j^{‘} > f(\Gamma_j)$ for every $j \in N$. Using Theorem 8, we can find a constant $C > 0$ so that $M_N(\Gamma_j^{‘}) > C > 0$ for every $j \geq j_0$, and $C = C(q,r,s,a,b)$ if $n = 2$, $C = C(n,a,b)$ if $n \geq 3$ and $M_N = M_n$ and $C = C(n,p,a,b)$ if $n \geq 3$ and $M_N = M_p$ with $p > n-1$, $p \neq n$. It results that $0 < C < M_N(\Gamma_j^{‘}) \leq M_N(f(\Gamma_j)) \leq \gamma(M(\Gamma_j)) = \gamma(M(\Gamma_{1j} \cup \Gamma_{2j})) \leq \gamma(M(\Gamma_{1j}) + M(\Gamma_{2j})) = \gamma(M(\Gamma_{2j})) \leq \gamma(M(\Gamma_{x,r_j,r_x})) \to 0$ if

$j \to \infty$ and we reached a contradiction.

It results that $CardC(f,x) = 1$ and hence there exists $\lim_{z\to x} f(z) \in \overline{\mathbf{R}}^n$.

Remark 9. The most important case in Theorem 17 is when x is an isolated singularity of D, i.e. when $E = \{x\}$. In this case, due to the condition "$\lim_{a\to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \subset D$", it results that $M(\{x\}) = 0$.

Definition. If $D \subset \mathbf{R}^n$ is a domain, $x \in \partial D$ is an isolated point of ∂D and $f : D \to \mathbf{R}^n$ is continuous, open, discrete, we say that x is an essential singularity of f if there exists not $\lim_{z\to x} f(z) \in \mathbf{R}^n$.

Using Theorem 17, we obtain as in the classical case of quasiregular mappings the characterization of the behavior of a continuous, open, discrete mapping satisfying a modular inequality near an essential singularity.

Theorem 18. Let $n \geq 2$, $D \subset \mathbf{R}^n$ be a domain, x an isolated point of ∂D, M a modulus on D such that $\lim_{a\to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \setminus \{x\} \subset D$, $\gamma : [0,\infty) \to [0,\infty)$ be increasing with $\lim_{t\to 0} \gamma(t) = 0$, let $f : D \to \mathbf{R}^n$ be continuous, open, discrete so that x is essential singularity of f and suppose that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from D. Then $\overline{Cf(B(x,b) \setminus \{x\})}$ is totally disconnected for every $b > 0$ so that $\overline{B}(x,b) \setminus \{x\} \subset D$.

If in Theorem 17 the map $f : D \setminus E \to \mathbf{R}^n$ is a homeomorphism (or even a map of finite multiplicity), we have the following eliminability result, which extends a known result from the theory of quasiconformal mappings from [30], Theorem 17.3, page 52.

Theorem 19. Let $n \geq 2$, $D \subset \mathbf{R}^n$ be a domain, $x \in D$, M a modulus on D so that $\lim_{a\to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \subset D$, let $\gamma : [0,\infty) \to [0,\infty)$ be increasing so that $\lim_{t\to 0} \gamma(t) = 0$ and let $E \subset D$ be closed in D and nowhere disconnecting so that $x \in E$ and $M(E) = 0$. Let $f : D \setminus E \to \mathbf{R}^n$ be continuous, open, discrete so that there exists $U_x \in \mathcal{V}(x)$ and $n_x \in N$ so that $N(f|U_x \cap (D \setminus E)) \leq n_x$ and so that $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from $D \setminus E$. Then there exists $\lim_{z\to x} f(z) \in \overline{R}^n$.

Proof: Let $b > 0$ be so that $\overline{B}(x,b) \subset U_x \cap D$, and we can suppose that there exists points $x_1, x_2, ..., x_{n_x} \in B(x,b) \setminus E$ so that $f(x_1) = f(x_2) = ... = f(x_{n_x})$. Let $U_i \in \mathcal{V}(x_i)$ be so that $\overline{U}_i \subset B(x,b) \setminus E$ and $f(x_i) \notin f(\partial U_i)$ for $i = 1,..,n_x$. Let $r > 0$ be so that $B(f(x_1), r) \cap f(\partial U_i) = \phi$ for $i = 1,...,n_x$, and let $V = B(f(x_1), r)$. Let Q_i be the component of $f^{-1}(V)$ containing x_i for $i = 1,...,n_x$. Then $Q_i \subset U_i$ and $f(Q_i) = V$ for $i = 1,...,n_x$. Let $g = f|((U_x \cap (D \setminus E)) \setminus \bigcup_{i=1}^{n_x} Q_i)$. Since n_x is the maximal multiplicity of f on $U_x \cap (D \setminus E)$, we see that $Img \cap V = \phi$ and from Theorem 17 we find that there exists $\lim_{z\to x} g(z) \in \overline{\mathbf{R}}^n$. It results that there exists $\lim_{z\to x} f(z) \in \overline{\mathbf{R}}^n$.

Remark 10. We worked in this paper with continuous, open, discrete mappings $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ satisfying a modular inequality $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ for every path family Γ from the domain D, where $\gamma : [0,\infty) \to [0,\infty)$ is increasing with $\lim_{t\to 0} \gamma(t) = 0$ and M is a modulus on D so that $\lim_{a\to 0} M(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x,b) \subset D$. The operator $N : \mathcal{M}(D) \to [0,\infty]$ is given by $N(u) = \int_D \Psi_{p(x)}(\Phi(|u|(x)))dx$ for every $u \in \mathcal{M}(D)$, where $p : D \to [1,\infty]$ is measurable and finite a.e., $\Phi : [0,\infty) \to [0,\infty)$ a Young function, $\Psi : [0,\infty) \times [1,\infty) \to [0,\infty)$ a Borel map so that all the mappings $\Psi_c : [0,\infty) \to [0,\infty)$ given by $\Psi_c(t) = \Psi(t,c)$ for $t \geq 0$, $c \geq 1$ are Young functions for every fixed $c \geq 1$ and so that there exists $q, r, s > 0$ so that $q \leq \Psi_{p(x)}(1)$, $1 \leq r \leq p(\Psi_{p(x)}) \leq q(\Psi_{p(x)}) \leq s \leq \infty$ for every $x \in D$. The modulus M_N is given by $M_N(\Gamma) = \inf_{\rho \in F(\Gamma)} N(\rho)$ if $\Gamma \in \mathcal{A}(D)$.

The stronger results were established in the case $n = 2$ or if $n \geq 3$ and $M_N = M_p$ with $p > n - 1$, and a key result used for proving the geometric properties of our generalized quasiregular mappings is that from Theorem 8 which says that $M_N(\Delta(E, F, B(x, b) \setminus \overline{B}(x, a))) \geq C(q, r, s, a, b) > 0$ if $x \in D$ and $S(x, t) \cap E \neq \phi$, $S(x, t) \cap F \neq \phi$ for $a < t < b$.

It is obvious that if we can have such a result in dimension $n \geq 3$ for more general modulus M_N, then all the theorems from this paper hold also in this cases.

In fact, for a quasiregular map $f : D \to \mathbf{R}^n$ we have the known modular inequality of Poleckii which says that $M_n(f(\Gamma)) \leq K M_n(\Gamma)$ for every path family Γ from D and every $K \geq K_I(f)$, i.e. we can take $M_N = M_n$, $M = M_n$ and the function $\gamma : [0, \infty) \to [0, \infty)$ given by $\gamma(t) = Kt$ for $t \geq 0$.

Also, in [5], [6], [12], [13], [14], [15], [20], [21] are considered mappings $f : D \subset \mathbf{R}^n \to \mathbf{R}^n$ of finite distortion so that $\exp(\mathcal{A} \circ K_0(f)) \in L^1_{loc}(D)$ for some Orlicz map $\mathcal{A}$, in [18], [19],, [24], [25] are considered mappings of finite distortion so that $K_0(f) \in BMO(D)$ and in [7], [8] are considered open, discrete mappings having local ACL^n inverses. All this mappings satisfy a modular inequality of type $M_N(f(\Gamma)) \leq M^n_\omega(\Gamma)$ for every path family Γ from D and some weight ω so that $\lim_{a \to 0} M^n_\omega(\Gamma_{x,a,b}) = 0$ for every $b > 0$ so that $\overline{B}(x, b) \subset D$, where $\omega = K_0(f)^{n-1}$ or $\omega = K_I(f)$, hence in all this cases we can take $M_N = M_n$ and $M = M^n_\omega$. In all this papers the whole work needed for proving boundary extension theorems, equicontinuity, eliminability and modulus of continuity theorems is done using mainly the modular inequality $M_N(f(\Gamma)) \leq \gamma(M(\Gamma))$ together with the fact that $\lim_{a \to 0} M(\Gamma_{x,a,b,D}) = 0$, and following the methods from the present paper. It results that the facts from the above mentioned papers are particular cases of our theory.

Coming back to the classes of homeomorphisms with finite mean dilatations $f : D \to D^{‘}$ between two domains from $\mathbf{R}^n$ studied in [9] and presented in Proposition 2, for which the modular inequality $M_q(f(\Gamma)) \leq \gamma(M_p(\Gamma))$ holds for every path family Γ from D, some $1 < q < p$ and some continuous, increasing function $\gamma : [0, \infty) \to [0, \infty)$ with $\lim_{t \to 0} \gamma(t) = 0$, it results that the properties established in this paper are valid in this class of mappings if $1 \leq n - 1 < q \leq p \leq n$.

Let us see for instance the particular case of Theorem 12 in this class of mappings.

Theorem 20. Let $n \geq 2$, $n - 1 < q < p \leq n$, D a domain in $\mathbf{R}^n$, $f : D \to \mathbf{R}^n$ continuous, open, discrete so that there exists $\gamma : [0, \infty) \to [0, \infty)$ continuous, increasing with $\lim_{t \to 0} \gamma(t) = 0$ so that $M_q(f(\Gamma)) \leq \gamma(M_p(\Gamma))$ for every path family Γ from D. Then there exists $F : (0, \infty) \to (0, \infty)$ continuous, increasing with $\lim_{t \to 0} F(t) = 0$ so that $|f(y) - f(x)| \leq F(1/\ln\ln(be/|y - x|))$ for every $x \in D$ so that $\overline{B}(x, b) \subset D$ and every $0 < |y - x| < b$.

Theorem 21. Let $n \geq 2$, $n - 1 < q < p \leq n$, D, $D^{‘}$ domains in $\mathbf{R}^n$, $h : D^{‘} \to D$ a homeomorphism, $f = h^{-1}$ so that $f \in ACL^q(D, D^{‘})$, f is a.e. differentiable on D, $J_f(x) \neq 0$ a.e. in D and $H_{I,q}(h) \leq L^{p/(p-q)}(D^{‘})$. Then there exists $F : (0, \infty) \to (0, \infty)$ continuous, increasing with $\lim_{t \to 0} F(t) = 0$ so that $|h(y) - h(x)| \leq F(1/\ln\ln(be/|y - x|))$ for every $x \in D^{‘}$ so that $\overline{B}(x, b) \subset D^{‘}$ and every $0 < |y - x| < b$.

In a future paper we shall study the boundary behavior of the generalized quasiregular mappings.

Natural extensions of our results can be established on abstract metric measure spaces. For instance, the condition $M_p(\Delta(E, F, (B(x, b) \setminus \overline{B}(x, a)))) \geq C(p, a, b) > 0$ holds if $S(x, t) \cap E \neq \phi$, $S(x, t) \cap F \neq \phi$ for $a < t < b$ and if $p > n - 1$ in some abstract metric spaces, as we can see from [1], Proposition 4.7. See also Chapter 13 from [19] and Chapter 2 from the present paper.

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