THE SMALLEST NON-ABELIAN NILPOTENT QUASIVARIETIES OF MOUFANG LOOPS

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Abstract. This paper describes the smallest non-abelian quasivarieties for nilpotent Moufang loops.

1. Introduction

The theory of quasivarieties is one of the most important compartments of the universal algebra. The basis of this theory was set by A.I. Mal'cev ([1], [2], [3], [4]). The special attention is paid to two important problem: description of the lattice of quasivarieties of algebras.
The research of these problem in the class of nilpotent Moufang loops is the goal of this paper.

In this paper we describe all minimal non-abelian quasivarieties for nilpotent Moufang loops:

- minimal non-associative quasivarieties of commutative Moufang loops;
- minimal non-associative and non-commutative quasivarieties of Moufang A-loops with one proper minimal non-associative sub-quasivariety of commutative Moufang loops and one proper minimal non-commutative subquasivariety of groups;
- minimal non-associative and non-commutative quasivarieties of Moufang loops with the only proper non-commutative subquasivariety of groups;
- minimal non-commutative quasivarieties of groups.

For some of these quasivarieties, examples of non-associative Moufang loops are constructed. For instance, the smallest non-associative and non-commutative nilpotent Moufang loop has 16 elements and forms the basis of the algebra of Cayley-Dixon numbers.

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2. Definitions, preliminary results, observations and notations

We shall use some notions and results from the monograph of R.H. Bruck [5].

The Moufang Loop (ML) is an $\langle L, \cdot, ^{-1} \rangle$ algebra of type $\langle 2, 1\rangle$ whose operations and elements satisfy the following identities:

$$x\,(y\cdot xz) = (xy\cdot x)z \tag{1}$$

$$x^{-1}\cdot xy = y = yx\cdot x^{-1} \tag{2}$$

where by x^{-1} we denote the result of the unary operation applied to the x element.

We observe that the (2) involves the identity $y\cdot (x^{-1})^{-1} = yx$, which in its turn involves the identity $(x^{-1})^{-1} = x$, that helps to deduce the identity

$$x\cdot x^{-1}y = y = yx^{-1}\cdot x \tag{3}$$

For a certain $x \in L$ element we denote $e = x^{-1}\cdot x$. Then, according to the identities (1)–(3) we will have

$$ye = x^{-1}\cdot x\,(ye) = x^{-1}\left[x\cdot y\left(xx^{-1}\right)\right] = x^{-1}\left[(xy\cdot x)\,x^{-1}\right] = x^{-1}\cdot xy = y$$

for any $y \in L$. It results that $e = y^{-1}\cdot y$ and, therefore, e doesn't depend on the x element. Then, taking in consideration that (3)

$$e\cdot y = yy^{-1}\cdot y = y$$

for any $y \in L$, it follows that e is a unit element of ML L. Further on ML L will be studied with the signature $\langle ., -1, e\rangle$ made up of three operational symbols, which will be simply noted as L.

ML is dissociative, in the sense that any of its subloops generated by two elements is associative (Moufang theorem [5]).

For x, y and z elements from ML L the associator $[x, y, z]$ and the commutator $[x, y]$ are defined by the equalities $[x, y, z]$=$(x\cdot yz)^{-1}\cdot(xy\cdot z)$ and $[x, y] = x^{-1}\cdot y^{-1}(xy)$, respectively.

The ML L associant-commutator is the subloop generated in L by all the associators and commutators of L and we shall denote it as L' or $[L, L]$. The set

$$Z\,(L) = \left\{x \in L \,\middle|\, [x, y, z] = e,\ [x, y] = e \text{ for any } y, z \in L\right\}$$

is called the ML L center.

For any H subloop of the L subloop by $[H, L]$ we shall denote the subloop generated in L by all the elements of type $[h, x, y]$ and $[h, x]$ where $h \in H, x, y \in L$.

The H subloop of the ML L is called normal in L if $xH = Hx,\ x \cdot yH = xy \cdot H$ for any $x, y \in L$. It is easy to verify that the associant-commutator L' and any subloop of ML L, contained in the center $Z(L)$, are normal in L.

With the help of the induction hypothesis special associant-commutator of the n multiplicity its defined; x_1 is a special associant-commutator of 1 multiplicity; if u is a special associant of n multiplicity which the includes exactly i_n variables, then $\left[u, x_{i_{n+1}}\right],\ \left[u, x_{i_{n+1}}, x_{i_{n+2}}\right]$ is a special associant-commutator of $n+1$ multiplicity.

ML L is called (central-)nilpotent (NML) of n class or n-nilpotent if for any values of the variables in L the value of any special associant-commutator of $n+1$ multiplicity is equal to the $e \in L$ unity element, but the value of at least one special associant-commutator of n multiplicity is different from e.

According to [6] in any nilpotent Moufang loop of class 2 the following identities are true

$$[x, y, z] = [y, z, x] = [y, x, z]^{-1}, \tag{4}$$

$$[x \cdot y, z, t] = [x, z, t]\,[y, z, t], \tag{5}$$

$$[x^m, y, z] = [x, y, z]^m, \tag{6}$$

$$[x, y, z]^6 = e, \tag{7}$$

$$[x \cdot y, z] = [x, z]\,[y, z]\,[x, y, z]^3, \tag{8}$$

and

$$[x^m, y] = [x, y]^m, \tag{9}$$

$$[x, y] = [y, x]^{-1}, \tag{10}$$

because ML is dissociative.

3. The smallest nilpotent non-abelian quasivarieties of Moufang loops

The following varieties are defined in the class of all 2-nilpotent Moufang loops:

$$K_{1,0,0} = mod\{[x,y,z] = e\},$$

$$K_{1,p,0} = mod\{[x,y,z] = e,\ [x,y]^p = e\},$$

$$K_{1,p,p^m} = mod\{[x,y,z] = e,\ [x,y]^p = e,\ x^{p^m} = e\},$$

where $m = 2, 3, \ldots$ for $p = 2$ and $m = 1, 2, \ldots$ for any prime number $p \geq 3$,

$$K_{2,0,0} = mod\{[x,y,z]^2 = e\},$$

$$K_{2,2,0} = mod\{[x,y,z]^2 = e,\ [x,y]^2 = e\},$$

$$K_{2,2,2^m} = mod\{[x,y,z]^2 = e,\ [x,y]^2 = e,\ x^{2^m} = e\},\ m = 2, 3, \ldots,$$

$$K_{3,0,0} = mod\{[x,y,z]^3 = e\},$$

$$K_{3,1,0} = mod\{[x,y,z]^3 = e,\ [x,y] = e\},$$

$$K_{3,1,3^m} = mod\{[x,y,z]^3 = e\},\ [x,y] = e,\ x^{3^m} = e\},\ m = 1, 2, \ldots,$$

$$K_{3,3,0} = mod\{[x,y,z]^3 = e,\ [x,y]^3 = e\},$$

$$K_{3,3,3^m} = mod\{[x,y,z]^3 = e\},\ [x,y]^3 = e,\ x^{3^m} = e\},\ m = 1, 2, \ldots$$

Denote by $\Re$ the set of all varieties defined above.

Lemma 1. *If a 2-nilpotent Moufang loop N is finite, then there exists such a variety $K \in \Re$ that $F_3(K) \in qN$.*

Proof. Since N is nilpotent we can regard N as a p-loop. Let $\exp(N) = p^m$. We consider the following possible cases.

1) N is non-associative and $p = 2$. In this case $m > 1$. Then according to the identity (7) in N the identity $(x,y,z)^2 = e$ holds true. For a certain integer k, $1 \leq k \leq m$, in N the identity $[x,y]^{2^k} = e$ also holds. Let $F_3 = F_3(x,y,z)$ be a $v(N)$-free loop of the third rank with free generators x, y, z, and $H = lp(a,b,c)$ be the subloop of F_3^4 generated by the elements

$$a = (x,x,e,e),\ b = (e, y^{2^{k-1}}, y, e),\ c = (e, z^{2^{k-1}}, z^{2^{m-1}}, z).$$

Then it becomes obvious that

$$a^{2^m} = b^{2^m} = c^{2^m} = e,\ [a,b] = (e, [x,y]^{2^{k-1}}, e, e),\ [a,c] = (e, [x,z]^{2^{k-1}}, e, e),$$

$$[b,c]=(e,\ [y,z]^{2^{2(k-1)}},\ [y,z]^{2^{m-1}},e),\ (a,b,c)=(e,\ (x,y,z)^{2^{2(k-1)}},\ e,\ e).$$

From here it follows that for $k=1$ the loop H is non-associative and non-commutative and the identities

$$(x_1,x_2,x_3)^2=e,\quad [x_1,x_2]^2=e \ \text{ and } \ H\in K_{2,2,2^m}$$

hold and for $k>1$ H is a non-commutative group and in it the identity holds true

$$[x_1,x_2]^2=e \ \text{ and } \ H\in K_{1,2,2^m}.$$

We will show that any equality relation in H between the elements a,b and c is a trivial equality. Indeed, let

$$(a^\alpha b^\beta\cdot c^\gamma)\cdot[a,b]^\delta[a,c]^\lambda[b,c]^\mu(a,b,c)^\nu=e \tag{1}$$

be such an equality relation in H. Then we have

$$\Big(x^\alpha,(x^\alpha y^{2^{k-1}\beta}\cdot z^{2^{k-1}\gamma})\cdot[x,y]^{2^{k-1}\delta}[x,z]^{2^{k-1}\lambda}[y,z]^{2^{2(k-1)}\mu}(x,y,z)^{2^{2(k-1)}\nu},$$
$$y^\beta z^{2^{k-1}\gamma}[y,z]^{2^{k-1}\mu},z^\gamma\Big)=(e,e,e,e),$$

from where in F_3 it results that the equality relations

$$x^\alpha=e,\ y^\beta[y,z]^{2^{m-1}\mu}=e,\ z^\gamma=e, \tag{2}$$

$$[x,y]^{2^{k-1}\delta}[x,z]^{2^{k-1}\lambda}(x,y,z)^{2^{2(k-1)}\nu}=e, \tag{3}$$

hold true in the $\nu(N)$-free loop F_3. But any equality relation between the free generators x,y,z is a true identity in F_3. That is why the equalities from (2) and (3) are true identities in F_3. But the first and the last identity from (2) are true in F_3 only if

$$\alpha\equiv 0\ mod\ 2^m,\ \gamma\equiv 0\ mod\ 2^m.$$

From the second identity of (2), substituting in it $z=e$, and from identity (3), substituting in it alternatively $z=e$, $y=e$, we obtain

$$y^\beta=e,\ [y,z]^{2^{k-1}\mu}=e,\ [x,y]^{2^{k-1}\delta}=e,\ [x,z]^{2^{k-1}\lambda} \tag{4}$$

and

$$(x,y,z)^{2^{2(k-1)}\nu}=e. \tag{5}$$

But the identities from (4) are true in $F_3(x,y,z)$ only if

$$\beta\equiv 0\ mod\ 2^m,\quad \mu\equiv 0\ mod\ 2,\quad \delta\equiv 0\ mod\ 2,\quad \lambda\equiv 0\ mod\ 2$$

and identity (5), when $k=1$, holds true in $F_3(x,y,z)$ only if $\nu \equiv 0 \; mod \; 2$ and when $k>1$ for any positive integer ν. From which we can easily conclude that (1) is a trivial equality. Therefore, for $k=1$ in the variety $K_{2,2,2^m}$, and for $k>1$ in the variety $K^m_{1,2}$, the loop H has a finite representation formed of three generators without any equality relation. Hence for $k=1$ the loop H is $K^m_{2,2}$-free of the third rank and $H \in q(N)$.

2) N is non-associative and $p=3$. In this case the identity $(x,y,z)^3=e$ holds true in N. Assume that for a certain integer $k,\ 0 \le k \le m$ the identity $[x,y]^{3^k}=e$ holds true in N.

If $k=0$, then in N the identity $[x,y]=e$ holds true and N is a commutative Moufang loop. Then the $\nu(N)$-free commutative Moufang loop $F_3(x,y,z)$ is free in any variety of Moufang loops with the exponent 3^m. Hence $F_3(K_{3,1,3^m}) \cong F_3(x,y,z) \in q(N)$.

Let $k \ge 1,\ F_3 = F_3(x,y,z)$ be a $\nu(N)$-free loop of the third rank with free generators x,y,z, and $H = lp(a,b,c)$ be the subloop generated in F_3^4 by the elements

$$a=(x,x,e,e),\ b=(e,y^{2^{k-1}},y,e),\ c=(e,z^{2^{k-1}},z^{2^{m-1}},z).$$

Then, obviously

$$a^{3^m}=b^{3^m}=c^{3^m}=(e,e,e,e),\ [a,b]=(e,[x,y]^{3^{k-1}},e,e),\ [a,c]=(e,[x,z]^{3^{k-1}},e,e),$$

$$[b,c]=(e,[y,z]^{3^{2(k-1)}},[y,z]^{3^{k-1}},e),\ (a,b,c)=(e,(x,y,z)^{3^{2(k-1)}},e,e).$$

From here it follows that for $k=1$ the loop H is non-associative and non-commutative and the following identities hold true in it

$$(x_1,x_2,x_3)^3=e,\ [x_1,x_2]^3=e \ \text{ and } \ H \in K_{3,3,3^m},$$

and for $k>1$ H is a non-commutative group and in it the identities

$$[x_1,x_2]^3=e \ \text{ and } \ H \in K_{1,3,3^m}.$$

hold true. By analogy with case 2) we show that for $k=1$ the loop H is $K_{3,3,3^m}$-free of the third rank and for $k>1$ the loop H is $K_{1,3,3^m}$-free of the third rank and $H \in q(N)$.

3. N is associative and p is any prime number. Similarly to the previous cases we can show that if in the group N the identity $[x,y]^{p^k}$ holds true for a certain natural number $k,\ 1 \le k \le m$, then for $k=1$ we have $F_3(K_{1,p,p^m}) \in q(N)$. $\square$

Lemma 2. *If the 2-nilpotent Moufang loop N, generated by three elements, is infinite, then there exists such a variety $K \in \Re$ that $F_3(K) \in qN$.*

Proof. Since the loop N is not finite, then $\exp(N) = 0$. We will consider the following possible cases.

1) Let N be non-associative, in N the identity $(x, y, z)^2 = e$ holds true and $\exp(lp([u, v] \mid u, v \in N)) = 2^m s$, where m is a non-negative integer and 2 does not divide s.

We will show, first, that $m > 0$. Indeed, let $m = 0$. Then, according to (8) and the identities $(x, y, z)^2 = e$, $[x, y]^s = e$ we can deduce $e = [x, y, z]^s = [x, y]^s[x, z]^s(x, y, z)^{3s} = (x, y, z)^s$. Hence in N the identity $(x, y, z)^s = e$ holds true and, since 2 does not divide s, we conclude that in N the identity $(x, y, z) = e$ is also true. That is N is associative - a contradiction.

Hence, $m \geq 1$. Let $F_3 = F_3(x, y, z)$ be a $\nu(N)$-free loop of the third rank with free generators x, y, z, and $H = lp(a, b, c)$ be a subloop generated in F_3^4 by the elements

$$a = (x, x, e, e),\ b = (e, y^{2^{m-1}s}, y, e),\ c = (e, z^{2^{m-1}s}, z^{2^{m-1}s}, z).$$

Then, obviously, $\exp(H) = 0$ and the following equalities hold true

$$\begin{aligned} &[a, b] = (e, [x, y]^{2^{m-1}}, e, e),\ [a, c] = (e, [x, z]^{2^{m-1}}, e, e), \\ &[b, c] = (e, [y, z]^{2^{2(m-1)}}, [y, z]^{2^{m-1}}, e),\ (a, b, c) = (e, (x, y, z)^{2^{2(m-1)} \cdot s^2}, e, e). \end{aligned} \tag{6}$$

From where it results that for $m = 1$ the loop H is non-associative and non-commutative and for it the identities $(x_1, x_2, x_3)^2 = e$, $[x_1, x_2]^2 = e$ hold true. For $m > 1$ H is a non-commutative group and the identity $[x_1, x_2]^2 = e$ holds true in it. Therefore, for $m = 1$ the loop $H \in K_{2,2,0}$, and for $m > 1$ the loop $H \in K_{1,2,0}$.

We will show now that any equality relation in H between the elements a, b and c is a trivial equality. Indeed, let

$$(a^\alpha b^\beta \cdot c^\gamma) \cdot [a, b]^\delta [a, c]^\lambda [b, c]^\mu (a, b, c)^\nu = e \tag{7}$$

be such an equality relation. Then we have

$$\begin{aligned} \Big(x^\alpha, (x^\alpha y^{2^{m-1}s\,\beta} \cdot z^{2^{m-1}s\,\gamma}) \cdot [x, y]^{2^{m-1}s\,\delta} [x, z]^{2^{m-1}s\,\lambda} [y, z]^{2^{2(m-1)}s^2\mu} \\ (x, y, z)^{2^{2(m-1)}s^2\nu}, y^\beta z^{2^{m-1}s\,\gamma} [y, z]^{2^{m-1}s\,\mu}, z^\gamma\Big) = (e, e, e, e). \end{aligned} \tag{8}$$

Similarly to Lemma 1 we can show the identities

$$x^{\alpha} = e,\ y^{\beta} = e,\ z^{\gamma} = e, \tag{9}$$

$$[x,y]^{2^{m-1}s\,\delta} = e,\ [x,z]^{2^{m-1}s^2\lambda} = e,\ [y,z]^{2^{m-1}s^2\mu} = e, \tag{10}$$

$$(x,y,z)^{2^{2(m-1)}s^2\nu} = e. \tag{11}$$

Because $\exp(N) = exp\,(F_3) = 0$, the identities from (9) hold true in $F_3(x,y,z)$ only if

$$\alpha = 0,\ \beta = 0,\ \gamma = 0.$$

The identities from (10) are true only if $\delta \equiv 0\ mod\ (2)$, $\lambda \equiv 0\ mod\ 2$ and $\mu \equiv 0\ mod\ 2$ and the identity (11), when $m = 1$ is true in F_3 only if $\nu \equiv 0\ mod\ 2$ and when $m > 1$ – for any positive integer ν. We can easily conclude that (7) is a trivial equality. Therefore, for $m = 1$ in the variety $K_{2,2,0}$ and for $m > 1$ in the variety $K_{1,2,0}$, the Moufang loop H has a finite representation formed of three generators without any equality relation. Hence, for $m = 1$ the loop H is $K_{2,2,0}$-free of the third rank and for $m > 1$ the loop H is $K_{1,2,0}$-free of the third rank and $H \in q(N)$.

2) N is non-associative, the identities $(x,y,z)^3 = e$ and $\exp(lp\,([u,v]\,|\,u,\ v \in N)) = 3^m s$, hold true in it, where m is a non-negative integer and 3 does not divide s. Let $m = 0$, then we consider the subloop $H = lp(a,b,c)$, generated in the $\nu(H)$-free loop $F_3(x,y,z)$ by the elements $a = x,\ b = y^s,\ c = z^s$. We can notice that in the loop $F_3(x,y,z)$ the following equalities hold true

$$(a,b,c) = (x,y,z)^{s^3},\ [a,b] = [x,y]^s = e,\ [a,c] = [x,z]^s = e,\ [b,c] = [y,z]^{s^2} = e,$$

which implies that H is a commutative Moufang loop. As $\exp(H) = 0$, it results that H is a free 2-nilpotent commutative Moufang loop, which is contained in the variety $K_{3,1,0}$. Therefore $F_3(K_{3,1,0}) \cong H \in q(N)$.

Now assume that $m \geq 1$. Let $F_3 = F_3(x,y,z)$ be a $\nu(N)$-free loop of the third rank and $H = lp(a,b,c)$ be the subloop generated in F_3^4 by the elements

$$a = (x,x,e,e),\ b = (e,y^{3^{m-1}s},y,e),\ c = (e,z^{3^{m-1}s},z^{3^{m-1}s},z).$$

Then, obviously, $\exp(H) = 0$ and the following equalities hold true

$$[a,b] = (e,[x,y]^{3^{m-1}s},e,e),\quad [a,c] = (e,[x,z]^{3^{m-1}s},e,e),$$

$$[b,c] = (e,[y,z]^{3^{(m-1)}s^2},[y,z]^{3^{m-1}s},e),\quad (a,b,c) = (e,(x,y,z)^{3^{2(m-1)}s^2},e,e).$$

From here it results that for $m = 1$ the loop H is non-associative and non-commutative and the identities $(x_1, x_2, x_3)^3 = e$, $[x_1, x_2]^3 = e$ hold true in it, while for $m > 1$ H is a non-commutative group and the identity $[x_1, x_2]^3 = e$ holds in it. Therefore, for $m = 1$ the loop $H \in K_{3,3,0}$ and for $m = 1$ the loop $H \in K_{1,3,0}$. Then, similarly to case 1) we can show that for $k = 1$ the loop H is $K_{3,3,0}$-free of the third rank and for $k > 1$ the H is $K_{1,3,0}$-free of the third rank and $H \in q(N)$.

3) N is non-associative, the identities $(x, y, z)^3 = e$ (respectively, $(x,y, z)^2 = e$) and $\exp(lp\,([u, v] \,|\, u, v \in N)) = 0$ hold true in it. Let $F_3(x, y, z)$ be a $\nu(N)$-free loop with free generators x, y and z. It is clear that $F_3(x, y, z) \in K_{3,0,0}$ (respectively, $F_3(x, y, z) \in K_{2,0,0}$).

Let an arbitrary equality relation hold true in the $\nu(N)$-free loop $F_3(x, y, z)$

$$(x^\alpha y^\beta \cdot y^\gamma) \cdot [x, y]^\delta [x, z]^\lambda [y, z]^\mu (x, y, z)^\nu = e. \tag{12}$$

This equality relation is the identity true in $F_3(x, y, z)$. Then we can easily deduce that it implies the identities

$$x^\alpha = e,\ y^\beta = e,\ y^\gamma = e,\ [x, y]^\delta = e,\ [x, z]^\lambda = e,\ [y, z]^\mu = e,\ (x, y, z)^\nu = e,$$

which are true in $F_3(x, y, z)$ only if

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \delta = 0,\ \lambda = 0,\ \mu = 0,\ \nu \equiv 0\, mod\, 3$$

$$\text{(respectively}\ \ \nu \equiv 0\, mod\, 2)$$

From here we obtain that (12) is a trivial equality in $F_3(x, y, z)$. Therefore, $F_3(x, y, z)$ is a free loop in the variety $K_{3,0,0}$ (respectively, $K_{2,0,0}$). From here it results that $F_3(x, y, z) \in q(N)$.

4) N is non-associative, the identities $(x, y, z)^2 = e$ and $(x, y, z)^3 = e$ do not hold true in it. We consider one of the non-associative subloops $N_1 = lp\,(u^2 \,|\, u \in N)$, $N_2 = lp\,(u^3 \,|\, u \in N)$. The loops N_1 and N_2 are non-associative loops from the N. Because the identity $(x, y, z)^6 = e$ holds true in N, then the identities $(x, y, z)^3 = e$ and $(x, y, z)^2 = e$, respectively, hold true in the non-associative loops N_1 and N_2, respectively. Thus we obtain one of the situations studied above.

5) N is associative and $\exp(lp\,([u, v] \,|\, u, v \in N)) = p^m s$, where p is a prime number and p does not divide s and $m \geq 1$. In this case we consider in the $\nu(N)$-free group $F_3(x, y, z)$ the elements $a = x^s$, $b = y^{p^{m-1}s}$, $c = z^{p^{m-1}s}$ and

$H = lp(a, b, c)$. Then it is obvious that the loop H with exponent zero is non-commutative and the following equalities hold true

$$[a, b]^p = e, \quad [a, c]^p = e, \quad [b, c]^p = e.$$

Then in the non-commutative group H the identity $[x, y]^p = e$ is true. Applying the same reasoning as in 1) or 2) we obtain $F_3(K_{1,p,0}) \cong H \in q(N)$.

6) N is associative and $\exp(lp\,([u, v] \mid u, v \in N)) = 0$. Similarly to previous cases we can easily deduce that $F_3(K_{1,0,0}) \in q(N)$. □

Lemma 3. *For any variety $K \in \Re$ the following equalities $q(F_3(K)) = q(F_\omega(K))$, $q(F_3(K)) = q(F_n(K))$, $n = 4, 5, \ldots$, hold.*

Proof. It is enough to show that for any natural number n the K-free loop $F_n(K)$, of the finite rank n, belongs to the quasivariety Q. Because $F_3 \in Q$, and also for $F_1, F_2 \in Q$, we assume that $n > 3$ and let $F_n = F_n(x_1, \ldots, x_n)$ be a K-free loop of the rank n with free generators $x_1, \ldots, x_n$. We will show first that the K-free loop F_n is approximated by the subloops of the K-free loop $F_3(x, y, z)$, i.e. for any element $u \neq e$ from F_n there exists such a homomorphism φ from F_n in F_3 that $\varphi(u) \neq e$. If we admit that it is impossible, then in F_n there exists such an element $u = u(x_1, \ldots, x_n) \neq e$ that any homomorphism φ from F_n in F_3 we have $\varphi(u) \neq e$. We will represent the element u in its canonic form

$$u = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \cdot \prod_{1 \le i < j \le n} [x_i, x_j]^{\beta_{ij}} \prod_{1 \le i < j < k \le n} (x_i, x_j, x_k)^{\gamma_{ijk}},$$

where the multiplication of factors from parenthesis is performed in a certain established order, for instance, from the left to the right. Assume for a certain i, $1 \le i \le n$ that $x_i^{\alpha_i} \neq e$. The mapping $x_j \to e$, $j \in \{1, \ldots, n\} \backslash \{i\}, x_i \to x$ extends to a homomorphism ψ from F_n in F_3. That is why $\psi(u) = \psi(x_i)^{\alpha_i} = x^{\alpha_i}$ and in F_3 we get the equality $x^{\alpha_i} = e$. But the last expression is a true identity in the K-free loop $F_n(x, y, z)$, hence in F_n as well. But in this case we came to a contradiction with $x_i^{\alpha_i} \neq e$. Hence, we can say that $x_1^{\alpha_1} = e, \ldots, x_n^{\alpha_n} = e$ and

$$u = \prod_{1 \le i < j \le n} [x_i, x_j]^{\beta_{ij}} \prod_{1 \le i < j < k \le n} (x_i, x_j, x_k)^{\gamma_{ijk}}.$$

Assume for a certain pair (i, j), $1 \le i < j \le n$, that $[x_i, x_j]^{\beta_{ij}} \neq e$. The mapping $x_k \to e$, $k \in \{1, \ldots, n\} \backslash \{i, j\}$, $x_i \to x$, $x_j \to y$ extends to a homomorphism ψ from F_n in F_3. That is why $\psi(u) = [\psi(x_i), \psi(x_j)]^{\beta_{ij}} = [x, y]^{\beta_{ij}}$ and we get that in F_3 the identity $[x,y]^{\beta_{ij}} = e$ holds true. But then this identity also holds true in F_n, which contradicts the inequality $[x_i,x_j]^{\beta_{ij}} \neq e$. Hence, we can say that $\prod_{1\leq i<j\leq n} [x_i,x_j]^{\beta_{ij}} = e$ and

$$u = \prod_{1\leq i<j<k\leq n} (x_i,x_j,x_k)^{\gamma_{ijk}}.$$

Now assume for a certain triplet (i,j,k), $1 \leq i < j < k \leq n$ that $(x_i,x_j,\, x_k)^{\gamma_{ijk}}$. The mapping $x_l \to e$, $l \in \{1,\dots,n\}\backslash\{i,j,k\}$, $x_i \to x$, $x_j \to y$, $x_k \to z$ extends to a homomorphism ψ from F_n in F_3. That is why $\psi(u) = [\psi(x_i),\psi(x_j),\psi(x_k)]^{\gamma_{ijk}} = [x,y,z]^{\gamma_{ijk}}$ and we get that in F_3 the identity $[x,y,z]^{\gamma_{ijk}} = e$ holds true. But then this identity is also true in F_n, which contradicts the inequality $[x_i,x_j,x_k]^{\gamma_{ijk}} \neq e$. Then we can say that

$$\prod_{1\leq i<j<k\leq n} [x_i,x_j,x_k]^{\gamma_{ijk}} = e$$

and $u = e$. We came to a contradiction with the assumption that $u \neq e$. From here we can conclude that the loop F_n is approximated by the subloops of the loop F_3, hence it is included isomorphically in a Cartesian product of subloops of the loop F_3. Therefore, F_n belongs to the quasivariety $q(F_3)$ and, hence, F_n also belongs to the quasivariety Q. □

According the Lemmas 1, 2 and 3 we can formulate the following theorem.

Theorem 1. *If Q is a quasivariety that contains a nilpotent non-associative or non-commutative Moufang loop, then there exists at least one variety $K \in \Re$ so that $F_\omega(K) \in Q$.*

Corollary 1. *For any variety $K \in \Re$ the following statements are true:*

a) *if $q(F_\omega(K))$ contains a non-associative and non-commutative loop H, then $q(H) = q(F_\omega(K))$;*

b) *if $q(F_\omega(K))$ contains only commutative Moufang loops (respectively, groups) and H is a non-associative (respectively, non-commutative) loop from $q(H) = q(F_\omega(K))$.*

Remark 1. Since the following inclusions holds true

$$K_{3,1,0} \subset K_{3,3,0},\ K_{1,3,0} \subset K_{3,3,0},\ K_{3,1,3^m} \subset K_{3,3,3^m},\ \ m = 1,2,\dots,$$

then each of the quasivarieties $q(F_\omega(K_{3,3}),\ q(F_\omega(K_{3,3}^m)),\ m = 1,2,\dots,$ contains only two non-abelian subquasivarieties: one formed of commutative Moufang loops and another formed of groups.

Remark 2. According to identity (5) and (8) internal substitutions of any loop in $K_{3,0,0}$ are automorphisms variety. Loops so these varieties are A-loops (see the research nilpotent A-loops in [7]).

Remark 3. Each of the set $\{q(F_\omega(K_{0,2,2})),\ q(F_\omega(K_{2^m,2,2})),\ m = 2,3,\dots\}$ cvasivariete has only one non-abelian subcvasivariete own being generated free group of rank 2 of this cvasivariete.

From Theorem 1, Corollar 1 and Remarks 1–3 results the following sentence:

Corollary 2. *Cvasivarietaties us all non-abelian minimal of the lattice cvasivarietaties of 2-nilpotent Moufang loops are: - minimal non-associative quasivarieties of commutative Moufang loops*

$$q(F_\omega(K_{3,1,0})),\ q(F_\omega(K_{3,1,3^m}))\quad (m = 1,2,\dots);$$

- minimal non-associative and non-commutative quasivarieties of Moufang A-loops with one proper minimal non-associative sub-quasivariety of commutative Moufang loops and one proper minimal non-commutative subquasivariety of groups;

$$q(F_\omega(K_{3,0,0})),\ q(F_\omega(K_{3,3,0})),\ q(F_\omega(K_{3,3,3^m}))\quad (m = 1,2,\dots);$$

- minimal non-associative and non-commutative quasivarieties of Moufang loops with the only proper non-commutative subquasivariety of groups

$$q(F_\omega(K_{2,0,0})),\ q(F_\omega(K_{2,2,0})),\ q(F_\omega(K_{2,2,2^m}))\quad (m = 2,3,\dots);$$

- minimal non-commutative quasivarieties of groups

$$q(F_\omega(K_{1,0,0})),\ q(F_\omega(K_{1,p,0}))\quad (p = 2,3,\dots),$$

$$q(F_\omega(K_{1,2,2^m}))\,(m = 2,3,\dots),\ q(F_\omega(K_{1,p,p^m}))\,(p \geq 3,\ m = 2,3,\dots).$$

Further, we whole show a few concrete examples of nilpotent Moufang loops. First, we will prove the following important statement.

Theorem 2. *If the alternative ring K with a unit contains a nilpotent sub-ring R with index $n \geq 2$ (i.e. any product of n factors $a_1 a_2 \ldots a_n = 0$ for any $a_1, \ldots, a_n \in K$), then the set L of all elements of the form $1+x$, where $x \in R$, forms a-nilpotent Moufang loop of class $n-1$.*

Proof. The equality

$$(1+r)(1-r+r^2-\ldots+(-1)^{n-1}r^{n-1}=1$$

where $x \in R$, shows that any element from the set L is reversible and, therefore, L is a Moufang loop. Let now R^k the set of all linear combinations of all products of k elements from R. Then fir any $x \in R^k$ we have the equality

$$(1+x)^{-1} = 1-x+x^2-\ldots+(-1)^{n-1}x^{n-1}$$

or, in brief,

$$(1+x)^{-1} = 1+x^*, \tag{13}$$

where $x^* = -x+x',\ x \in R^{2k}$. Now, if $x, y \in r$ and $z \in R^k$, then, according to the Moufang theorem and the equality (13),

$$\begin{aligned}
&[1+x,\,1+y,\,1+z] = ((1+x)\cdot(1+y)(1+z))^{-1}\cdot((1+x)(1+y)\cdot(1+z)) = \\
&((1+z)^{-1}(1+y)^{-1}\cdot(1+x)^{-1})\cdot((1+x)(1+y)\cdot(1+z)) = \\
&((1+z^*)(1+y)^{-1}\cdot(1+x)^{-1})\cdot((1+x)(1+y)\cdot(1+z)) = \\
&((1+y)^{-1}(1+x)^{-1}+z^*\cdot(1+y)^{-1}(1+x)^{-1}\cdot((1+x)(1+y)\cdot(1+z)) = \\
&1+z+(z^*\cdot(1+y)^{-1}(1+x)^{-1})\cdot((1+x)(1+y)\cdot(1+z)) = \\
&1+z+(z^*\cdot(1+y^*)(1+x^*))\cdot((1+x)(1+y)\cdot(1+z)) = \\
&1+z+(z^*+z^*\cdot x^*+z^*\cdot y^*(1+x^*))\cdot(1+z+y+x+xz+yz+xy\cdot z) = \\
&1+z+z^*+z^*x^*+z^*\cdot y^*(1+x^*)+ \\
&(z^*(1+x^*+y^8(1+x^*)))\cdot(z+y+x+xz+yz+xy\cdot z) = \\
&1+z-z+z'+z^*x^*+ \\
&z^*\cdot y^*(1+x^*)+(z^*(1+x^*+y^8(1+x^*)))\cdot(z+y+x+xz+yz+xy\cdot z) = 1+z_0,
\end{aligned}$$

where

$$z_0 = z'+z^*x^*+z^*\cdot y^*(1+x^*)+(z^*(1+x^*+y^*(1+x^*))\cdot(z+y+x+xz+yz+xy\cdot z) \in R^{k+1}.$$

Hence, we have

$$[1+x,\,1+y,\,1+z] = 1+z_0 \in 1+R^{k+1}. \tag{14}$$

Similarly for $x \in R^k$ and $y \in R$ we can deduce

$$\begin{aligned}
&[1+x,\,1+y] = (1+x)^{-1}\cdot(1+y)^{-1}\cdot((1+x)(1+y)) = \\
&(1+x)^{-1}(1+(1+y)^{-1}\cdot x(1+y)) = \\
&(1+x^*)(1+(1+y^*)\cdot x(1+y)) = 1+x^*+(1+x^*)\cdot(1+y^*)(x+xy) = \\
&1+x^*+(1+x^*)\cdot(x+xy+y^*x+y^*\cdot xy) = \\
&1-x+x'+x+x^*x+(1+x^*)(xy+y^*x+y^*\cdot xy) = \\
&1+x'+x^*x+(1+x^*)(xy+y^*x+y^*\cdot xy) = 1+x_0
\end{aligned}$$

where

$$x_0 = x'+x^*x+(1+x^*)(xy+y^*x+y^*\cdot xy) \in Rk+1.$$

Hence, we have

$$[1+x,\,1+y] = 1+x_0 \in 1+R^{k+1}. \tag{15}$$

Now, if $x \in R^k$ and $y, z \in R$, then $x^* \in R^k$ and $y^*, z^* \in R$, which results in

$$[1+z^*,\,1+y^*,\,1+x^*] \in 1+R^{k+1} \quad \text{and} \quad [1+z^*,\,1+y^*,\,1+x^*]{-1} \in 1+R^{k+1}.$$

Then, according to (14) and (15), we will have

$$\begin{aligned}
&(1+x)\cdot(1+y)(1+z) = (1+x^*)^{-1}\cdot(1+y^*)^{-1}(1+z^*)^{-1} = \\
&((1+z^*)^{-1}(1+y^*)^{-1}\cdot(1+x^*)^{-1}) = \\
&(((1+z^*)\cdot(1+y^*)^{-1}(1+x^*))[1+z^*,\,1+y^*,\,1+x^*]) = \\
&[1+z^*,\,1+y^*,\,1+x^*]^{-1}\,((1+x^*)^{-1}(1+y^*)^{-1}\cdot(1+z^*)^{-1}) = \\
&[1+z^*,\,1+y^*,\,1+x^*]^{-1}\,((1+x)(1+y)\cdot(1+z)) = (1+x)(1+y)\cdot(1+z) \\
&[1+z^*,\,1+y^*,\,1+x^*v]^{-1}\,[[1+z^*,\,1+y^*,\,1+x^*]^{-1},(1+x)\cdot(1+y)(1+z)] \\
&\in (1+x)(1+y)\cdot(1+z)(1+R^{k+1}),
\end{aligned}$$

which shows that the associator

$$[1+x,\,1+y,\,1+z] \in 1+R^{k+1}. \tag{16}$$

Now, taking into account (15) and (16), it is not difficult to deduce that all associator-commutators of multiplicity $n-1$ of the elements from Moufang loop L are equal tj the unit, which wqs required. □

Example 1. Let R be an alternative n-nilpotent ring and $\mathbb{Z}$ the ring of integers. On the set $K = R \times \mathbb{Z}$ we define operations +" and ·" as follows:

$$(a,k)+(b,l) = (a+b,\,k+l),$$

$$(a,k)\cdot(b,l) = (a\cdot b+la+kb,\,k\cdot l),$$

where (a,k), $(b,l) \in K$. It is easy to see that K together with operations defined is an alternative ring with the unit $l = (0,1)$ and the set L' of all elements of the form $(a,0)$ is a subring K isomorphic to R. From here, due to Theorem 2, the set $L = e + L'$ forms an $(n-1)$-nilpotent Moufang loop. In Particular, if $n = 3$ and the ring R has characteristic 3, then L is a 2-nilpotent Moufang loop with exponent 3.

Example 2. The basis of Cayley-Dixon algebra K [4], over the field of real numbers R, consists of the elements $e_1 = 1$, $e_2 = i$, $e_3 = j$, $e_4 = k$, $e_5 = e$, $e_6 = ie$, $e_7 = je$, $e_8 = ke$ the first of which is a unit for algebra K and the following four form the basis of the sub-algebra of quaternions that are multiplied by the others in accordince with the relations:

$$\begin{aligned} &i^2 = j^2 = k^2 = e^2 = -1, \\ &ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \\ &eq = \overline{q}e,\ p \cdot qe = qp \cdot e,\ pe \cdot q = p\overline{q},\ pe \cdot qe = -\overline{q}p, \end{aligned} \tag{17}$$

where $\overline{q} = -q$, $p, q \in \{i, j, k\}$.

The Cayley numbers K are multiplied according to the distributive laws and relations (17), it is easy to verify that

$$[e_i, e_j] = 1 \text{ or } [e_i, e_j] = -1, \quad [e_i, e_j, e_k] = 1 \text{ or } [e_i, e_j, e_k] = -1. \tag{18}$$

From (17) and (18) we can see that the subsets

$$L = \{\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \pm e,\ \pm ie,\ \pm je,\ \pm ke\},$$

$$\overline{L} = \overline{R} \cup \overline{R}i \cup \overline{R}j \cup \overline{R}e \cup \overline{R}ie \cup \overline{R}je \cup \overline{R}ke\,(\overline{R} = R \setminus \{0\})$$

with respect yo the multiplication are Moufang loops with the associators and commutators equal to 1 or -1, hence, they belong to the center of this loop. therefore, Moufang loops L and $\overline{L}$ are non-associative, non-commutative and 2-nilpotent. It is easy to verify that the exponent of $\overline{L}$ is 4 and the exponent of $\overline{L}$ is infinite and in both loops the following identities hold

$$(x, y, z)^2 = 1,\ [x, y]^2 = 1.$$

Therefore, $L \in K_{2,2,2^2}$ and $\overline{L} \in K_{2,2,0}$.

Example 3. In the ring of all square matrices of order $n \geq 3$ over the Cayley-Dixon algebra we study the set of all matrices of the form $E + A \cdot q$, where E is the unit matrix of order n, q is the Cayley number and A is an upper triangular (respectively, lower triangular) matrix of order n that have the unit on the main diagonal and the other elements, above it respectively, below it) are real numbers the totality of these matrices A forms a $(n-1)$-nilpotent group (see [8]). Using direct calculations we can see that for any elements $E + Ap$, $E + Bq$, $E + Cr \in L$ the following relations hold

$$[E + Ap,\ E + Bq,\ E + Cr] = [Ap,\ Bq,\ Cr] = [A, B, C][p, q, r] = \pm E.$$

From here we can deduce

$$\begin{aligned}&[[E + A_1p_1,\ E + A_2p_2,\ E + A_3p_3], E + A_4p_4,\ E + A_5p_5] =\\ &[[A_1p_1,\ A_2p_2,\ A_3p_3], A_4p_4,\ A_5p_5] = [\pm E, A_4p_4, A_5p_5] =\\ &[E, A_4, A_5][\pm 1, p_4, p_5] = E,\end{aligned}$$

$$\begin{aligned}&[[E + A_1p_1,\ E + A_2p_2,\ E + A_3p_3], E + A_4p_4] =\\ &[[A_1p_1,\ A_2p_2,\ A_3p_3], A_4p_4] = [\pm E, A_4p_4] = [E, A_4][\pm 1, p_4] = E,\end{aligned}$$

$$\begin{aligned}&[[E + A_1p_1,\ E + A_2p_2], E + A_3p_3,\ E + A_4p_4] =\\ &[[A_1p_1,\ A_2p_2], A_3p_3,\ A_4p_4] = [[A_1, A_2][p_1, p_2], A_3, A_4] =\\ &[\pm[A_1, A_2], A_3p_3, A_4p_4] = [[A_1, A_2], A_3, A_4][\pm 1, p_3, p_4] = \pm E,\end{aligned}$$

$$\begin{aligned}&[[\dots[[E + A_1p_1,\ E + A_2p_2], E + A_3p_3], \dots], E + A_np_n] =\\ &[\dots[[A_1p_1,\ A_2p_2], A_3p_3], \dots], A_np_n] = E.\end{aligned}$$

Therefore, we can conclude that the set L forms a relation with the multiplication by a $(n-1)$-nilpotent Moufang loop. in particular, if $n = 3$ the Moufang loop L is 2-nilpotent, i.e. all commutators of the loop belong to the center. The center of loop L consists of all matrices on whose main diagonal there is a unit, in its upper respectively, lower) corner there is a real number and its other elements are equal to zero. from here we can easy understand that any element from the center that is different from a unit is not a finite order. in particular, any non-trivial commutator has a finite order. Therefore, the non-associative and non-commutative Moufang loop L belongs to the variety $K_{2,0,0}$.

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