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The structure of shock and interphase layers for a heat conducting Maxwellian rate-type approach to solid-solid phase transitions

Part I: Thermodynamics and admissibility

Abstract We consider a thermoelastic model for phase transforming materials which can adequately describe the evolution with respect to the temperature of the hysteresis loop both in compression and tension tests. The specificity of this model is that the Grüneisen coefficient changes its sign. The model is augmented by considering a dissipative mechanism governed by a Maxwellian rate-type constitutive equation. Existence and uniqueness of traveling wave solutions are investigated. One derives that the admissibility condition induced by the Maxwellian rate-type approach, coupled or not with Fourier heat conduction law is related to the chord criterion with respect to the Hugoniot locus. We investigate the structure of profile layers and their energetic properties. The influence of the exothermic or endothermic character of phase transitions on the inner structure of profile layers is captured.

Keywords Phase transitions · Thermoelasticity · Shocks · Entropy · Traveling waves · Admissibility

1 Introduction

The subject of non-linear wave propagation which causes changes not only in stress or motion, but also in heat and temperature has attracted the interest of both theoreticians and experimentalist. We mention here for example, as background references, the comprehensive analysis of Drumheller [7] and the extensive review article by Menikoff and Plohr [23].

It is known that the temperature variation is especially important during phase transitions even in quasistatic tests (Shaw and Kiriakides [29]). The more this happens during impact-induced experiments on materials capable of undergoing phase transformations, like shape memory alloys (SMAs) and many ceramics. Unfortunately, temperature measurements are exceedingly difficult to be obtained during shock wave experiments and too little is known about the structure of propagating phase boundaries. Nevertheless, relevant theoretical studies that take proper account of thermal as well as mechanical constitutive response of thermoelastic and phase transforming materials in dynamic problems have been done (James[17], Pego [26], Dunn and Fosdick [6], Abeyaratne and Knowles [2], Knowles [18] and the literature therein).

The study of steady, structured shock waves or traveling waves is an important subject in the theory of waves both from practical and theoretical point of view. Thus, the study of traveling waves provides admissibility criteria for discontinuous solutions of adiabatic thermoelastic theories which derives from associated dissipative systems (Liu [21], Slemrod [30], Pego [26]). Steady shock waves were first analyzed in Newtonian fluids (Weyl [34], Gilbarg [15] and the literature therein). In metallic materials, they have been experimentally

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observed in the 19060s (see for instance Barker [4]). The structure of these steady shock waves, which is due to the viscous effects governing the viscoplastic flow of metals, has been recently investigated and revisited by Molinari and Ravichandran [24].

Since stable phases of materials which suffer phase transformations are modelled, in general, by thermoelastic constitutive relations, and the transition from one stable phase to another does not occur instantaneously, but there exists a phase transition time, it appears as natural to question if a propagating phase boundary has a structure and which are its thermomechanical features. We investigate in the following this issue for an augmented model of a thermoelastic material which can change both the phase and the sign of the Grüneisen coefficient.

We briefly introduce in Sect. 2 the dynamic thermomechanic bar theory in lagrangian description. Based on experimental observations on pseudoelastic NiTi (Shaw [28]), in Sect. 3 we describe the constitutive assumptions for a thermoelastic three phase materials, i.e. a material which can exist in the austenitic phase $\mathscr{A}$ and in two variants of martensite $\mathscr{M}^{\pm}$, one obtained in tension ($\sigma > 0$) and the other in compression ($\sigma < 0$) tests. As usual for phase transforming materials (Ericksen [8], Abeyaratne et *al* [3]) the stress-strain relation $\sigma = \sigma_{eq}(\varepsilon, \theta)$ at fixed temperature θ is non-monotone on certain strain intervals. The particular feature of our assumptions, in agreement with the experimental behavior in [28], is that $\frac{\partial \sigma_{eq}}{\partial \theta}$ is positive on that part of the constitutive domain in the $\varepsilon - \theta$ plane associated with $\mathscr{A} \leftrightarrow \mathscr{M}^{+}$ expansive phase transformation and it is negative on the complementary part associated with $\mathscr{A} \leftrightarrow \mathscr{M}^{-}$ compressive phase transformation. This behavior reflects experimental observations related with the shape memory effect and the fact that in traction tests the hysteresis loop moves upwards, while in compression tests it moves downwards in the $\varepsilon - \sigma$ plane, as the temperature grows. Therefore, it follows that the Grüneisein coefficient which typically is positive, changes its sign in the $\varepsilon - \theta$ plane and this behavior has an important effect on the structure of profile layers. Further we remind the thermodynamic relations arising from the Clausius-Duhem inequality, the Gibbsian thermostatic stability conditions, and on the other side the dynamic stability condition which ensures the existence of a real sound speed for the thermoelastic material. Consequently, we associate the stable/unstable phases of the material with the domain of hyperbolicity/ellipticity of the dynamic thermoelastic PDE system. We end this section by describing the jump relations across a first order discontinuity for the adiabatic system of thermoelasticity and by characterizing the Hugoniot locus in the $\varepsilon - \theta$ plane and in the $\varepsilon - \sigma$ plane.

It is known that the Riemann problem for a thermoelastic bar capable of undergoing non-isothermal phase transitions may not have a unique solution even if the requirement that the entropy has to increase after the passage of the wave discontinuity is satisfied. One way to identify meaningful weak solutions for the quasilinear adiabatic thermoelastic system is to augment the constitutive law $\sigma = \sigma_{eq}(\varepsilon, \theta)$ in such a way that the stress depends additionally on other physical mechanisms. Usually, these are obtained by introducing a dependence on strain rate and spatial gradients of strain (James [16], Slemrod [30], [31], Truskinovski [33]). We adopt in Sect. 4 another point of view and we augment the thermoelastic constitutive equation by assuming that the stress depends additionally not only on strain rate $\dot{\varepsilon}$, but also on the stress rate $\dot{\sigma}$. Thus, we introduce a dissipative regularizing term which includes stress relaxation phenomena toward equilibrium between phases. Hence, we consider the following approach which combines aspects of both the Maxwell and KelvinVoigt models, i.e. $\sigma = \sigma_{eq}(\varepsilon, \theta) + \mu \dot{\varepsilon} - \tau \dot{\sigma}$, where $\mu > 0$ is a Newtonian viscosity and $\tau > 0$ is a time of relaxation. This relaxation time could be related with a phase transition time. Next, we describe the necessary and sufficient restrictions imposed by the Clausius-Duhem inequality on this Maxwellian rate-type model. This constitutive model has been successfully used to describe quasistatic strain-, stress- and temperature-controlled austenitic-martensitic phase transformation in shape memory alloys in [11], [12], [13], while impact induced phase transformation for the isothermal case in [9] .

In order to exhibit the inner structure of shock and interphase layers corresponding to this augmented thermomechanic theory, Sect. 5 is devoted to a detailed analysis of traveling wave solutions. Since for $\tau = 0$ our rate-type constitutive equation reduces to the Kelvin-Voigt model in solid mechanics, which is equivalent with the Navier-Stokes equation for one-dimensional flows, from our analysis one retrieves and thus one revisits classical results obtained in studying steady wave solutions for viscous, heat conducting fluids. Let us remind that Gilbarg [15] has given a sufficient set of conditions on the equation of state, which includes Weyl's fluids [34], to prove the existence of one-dimensional shock layers and has investigated their limit behavior for small viscosity and heat conductivity. His constitutive restrictions correspond to a convex relation between pressure and specific volume and to a positive Grüneisen coefficient. A direct consequence of these constitutive assumptions is that the admissible shocks are of compressive heating type. The non-convex case in non-isothermal gas dynamics has been considered later by Liu [22] and it leads to the occurrence of shocks of expansive cooling type. He proposed an admissibility condition of Oleinik type [25], called extended entropy condition, which is just a chord criterion with respect to the Hugoniot locus. His proof assumes there is no heat conduction. When both viscosity and heat conduction are considered in the structure of the profile layer Gilbarg's result for the non-convex case has been extended by Pego [26].

Starting from our special constitutive assumptions we pursue two main objectives in this section. First, we discuss the existence and uniqueness of traveling wave solutions for the augmented PDEs system. In this way we answer the question, which is the admissibility condition induced by the Maxwellian rate-type approach, coupled or not with Fourier heat conduction law. Second, we investigate the inner structure of profile layers, the capacity of heat conduction and/or relaxation (viscous) dissipative mechanisms to structure shock waves and phase transition fronts and the effect of Grüneisen coeffcient. The questions to be answered are: 1) what happens when the viscous added effect and the heat conductivity effect vanish ?; 2) inherits the adiabatic thermoelastic wave structure with sharp interfaces the wave structure of the augmented theory ?

We find that the chord criterion with respect to the Hugoniot locus in the $\varepsilon-\sigma$ plane is, in general, a necessary and sufficient condition for the existence and uniqueness of viscous, heat conducting profile layers. It should be noted that this is an extremely practical admissibility condition for discontinuous solutions of the adiabatic thermoelastic system because it does not depend on the considered dissipative mechanisms.

We also show that there may exist a non-physical situation, and we characterize it from thermodynamical point of view, when a strong discontinuity satisfies the chord criterion, but a viscous, heat-conducting profile layer does not exist if the viscosity effect is dominated by the heat conductivity effect, like in the example given by Pego [26].

We consider separately the cases when the Grüneisen coefficient is positive, negative or changes its sign inside the profile layer. We find that when the Grüneisen coefficient changes sign inside the layer, the temperature variation is non-monotone, and even more, it reaches lower/larger values than the initial and final temperature for compressive/expansive wave discontinuity. This finding could be also important in terms of experimental. Thus, the profile layer of the temperature displays an asymmetric spike-layer form which is in agreement with the exothermic or endothermic character of phase transformation. On the other side, this behavior implies that, in this case, the adiabatic thermoelastic temperature structure with sharp interface does not inherit the structure of the augmented theory.

We investigate the basic difference between the effect of viscosity and heat conduction on the structure of the profile layers. We illustrate that when the only structuring mechanism is the heat conduction the temperature is continuous, but the strain may be discontinuous having isothermal jumps inside the profile layer.

We show that the entropy production in a viscous, thermally conducting profile layer of Maxwells type is independent of viscosity or heat conduction. Moreover, we show that when the viscosity effect dominates the heat conductivity effect then the variation of entropy inside the profile layer is monotone, while in the opposite case the entropy variation is non-monotone and even more its values can become inside the profile layer lower than the entropy front state and/or larger than the entropy of the Hugoniot back state.

For a quantitative analysis of these features an explicit piecewise linear model based on experimental results of the type obtained by Shaw [28] will be given in Part II [10] and investigated numerically.

2 Thermomechanic bar theory. Lagrangian description

We consider a thin cylindrical bar with length L, constant cross-sectional area, constant mass density ρ (mass per unit length) in an unstressed reference configuration, which corresponds to a defined phase of the material. Let the function $x=\chi(X,t)$ express the *longitudinal motion* of the bar and $\theta=\theta(X,t)>0$ express the *absolute temperature*. The first gives the actual position x occupied at the time t by a particle labelled $X\in[0,L]$ in the reference configuration. The function $\chi(X,t)$ is assumed to be injective and bi-continuous with respect to X. Whenever $\chi(X,t)$ is differentiable we denote by $\varepsilon(X,t)=\frac{\partial\chi}{\partial X}-1>-1$ the *strain* at point X and by $v=v(X,t)=\frac{\partial\chi}{\partial t}$ the *particle velocity*. We denote by $\sigma=\sigma(X,t)$ the *nominal stress* (longitudinal force per unit area in the reference configuration), by $e=e(X,t)$ the *specific internal energy* per unit mass, by $\eta=\eta(X,t)$ the *specific entropy* per unit mass, by $q=q(X,t)$ the *axial heat flux*, by $r=r(X,t)$ the *lateral heat exchange* of the bar with its surrounding. At points (X,t) where v, ε, θ, σ, e, q and r are smooth functions the compatibility equation between strain and particle velocity, the balance of momentum and the balance of energy become

$$\frac{\partial\varepsilon}{\partial t}-\frac{\partial v}{\partial X}=0,\quad \rho\frac{\partial v}{\partial t}-\frac{\partial\sigma}{\partial X}=0,\quad \rho\frac{\partial e}{\partial t}-\sigma\frac{\partial\varepsilon}{\partial t}+\frac{\partial q}{\partial X}=r. \tag{1}$$

If we suppose that across a curve $X = S(t)$ in the $X - t$ plane at least one of the quantities v, ε, θ, σ, e, q experience jumps then the continuity of the motion χ, the balance of momentum and energy require

$$\dot{S}[\varepsilon] + [v] = 0, \quad \rho\dot{S}[v] + [\sigma] = 0, \quad \rho\dot{S}[e] + <\sigma> [v] - [q] = 0. \tag{2}$$

Such a curve is usually called a *strong discontinuity*, or a *first order discontinuity*. Here $\dot{S}(t)$ denotes the speed of propagation of the discontinuity and for any quantity $f = f(X,t)$ we have used the notations $[f](t) = f^+(t) - f^-(t)$=$f(S(t)+,t) - f(S(t)-,t)$ and $<f>(t) = \frac{1}{2}(f^+(t) + f^-(t))$. We name $X > S(t)$ as the + side and $X < S(t)$ as the $-$ side of the discontinuity.

The second law of thermodynamics in the form of the Clausius-Duhem inequality reads

$$\rho\frac{\partial\eta}{\partial t} \geq -\frac{\partial}{\partial X}\left(\frac{q}{\theta}\right) + \frac{r}{\theta}, \tag{3}$$

while for jump discontinuities it becomes

$$-\rho\dot{S}[\eta] + \left[\frac{q}{\theta}\right] \geq 0. \tag{4}$$

We note that for smooth processes the Clausius-Duhem inequality is used to restrict the form of the constitutive relations, while for non-smooth processes, i.e. solutions with jump discontinuities, it becomes an additional constraint that weak solutions must satisfy.

Let us consider the case when $q^+ = q^- = 0$ across a propagating strain discontinuity. Then the energy balance $(2)_3$ and the entropy inequality (4) across the discontinuity become

$$\rho[e] - <\sigma> [\varepsilon] = 0, \quad \text{and} \quad \rho\dot{S}(\eta^+ - \eta^-) \leq 0. \tag{5}$$

The first relation is known as the *Rankine-Hugoniot equation*. The second one asserts that after the passage of a strong discontinuity the entropy of a particle will not decrease.

3 Three phase materials - the thermoelastic case

3.1 Constitutive assumptions

Solid-solid phase transformations are responsible for the remarkable properties of SMAs. They are well understood and explained at crystallographic level. Basically, there are two relevant phases associated with SMAs, the austenite (stable at high temperatures) and the martensite (stable at low temperatures). While the austenite has a well-ordered body-centered cubic structure that presents only one variant, the martensite can form even twenty four variants. For an uniaxial test at a given temperature, it is enough to consider a material which exists in the austenite phase $\mathscr{A}$, for sufficiently small values of strain, and in two variants of martensite $\mathscr{M}^+$ and $\mathscr{M}^-$. One variant is obtained for sufficiently large tensile strain and the other variant for sufficiently large compressive strain, respectively. In general this deformation behavior for single crystal and polycrystalline NiTi was observed to be asymmetric in tension and in compression (Gall et *al* [14]).

From phenomenological point of view, starting with the paper by Ericksen [8], the reversible phase transformations in crystalline solids have been successfully studied using the theory of thermoelasticity with non-convex free-energy or, equivalently, in the one-dimensional context, with non-monotone stress-strain relation for certain interval of temperature. In this paper we consider such a stress-strain-temperature relation

$$\sigma = \sigma_{eq}(\varepsilon, \theta), \tag{6}$$

in order to characterize the response of a three phase shape memory alloy in traction and compression tests. This phenomenological constitutive equation can be determined starting from isothermal stress-strain curves obtained experimentally at very low strain-rates over an interval of temperature and from the macroscopic observations which accompany the evolution of inhomogeneous deformation. A typical example is given by the pseudoelastic responses of a nearly equiatomic polycrystalline NiTi alloy under uniaxial traction tests reported by Shaw and Kyriakides [29] and Shaw [28, Fig.3] for temperatures between 15 °C and 55 °C.

The above mentioned set of uniaxial displacement controlled tests conducted in nearly isothermal conditions are characterized by hysteresis loops having the following characteristics. The bar, initially in the phase of low stretch (austenite), starts to deform elastically in a homogeneous manner. This homogeneity is lost shortly after the maximum stress $\sigma = \sigma_M^+(\theta)$, which corresponds to the strain level $\varepsilon = \varepsilon_M^+(\theta)$, is reached (see Fig. 1). Thus the beginning of a stress decay is followed immediately by a significant stress-drop which accompanies the first nucleation of martensite. The forward austenite - martensite phase transformation produces a well defined upper stress plateau with small oscillations. Along it the transformation occurs in a localized way, i.e through nucleation events and subsequent growth of the high stretch phase (martensite) into the austenite phase. Once the transformation is complete the specimen starts again to deforms elastically and homogeneously while the slope of the stress-strain relation $\sigma = \sigma_{eq}(\varepsilon, \theta)$ is again positive.

During unloading the stress decreases nonlinearly while the specimen deforms homogeneously in the new martensite phase. This homogeneity is lost shortly after a minimum stress $\sigma = \sigma_m^+(\theta)$ has been reached (see Fig. 1), which corresponds to the strain $\varepsilon = \varepsilon_m^+(\theta)$. After a sudden stress rise, unstable transformation from martensite to austenite proceeds along a lower stress plateau by the propagation of distinct phase fronts along the length of the unloaded specimen.

Since along the loading and unloading stress plateaus the coexistence of two solid phases is allowed and, in general, multiple co-existent phase distributions are possible for a single axial stress state it is natural and common to consider the slope of the stress-strain relation $\sigma = \sigma_{eq}(\varepsilon, \theta)$ negative for $\varepsilon \in (\varepsilon_M^+(\theta), \varepsilon_m^+(\theta))$. We will see later in what way the monotone increasing/decreasing stress-strain relations $\sigma = \sigma_{eq}(\varepsilon, \theta)$ are associated with the so called stable/unstable states of the material.

While the monotone increasing parts of the stress-strain relations $\sigma = \sigma_{eq}(\varepsilon, \theta)$ can be chosen in such a way to fit known quasi-static isothermal experiments like in Shaw [28, Fig.3], the monotone decreasing part of these curves cannot be determined in a direct way from such experiments. Consequently, in general, they are chosen in a conventional way which is illustrated in the example considered in Part II of this paper [10].

Let us note that for theories like those developed by Abeyaratne and Knowles [1] based on additional constitutive information in the form of driving force and nucleation criteria an explicit form for $\sigma = \sigma_{eq}(\varepsilon, \theta)$ on the unstable interval $\varepsilon \in (\varepsilon_M^+(\theta), \varepsilon_m^+(\theta))$ is necessary only in so far as the Maxwell stress need to be determined. On the other side, theories which include rate-effects like in our case possess their own kinetics due to the (viscous) dissipation mechanisms incorporated. Thus, the form of the descending part of the constitutive equation $\sigma = \sigma_{eq}(\varepsilon, \theta)$ will only affect the kinetics of phase transformation, i.e. the rate at which the transformation takes place in the unstable interval. Indeed, it was shown, for the isothermal case, in Făciu and Molinari [9, Part II, Sect. 2, relations (11)-(12)] how the slope of the equilibrium curve influences the growth/decay of a perturbation of an equilibrium state.

The same type of deformation behavior, but in general asymmetric, can be observed in compression tests. Therefore, we suppose in the following that the pairs of stress and strain $(\sigma = \sigma_M^\pm(\theta), \varepsilon = \varepsilon_M^\pm(\theta))$ and $(\sigma = \sigma_m^\pm(\theta), \varepsilon = \varepsilon_m^\pm(\theta))$ associated with the changes of slopes of the equilibrium stress-strain relation at constant temperature (see Fig. 1) can be determined experimentally. Using this information we can plot a phase diagram in the $\varepsilon - \theta$ plane, like in Fig. 2, which contains essential constitutive information on phase transformation.

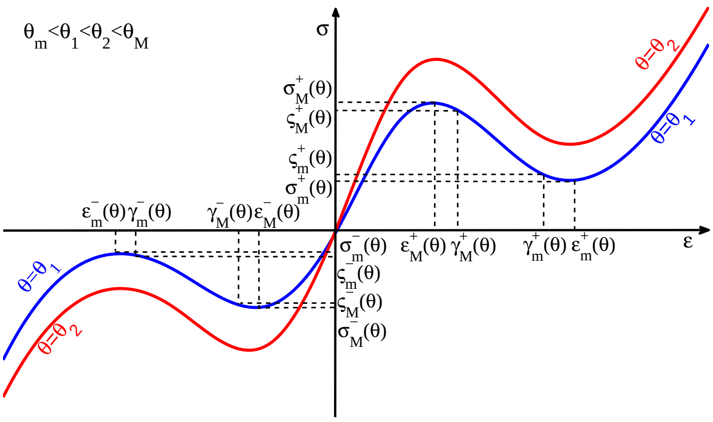

Fig. 1 Evolution of the stress-strain curves with respect to temperatures: $\theta \in (\theta_m, \theta_M)$ - pseudoelastic range.

In the following we assume (see also Abeyaratne et *al* [3]) there are two critical temperatures θ_m and θ_M such as, for $\theta > \theta_M$ the material only exists in its austenite form no matter what the stress level is, whereas

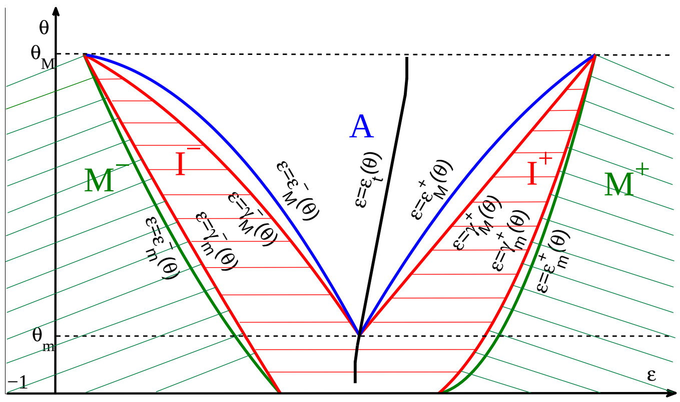

Fig. 2 Phase diagram in the $\varepsilon-\theta$ plane.

for $\theta<\theta_m$ the material only exists in its martensitic forms. For $\theta\in[\theta_m,\theta_M]$ all three phases are available to the material. The thermomechanical assumptions we consider here are.

H1) The boundary curves $\varepsilon=\varepsilon_m^{\pm}(\theta)$, $\varepsilon=\varepsilon_M^{\pm}(\theta)$ of the phase diagram in the $\varepsilon-\theta$ plane (see Fig. 2) are *continuously differentiable* and have the following properties:

$$\frac{\mathrm{d}\varepsilon_M^+(\theta)}{\mathrm{d}\theta}>0,\quad \frac{\mathrm{d}\varepsilon_M^-(\theta)}{\mathrm{d}\theta}<0\quad \text{for } \theta\in(\theta_m,\theta_M);\qquad \frac{\mathrm{d}\varepsilon_m^+(\theta)}{\mathrm{d}\theta}>0,\quad \frac{\mathrm{d}\varepsilon_m^-(\theta)}{\mathrm{d}\theta}<0\quad \text{for } \theta<\theta_M$$
$$\varepsilon_M^+(\theta_m)=\varepsilon_M^-(\theta_m),\quad \varepsilon_m^-(\theta_M)=\varepsilon_M^-(\theta_M),\quad \varepsilon_m^+(\theta_M)=\varepsilon_M^+(\theta_M). \tag{7}$$

H2) The stress response function $\sigma_{eq}(\varepsilon,\theta)$ is *continuous*, *piecewise-smooth* and satisfies the following properties. **a**) At each temperature $\theta>\theta_M$, $\sigma_{eq}(\varepsilon,\theta)$ is a monotonically *increasing* function of strain. **b**) At each temperature $\theta\in[\theta_m,\theta_M]$ (see Fig. 1) the function $\sigma_{eq}(\varepsilon,\theta)$ is: a monotonically *increasing* function of strain for $\varepsilon<\varepsilon_m^-(\theta)$, for $\varepsilon\in(\varepsilon_M^-(\theta),\varepsilon_M^+(\theta))$ and for $\varepsilon>\varepsilon_m^+(\theta)$; a monotonically *decreasing* function of strain over the intervals $(\varepsilon_m^-(\theta),\varepsilon_M^-(\theta))$ and $(\varepsilon_M^+(\theta),\varepsilon_m^+(\theta))$. **c**) At each temperature $\theta<\theta_m$, $\sigma_{eq}(\varepsilon,\theta)$ is a monotonically *increasing* function of strain for $\varepsilon<\varepsilon_m^-(\theta)$ and $\varepsilon>\varepsilon_m^+(\theta)$ while on the remaining interval $(\varepsilon_m^-(\theta),\varepsilon_m^+(\theta))$ it is monotonically *decreasing*.

It is well known that the pseudoelastic hysteresis is strongly influenced by the temperature. Indeed, according to the traction tests reported by Shaw [28] the hysteresis loop moves *upward* as the temperature grows. On the other side, for compression tests, the hysteresis loop moves *downward* as the temperature grows (Tobushi et *al* [32]). Consequently, we consider the following natural assumption.

H3) There exists a monotone curve $\varepsilon=\varepsilon_t(\theta)$ across which $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}$ changes the sign (Fig. 2), i.e.

$$\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}>0,\ \text{for } \varepsilon>\varepsilon_t(\theta);\qquad \frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}<0,\ \text{for } \varepsilon<\varepsilon_t(\theta),$$
$$\varepsilon_M^-(\theta)<\varepsilon_t(\theta)<\varepsilon_M^+(\theta),\ \text{for } \theta\in(\theta_m,\theta_M)\quad \text{and}\quad \varepsilon_m^-(\theta)<\varepsilon_t(\theta)<\varepsilon_m^+(\theta),\ \text{for } \theta<\theta_m. \tag{8}$$

Concerning the *smoothness assumptions* of relation $\sigma=\sigma_{eq}(\varepsilon,\theta)$ we distinguish two cases.

S1) First, we consider $\sigma=\sigma_{eq}(\varepsilon,\theta)$ a *smooth function* (at least of class C^2) on its domain of definition.

S2) Second, we suppose $\sigma=\sigma_{eq}(\varepsilon,\theta)$ a *continuous and piecewise smooth function* on its domain of definition. More precisely, it is smooth (at least of class C^2) on each domain delimitate by the curves $\varepsilon=\varepsilon_M^{\pm}(\theta)$, $\varepsilon=\varepsilon_m^{\pm}(\theta)$, and $\varepsilon=\varepsilon_t(\theta)$ across which $\frac{\partial\sigma_{eq}}{\partial\varepsilon}$, $\frac{\partial\sigma_{eq}}{\partial\theta}$, $\frac{\partial^2\sigma_{eq}}{\partial\theta^2}$ may have jump discontinuities. A typical example is given in Part II [10] where a piecewise linear relation is considered.

3.2 Thermodynamic considerations for the thermoelastic model

It is well known that the second law of thermodynamics (31) imposes the following restrictions on the free energy $\psi=\psi_{eq}(\varepsilon,\theta)$, entropy $\eta=\eta_{eq}(\varepsilon,\theta)$ and dissipation of the thermoelastic model (6)

$$\sigma_{eq}(\varepsilon,\theta)=\rho\frac{\partial\psi_{eq}(\varepsilon,\theta)}{\partial\varepsilon},\quad \eta_{eq}(\varepsilon,\theta)=-\frac{\partial\psi_{eq}}{\partial\theta}(\varepsilon,\theta),\quad D_{th}=-\frac{q}{\theta}\frac{\partial\theta}{\partial X}\geq 0. \tag{9}$$

Indeed, in this case, for any smooth fields ε and θ there exists only thermal dissipation. Since we consider the Fourier law for axial heat conduction, i.e. $q=-\kappa\frac{\partial\theta}{\partial X}$, we remind that $(9)_3$ requires $\kappa>0$.

Let us first consider the smooth case **S1**). The stress response function $\sigma=\sigma_{eq}(\varepsilon,\theta)$, determined mainly from quasistatic experiments, defines a unique free energy function $\psi_{eq}(\varepsilon,\theta)$, modulo an additive function of temperature $\phi=\phi(\theta)$, as well as, the entropy $\eta=\eta_{eq}(\varepsilon,\theta)$, the internal energy $e=e_{eq}(\varepsilon,\theta)=\psi_{eq}+\theta\eta_{eq}$ and the specific heat at constant strain $C=C_{eq}(\varepsilon,\theta)$ by relations

$$\psi_{eq}(\varepsilon,\theta)=\int_{\varepsilon_0}^{\varepsilon}\frac{1}{\rho}\sigma_{eq}(s,\theta)\mathrm{d}s+\phi(\theta),\qquad \eta_{eq}(\varepsilon,\theta)=-\int_{\varepsilon_0}^{\varepsilon}\frac{1}{\rho}\frac{\partial\sigma_{eq}(s,\theta)}{\partial\theta}\mathrm{d}s-\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta},\tag{10}$$

$$C_{eq}(\varepsilon,\theta)\equiv\frac{\partial e_{eq}}{\partial\theta}\equiv\theta\frac{\partial\eta_{eq}}{\partial\theta}\equiv-\theta\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\theta^2}=-\theta\int_{\varepsilon_0}^{\varepsilon}\frac{1}{\rho}\frac{\partial^2\sigma_{eq}(s,\theta)}{\partial\theta^2}\mathrm{d}s-\theta\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2},\tag{11}$$

where ε_0 is an arbitrary reference strain.

It is known that from calorimetric measurements it is possible to determine the specific heat at a constant strain ε_0 over an interval of temperatures, i.e. $C_{eq}(\varepsilon_0,\theta)$. This information is sufficient to determine the additive function $\phi=\phi(\theta)$ as solution of the differential equation

$$\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2}=-\frac{C_{eq}(\varepsilon_0,\theta)}{\theta},\tag{12}$$

up to an arbitrary linear function of θ, which can be established once the free energy and the entropy at a given state, respectively $\psi_{eq}(\varepsilon_0,\theta_0)$ and $\eta_{eq}(\varepsilon_0,\theta_0)$ are given.

Moreover, according to assumption **S1**) the free energy $\psi_{eq}(\varepsilon,\theta)$ and the entropy $\eta_{eq}(\varepsilon,\theta)$ are at least of C^1 class and the specific heat $C_{eq}(\varepsilon,\theta)$ is at least of C^0 class on the domain of definition. If the weaker assumption **S2**) is fulfilled one shows that the free energy is of class C^1, the entropy as well as the internal energy are of class C^0, while the specific heat is a discontinuous function on its domain of definition.

We also remind the following energy identity for smooth fields

$$\rho\frac{\partial e_{eq}(\varepsilon,\theta)}{\partial t}=\sigma_{eq}(\varepsilon,\theta)\frac{\partial\varepsilon}{\partial t}-\theta\frac{\partial\sigma_{eq}}{\partial\theta}\frac{\partial\varepsilon}{\partial t}+\rho C_{eq}\frac{\partial\theta}{\partial t},\tag{13}$$

where the first right-term with minus sign is the working, while the second and the third right-term describe the contribution of the latent heat and specific heat, repectively, to the heating of the thermoelastic material.

Often are employed the strain ε and the entropy η, rather than ε and the temperature θ, as independent variables. This is possible because the specific heat at constant strain $C_{eq}(\varepsilon,\theta)$ is always strictly positive and, according to (11), $\eta_{eq}(\varepsilon,\theta)$ must be a strictly increasing function of θ for each fixed ε. It follows that the equation $(10)_2$ can be solved for θ in a unique manner as $\theta=\tilde\theta(\varepsilon,\eta)$. The internal energy is then defined by $e=\tilde e(\varepsilon,\eta)=e_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))$ and the stress by $\sigma=\tilde\sigma(\varepsilon,\eta)=\sigma_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))$. Moreover, in this case the internal energy is a thermodynamic potential for the stress and temperature, i.e. $\sigma=\tilde\sigma(\varepsilon,\eta)=\rho\frac{\tilde e(\varepsilon,\eta)}{\partial\varepsilon}$ and $\theta=\tilde\theta(\varepsilon,\eta)=\frac{\tilde e(\varepsilon,\eta)}{\partial\eta}$. The specific heat at constant strain is given by $\tilde C(\varepsilon,\eta)$=$C_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))$=$\tilde\theta(\varepsilon,\eta)\frac{\partial\eta_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))}{\partial\theta}$ =$\tilde\theta(\varepsilon,\eta)\left(\frac{\partial\tilde\theta(\varepsilon,\eta)}{\partial\eta}\right)^{-1}$. Let us note that by using the chain rule we get

$$\frac{\partial\tilde\sigma(\varepsilon,\eta)}{\partial\varepsilon}=\frac{\partial\sigma_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))}{\partial\varepsilon}+\frac{\tilde\theta(\varepsilon,\eta)}{\rho C_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))}\left(\frac{\partial\sigma_{eq}(\varepsilon,\tilde\theta(\varepsilon,\eta))}{\partial\theta}\right)^2.\tag{14}$$

Since we will use as independent variables the strain ε and the temperature θ it is useful to remind here the equation of an *isentrope*. By differentiating relation $\eta_{eq}(\varepsilon,\theta)=\eta^*$ =const. and by using the relations (9) we get that an isentrope in the $\varepsilon-\theta$ plane is a solution $\theta=\theta_I(\varepsilon)$ of the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\varepsilon}=\frac{\theta}{\rho C_{eq}(\varepsilon,\theta)}\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}.\tag{15}$$

Let us note that if $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}<0$ the temperature *decreases* along the isentrope, while if $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}>0$ the temperature *increases* along the isentrope.

Some dimensionless combinations are often used. For instance, sometime it is convenient to introduce the *Grüneisen coefficient* which is defined as

$$\Gamma = \Gamma(\varepsilon,\theta) = -\frac{1+\varepsilon}{\rho C_{eq}(\varepsilon,\theta)}\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}, \tag{16}$$

and characterizes the temperature changes along an isentrope. Indeed, according to (15) we have $\frac{\mathrm{d}\theta}{\theta} = -\Gamma(\varepsilon,\theta)\frac{\mathrm{d}\varepsilon}{1+\varepsilon}$, i.e it is the negative slope of the isentrope in the $\log\theta - \log(1+\varepsilon)$ plane.

The coefficient of thermal expansion at constant stress is defined as

$$\alpha(\varepsilon,\theta) = -\Big(\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}\Big)\Big(\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon}\Big)^{-1}. \tag{17}$$

and characterizes the temperature changes along an isobar (σ =const.) in the $\varepsilon-\theta$ plane.

Usually Γ and α are positive for most metals although there are known exceptions. Let us note that according to our assumption **H3**, Γ changes its sign across the curve $\varepsilon = \varepsilon_t(\theta)$ (Fig. 2). Moreover, α changes also its sign in the $\varepsilon-\theta$ plane. Such behavior, when the thermal expansion coefficient is negative during martensitic - austenitic transformation has been reported by Uchil et *al* [35] in near-equiatomic, cold-worked Nitinol exhibiting shape memory effect.

3.3 Stability conditions. Constitutive domains of stable/unstable phases.

According to the Gibbsian thermostatics (Coleman and Noll [5]), a necessary condition for a point (ε,η) to be *thermostatically stable* is that $\tilde{e}(\varepsilon^*,\eta^*) - \tilde{e}(\varepsilon,\eta) - \frac{\partial\tilde{e}(\varepsilon,\eta)}{\partial\varepsilon}(\varepsilon^*-\varepsilon) - \frac{\partial\tilde{e}(\varepsilon,\eta)}{\partial\eta}(\eta^*-\eta) \geq 0$, for any (ε^*,η^*) in the domain of $\tilde{e}(\cdot,\cdot)$. One shows that conditions $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon} \geq 0$ and $C_{eq(\varepsilon,\theta)} = -\theta\frac{\partial^2\psi_{eq}(\varepsilon,\theta)}{\partial\theta^2} > 0$ are necessary and sufficient to ensure the Gibbsian thermostatic stability.

A natural physical condition to be imposed on the constitutive functions is to require the existence of real and finite sound speeds (acceleration waves) in the adiabatic case. We call it *dynamic stability condition* since it ensures the stability of the solutions of the equations of motion. One shows that it is a weaker condition on $\sigma_{eq}(\varepsilon,\theta)$ than the *thermostatic stability condition*.

The system of equations (1) describing the motion of an isolated ($r=0$) thermoelastic bar (6) in the absence of conductivity ($\kappa=0$) is called the *adiabatic thermoelastic system*, and can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ \varepsilon \\ \theta \end{pmatrix} - \begin{pmatrix} 0 & \frac{1}{\rho}\frac{\partial\sigma_{eq}}{\partial\varepsilon} & \frac{1}{\rho}\frac{\partial\sigma_{eq}}{\partial\theta} \\ 1 & 0 & 0 \\ \frac{\theta}{\rho C_{eq}}\frac{\partial\sigma_{eq}}{\partial\theta} & 0 & 0 \end{pmatrix}\frac{\partial}{\partial X}\begin{pmatrix} v \\ \varepsilon \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

This system is appropriate for the description of wave propagation since the heat conductivity can be ignored outside the narrow transition zones. The type of system is given by the eigenvalues and the right eigenvectors of the above matrix. The eigenvalues are solution of the equation $\lambda\big[\lambda^2 - \big(\frac{1}{\rho}\frac{\partial\sigma_{eq}}{\partial\varepsilon} + \frac{\theta}{\rho^2 C_{eq}}\big(\frac{\partial\sigma_{eq}}{\partial\theta}\big)^2\big)\big] = 0$. This system is strictly hyperbolic if the three eigenvalues are real and distinct, and the corresponding right eigenvectors are linearly independent. One shows that this happens if and only if

$$U^2(\varepsilon,\theta) \equiv \frac{1}{\rho}\frac{\partial\sigma_{eq}}{\partial\varepsilon} + \frac{\theta}{\rho^2 C_{eq}}\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2 > 0. \tag{19}$$

In this case $U(\varepsilon,\theta)$ is called the *sound speed* at the state (ε,θ).

It is obvious that the hyperbolicity condition (19) is fulfilled for any pair (ε,θ) such that $\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon} \geq 0$, i.e. for $\varepsilon \in (\infty,\varepsilon_m^-(\theta)) \cup (\varepsilon_M^-(\theta),\varepsilon_M^+(\theta)) \cup (\varepsilon_m^+(\theta),\infty)$. When the slope of the isotherm $\sigma = \sigma_{eq}(\varepsilon,\theta)$ becomes negative the system may changes type becoming an elliptic one. Therefore, we consider an additional assumption which allows to define the constitutive domain of phases.

H4) Let us suppose that *at each temperature* $\theta \in [\theta_m, \theta_M]$ there exists at least a value $\varepsilon^* \in (\varepsilon_m^-(\theta), \varepsilon_M^-(\theta))$ and at least a value $\varepsilon^* \in (\varepsilon_M^+(\theta), \varepsilon_m^+(\theta))$ such that $U^2(\varepsilon^*, \theta) < 0$, i.e. the hyperbolicity condition is violated. Then one proves there exists $\varepsilon = \gamma_M^\pm(\theta)$ and $\varepsilon = \gamma_m^\pm(\theta)$ (Fig. 2) such that

$$\varepsilon_m^-(\theta) < \gamma_m^-(\theta) < \gamma_M^-(\theta) < \varepsilon_M^-(\theta) < \varepsilon_M^+(\theta) < \gamma_M^+(\theta) < \gamma_m^+(\theta) < \varepsilon_m^+(\theta) \tag{20}$$

with the property that

$$U^2(\varepsilon, \theta) > 0 \text{ for } \varepsilon \in (\infty, \gamma_m^-(\theta)) \cup (\gamma_M^-(\theta), \gamma_M^+(\theta)) \cup (\gamma_m^+(\theta), \infty),$$
$$U^2(\varepsilon, \theta) < 0 \text{ for } \varepsilon \in (\gamma_m^-(\theta), \gamma_M^-(\theta)) \cup (\gamma_M^+(\theta), \gamma_m^+(\theta)).$$

We also suppose that *at each temperature* $\theta < \theta_m$ there exists at least an $\varepsilon^* \in (\varepsilon_m^-(\theta), \varepsilon_m^+(\theta))$ such that the hyperbolicity condition is violated. Then one proves there exists $\varepsilon = \gamma_m^-(\theta)$ and $\varepsilon = \gamma_m^+(\theta)$ such that

$$U^2(\varepsilon, \theta) > 0 \text{ for } \varepsilon \in (\infty, \gamma_m^-(\theta)) \cup (\gamma_m^+(\theta), \infty), \quad \text{and} \quad U^2(\varepsilon, \theta) < 0 \text{ for } \varepsilon \in (\gamma_m^-(\theta), \gamma_m^+(\theta)).$$

Moreover, we suppose in the following that the boundary curves $\varepsilon = \gamma_m^\pm(\theta)$, $\varepsilon = \gamma_M^\pm(\theta)$ are *continuously differentiable* and have the following properties:

$$\frac{\mathrm{d}\gamma_M^+(\theta)}{\mathrm{d}\theta} > 0, \quad \frac{\mathrm{d}\gamma_M^-(\theta)}{\mathrm{d}\theta} < 0, \quad \text{for } \theta \in (\theta_m, \theta_M); \qquad \frac{\mathrm{d}\gamma_m^+(\theta)}{\mathrm{d}\theta} > 0, \quad \frac{\mathrm{d}\gamma_m^-(\theta)}{\mathrm{d}\theta} < 0, \quad \text{for } \theta < \theta_M$$
$$\gamma_M^+(\theta_m) = \gamma_M^-(\theta_m), \quad \gamma_m^-(\theta_M) = \gamma_M^-(\theta_M), \quad \gamma_m^+(\theta_M) = \gamma_M^+(\theta_M). \tag{21}$$

If we denote $\varsigma_M^\pm(\theta) = \sigma_{eq}(\gamma_M^\pm(\theta), \theta)$ and $\varsigma_m^\pm(\theta) = \sigma_{eq}(\gamma_m^\pm(\theta), \theta)$ we remark that $\varsigma_M^+(\theta) < \sigma_M^+(\theta)$, $\varsigma_m^+(\theta) > \sigma_m^+(\theta)$, $\varsigma_M^-(\theta) > \sigma_M^-(\theta)$, $\varsigma_m^-(\theta) < \sigma_m^-(\theta)$ (Fig. 1).

Therefore, the functions $\varepsilon = \varepsilon_m^\pm(\theta)$ and $\varepsilon = \varepsilon_M^\pm(\theta)$ associated with the change of monotonicity of the isotherms $\sigma = \sigma_{eq}(\varepsilon, \theta)$ delimitate the *domains of thermostatic stability* in the $\varepsilon - \theta$ plane. On the other side, the constitutive functions $\varepsilon = \gamma_m^\pm(\theta)$ and $\varepsilon = \gamma_M^\pm(\theta)$ bound the *regions of hyperbolicity/ellipticity* of the adiabatic thermoelastic system in the same plane, i.e the *domains of dynamic stability* (Fig. 2). Indeed, it is known that if the initial boundary-value data belong to the domains of hyperbolicity of the adiabatic thermoelastic system the problems are well-posed and even more they are stable according to a linearized stability analysis. In the domains of ellipticity the initial-boundary data are ill-posed in the sense of Hadamard. Thus, possible solutions belonging to these regions will be dismissed in a pure thermoelastic approach of phase transitions.

We can identify the *stable phases* of the material, denoted by $\mathscr{A}$ phase and $\mathscr{M}^\pm$ phases, with the domains of hyperbolicity of the adiabatic system, while the *unstable phases*, denoted by $\mathscr{I}^\pm$ phases, with the domains of ellipticity. For instance, in the case $\theta \in [\theta_m, \theta_M]$, we say that a particle X at a time t is in the austenitic phase $\mathscr{A}$ if the pair $(\varepsilon, \theta)(X,t) \in \mathscr{A}$ where $\mathscr{A} = \{(\varepsilon, \theta) | \gamma_M^-(\theta) < \varepsilon < \gamma_M^+(\theta)\}$. The other stable phases in which the material may exist are $\mathscr{M}^+ = \{(\varepsilon, \theta) | \ \varepsilon > \gamma_M^+(\theta)\}$, $\mathscr{M}^- = \{(\varepsilon, \theta) | \ \varepsilon < \gamma_M^-(\theta)\}$, while the so-called unstable phases are $\mathscr{I}^+ = \{(\varepsilon, \theta) | \ \gamma_M^+(\theta) < \varepsilon < \gamma_m^+(\theta)\}$, $\mathscr{I}^- = \{(\varepsilon, \theta) | \ \gamma_m^-(\theta) < \varepsilon < \gamma_M^-(\theta)\}$ (Fig. 2).

3.4 Jump relations for thermoelastic materials

If $\dot{S} > 0$ we call the material at the $+$ side of the discontinuity to be *in front* of the wave, while the material at the $-$ side to be *in back* of the wave. The wave discontinuity is said to be *compressive* if the deformation decreases after the passage of the wave ($\varepsilon^- < \varepsilon^+$), and *expansive* if the deformation increases ($\varepsilon^- > \varepsilon^+$). If $\dot{S} < 0$ we have to change only $+$ to $-$ and correspondingly the terminology. In the present setting a strain discontinuity is called either a *thermoelastic shock wave*, or a *phase boundary*, according to whether the particles separated by the discontinuity are in the same phase, or in distinct phases. We only consider the case when $q^+ = q^- = 0$. According to (2) and (4) the relations between the front and back state read

$$v^- - v^+ = -\dot{S}(\varepsilon^- - \varepsilon^+), \qquad \sigma_{eq}(\varepsilon^-, \theta^-) - \sigma_{eq}(\varepsilon^+, \theta^+) = \rho \dot{S}^2(\varepsilon^- - \varepsilon^+), \tag{22}$$

$$\rho\left(e_{eq}(\varepsilon^-, \theta^-) - e_{eq}(\varepsilon^+, \theta^+)\right) = \frac{1}{2}(\sigma_{eq}(\varepsilon^-, \theta^-) + \sigma_{eq}(\varepsilon^+, \theta^+))(\varepsilon^- - \varepsilon^+), \tag{23}$$

$$\rho \dot{S}\big(\eta_{eq}(\varepsilon^-, \theta^-) - \eta_{eq}(\varepsilon^+, \theta^+)\big) \geq 0. \tag{24}$$

Let us suppose that the *front state* $(\varepsilon^+,\theta^+,v^+)$ is known. Then, relations (22)-(23) represent an algebraic non-linear system for the unknown *back state* $(\varepsilon^-,\theta^-,v^-)$ and the speed of the discontinuity $\dot{S}$. Depending on the thermoelastic constitutive assumptions this system may generally be solved if one of this four quantities is prescribed. In addition such a solution have to satisfy the entropy inequality (24). Let us note that, the Rankine-Hugoniot equation (23) provides only restrictions, on the back states (ε,θ) which can be reached in a shock process which has (ε^+,θ^+) as a front state. Moreover, this restriction does not depend on the shock speed $\dot{S}$. We denote by

$$H(\varepsilon,\theta;\varepsilon^+,\theta^+) = \rho e_{eq}(\varepsilon,\theta) - \rho e^+ - \frac{1}{2}(\sigma_{eq}(\varepsilon,\theta)+\sigma^+)(\varepsilon-\varepsilon^+) \tag{25}$$

the *Hugoniot function based at* (ε^+,θ^+) where $e^+ = e_{eq}(\varepsilon^+,\theta^+)$ and $\sigma^+ = \sigma_{eq}(\varepsilon^+,\theta^+)$. The set $\{(\varepsilon,\theta) \mid H(\varepsilon,\theta;\varepsilon^+,\theta^+) = 0\}$ is called the *Hugoniot set (locus) based* at (ε^+,θ^+) in the $\varepsilon-\theta$ plane.

In the smooth case **S1**, the Hugoniot function is at least of C^1 class. If $\sigma_{eq}(\varepsilon,\theta)$ satisfies the weaker smoothness assumption **S2** then it is continuous, piecewise smooth, and

$$\frac{\partial H(\varepsilon,\theta)}{\partial\theta} = \rho C_{eq}(\varepsilon,\theta) - \frac{1}{2}\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}(\varepsilon-\varepsilon^+) > 0, \tag{26}$$

at the points where the derivative makes sense. The positivity is here an assumption justified by the fact that we do not consider shocks of arbitrary intensity, and in general, for real materials $|\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}| << \rho C_{eq}(\varepsilon,\theta)$. Situations when the Hugoniot set is not curve-like and can bifurcate has been considered by Dunn and Fosdick [6]. According to (26) the implicit function theorem ensures that the equation $H(\varepsilon,\theta;\varepsilon^+,\theta^+) = 0$ can be solved (at least locally) with respect to θ. We suppose in the following that this unique solution

$$\theta = \Theta_H(\varepsilon;\varepsilon^+,\theta^+), \tag{27}$$

called the *temperature-strain Hugoniot curve (locus) based at* (ε^+,θ^+) exists globally and has the properties that $\theta^+ = \Theta_H(\varepsilon^+;\varepsilon^+,\theta^+)$ and $H(\varepsilon,\Theta_H(\varepsilon;\varepsilon^+,\theta^+);\varepsilon^+,\theta^+) = 0$ on its domain of definition. If the smoothness assumption **S1** is satisfied, the function (27) is at least of C1 class, while if the smoothness assumption **S2** is fulfilled, it is continuous and piece-wise smooth. This function describes all those states in the $\varepsilon-\theta$ plane that are potentially attainable as back states in a shock process which has (ε^+,θ^+) as a front state.

The image of (27) through the function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ in the $\varepsilon-\sigma$ plane is

$$\sigma = \sigma_H(\varepsilon;\varepsilon^+,\theta^+) \overset{\text{def}}{=} \sigma_{eq}(\varepsilon,\Theta_H(\varepsilon;\varepsilon^+,\theta^+)), \tag{28}$$

and is called the *stress-strain Hugoniot curve (locus) based at* (ε^+,σ^+). This function describes all reachable (σ,ε) back states in a wave discontinuity which has (ε^+,σ^+) as a front state.

4 A thermal Maxwellian rate-type approach to phase transitions

It is well known that initial-boundary value problems for the adiabatic system of thermoelasticity (18) can lead to non-unique discontinuous solutions. We therefore need a *selection criterion* to identify meaningful solutions. We use in the following a standard procedure to establish such a criterion. This procedure asserts that: a strong propagating discontinuity is admissible within the thermoelastic theory if and only if the limit values $(\varepsilon^\pm,\theta^\pm,v^\pm)$ on either side of the discontinuity can be connected by a traveling wave solution constructed within an *augmented theory*.

We introduce in the following an augmented theory whose dissipative mechanisms are described by regularizing terms characterizing stress relaxation and pseudo-creep processes toward equilibrium between phases and by heat conduction. Thus we consider in this paper the following Maxwellian rate-type constitutive relation

$$\frac{\partial\sigma}{\partial t} - E\frac{\partial\varepsilon}{\partial t} = -\frac{E}{\mu}(\sigma-\sigma_{eq}(\varepsilon,\theta)), \tag{29}$$

where $E =$ const. > 0 is the *dynamic Young modulus*, $\mu =$ const. > 0 is a *Newtonian viscosity* coefficient and $\sigma = \sigma_{eq}(\varepsilon,\theta)$ is the equilibrium state equation satisfying assumptions **H1 - H4**. Let us note that $\tau = \frac{\mu}{E}$ is a *relaxation time* and $k = \frac{E}{\mu}$ is called *Maxwellian viscosity* coefficient. When $\mu \to 0$ (or, $k \to \infty$) this constitutive equation is seen as a rate-type approximation of the thermoelastic model.

The rate-type constitutive equation (29) includes as a limiting case for $E \to \infty$ the Kelvin-Voigt model

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\mu}(\sigma - \sigma_{eq}(\varepsilon, \theta)). \tag{30}$$

We assume here that the Fourier law of heat conduction $q = -\kappa \frac{\partial \theta}{\partial X}$ holds, where $\kappa = \text{const.} > 0$ is the heat conductivity coefficient.

4.1 Thermodynamic considerations for the augmented theory.

If one uses the Helmholtz free energy $\psi = e - \theta\eta$, the entropy inequality (3) takes the form

$$-\rho\frac{\partial \psi}{\partial t} + \sigma\frac{\partial \varepsilon}{\partial t} - \rho\eta\frac{\partial \theta}{\partial t} - \frac{q}{\theta}\frac{\partial \theta}{\partial X} \geq 0. \tag{31}$$

By investigating the compatibility with the Clausius-Duhem inequality (31) of the Maxwellian rate-type material (29) endowed with Fourier heat conduction law one obtains the following results (see also [11]). The constitutive equation (29) admits a unique free energy function $\psi = \psi_{Mxw}(\varepsilon, \sigma, \theta)$ (modulo an additive function of temperature) if and only if the slope of the straight line connecting any two points of an equilibrium isotherm is bounded from above by the instantaneous Young modulus E. Moreover, in what follows we suppose there are two positive constants E_* and E^* such that

$$-E_* \leq \frac{\sigma_{eq}(\varepsilon_1, \theta) - \sigma_{eq}(\varepsilon_2, \theta)}{\varepsilon_1 - \varepsilon_2} \leq E^* < E, \quad \text{for any } \varepsilon_1,\ \varepsilon_2 \text{ and any } \theta. \tag{32}$$

The free energy has to satisfy the following Cauchy problem for a first order PDE, i.e.

$$\frac{\partial \psi_{Mxw}}{\partial \varepsilon} + E\frac{\partial \psi_{Mxw}}{\partial \sigma} = \frac{\sigma}{\rho}, \quad \frac{\partial \psi_{Mxw}}{\partial \sigma}(\varepsilon, \sigma_{eq}(\varepsilon, \theta), \theta) = 0, \tag{33}$$

while the entropy, the intrinsic dissipation and the thermal dissipation are given, respectively by

$$\eta = -\frac{\partial \psi_{Mxw}}{\partial \theta}(\varepsilon, \sigma, \theta), \quad D_{Mxw} \equiv \frac{E}{\mu}\rho\frac{\partial \psi_{Mxw}}{\partial \sigma}(\varepsilon, \sigma, \theta)(\sigma - \sigma_{eq}(\varepsilon, \theta)) \geq 0, \quad D_{th} = \kappa\left(\frac{\partial \theta}{\partial X}\right)^2 \geq 0. \tag{34}$$

According to (33) the general form of the free energy function is

$$\rho\psi_{Mxw}(\varepsilon, \sigma, \theta) = \frac{\sigma^2}{2E} + \varphi(\sigma - E\varepsilon, \theta), \tag{35}$$

where $\varphi = \varphi(\tau, \theta)$ satisfies relation

$$\frac{\partial \varphi}{\partial \tau}(\,\sigma_{eq}(\varepsilon, \theta) - E\varepsilon\,, \theta) = -\frac{\sigma_{eq}(\varepsilon, \theta)}{E}. \tag{36}$$

Let us denote by $h(\varepsilon, \theta) = \sigma_{eq}(\varepsilon, \theta) - E\varepsilon$. Condition (32) ensures that function h is invertible with respect to ε for any fixed θ. We denote by $h^{-1}(\cdot, \theta)$ this function. Therefore, for any triplet $(\varepsilon, \sigma, \theta)$ there is a unique $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, \sigma, \theta) = h^{-1}(\sigma - E\varepsilon, \theta)$ such that

$$\sigma - E\varepsilon = h(\tilde{\varepsilon}, \theta) = \sigma_{eq}(\tilde{\varepsilon}, \theta) - E\tilde{\varepsilon}. \tag{37}$$

Thus, the free energy function of the Maxwellian rate-type constitutive equation (29) is explicitly determined (up to an additive function of θ) by the equilibrium relation $\sigma = \sigma_{eq}(\varepsilon, \theta)$ and the Young modulus E through relation

$$\rho\psi_{Mxw}(\varepsilon, \sigma, \theta) = \frac{\sigma^2}{2E} - \frac{\sigma_{eq}^2(\tilde{\varepsilon}, \theta)}{2E} + \int_{\varepsilon_0}^{\tilde{\varepsilon}} \sigma_{eq}(s, \theta)\mathrm{d}s + \rho\phi(\theta), \tag{38}$$

where $\phi(\theta)$ is a smooth function. The entropy function is given by

$$\rho\eta_{Mxw}(\varepsilon, \sigma, \theta) = -\int_{\varepsilon_0}^{\tilde{\varepsilon}} \frac{\partial \sigma_{eq}(s, \theta)}{\partial \theta}\mathrm{d}s - \rho\frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, \tag{39}$$

and the specific heat by

$$C_{Mxw}(\varepsilon,\sigma,\theta)=\theta\frac{\partial\eta_{Mxw}}{\partial\theta}(\varepsilon,\sigma,\theta)=-\frac{\theta}{\rho}\Big(\int_{\varepsilon_0}^{\tilde{\varepsilon}}\frac{\partial^2\sigma_{eq}(s,\theta)}{\partial\theta^2}\mathrm{d}s+\frac{(\frac{\partial\sigma_{eq}(\tilde{\varepsilon},\theta)}{\partial\theta})^2}{E-\frac{\partial\sigma_{eq}(\tilde{\varepsilon},\theta)}{\partial\varepsilon}}+\rho\frac{\mathrm{d}^2\phi(\theta)}{\mathrm{d}\theta^2}\Big). \tag{40}$$

If the equilibrium curve $\sigma=\sigma_{eq}(\varepsilon,\theta)$ satisfies the smoothness assumptions **S1** then the free energy $\psi_{Mxw}(\varepsilon,\sigma,\theta)$ and the entropy $\eta_{Mxw}(\varepsilon,\sigma,\theta)$ are at least of C^1 class, while the specific heat $C_{Mxw}(\varepsilon,\sigma,\theta)$ is at least of C^0 class on the domain of definition. If the smoothness assumptions **S2** are satisfied, i.e. $\sigma=\sigma_{eq}(\varepsilon,\theta)$ is continuous and piecewise smooth, then the free energy $\psi_{Mxw}(\varepsilon,\sigma,\theta)$ is still of C^1 class, the entropy is of C^0 class and piecewise smooth, while the specific heat $C_{Mxw}(\varepsilon,\sigma,\theta)$ is a discontinuous and piecewise smooth function.

One can show that the following two relations hold

$$\rho E\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma,\theta)=\sigma-\sigma_{eq}(\tilde{\varepsilon},\theta)=E(\varepsilon-\tilde{\varepsilon})=E(\varepsilon-h^{-1}(\sigma-E\varepsilon,\theta)), \tag{41}$$

$$\frac{E}{E+E_*}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2\le E\rho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma,\theta)(\sigma-\sigma_{eq}(\varepsilon,\theta))\le\frac{E}{E-E^*}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2. \tag{42}$$

Thus, from $(34)_2$ one gets the following estimate on the intrinsic dissipation generated by the Maxwellian rate-type model

$$\frac{E}{\mu(E+E_*)}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2\le D_{Mxw}(\varepsilon,\sigma,\theta)\le\frac{E}{\mu(E-E^*)}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2. \tag{43}$$

Let us note that the free energy, entropy and internal energy of the Maxwellian model *at equilibrium* are just the free energy, entropy and internal energy of the thermoelastic model $\sigma=\sigma_{eq}(\varepsilon,\sigma)$, that is, $\psi_{Mxw}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta)=\psi_{eq}(\varepsilon,\sigma)$, $\eta_{Mxw}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta)=\eta_{eq}(\varepsilon,\sigma)$ and $e_{Mxw}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta)=e_{eq}(\varepsilon,\sigma)$. Indeed, from (37) and (38) we get that $\sigma=\sigma_{eq}(\varepsilon,\theta)$ involve $\varepsilon=\tilde{\varepsilon}$, wherefrom by using (38) and (39) we obtain relations (10). Concerning the relation between the specific heat of the Maxwellian model (40) at equilibrium and the specific heat of the thermoelastic model (11) we get by using notation (19)

$$C_{Mxw}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta)=C_{eq}(\varepsilon,\theta)-\frac{\theta}{\rho}\frac{\left(\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta}\right)^2}{\left(E-\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\varepsilon}\right)}=C_{eq}(\varepsilon,\theta)\frac{E-\rho\lambda^2(\varepsilon,\theta)}{E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon,\theta)}. \tag{44}$$

Therefore, a necessary condition on the constitutive functions $\sigma=\sigma_{eq}(\varepsilon,\theta)$ and E to ensure the positiveness of the specific heat of the rate-type Maxwellian model is

$$\rho\lambda^2(\varepsilon,\theta)=\frac{\partial\sigma_{eq}}{\partial\varepsilon}+\frac{\theta}{\rho C_{eq}(\varepsilon,\theta)}\left(\frac{\partial\sigma_{eq}}{\partial\theta}\right)^2<E. \tag{45}$$

In order to determine the unknown function $\phi(\theta)$ in (38) we suppose again that the specific heat of the thermoelastic model at a constant strain ε_0 is known over an interval of temperatures, i.e we may use again the equation (12).

By investigating the properties of the thermodynamic functions of the Maxwellian model when $E\to\infty$ we obtain

$$\lim_{E\to\infty}\psi_{Mxw}(\varepsilon,\sigma,\theta)=\psi_{eq}(\varepsilon,\theta),\ \lim_{E\to\infty}\eta_{Mxw}(\varepsilon,\sigma,\theta)=\eta_{eq}(\varepsilon,\theta),\ \lim_{E\to\infty}C_{Mxw}(\varepsilon,\sigma,\theta)=C_{eq}(\varepsilon,\theta), \tag{46}$$

that means, the free energy, entropy, internal energy and specific heat of the Kelvin-Voigt model coincide with the free energy, entropy, internal energy and specific heat of the thermoelastic model (6).

Consequently, the internal dissipation generated in a smooth process by the Kelvin-Voigt model is obtained from (43) and (30) as

$$D_{KV}(\varepsilon,\sigma,\theta)=\lim_{E\to\infty}D_{Mxw}(\varepsilon,\sigma,\theta)=\frac{1}{\mu}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2=\mu\frac{\partial\varepsilon}{\partial t}^2. \tag{47}$$

By using the balance laws (1), the constitutive equations (29)-(30) and relations (33)-(34) we can establish the following energy identities. For the thermal Maxwellian rate-type material with Fourier heat conduction law the smooth solutions of the corresponding PDEqs system satisfies

$$\rho\frac{\partial e_{Mxw}(\varepsilon,\sigma,\theta)}{\partial t}=\sigma\frac{\partial\varepsilon}{\partial t}-\frac{E}{\mu}\rho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\sigma-\sigma_{eq}(\varepsilon,\theta))+\frac{E}{\mu}\rho\theta\frac{\partial^2\psi_{Mxw}}{\partial\sigma\partial\theta}(\sigma-\sigma_{eq}(\varepsilon,\theta))+\rho C_{Mxw}\frac{\partial\theta}{\partial t}, \tag{48}$$

$$\rho\frac{\partial\eta_{Mxw}(\varepsilon,\sigma,\theta)}{\partial t}+\frac{\partial}{\partial X}\left(\frac{q}{\theta}\right)=\frac{1}{\theta}\big(D_{Mxw}(\varepsilon,\sigma,\theta)+D_{th}(\theta)\big)\equiv P_{Mxw}(\varepsilon,\sigma,\theta)\geq 0. \tag{49}$$

For a non-isothermal Kelvin-Voigt material with Fourier heat conduction law the smooth solutions of the corresponding PDEqs system satisfies

$$\rho\frac{\partial e_{eq}(\varepsilon,\theta)}{\partial t}=\sigma_{eq}(\varepsilon,\theta)\frac{\partial\varepsilon}{\partial t}-\frac{1}{\mu}(\sigma-\sigma_{eq}(\varepsilon,\theta))^2-\frac{1}{\mu}\rho\theta\frac{\partial^2\psi_{eq}}{\partial\varepsilon\partial\theta}(\sigma-\sigma_{eq}(\varepsilon,\theta))+\rho C_{eq}\frac{\partial\theta}{\partial t}, \tag{50}$$

$$\rho\frac{\partial\eta_{eq}(\varepsilon,\theta)}{\partial t}+\frac{\partial}{\partial X}\left(\frac{q}{\theta}\right)=\frac{1}{\theta}\big(D_{KV}(\varepsilon,\sigma,\theta)+D_{th}(\theta)\big)\equiv P_{KV}(\varepsilon,\theta)\geq 0. \tag{51}$$

Let us note that in relations $(48)_1$ and $(50)_1$ the first right term with minus sign is the working, while the second, the third, and the forth right-term represents the contribution to the heating of the intrinsic dissipation, *latent heat* and specific heat, respectively. We also note that the right-hand terms in (49) and (51), denoted by $P_{Mxw}(\varepsilon,\sigma,\theta)$ and $P_{KV}(\varepsilon,\theta)$, represent the *total entropy production* corresponding to a heat conducting smooth process of a Maxwellian material and a Kelvin-Voigt material, respectively.

The system of equations (1) and (29), with $e=e_{Mxw}(\varepsilon,\sigma,\theta)$, describing the adiabatic motion ($q=0$) of a Maxwellian rate-type bar can be written as a relaxation system (very small μ) with sources

$$\frac{\partial}{\partial t}\begin{pmatrix}v\\ \varepsilon\\ \theta\\ \sigma\end{pmatrix}-\begin{pmatrix}0&0&0&1/\rho\\ 1&0&0&0\\ 0&0&0&0\\ E&0&0&0\end{pmatrix}\frac{\partial}{\partial X}\begin{pmatrix}v\\ \varepsilon\\ \theta\\ \sigma\end{pmatrix}=\frac{E}{\mu}\big(\sigma-\sigma_{eq}(\varepsilon,\theta)\big)\begin{pmatrix}0\\ 0\\ \frac{1}{C_{Mxw}}\left(\frac{\partial\psi_{Mxw}}{\partial\sigma}-\theta\frac{\partial^2\psi_{Mxw}}{\partial\sigma\partial\theta}\right)\\ -1\end{pmatrix}. \tag{52}$$

It is easy to verify that this system is always hyperbolic semi-linear irrespective of the slope with respect to ε of the equilibrium curve $\sigma=\sigma_{eq}(\varepsilon,\theta)$ as long as the dynamic Young's modulus E is strictly positive and finite. Indeed, this system is semi-linear since all non-linear terms are included in the right part of (52) and the eigenvalues of the matrix are given by $\lambda=\pm\sqrt{\frac{E}{\rho}}$ and $\lambda=0$ (twice). Therefore, initial-boundary value problems are now well-posed even in the unstable domains $\mathscr{I}^{\pm}$ where phase transformations occur. One expects that when $\mu\to 0$ solutions of the rate-type system (52) "approach" solutions of the adiabatic thermoelastic system (18) in the sense that the stress σ is rapidly driven back to the equilibrium $\sigma_{eq}(\varepsilon,\theta)$, except perhaps in narrow phase transition time intervals where σ, ε, θ and v have a very steep variation.

The adiabatic Kelvin-Voigt rate-type system (1) and (30), where $e=e_{eq}(\varepsilon,\theta)$, can be viewed as a limiting case of the Maxwellian rate-type system for $E\to\infty$. In this case the characteristic directions of the hyperbolic system in the $X-t$ plane tend to infinite, i.e. the hyperbolic system (52) transforms into a parabolic one.

5 Traveling wave solutions.

In the following we seek steady wave solutions for the system of six equations composed by the Maxwellian rate-type constitutive equation (29) (or, the Kelvin-Voigt model (30)), the corresponding internal energy law $e=e_{Mxw}(\varepsilon,\sigma,\theta)$, (or, $e=e_{eq}(\varepsilon,\theta)$), the Fourier law and the balance laws (1) when $r=0$. These solutions, sought in the form $(\varepsilon,\sigma,\theta,v,q,e)=(\hat\varepsilon,\hat\sigma,\hat\theta,\hat v,\hat q,\hat e)(\xi)$, where $\xi=X-\dot{S}t$, $\dot{S}=$ const. satisfy the boundary conditions

$$\lim_{\xi\to\pm\infty}(\hat\varepsilon,\hat\sigma,\hat\theta,\hat v,\hat q,\hat e)(\xi)=(\varepsilon^{\pm},\sigma^{\pm}=\sigma_{eq}(\varepsilon^{\pm},\theta^{\pm}),\theta^{\pm},v^{\pm},0,e^{\pm}=e_{eq}(\varepsilon^{\pm},\theta^{\pm})), \tag{53}$$

where ε^+, v^+, θ^+, ε^-, v^-, θ^- are given values.

In general, such traveling wave solutions of the rate-type systems represent a profile layer which connects two thermomechanical equilibrium states of the material and approximates a strong discontinuity of the adiabatic thermoelastic system propagating with a constant velocity $\dot{S}$.

Let us look first *smooth* steady wave solutions. We denote by prime the derivative with respect to ξ. Independently of any constitutive assumptions we get from balance laws (1) and entropy inequality (3) relations

$$\hat{v}'(\xi)+\dot{S}\hat{\varepsilon}'(\xi)=0,\quad \hat{\sigma}'(\xi)+\rho\dot{S}\hat{v}'(\xi)=0,\quad \dot{S}\big(\rho\hat{e}'(\xi)-\hat{\sigma}(\xi)\hat{\varepsilon}'(\xi)\big)=\hat{q}'(\xi),\quad \rho\dot{S}\eta'\leq\big(\frac{q}{\theta}\big)',\tag{54}$$

wherefrom, by using the boundary conditions (53) one gets

$$\begin{aligned}&\hat{v}(\xi)=v^+-\dot{S}(\hat{\varepsilon}(\xi)-\varepsilon^+),\quad \hat{\sigma}(\xi)=\sigma_R(\hat{\varepsilon}(\xi))\overset{\text{def}}{=}\sigma^++\rho\dot{S}^2(\hat{\varepsilon}(\xi)-\varepsilon^+),\\ &\hat{q}(\xi)=\dot{S}\big(\rho\hat{e}(\xi)-\rho e^+-\frac{1}{2}(\hat{\varepsilon}(\xi)-\varepsilon^+)(\hat{\sigma}(\xi)+\sigma^+)\big),\quad \hat{q}(\xi)\leq\rho\dot{S}\hat{\theta}(\xi)(\hat{\eta}(\xi)-\eta^+).\end{aligned}\tag{55}$$

If we set $\xi\to-\infty$ we recover the Rankine-Hugoniot relations (22)-(23) and the entropy jump inequality (24) for the adiabatic thermoelastic system . Therefore, if $\dot{S}>0$ and (ε^+,θ^+) is a given front state of a wave discontinuity then the pair (ε^-,θ^-) has to belong to the Hugoniot set based at (ε^+,θ^+) given by (25), i.e. $H(\varepsilon^-,\theta^-;\varepsilon^+,\theta^+)=0$ or equivalently $\theta^-=\Theta_H(\varepsilon^-;\varepsilon^+,\theta^+)$. If $\dot{S}<0$ and (ε^-,θ^-) is a given front state of a wave discontinuity then the pair (ε^+,θ^+) has to belong to the Hugoniot set based at (ε^-,θ^-), i.e. $H(\varepsilon^+,\theta^+;\varepsilon^-,\theta^-)=0$ or equivalently $\theta^+=\Theta_H(\varepsilon^+;\varepsilon^-,\theta^-)$. Moreover, the constant steady wave speed $\dot{S}$ is determined by the equilibrium states to be connected through relation

$$\rho\dot{S}^2=\frac{\sigma_{eq}(\varepsilon^+,\theta^+)-\sigma_{eq}(\varepsilon^-,\theta^-)}{\varepsilon^+-\varepsilon^-}<E.\tag{56}$$

Let us note that relation $(55)_2$ asserts that in a steady structured wave the strain-stress pairs $(\hat{\varepsilon}(\xi),\hat{\sigma}(\xi))$ belong to a straight line of slope $\rho\dot{S}^2$ in the $\sigma-\varepsilon$ plane. This is called the *Rayleigh line construction*. Therefore, the function $\sigma=\sigma_R(\varepsilon)$ defined above is called the *Rayleigh line*.

By using the Maxwellian rate-type constitutive equation (29) and the Fourier law we get that $\varepsilon=\hat{\varepsilon}(\xi)$ and $\theta=\hat{\theta}(\xi)$ have to satisfy the non-linear autonomous system with boundary conditions

$$\begin{aligned}&\hat{\varepsilon}'=-\frac{E}{\mu\dot{S}(E-\rho\dot{S}^2)}R(\hat{\varepsilon},\hat{\theta}),\qquad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi)=\varepsilon^\pm\\ &\hat{\theta}'=-\frac{\dot{S}}{\kappa}H_{Mxw}(\hat{\varepsilon},\hat{\theta}),\qquad \lim_{\xi\to\pm\infty}\hat{\theta}(\xi)=\theta^\pm,\end{aligned}\tag{57}$$

where, if $\dot{S}>0$,

$$R(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-)\equiv\sigma_R(\varepsilon)-\sigma_{eq}(\varepsilon,\theta)=\sigma^++\rho\dot{S}^2(\varepsilon-\varepsilon^+)-\sigma_{eq}(\varepsilon,\theta),\tag{58}$$

$$H_{Mxw}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-)\equiv\rho e_{Mxw}\big(\varepsilon,\sigma_R(\varepsilon),\theta\big)-\rho e^+-\frac{1}{2}\big(\varepsilon-\varepsilon^+\big)\big(\sigma_R(\varepsilon)+\sigma^+\big).\tag{59}$$

and (ε^+,θ^+) represents the front state, while (ε^-,θ^-) is the Hugoniot state, i.e. $\theta^-=\Theta_H(\varepsilon^-;\varepsilon^+,\theta^+)$.

If $\dot{S}<0$, the initial front state is (ε^-,θ^-) and (ε^+,θ^+) is the Hugoniot state, i.e. $\theta^+=\Theta_H(\varepsilon^+;\varepsilon^-,\theta^-)$, then the superscripts + and - have to be invert in (59). For simplicity, when there are no ambiguities we will drop from the notations of R and H_{Mxw} their dependence on (ε^+,θ^+) and (ε^-,θ^-).

According to (46), by making $E\to\infty$ in the system (57) we obtain the non-linear autonomous system describing the traveling wave solutions for the Kelvin-Voigt model (30) with Fourier heat conduction law,

$$\begin{aligned}&\hat{\varepsilon}'=-\frac{1}{\mu\dot{S}}R(\hat{\varepsilon},\hat{\theta}),\quad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi)=\varepsilon^\pm\\ &\hat{\theta}'=-\frac{\dot{S}}{\kappa}H_{KV}(\hat{\varepsilon},\hat{\theta}),\ \lim_{\xi\to\pm\infty}\hat{\theta}(\xi)=\theta^\pm,\end{aligned}\tag{60}$$

where, if $\dot{S}>0$,

$$H_{KV}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-)\equiv\rho e_{eq}\big(\varepsilon,\theta\big)-\rho e^+-\frac{1}{2}\big(\varepsilon-\varepsilon^+\big)\big(\sigma_R(\varepsilon)+\sigma^+\big).\tag{61}$$

It is useful to note that the pairs $(\varepsilon^\pm,\theta^\pm)$ are fixed points for both dynamical systems (57), and (60). Indeed, according to (56) we have $R(\varepsilon^\pm,\theta^\pm)=0$. On the other side, since $e_{Mxw}(\varepsilon^\pm,\sigma_{eq}(\varepsilon^\pm,\theta^\pm),\theta^\pm)=e_{eq}(\varepsilon^\pm,\theta^\pm)$ it follows $H_{Mxw}(\varepsilon^\pm,\theta^\pm)=H_{KV}(\varepsilon^\pm,\theta^\pm)=H(\varepsilon^\pm,\theta^\pm)=0$. In the $\varepsilon-\sigma$ plane that means the pairs $(\varepsilon^\pm,\sigma^\pm)$ represent the intersection of the Hugoniot locus (28) with the Rayleigh line.

Remark. Let us note that functions $H_{Mxw}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-)$ and $H_{KV}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-)$ resemble with the Hugoniot function $H(\varepsilon,\theta;\varepsilon^+,\theta^+)$ based at (ε^+,θ^+) defined in (25). In both cases the equilibrium response function $\sigma=\sigma_{eq}(\varepsilon,\theta)$ is replaced by the Rayleigh line $\sigma=\sigma_R(\varepsilon)$. A second difference appears only for H_{Mxw}. In this case the internal energy $e_{eq}(\varepsilon,\theta)$ of the thermoelastic model is replaced with the internal energy of the rate-type Maxwellian model (29) along the Rayleigh line, i.e. $e_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\theta)$. For future use one shows that

$$H_{Mxw}(\varepsilon,\theta)=H(\varepsilon,\theta)+\rho e_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\theta)-\rho e_{eq}(\varepsilon,\theta)-\frac{1}{2}(\varepsilon-\varepsilon^+)R(\varepsilon,\theta), \tag{62}$$

$$H_{KV}(\varepsilon,\theta)=H(\varepsilon,\theta)-\frac{1}{2}(\varepsilon-\varepsilon^+)R(\varepsilon,\theta). \tag{63}$$

The question to be answered in the following concerns the conditions which ensure the existence and uniqueness of traveling wave solutions for the system (57), or (60), when the equilibrium state equation $\sigma=\sigma_{eq}(\varepsilon,\theta)$ satisfies our special constitutive assumptions **H1**-**H4**. These conditions provide also a selection criterion for weak solutions of the adiabatic thermoelastic system (18).

5.1 Structuring mechanism: only viscous dissipation

We are first interested to determine the admissibility condition induced by the Maxwellian rate-type approach (29), or by the Kelvin-Voigt model (30), in the absence of heat conduction.

5.1.1 Admissibility condition: chord criterion with respect to the Hugoniot locus in the strain-stress plane

We say that an elastic shock wave or a phase boundary is an *admissible* weak solution for the adiabatic thermoelastic system if there exists a unique traveling wave $(\varepsilon(\xi),\theta(\xi),v(\xi))$ provided by the augmented constitutive approach which connects the limit values $(\varepsilon^\pm,\theta^\pm,v^\pm)$. We say that the traveling wave describes a *shock layer* if $(\varepsilon^\pm,\theta^\pm)$ are in the same phase, or an *interphase transition layer* if $(\varepsilon^\pm,\theta^\pm)$ are in different phases. We designate them in a generic way as a *profile layer*.

We derive in the following an useful and practical criterion which selects physically admissible shock waves or phase boundaries for the thermoelastic adiabatic system. This selection criterion is just *the chord criterion with respect to the Hugoniot locus in the strain-stress plane* $\sigma=\sigma_H(\varepsilon;\varepsilon^+,\theta^+)$ defined by (28).

In order to clarify our result we briefly remind the case of *isothermal elasticity* with non-monotone stress-strain relation. In this case the PDEs system is composed by equations $(1)_{1-2}$ and an *isothermal* equilibrium curve $\sigma=\sigma_{eq}(\varepsilon)$. We have shown in Făciu and Molinari [9, Part II] that by considering the Maxwellian rate-type approach as an augmented theory for the non-monotone elastic model we obtain the same viscosity admissibility criterion as that obtained by Pego [27] using Kelvin-Voigt isothermal viscoelastic constitutive equation (see also Slemrod [30]). According to the traveling wave analysis for the rate-type system one has shown that the *Maxwellian viscosity criterion* in the isothermal case is equivalent with the *chord criterion with respect to the elastic constitutive equation* $\sigma=\sigma_{eq}(\varepsilon)$ which claims that: a *compressive wave discontinuity*, i.e. $(\varepsilon^+-\varepsilon^-)\dot{S}>0$, is admissible iff the chord which joins $(\varepsilon^+,\sigma^+=\sigma_{eq}(\varepsilon^+))$ to $(\varepsilon^-,\sigma^-=\sigma_{eq}(\varepsilon^-))$ lies *below* the graph of the function $\sigma=\sigma_{eq}(\varepsilon)$ for ε between ε^+ and ε^-, while an *expansive wave discontinuity*, i.e. $(\varepsilon^+-\varepsilon^-)\dot{S}<0$, is admissible iff the chord lies *above* the graph in the same interval.

The results obtained below extends the above condition within the adiabatic thermoelastic theory.

Proposition 1 *Let us suppose that the thermoelastic constitutive equation $\sigma=\sigma_{eq}(\varepsilon,\theta)$ satisfies the general assumptions **H1**-**H4**, which includes the case of negative Grüneisen coefficients. Then the viscosity criterion generated by the Maxwellian rate type approach (29), or by the Kelvin-Voigt model (30), in the absence of heat conduction is equivalent with the following selection criterion.*

Chord criterion with respect to the Hugoniot locus in the strain-stress plane.

If $\dot{S}>0$, the front state is (ε^+,θ^+) and the Hugoniot back state is (ε^-,θ^-) then a *compressive wave discontinuity* is admissible iff the Rayleigh line lies *below* the Hugoniot locus, i.e.

$$\sigma_R(\varepsilon)=\sigma^++\rho\dot{S}^2(\varepsilon-\varepsilon^+)<\sigma_H(\varepsilon;\varepsilon^+,\theta^+),\ \text{for any}\ \varepsilon\in(\varepsilon^-,\varepsilon^+), \tag{64}$$

and an *expansive wave discontinuity* is admissible iff the Rayleigh line lies *above* the Hugoniot locus, i.e.

$$\sigma_R(\varepsilon) = \sigma^+ + \rho\dot{S}^2(\varepsilon - \varepsilon^+) > \sigma_H(\varepsilon;\varepsilon^+,\theta^+), \text{ for any } \varepsilon \in (\varepsilon^+,\varepsilon^-). \tag{65}$$

If $\dot{S} < 0$, the front state is (ε^-,θ^-) and the Hugoniot back state is (ε^+,θ^+) then the above statement remains valid if we invert superscripts + with - in relations (64) and (65).

This result is related with the extended entropy condition for gas dynamic equations of Liu [21]. When the Maxwellian approach (29) is coupled with the Fourier heat we show in Sect. 5.2 that the chord criterion with respect to the Hugoniot locus is also an admissibility condition if additional constitutive assumptions are fulfilled, or if the viscosity effects dominate the heat conductivity effects (see also Pego [26]).

We have to prove in the following that conditions of the type (64)-(65) are necessary and sufficient for the existence of a unique profile layer connecting the limit values $(\varepsilon^\pm,\theta^\pm)$.

5.1.2 Traveling waves for Maxwellian rate-type model (29) without heat conduction.

The only structuring parameters of these layers are the viscosity μ and the dynamic Young's modulus E. Such traveling waves are solutions of the problem

$$\begin{aligned} \hat{\varepsilon}' &= -\frac{E}{\mu\dot{S}(E-\rho\dot{S}^2)}R(\hat{\varepsilon},\hat{\theta}), \quad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi) = \varepsilon^\pm, \\ 0 &= H_{Mxw}(\hat{\varepsilon},\hat{\theta}). \end{aligned} \tag{66}$$

Let us consider $\dot{S} > 0$ and (ε^+,θ^+) a fixed front state and (ε^-,θ^-) a Hugoniot state, i.e. $\theta^- = \Theta_H(\varepsilon^-;\varepsilon^+,\theta^+)$. The strain-temperature pair $(\hat{\varepsilon}(\xi),\hat{\theta}(\xi))$ has to satisfy the algebraic equation $(66)_2$ where function $H_{Mxw}(\hat{\varepsilon},\hat{\theta})$ is given by (59). The set $\{(\varepsilon,\theta) \mid H_{Mxw}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-) = 0\}$ describes the trajectory in the $\varepsilon-\theta$ plane of a traveling wave governed by a Maxwellian rate-type dissipative mechanism in the absence of heat conduction. Let us note that $H_{Mxw}(\varepsilon^\pm,\theta^\pm;\varepsilon^+,\theta^+,\varepsilon^-) = 0$. The function $H_{Mxw}(\varepsilon,\theta)$ is at least of C^1 class if the smoothness assumption **S1** is satisfied and it is a continuous and piecewise C^1 function on its domain of definition for the weaker assumption **S2**. Since $\frac{\partial H_{Mxw}}{\partial\theta}(\varepsilon,\theta) = \rho\frac{\partial e_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\theta)}{\partial\theta} = \rho C_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\theta) > 0$ at the points where the derivative makes sense, by using the theorem of implicit function it can be shown that the equation $H_{Mxw}(\varepsilon,\theta) = 0$ can be solved at least locally with respect to ε. In the following we suppose that it can be solved globally, that means, there exists a unique function

$$\theta = \Theta_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-), \tag{67}$$

with the property that $H_{Mxw}(\varepsilon,\Theta_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)) = 0$ for ε belonging to an interval which contains ε^+ and ε^- and $\Theta_{Mxw}(\varepsilon^\pm;\varepsilon^+,\theta^+,\varepsilon^-) = \theta^\pm$. This function is at least of C^1 class if assumption **S1** is satisfied and it is continuous and piecewise C^1 for the weaker assumption **S2**. Its image through the function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ in the $\varepsilon-\sigma$ plane is given by

$$\sigma = \sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-) \stackrel{\text{def}}{=} \sigma_{eq}\big(\varepsilon,\Theta_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)\big), \tag{68}$$

and connects the states $(\varepsilon^\pm,\sigma^\pm)$. It is useful to note that $\sigma^\pm = \sigma_{Mxw}(\varepsilon^\pm) = \sigma_H(\varepsilon^\pm) = \sigma_{eq}(\varepsilon^\pm,\theta^\pm)$.

By using the above notations we get from (66) that $\varepsilon = \hat{\varepsilon}(\xi)$ has to be solution of the problem

$$\hat{\varepsilon}' = -\frac{E}{\mu\dot{S}(E-\rho\dot{S}^2)}\big(\sigma^+ + \rho\dot{S}^2(\hat{\varepsilon}-\varepsilon^+) - \sigma_{Mxw}(\hat{\varepsilon};\varepsilon^+,\theta^+,\theta^-)\big), \quad \lim_{\xi\to\pm}\hat{\varepsilon}(\xi) = \varepsilon^\pm \tag{69}$$

It is already known from the Maxwellian isothermal case studied in [9, Part II] that a solution of the problem (69) exists if and only if the *chord criterion with respect to the curve* $\sigma = \sigma_{Mxw}(\varepsilon)$ is fulfilled.

Thus, for a right-facing discontinuity $\dot{S} > 0$, in the *compressive* case $(\varepsilon^- < \varepsilon^+)$, the Rayleigh line has to lie *below* the curve $\sigma_{Mxw}(\varepsilon)$, i.e. $\sigma_R(\varepsilon) = \sigma^+ + \rho\dot{S}^2(\varepsilon-\varepsilon^+) < \sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$, for any $\varepsilon \in (\varepsilon^-,\varepsilon^+)$, while for the *expansive* case $(\varepsilon^+ < \varepsilon^-)$, the Rayleigh line has to lie *above*, i.e. $\sigma_R(\varepsilon) = \sigma^+ + \rho\dot{S}^2(\varepsilon-\varepsilon^+) > \sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$, for any $\varepsilon \in (\varepsilon^+,\varepsilon^-)$.

For a left-facing wave $\dot{S} < 0$, when the front state is (ε^-,θ^-) and the Hugoniot state is (ε^+,θ^+) the admissibility condition is obtained by inverting the superscripts + and - in the above relations.

Lemma 1 *The chord criterion with respect to the curve $\sigma = \sigma_{Mxw}(\varepsilon)$ is equivalent with the chord criterion with respect to the Hugoniot curve $\sigma = \sigma_H(\varepsilon)$.*

Proof. Let us consider for instance the expansive case of a forward propagating discontinuity, that is $\dot{S} > 0$ and $\varepsilon^+ < \varepsilon^-$. We suppose first that the chord criterion with respect to the curve $\sigma = \sigma_{Mxw}(\varepsilon)$ is fulfilled, that is $\sigma_R(\varepsilon) = \sigma^+ + \rho\dot{S}^2(\varepsilon - \varepsilon^+) > \sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$, for any $\varepsilon \in (\varepsilon^+,\varepsilon^-)$. We prove that the chord criterion with respect to the Hugoniot curve has to be also satisfied, that is $\sigma_R(\varepsilon) = \sigma^+ + \rho\dot{S}^2(\varepsilon-\varepsilon^+) > \sigma_H(\varepsilon;\varepsilon^+,\theta^+)$ for any $\varepsilon \in (\varepsilon^+,\varepsilon^-)$. The proof is based by reduction to the absurd. Let us suppose there is an $\varepsilon^* \in (\varepsilon^+,\varepsilon^-)$ such that $\sigma_R(\varepsilon^*) = \sigma_H(\varepsilon^*;\varepsilon^+,\theta^+)$. We denote by $\theta^* = \theta_H(\varepsilon^*;\varepsilon^+,\theta^+)$. Therefore, $H(\varepsilon^*,\theta^*;\varepsilon^+,\theta^+) = 0$ and $\sigma_R(\varepsilon^*) = \sigma_H(\varepsilon^*;\varepsilon^+,\theta^+) \equiv \sigma_{eq}(\varepsilon^*,\theta_H(\varepsilon^*;\varepsilon^+,\theta^+)) = \sigma_{eq}(\varepsilon^*,\theta^*)$. By using (62) and the fact that $e_{Mxw}(\varepsilon,\sigma_{eq}(\varepsilon,\theta),\theta) = e_{eq}(\varepsilon,\theta)$ we get that $H_{Mxw}(\varepsilon^*,\theta^*;\varepsilon^+,\theta^+) = 0$. Therefore, $\theta^* = \theta_{Mxw}(\varepsilon^*;\varepsilon^+,\theta^+)$, which implies $\sigma_{Mxw}(\varepsilon^*;\varepsilon^+,\theta^+) \equiv \sigma_{eq}(\varepsilon^*,\theta_{Mxw}(\varepsilon^*;\varepsilon^+,\theta^+)) = \sigma_{eq}(\varepsilon^*,\theta^*) = \sigma_R(\varepsilon^*)$. Thus, it results a contradiction with our initial assumption that the chord criterion with respect to $\sigma = \sigma_{Mxw}(\varepsilon)$ is satisfied for any $\varepsilon \in (\varepsilon^+,\varepsilon^-)$. The proof is similar for the compressive case and for a back propagating discontinuity $\dot{S} < 0$.

In order to prove that the chord criterion with respect to the Hugoniot curve $\sigma = \sigma_H(\varepsilon)$ implies the chord criterion with respect to the curve $\sigma = \sigma_{Mxw}(\varepsilon)$ we use in a similar way the reduction to the absurd, the definitions of these curves and relation (62). □

Remark. This equivalence between the two chord criteria transfers the admissibility condition from a relation which depends on the energetic properties of the rate-type dissipative model, namely $\sigma = \sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$, to a relation which depends only on the energetic properties of the thermoelastic constitutive model, namely $\sigma = \sigma_H(\varepsilon;\varepsilon^+,\theta^+)$. That is way the chord criterion with respect to the Hugoniot locus is extremely useful in practice.

The entropy production in a profile layer of Maxwell's type, thermally non-conducting. Let us denote by $\hat{\psi}(\varepsilon) \equiv \psi_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon))$, $\hat{\eta}(\varepsilon) \equiv \eta_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon))$ and $\hat{e}(\varepsilon) \equiv e_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon))$, where $\varepsilon = \hat{\varepsilon}(\xi)$ is solution of (69), the free energy, entropy and internal energy along a viscous, heat non-conducting profile layer generated by the Maxwellian rate-type model. By using relation (33) we get

$$\sigma_R(\varepsilon) = \rho\frac{\mathrm{d}\hat{e}(\varepsilon)}{\mathrm{d}\varepsilon} + (E - \rho\dot{S}^2)\rho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon)) - \rho\Theta_{Mxw}(\varepsilon)\frac{\mathrm{d}\hat{\eta}(\varepsilon)}{\mathrm{d}\varepsilon}. \quad (70)$$

Since $H_{Mxw}(\varepsilon,\Theta_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)) = 0$ for any ε between ε^+ and ε^- one obtains from (59) the following identities

$$\rho\hat{e}(\varepsilon) - \rho e^+ = \frac{1}{2}(\varepsilon-\varepsilon^+)(\sigma^+ + \sigma_R(\varepsilon)), \quad \text{and} \quad \rho\frac{\mathrm{d}\hat{e}(\varepsilon)}{\mathrm{d}\varepsilon} = \sigma_R(\varepsilon). \quad (71)$$

From (70) and $(71)_2$ we derive

$$\rho\frac{\mathrm{d}\hat{\eta}(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{E-\rho\dot{S}^2}{\Theta_{Mxw}(\varepsilon)}\rho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon)), \quad (72)$$

wherefrom, by integration, we obtain

$$\rho(\eta_{eq}(\varepsilon^+,\theta^+) - \eta_{eq}(\varepsilon^-,\theta^-)) = \rho(\hat{\eta}(\varepsilon^+) - \hat{\eta}(\varepsilon^-)) = \int_{\varepsilon^-}^{\varepsilon^+}\frac{E-\rho\dot{S}^2}{\Theta_{Mxw}(\varepsilon)}\rho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon))\mathrm{d}\varepsilon. \quad (73)$$

According to relations (34), (49), (69) and (73) the total entropy production induced by a traveling wave governed by a Maxwellian rate-type constitutive equation in the absence of heat conduction is given by

$$\begin{aligned} P_{Mxw}^{trav} &= \int_{-\infty}^{\infty}\frac{D_{Mxw}(\hat{\varepsilon},\hat{\sigma},\hat{\theta})}{\Theta_{Mxw}(\hat{\varepsilon})}\mathrm{d}\xi = \int_{-\infty}^{\infty}\frac{E}{\mu}\frac{\rho}{\Theta_{Mxw}(\hat{\varepsilon})}\frac{\partial\psi_{Mxw}}{\partial\sigma}(\hat{\varepsilon},\sigma_R(\hat{\varepsilon}),\Theta_{Mxw}(\hat{\varepsilon}))\big(\sigma_R(\hat{\varepsilon}) - \sigma_{Mxw}(\hat{\varepsilon})\big)\mathrm{d}\xi \\ &= -\dot{S}\int_{-\infty}^{\infty}\frac{(E-\rho\dot{S}^2)}{\Theta_{Mxw}(\hat{\varepsilon})}\rho\frac{\partial\psi_{Mxw}}{\partial\sigma}(\hat{\varepsilon},\sigma_R(\hat{\varepsilon}),\Theta_{Mxw}(\hat{\varepsilon}))\hat{\varepsilon}'\mathrm{d}\xi = -\dot{S}\rho(\eta_{eq}(\varepsilon^+,\theta^+) - \eta_{eq}(\varepsilon^-,\theta^-)) \geq 0. \quad (74) \end{aligned}$$

Therefore, in a Maxwellian thermally non-conducting profile layer, the entropy of the Hugoniot back state can not be lower than the entropy of the front state. Moreover, the total entropy production P_{Mxw}^{trav} of the traveling wave solution does not depend on the viscosity and is, according to (24), exactly the entropy production of a strong discontinuity compatible with the second law of thermodynamics for the associated thermoelastic constitutive equation $\sigma = \sigma_{eq}(\varepsilon,\theta)$.

5.1.3 Traveling waves for Kelvin-Voigt model (30) without heat conduction.

In this case, according to (60), the structure of the profile layer is only characterized by the viscosity μ and the solution is given by the corresponding reduced system

$$\begin{aligned} &\hat{\varepsilon}' = -\tfrac{1}{\mu\dot{S}}R(\hat{\varepsilon},\hat{\theta}), \quad \lim_{\xi\to\pm\infty}\hat{\varepsilon}(\xi) = \varepsilon^{\pm} \\ &0 = H_{KV}(\hat{\varepsilon},\hat{\theta}). \end{aligned} \tag{75}$$

Therefore, all the strain-temperature pairs $(\hat{\varepsilon}(\xi),\hat{\theta}(\xi))$ have to satisfy the algebraic equation $(75)_2$. The set $\{(\varepsilon,\theta) \mid H_{KV}(\varepsilon,\theta;\varepsilon^+,\theta^+,\varepsilon^-) = 0\}$ describes the trajectory in the $\varepsilon-\theta$ plane of the traveling wave governed by a Kelvin-Voigt dissipative mechanism in the absence of heat conduction. The function $H_{KV}(\varepsilon,\theta)$ is at least of C^1 class if the smoothness assumption **S1** is satisfied and it is a continuous and piecewise C^1 function on its domain of definition for the weaker assumption **S2**. Moreover, at the points where the derivative makes sense we have $\frac{\partial H_{KV}}{\partial\theta}(\varepsilon,\theta) = \rho\frac{\partial e_{eq}}{\partial\theta}(\varepsilon,\theta) = \rho C_{eq}(\varepsilon,\theta) > 0$. Therefore, the above algebraic equation can be solved with respect to ε. We suppose there exists a unique function

$$\theta = \Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-), \tag{76}$$

with the properties that $H_{KV}(\varepsilon,\Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)) = 0$ for ε belonging to an interval which contains $\varepsilon^{\pm}$, and $\Theta_{KV}(\varepsilon^+;\varepsilon^{\pm},\theta^+,\varepsilon^-) = \theta^{\pm}$. Its image through the function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ in the $\varepsilon-\sigma$ plane is given by

$$\sigma = \sigma_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-) \stackrel{\text{def}}{=} \sigma_{eq}\big(\varepsilon,\Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)\big), \tag{77}$$

and connects the states $(\varepsilon^{\pm},\sigma^{\pm})$. It is useful to note that $\sigma^{\pm} = \sigma_{KV}(\varepsilon^{\pm}) = \sigma_H(\varepsilon^{\pm}) = \sigma_{eq}(\varepsilon^{\pm},\theta^{\pm})$.

By using the above notations we get from (75) that $\varepsilon = \hat{\varepsilon}(\xi)$ is solution of the problem

$$\hat{\varepsilon}' = -\frac{1}{\mu\dot{S}}\big(\sigma^+ + \rho\dot{S}^2(\hat{\varepsilon}-\varepsilon^+) - \sigma_{KV}(\hat{\varepsilon};\varepsilon^+,\theta^+,\varepsilon^-)\big), \quad \lim_{\xi\to\pm}\hat{\varepsilon}(\xi) = \varepsilon^{\pm}. \tag{78}$$

A solution of this problem exists iff a chord criterion with respect to the curve $\sigma = \sigma_{KV}(\varepsilon)$ is fulfilled (see Slemrod [30] and Pego [27]). One proves finally in a similar way as in Lemma 1 that the chord criterion with respect to $\sigma = \sigma_{KV}(\varepsilon)$ is equivalent with the chord criterion with respect to the Hugoniot curve $\sigma = \sigma_H(\varepsilon;\varepsilon^+,\theta^+)$ given by relations (64)-(65).

The entropy production in a profile layer of Kelvin-Voigt's type, thermally non-conducting. If we denote by $\hat{\psi}(\varepsilon) = \psi_{eq}(\varepsilon,\Theta_{KV}(\varepsilon))$, $\hat{\eta}(\varepsilon) = \eta_{eq}(\varepsilon,\Theta_{KV}(\varepsilon))$ and $\hat{e}(\varepsilon) = e_{eq}(\varepsilon,\Theta_{KV}(\varepsilon))$ the free energy, entropy and internal energy along the trajectory in the $\varepsilon-\theta$ plane of a a viscous, heat non-conducting profile layer generated by the Kelvin-Voigt model we get immediately from $(9)_{1,2}$ that

$$\sigma_{KV}(\varepsilon) = \rho\frac{\mathrm{d}\hat{e}(\varepsilon)}{\mathrm{d}\varepsilon} - \rho\Theta_{KV}(\varepsilon)\frac{\mathrm{d}\hat{\eta}(\varepsilon)}{\mathrm{d}\varepsilon}. \tag{79}$$

Since $H_{KV}(\varepsilon,\Theta_{KV}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)) = 0$ for any ε between ε^+ and ε^- we obtain from (61) by using the above notations the following identities

$$\rho\hat{e}(\varepsilon) - \rho e^+ = \frac{1}{2}(\varepsilon-\varepsilon^+)(\sigma^+ + \sigma_R(\varepsilon)), \qquad \rho\frac{\mathrm{d}\hat{e}(\varepsilon)}{\mathrm{d}\varepsilon} = \sigma_R(\varepsilon). \tag{80}$$

From (79) and $(80)_2$ we derive

$$\rho\frac{\mathrm{d}\hat{\eta}(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon)}{\Theta_{KV}(\varepsilon)} \tag{81}$$

wherefrom, we obtain

$$\rho(\eta_{eq}(\varepsilon^+,\theta^+) - \eta_{eq}(\varepsilon^-,\theta^-)) = \rho(\hat{\eta}(\varepsilon^+) - \hat{\eta}(\varepsilon^-)) = \int_{\varepsilon^-}^{\varepsilon^+}\frac{\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon)}{\Theta_{KV}(\varepsilon)}\mathrm{d}\varepsilon. \tag{82}$$

On the other hand, according to relations (47), (51), (78) and (82) the total entropy production of a traveling wave governed by a Kelvin-Voigt material in the absence of heat conduction is given by

$$P_{KV}^{trav} = \int_{-\infty}^{\infty}\frac{\mu\dot{S}^2}{\hat{\theta}}(\hat{\varepsilon}')^2\mathrm{d}\xi = -\dot{S}\int_{\varepsilon^-}^{\varepsilon^+}\frac{\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon)}{\Theta_{KV}(\varepsilon)}\mathrm{d}\varepsilon = -\dot{S}\rho(\eta_{eq}(\varepsilon^+,\theta^+) - \eta_{eq}(\varepsilon^-,\theta^-)) \geq 0. \tag{83}$$

One derives the same conclusion like for the entropy production in a profile layer generated by the Maxwellian rate-type model in the absence of heat conduction.

5.2 Structuring mechanisms: Maxwellian viscosity coupled with heat conduction.

We are now interested to investigate the existence and uniqueness of the solutions of the non-linear autonomous system (57). Following the method proposed by Gilbarg [15] we first analyze the system behavior near its critical points. The linearization of (57) in a neighborhood of $(\varepsilon^\pm,\theta^\pm)$ leads to the system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\hat{\varepsilon}\\ \hat{\theta}\end{pmatrix} = J_{Mxw}(\varepsilon^\pm,\theta^\pm)\begin{pmatrix}\hat{\varepsilon}\\ \hat{\theta}\end{pmatrix}, \tag{84}$$

where

$$J_{Mxw}(\varepsilon,\theta) = -\begin{pmatrix} \dfrac{E}{\mu\dot{S}(E-\rho\dot{S}^2)}\dfrac{\partial R}{\partial\varepsilon} & \dfrac{E}{\mu\dot{S}(E-\rho\dot{S}^2)}\dfrac{\partial R}{\partial\theta} \\ \dfrac{\dot{S}}{\kappa}\dfrac{\partial H_{Mxw}}{\partial\varepsilon} & \dfrac{\dot{S}}{\kappa}\dfrac{\partial H_{Mxw}}{\partial\theta} \end{pmatrix}. \tag{85}$$

We show that

$$\frac{\partial R}{\partial\varepsilon}(\varepsilon^\pm,\theta^\pm) = \rho\dot{S}^2 - \frac{\partial\sigma_{eq}}{\partial\varepsilon}, \qquad \frac{\partial R}{\partial\theta}(\varepsilon^\pm,\theta^\pm) = -\frac{\partial\sigma_{eq}}{\partial\theta}, \tag{86}$$

$$\frac{\partial H_{Mxw}}{\partial\varepsilon}(\varepsilon^\pm,\theta^\pm) = -\theta^\pm\frac{(E-\rho\dot{S}^2)\dfrac{\partial\sigma_{eq}}{\partial\theta}}{\left(E-\dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\right)}, \qquad \frac{\partial H_{Mxw}}{\partial\theta}(\varepsilon^\pm,\theta^\pm) = \rho\frac{\partial e_{eq}}{\partial\theta} - \theta^\pm\frac{\left(\dfrac{\partial\sigma_{eq}}{\partial\theta}\right)^2}{\left(E-\dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\right)}. \tag{87}$$

To prove relations (87) we have to use the properties of the free energy function $\psi = \psi_{Mxw}(\varepsilon,\sigma,\theta)$ of the Maxwellian model from Section 4.1. Starting from (59), by using the properties $(33)_1$ and $(34)_1$ we get

$$\frac{\partial H_{Mxw}}{\partial\varepsilon}(\varepsilon,\theta) = -\rho(E-\rho\dot{S}^2)\Big(\frac{\partial\psi_{Mxw}}{\partial\sigma}(\varepsilon,\sigma_R(\varepsilon),\theta) - \theta\frac{\partial^2\psi_{Mxw}}{\partial\sigma\partial\theta}(\varepsilon,\sigma_R(\varepsilon),\theta)\Big). \tag{88}$$

By using (38) and (39) we prove that

$$\rho\frac{\partial^2\psi_{Mxw}}{\partial\sigma\partial\theta}(\varepsilon,\sigma,\theta) = -\frac{\partial\sigma_{eq}}{\partial\theta}(\tilde{\varepsilon},\theta)\left(E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\tilde{\varepsilon},\theta)\right)^{-1}, \tag{89}$$

where $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon,\sigma,\theta) = h^{-1}(\sigma - E\varepsilon,\theta)$ is solution of the equation (37). Moreover, since $\sigma^\pm = \sigma_{eq}(\varepsilon^\pm,\theta^\pm) = \sigma_R(\varepsilon^\pm)$ we get that $\tilde{\varepsilon}(\varepsilon^\pm,\sigma^\pm,\theta^\pm) = \varepsilon^\pm$. Relation $(87)_1$ is then obtained by using (88), (89) and $(33)_2$. Relation $(87)_2$ is obtained directly from (44).

Let us note that if we consider the non-linear autonomous system describing the traveling wave solutions for the Kelvin-Voigt model (60) we obtain the linearized system by using the function $H_{KV}(\varepsilon,\theta)$ instead $H_{Mxw}(\varepsilon,\theta)$ in (85). By a direct calculation, or by making $E \longrightarrow \infty$ in (85)-(87), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\hat{\varepsilon}\\ \hat{\theta}\end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\mu\dot{S}}\big(\rho\dot{S}^2 - \dfrac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon^\pm,\theta^\pm)\big) & \dfrac{1}{\mu\dot{S}}\dfrac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon^\pm,\theta^\pm) \\ \dfrac{\dot{S}}{\kappa}\theta^\pm\dfrac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon^\pm,\theta^\pm) & -\dfrac{\dot{S}}{\kappa}\rho\dfrac{\partial e_{eq}}{\partial\theta}(\varepsilon^\pm,\theta^\pm) \end{pmatrix}\begin{pmatrix}\hat{\varepsilon}\\ \hat{\theta}\end{pmatrix}. \tag{90}$$

The characteristic equation of the linearized system (84) at the critical points $(\varepsilon^\pm,\theta^\pm)$ is

$$r^2 + r\Big\{\frac{E\Big(\rho\dot{S}^2 - \dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}{\mu\dot{S}(E-\rho\dot{S}^2)} + \frac{\dot{S}}{\kappa}\Big(\rho\frac{\partial e_{eq}}{\partial\theta} - \frac{\theta^\pm\Big(\dfrac{\partial\sigma_{eq}}{\partial\theta}\Big)^2}{\Big(E-\dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}\Big)\Big\} + \frac{E\Big(\rho\dot{S}^2 - \dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}{\kappa\mu(E-\rho\dot{S}^2)}\Big\{\rho\frac{\partial e_{eq}}{\partial\theta} - \frac{\theta^\pm\Big(\dfrac{\partial\sigma_{eq}}{\partial\theta}\Big)^2}{\Big(\rho\dot{S}^2 - \dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}\Big\} = 0. \tag{91}$$

The discriminant of this equation

$$\Delta(\varepsilon^\pm,\theta^\pm) = \Big\{\frac{E\Big(\rho\dot{S}^2 - \dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}{\mu\dot{S}(E-\rho\dot{S}^2)} - \frac{\dot{S}}{\kappa}\Big(\rho\frac{\partial e_{eq}}{\partial\theta} - \frac{\theta^\pm\Big(\dfrac{\partial\sigma_{eq}}{\partial\theta}\Big)^2}{\left(E-\dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\right)}\Big)\Big\}^2 + \frac{4E\theta^\pm\Big(\dfrac{\partial\sigma_{eq}}{\partial\theta}\Big)^2}{\mu\kappa\Big(E-\dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}, \tag{92}$$

is positive and then both eigenvalues $r_{1,2}(\varepsilon^{\pm},\theta^{\pm})$ are real. Let us note that their product and their sum is

$$r_1 r_2 = \frac{\rho^2 E}{\mu\kappa(E-\rho\dot{S}^2)}\frac{\partial e_{eq}}{\partial\theta}\big(\dot{S}^2-\lambda^2\big), \tag{93}$$

$$r_1+r_2 = -\frac{1}{\dot{S}}\Big[\frac{\rho E(\dot{S}^2-\lambda^2)}{\mu(E-\rho\dot{S}^2)} + \frac{E\theta^{\pm}\Big(\dfrac{\partial\sigma_{eq}}{\partial\theta}\Big)^2}{\mu(E-\rho\dot{S}^2)\rho\dfrac{\partial e_{eq}}{\partial\theta}} + \frac{\dot{S}^2}{\kappa}\rho\frac{\partial e_{eq}}{\partial\theta}\frac{(E-\rho\lambda^2)}{\Big(E-\dfrac{\partial\sigma_{eq}}{\partial\varepsilon}\Big)}\Big] \tag{94}$$

where $\lambda^2(\varepsilon^{\pm},\theta^{\pm}) \equiv \frac{1}{\rho}\frac{\partial\sigma_{eq}}{\partial\varepsilon} + \theta\big(\frac{\partial\sigma_{eq}}{\partial\theta}\big)^2/\big(\rho^2\frac{\partial e_{eq}}{\partial\theta}\big)$, represents according to (19) the square of non-zero characteristic directions of the adiabatic thermoelastic system at the critical points. Let us note that the sign of the product of the eigenvalues is positive or negative according to whether the speed of the propagating discontinuity $\dot{S}$ is larger or smaller than the adiabatic sound speed at the critical point.

If $r_1r_2<0$, i.e $\dot{S}^2<\lambda^2(\varepsilon,\theta)$, (subsonic case) the eigenvalues have opposite signs and the critical point is a *saddle point*.

If $r_1r_2>0$, i.e $\dot{S}^2>\lambda^2(\varepsilon,\theta)$, (supersonic case) the eigenvalues have the same sign. According to (32), (45) and (56) we have $E>\frac{\partial\sigma_{eq}}{\partial\varepsilon}$, $E>\rho\lambda^2(\varepsilon,\theta)$ and $E>\rho\dot{S}^2$, respectively. Therefore, the sign of r_1+r_2 is equal to the sign of $-\dot{S}$. Thus, if $\dot{S}>0$ then both eigenvalues are negative and the critical point is an *attractive node* while if $\dot{S}<0$ both eigenvalues are positive and the critical point is a *repulsive* node.

If $r_1=0$, i.e. $\dot{S}^2=\lambda^2(\varepsilon,\theta)$ then the sign of r_2 is equal to the sign of $-\dot{S}$. In this case, in the neighborhood of the critical point the orbits are straight lines parallel with the eigenvector corresponding to the non-zero eigenvalue. The orientation being away ($r_2>0$), or towards ($r_2<0$) an axis of stationary points parallel with the eigenvector corresponding to the null eigenvalue.

5.2.1 Existence, uniqueness and structure of viscous, heat conducting profile layers.

We assume that the thermoelastic constitutive equation $\sigma=\sigma_{eq}(\varepsilon,\theta)$ satisfies the assumptions **H1**-**H4** corresponding to a phase transforming material, the smoothness assumption **S1**, and the dynamic Young's modulus E satisfies conditions (32) and (45).

Let us consider $\dot{S}>0$ and (ε^+,θ^+) a front state of a wave discontinuity for the adiabatic thermoelastic system and (ε^-,θ^-) a Hugoniot back state. Our goal is to determine the constitutive restrictions under which *the chord criterion (64)-(65) with respect to the Hugoniot locus* $\sigma=\sigma_H(\varepsilon;\varepsilon^+,\theta^+)$ is a necessary and sufficient condition for the existence of a unique profile layer structured by the Maxwellian rate-type constitutive equation and by the Fourier law. In fact, we first investigate when *the chord criterion with respect to the stress-strain curve* $\sigma=\sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$, given by (68), ensures the existence and uniqueness of a viscous, heat conducting profile layer for any given coefficients $\mu>0$ and $\kappa>0$, and then we apply Lemma 1 according to which this criterion is equivalent with with the chord criterion with respect to the Hugoniot locus.

The study of the behavior of the solutions of the system (57) is based on the idea of Gilbarg [15] and used later by Pego [26] to exploit the topological properties of the curves $H_{Mxw}(\varepsilon,\theta)=0$ and $R(\varepsilon,\theta)=0$ along which $\hat{\theta}'(\xi)$ and $\hat{\varepsilon}'(\xi)$ vanishes. We consider for instance the *compressive case* when $\varepsilon^-<\varepsilon^+$. We distinguish several situations depending on the sign of $\frac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon^{\pm},\theta^{\pm})$, i.e. on the sign of the Grüneisen coefficient (16) at the critical points. The *expansive case* when $\varepsilon^+<\varepsilon^-$ can be investigated in a similar way.

C. The compressive case ($\varepsilon^-<\varepsilon^+$).

In this case the *chord criterion with respect to the curve* $\sigma=\sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$ requires that

$$s(\varepsilon)\overset{\text{def}}{=}\sigma_R(\varepsilon)-\sigma_{Mxw}(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)<0, \quad \text{for any } \varepsilon\in(\varepsilon^-,\varepsilon^+). \tag{95}$$

We first establish some properties of the functions (67) and (68). By using the theorem of implicit functions and the thermodynamic properties established in Section 4.1 for the Maxwellian model we can show that

$$\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{(E-\rho\dot{S}^2)}{C_{Mxw}}\Big(\frac{\partial\psi_{Mxw}}{\partial\sigma} - \Theta_{Mxw}(\varepsilon)\frac{\partial^2\psi_{Mxw}}{\partial\theta\partial\sigma}\Big)(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon)). \tag{96}$$

Let us remind that $\theta = \Theta_{Mxw}(\varepsilon)$ is the trajectory in the $\varepsilon - \theta$ plane of the viscous, heat non-conducting profile layer described in Section 5.1.2. It is interesting to observe that relation (96) can be directly obtained by writing the energy identity (48) for the traveling wave solution $(\hat{\varepsilon}(\xi), \hat{\theta}(\xi) = \Theta_{Mxw}(\hat{\varepsilon}(\xi)))$ given by (67) and (69). The first term in the parenthesis is related with the contribution of the intrinsic dissipation, while the second one is related with the contribution of the latent heat, to the heating in a viscous, heat non-conducting profile layer. Therefore the sign of the above derivative reflects a balance between the intrinsic dissipation and the latent heat inside the viscous heat non-conducting profile layer.

By using the thermodynamic properties (41) and (89) we can write (96) as

$$\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{(E-\rho\dot{S}^2)}{\rho C_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon))}\left(\frac{\sigma_R(\varepsilon)-\sigma_{eq}(\tilde{\varepsilon},\Theta_{Mxw}(\varepsilon))}{E} + \Theta_{Mxw}(\varepsilon)\frac{\frac{\partial\sigma_{eq}}{\partial\theta}(\tilde{\varepsilon},\Theta_{Mxw}(\varepsilon))}{E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\tilde{\varepsilon},\Theta_{Mxw}(\varepsilon))}\right) \tag{97}$$

where $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$ is the unique solution of the equation (37) for $\sigma = \sigma_R(\varepsilon)$ and $\theta = \Theta_{Mxw}(\varepsilon)$, i.e. it satisfies

$$\sigma_R(\varepsilon) - E\varepsilon = \sigma_{eq}(\tilde{\varepsilon},\Theta_{Mxw}(\varepsilon)) - E\tilde{\varepsilon}. \tag{98}$$

Finally, one shows that at the critical points we have

$$\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^\pm)}{\mathrm{d}\varepsilon} = \frac{(E-\rho\dot{S}^2)\theta^\pm\frac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon^\pm,\theta^\pm)}{\rho C_{Mxw}(\varepsilon^\pm,\sigma^\pm,\theta^\pm)\left(E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon^\pm,\theta^\pm)\right)}. \tag{99}$$

Since $\frac{\mathrm{d}\sigma_{Mxw}(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon,\Theta_{Mxw}(\varepsilon)) + \frac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon,\Theta_{Mxw}(\varepsilon))\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon)}{\mathrm{d}\varepsilon}$, by using (99) and (44) we get that

$$s'(\varepsilon^\pm) = \frac{C_{eq}(\varepsilon^\pm,\theta^\pm)}{C_{Mxw}(\varepsilon^\pm,\sigma^\pm,\theta^\pm)}\rho\left(\dot{S}^2-\lambda^2(\varepsilon^\pm,\theta^\pm)\right). \tag{100}$$

Because $s(\varepsilon^\pm) = 0$, a direct consequence of the chord condition (95) is $s'(\varepsilon^-) \leq 0$ and $s'(\varepsilon^+) \geq 0$. By using (100) one obtains that $\dot{S}^2 - \lambda^2(\varepsilon^-,\theta^-) \leq 0$ and $\dot{S}^2 - \lambda^2(\varepsilon^+,\theta^+) \geq 0$. If the inequalities are strict, from (93)-(94), one gets that (ε^-,θ^-) is a *saddle node (subsonic critical point)*, while (ε^+,θ^+) is an *attractive node (supersonic critical point)*. Therefore, the chord criterion is consistent with the shock inequalities of Lax [20], which for a right-facing wave discontinuity read $0 < \lambda(\varepsilon^+,\theta^+) < \dot{S} < \lambda(\varepsilon^-,\theta^-)$. A degenerate case, is when $\dot{S} = \lambda(\varepsilon^\pm,\theta^\pm)$ which is considered separately.

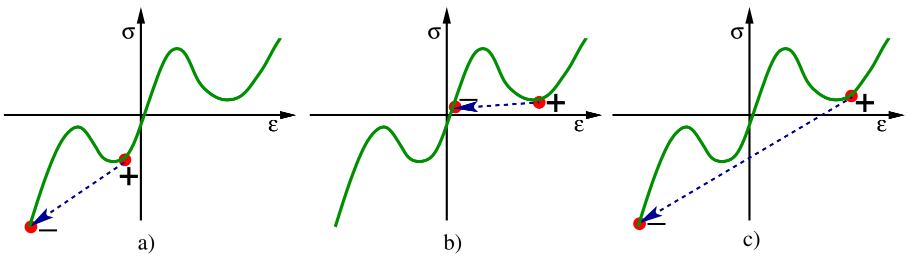

Fig. 3 Typical compressive jump discontinuities from state (ε^+,θ^+) to (ε^-,θ^-) satisfying the chord criterion with respect to the Hugoniot curve $\sigma = \sigma_H(\varepsilon;\varepsilon^+,\theta^+)$. Phase transformations: a) Case C1. $\mathscr{A} \to \mathscr{M}^-$; b) Case C2. $\mathscr{M}^+ \to \mathscr{A}$; c) Case C3. $\mathscr{M}^+ \to \mathscr{M}^-$.

Case C1. $\frac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon^\pm,\theta^\pm) < 0$, i.e. positive Grüneisen coefficients (16) at the critical points.

That means $(\varepsilon^\pm,\theta^\pm)$ belong to the region where $\frac{\partial\sigma_{eq}}{\partial\theta} < 0$, that is where $\varepsilon < \varepsilon_t(\theta)$ (Fig. 2). According to assumption **H3**-**H4** the front state and the Hugoniot state $(\varepsilon^\pm,\theta^\pm)$ lie in the austenitic phase $\mathscr{A}$ or in the martensitic variant $\mathscr{M}^-$. A typical compressive jump discontinuity from $\mathscr{A}$ to $\mathscr{M}^-$ is illustrated in Fig. 3a.

Since $\frac{\partial R(\varepsilon,\theta)}{\partial\theta} = -\frac{\partial\sigma_{eq}(\varepsilon,\theta)}{\partial\theta} > 0$ for $\varepsilon < \varepsilon_t(\theta)$, it follows that $R(\varepsilon,\theta) = 0$ is locally uniquely representable as a single valued function of ε. We assume there exists a function denoted $\theta = \Theta_R(\varepsilon;\varepsilon^+,\theta^+,\varepsilon^-)$ for ε belonging to an interval which contains $\varepsilon^\pm$ such that $R(\varepsilon,\Theta_R(\varepsilon)) = 0$ and $\theta^\pm = \Theta_R(\varepsilon^\pm;\varepsilon^+,\theta^+,\varepsilon^-)$. Its image through the function $\sigma = \sigma_{eq}(\varepsilon,\theta)$ in the $\varepsilon-\sigma$ plane is just the Rayleigh line, i.e. $\sigma_R(\varepsilon) = \sigma_{eq}(\varepsilon,\Theta_R(\varepsilon))$. Moreover, we have

$$\frac{\mathrm{d}\Theta_R(\varepsilon)}{\mathrm{d}\varepsilon} = \Big(\rho\dot{S}^2 - \frac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon,\Theta_R(\varepsilon))\Big)\Big(\frac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon,\Theta_R(\varepsilon))\Big)^{-1}. \tag{101}$$

Let us introduce the function $t(\varepsilon) \stackrel{\text{def}}{=} \Theta_R(\varepsilon) - \Theta_{Mxw}(\varepsilon)$ for $\varepsilon \in (\varepsilon^-,\varepsilon^+)$. We note that $t(\varepsilon^\pm) = 0$, and these are the only points where function $t = t(\varepsilon)$ vanishes, or equivalently, $(\varepsilon^\pm,\theta^\pm)$ are the only critical points of (57) in the interval $(\varepsilon^-,\varepsilon^+)$. Indeed, if we suppose there exists an $\varepsilon^* \in (\varepsilon^-,\varepsilon^+)$ such that $\Theta_R(\varepsilon^*) = \Theta_{Mxw}(\varepsilon^*)$ we get $\sigma_R(\varepsilon^*) = \sigma_{eq}(\varepsilon^*,\Theta_R(\varepsilon^*)) = \sigma_{eq}(\varepsilon^*,\Theta_{Mxw}(\varepsilon^*)) = \sigma_{Mxw}(\varepsilon^*)$ which is in contradiction with our assumption that the chord condition (95) is satisfied. By using (97) and (101) we get that

$$t'(\varepsilon^\pm) = \frac{\mathrm{d}\Theta_R(\varepsilon^\pm)}{\mathrm{d}\varepsilon} - \frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^\pm)}{\mathrm{d}\varepsilon} = s'(\varepsilon^\pm)\Big(\rho\frac{\partial\sigma_{eq}(\varepsilon^\pm,\theta^\pm)}{\partial\theta}\Big)^{-1}. \tag{102}$$

Since the chord criterion (95) requires $s'(\varepsilon^-) \le 0$ and $s'(\varepsilon^+) \ge 0$ one gets that $t'(\varepsilon^-) \ge 0$ and $t'(\varepsilon^+) \le 0$. Thus, it follows that $t(\varepsilon) = \Theta_R(\varepsilon) - \Theta_{Mxw}(\varepsilon) > 0$, for any $\varepsilon \in (\varepsilon^-,\varepsilon^+)$ (Fig. 4).

Let us note that function $\theta = \Theta_{Mxw}(\varepsilon)$ is a *strictly decreasing* function of $\varepsilon \in (\varepsilon^-,\varepsilon^+)$, and consequently, *the Hugoniot back state temperature has to be larger than the front state temperature*, i.e. $\theta^- > \theta^+$. Therefore, the corresponding *compressive discontinuity is of heating type*. The result is in agreement with the fact that the phase transformation $\mathscr{A} \to \mathscr{M}^-$ is exothermic. This behavior is a consequence of the fact that both terms in the parenthesis of the right part of relation (97) are negative. From physical point of view that means *both the intrinsic dissipation* and *the latent heat* contribute to the increase of temperature in the viscous, heat non-conducting profile layer.

To prove this assertion we have to note that $\frac{\partial\sigma_{eq}}{\partial\theta}(\tilde{\varepsilon}(\varepsilon),\Theta_{Mxw}(\varepsilon)) < 0$, and the chord condition (95) implies that $\sigma_R(\varepsilon) - \sigma_{eq}(\tilde{\varepsilon}(\varepsilon),\Theta_{Mxw}(\varepsilon)) < 0$, where $\tilde{\varepsilon}(\varepsilon)$ is given by (98), for any $\varepsilon \in (\varepsilon^-,\varepsilon^+)$. The last inequality follows from the identity $\big(\sigma_R(\varepsilon) - \sigma_{eq}(\tilde{\varepsilon},\Theta_{Mxw}(\varepsilon))\big)\big(E - \frac{\partial\sigma_{eq}}{\partial\varepsilon}(\varepsilon^*,\Theta_{Mxw}(\varepsilon))\big) = \big(\sigma_R(\varepsilon) - \sigma_{eq}(\varepsilon,\Theta_{Mxw}(\varepsilon))\big)E$, where ε^* lies between ε and $\tilde{\varepsilon}(\varepsilon)$.

Concerning the function $\theta = \Theta_R(\varepsilon)$, we note that it can be monotone decreasing, but it can be non-monotone, too. Indeed, the inequalities $\frac{\mathrm{d}\Theta_R(\varepsilon^+)}{\mathrm{d}\varepsilon} < \frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^+)}{\mathrm{d}\varepsilon} < 0$ and $\frac{\mathrm{d}\Theta_R(\varepsilon^-)}{\mathrm{d}\varepsilon} > \frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^-)}{\mathrm{d}\varepsilon}$, which follow from relation (102), require only that $\theta = \Theta_R(\varepsilon)$ is a decreasing function of ε in the neighborhood of ε^+ (Fig. 4).

The existence of a connecting orbit follows now from topological considerations similar with those used by Gilbarg [15]. The closed curve formed by $\theta = \Theta_{Mxw}(\varepsilon)$ and $\theta = \Theta_R(\varepsilon)$, for $\varepsilon \in (\varepsilon^-,\varepsilon^+)$, bounds a simply connected region P of the $\varepsilon-\theta$ plane. Since $H_{Mxw} > 0$ on the curve $R = 0$ and $R < 0$ on the curve $H_{Mxw} = 0$, for $\varepsilon \in (\varepsilon^-,\varepsilon^+)$, one concludes that everywhere in P, $H_{Mxw} > 0$ and $R < 0$. Let us note that on the boundaries $H_{Mxw} = 0$ and $R = 0$ all vector fields of the flow induced by (57) point toward the region P, horizontally and vertically, respectively.

Let us consider first the case of strict inequalities, i.e. $\dot{S}^2 > \lambda^2(\varepsilon^+,\theta^+)$ and $\dot{S}^2 < \lambda^2(\varepsilon^-,\theta^-)$. Since $\frac{\mathrm{d}\theta}{\mathrm{d}\varepsilon} = \frac{\mu(E-\rho\dot{S}^2)\dot{S}^2}{\kappa E}\frac{H_{Mxw}}{R}$, all integral curves of (57) must be monotone decreasing in P, and because they cannot leave P and there is no critical point in this region they must tend to the attractive point (ε^+,θ^+) (Fig. 4). Taking into account that (ε^-,θ^-) is a saddle point one obtains that a trajectory connecting (ε^+,θ^+) and (ε^-,θ^-) exists and lies inside the region P. Moreover, the temperature and the deformation vary *monotonously* across this *viscous, thermally conducting profile layer*.

Let us note that when $\dot{S}^2 = \lambda^2(\varepsilon^\pm,\theta^\pm)$, i.e. one eigenvalue (91) is zero and the another one is negative at a critical point, the two curves $H_{Mxw} = 0$ and $R = 0$ are tangent at $(\varepsilon^\pm,\theta^\pm)$. Moreover, they are tangent with the integral curve of (57) and with the isentrope (15) passing through this point, i.e. $\frac{\mathrm{d}\theta_{Mxw}(\varepsilon^\pm)}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\theta_R(\varepsilon^\pm)}{\mathrm{d}\varepsilon} = \theta^+\frac{\frac{\partial\sigma_{eq}}{\partial\theta}}{\rho C_{eq}}(\varepsilon^\pm,\theta^\pm) < 0$. The direction of this common tangent coincides with the direction of the eigenvector corresponding to the eigenvalue zero. Similar topological arguments prove the existence of a trajectory connecting (ε^+,θ^+) and (ε^-,θ^-).

Conversely, let us prove that the chord criterion is also a necessary condition for the existence of a profile layer. We suppose by absurd that a profile layer connecting $(\varepsilon^\pm,\theta^\pm)$ exists, but the chord criterion is violated. Let us assume there exists at least one point $\varepsilon^* \in (\varepsilon^-,\varepsilon^+)$ such that $\sigma_R(\varepsilon^*) = \sigma_{Mxw}(\varepsilon^*;\varepsilon^+,\theta^+,\varepsilon^-)$.

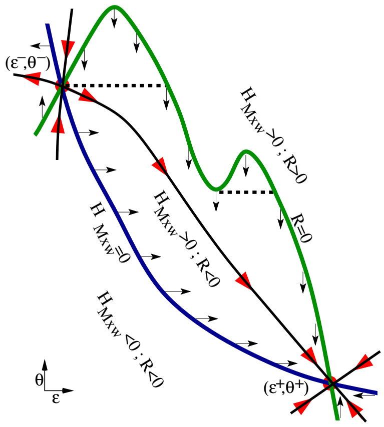

Fig. 4 Case C1 - phase portrait of (57) for $\mathscr{A} \to \mathscr{M}^-$ phase transformation illustrated in Fig. 3a.

According to relation

$$s(\varepsilon) = \sigma_R(\varepsilon) - \sigma_{Mxw}(\varepsilon) = \sigma_{eq}(\varepsilon, \Theta_R(\varepsilon)) - \sigma_{eq}(\varepsilon, \Theta_{Mxw}(\varepsilon)) = -\frac{\partial \sigma_{eq}(\varepsilon, \bar{\theta}(\varepsilon))}{\partial \theta}(\Theta_{Mxw}(\varepsilon) - \Theta_R(\varepsilon)), \quad (103)$$

where $\bar{\theta}(\varepsilon)$ lies between $\Theta_{Mxw}(\varepsilon)$ and $\Theta_R(\varepsilon)$, it follows that $\Theta_{Mxw}(\varepsilon^*) = \Theta_R(\varepsilon^*) \equiv \theta^*$, i.e. $R(\varepsilon^*, \theta^*) = 0$ and $H_{Mxw}(\varepsilon^*, \theta^*; \varepsilon^+, \theta^+, \varepsilon^-) = 0$. Therefore, $(\varepsilon^*, \theta^*)$ is a critical point of the system (57). On the other side, by using relation (62) we obtain $H(\varepsilon^*, \theta^*; \varepsilon^+, \theta^+) = 0$, that is $(\varepsilon^*, \theta^*)$ is also a Hugoniot state. Therefore, the curves $\theta = \Theta_{Mxw}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ and $\theta = \Theta_R(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ pass through the critical points between ε^- and ε^+. We also note that the Rayleigh line provides a natural ordering for these points. Considering the position of $\sigma = \sigma_{Mxw}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ with respect to the Rayleigh line one proves as before that these critical points alternate between being saddle points and attractive points. Because $H_{Mxw} > 0$ above $\theta = \Theta_{Mxw}(\varepsilon)$ and $R < 0$ below $\theta = \Theta_R(\varepsilon)$ one gets from a phase portrait diagram of type in Fig. 4 that $(\varepsilon^+, \theta^+)$ can be connected by a trajectory only with the first critical point with smaller strain than ε^+ and thus it is impossible to connect $(\varepsilon^+, \theta^+)$ by a trajectory with $(\varepsilon^-, \theta^-)$. Contradiction.

The uniqueness of the profile layer is based on the fact that a trajectory connecting $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$ can not lie outside P (see also Pego [26]).

Thus, for any $\mu > 0$ and $\kappa > 0$ there exists a unique profile layer $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \kappa)$ joining $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$. The limit behavior of such profile layer as $\mu \to 0$ and $\kappa \to 0$ can be studied in a similar way as was done by Gilbarg [15] for a viscous, thermally conducting fluid. One proves the existence of the iterated limits and their equality with the double limit. The limit is just a step wave discontinuity connecting $(\varepsilon^+, \theta^+)$ and $(\varepsilon^-, \theta^-)$. Moreover, Gilbarg [15] has put into evidence a basic difference in the effect of viscosity and of heat conduction on the structure of the profile layers which holds for the Maxwellian approach, too. Thus, if we consider a fixed viscosity $\mu = \bar{\mu}$ and $\kappa \to 0$, the trajectories in $\varepsilon - \theta$ plane of all profile layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \bar{\mu}, \kappa)$ are increasingly close to the decreasing curve $\theta = \Theta_{Mxw}(\varepsilon)$ and approach the solutions of the reduced system (66). This traveling wave solution is smooth with respect to ξ and describe a *viscous, heat non-conducting profile layer*.

If $\theta = \Theta_R(\varepsilon)$ is *monotone* decreasing and if we consider a fixed heat conductivity $\kappa = \bar{\kappa}$ and $\mu \to 0$, all shock layers curves $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ are increasingly close to the curve $\theta = \Theta_R(\varepsilon)$ and approaches the solutions of the following reduced system

$$\begin{aligned} &0 = R(\hat{\varepsilon}, \hat{\theta}), \\ &\hat{\theta}' = -\tfrac{\dot{S}}{\kappa} H_{Mxw}(\hat{\varepsilon}, \hat{\theta}), \quad \lim_{\xi \to \pm\infty} \hat{\theta}(\xi) = \theta^{\pm}, \end{aligned} \quad (104)$$

These profile layers describe *non-viscous, heat conducting profile layers*.

An important difference appears when $\theta = \Theta_R(\varepsilon)$ is *non monotone*. Since all integral curves of the system (57) are strictly decreasing in P one shows that as $\mu \to 0$ the trajectories in $\varepsilon - \theta$ plane of the profile layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ are increasingly close to the monotone decreasing curve $\theta = \bar{\Theta}_R(\varepsilon)$ defined by

$$\theta = \bar{\Theta}_R(\varepsilon) = \min_{\zeta \in [\varepsilon^-, \varepsilon]} \Theta_R(\zeta), \quad \text{for } \varepsilon \in [\varepsilon^-, \varepsilon^+]. \quad (105)$$

This function is represented with dotted line on those parts which do not coincide with $\theta = \Theta_R(\varepsilon)$ in Fig. 4. If $\theta = \Theta_R(\varepsilon)$ has a finite number of minima then $\theta = \bar{\Theta}_R(\varepsilon)$ has at most a finite number of intervals on which θ is constant, which correspond to what are called *isothermal jumps in strain inside the profile layer*. Therefore, in this case, as $\mu \to 0$ the profile layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ approaches a pair of functions denoted by $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu = 0, \bar{\kappa})$ with the property that $\hat{\varepsilon}(\xi; \mu = 0, \bar{\kappa})$ is *discontinuous* and $\hat{\theta}(\xi; \mu = 0, \bar{\kappa})$ is continuous and piecewise smooth. Thus, the notion of traveling wave solution must be enlarged in order to admit such discontinuous solutions for the reduced system (104).

Case C2. $\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^{\pm}, \theta^{\pm}) > 0$, i.e. negative Grüneisen coefficients at the critical points.

That means $(\varepsilon^{\pm}, \theta^{\pm})$ belong to the region where $\frac{\partial \sigma_{eq}}{\partial \theta} > 0$, that is where $\varepsilon > \varepsilon_t(\theta)$ (Fig. 2). According to assumptions **H3-H4** the front state and the Hugoniot state $(\varepsilon^{\pm}, \theta^{\pm})$ lie in the austenitic phase $\mathscr{A}$ or in the martensitic variant $\mathscr{M}^+$. A typical compressive jump discontinuity from $\mathscr{M}^+$ to $\mathscr{A}$ is illustrated in Fig. 3b.

Since $\frac{\partial R(\varepsilon,\theta)}{\partial \theta} = -\frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} < 0$ for $\varepsilon > \varepsilon_t(\theta)$, it follows that $R(\varepsilon, \theta) = 0$, given by (58), is representable as a single valued function of ε. We suppose there exists a function $\theta = \Theta_R(\varepsilon; \varepsilon^+, \theta^+, \theta^-)$, which satisfies $R(\varepsilon, \Theta_R(\varepsilon)) = 0$ and consequently, $\sigma_R(\varepsilon) = \sigma_{eq}(\varepsilon, \Theta_R(\varepsilon)))$, for ε belonging to an interval which contains ε^- and ε^+. Moreover, relations (97) and (101) are still valid.

By defining the function $t(\varepsilon) \stackrel{\text{def}}{=} \Theta_R(\varepsilon) - \Theta_{Mxw}(\varepsilon)$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$, taking into account that the Grüneisen coefficient is negative at the critical points, and using the same reasoning based on the chord condition (95) as in case *C1* we obtain from (102) that $t'(\varepsilon^-) \le 0$ and $t'(\varepsilon^+) \ge 0$, which involves that $t(\varepsilon) = \Theta_R(\varepsilon) - \Theta_{Mxw}(\varepsilon) < 0$, for any $\varepsilon \in (\varepsilon^-, \varepsilon^+)$.

From (99) one gets that $\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^{\pm})}{\mathrm{d}\varepsilon} \ge 0$. Therefore, $\theta = \Theta_{Mxw}(\varepsilon)$ is monotone increasing in the neighborhood of $\varepsilon^{\pm}$, but we cannot say anything, without additional constitutive assumptions, neither about its monotonicity, nor about the order relation between θ^- and θ^+. Indeed, in the present compressive case, the first term in the right part of relation (97) is negative, that means the intrinsic dissipation always contributes to the increase of the temperature inside the viscous, heat non-conducting profile layer, while the second term is positive, that is, the latent heat contributes to the decrease of the temperature inside this layer. Therefore, $\theta = \Theta_{Mxw}(\varepsilon)$ is monotone increasing on those intervals where the cooling due to latent heat dominates the heating due to intrinsic dissipation and it is monotone decreasing when the opposite case happens.

The following representative cases will be analyzed:

a) $\theta^- < \theta^+$ and $\theta = \Theta_{Mxw}(\varepsilon)$ *monotone increasing* (Fig. 5a).

b) $\theta^- < \Theta_{Mxw}(\varepsilon) < \theta^+$, but $\theta = \Theta_{Mxw}(\varepsilon)$ is *non-monotone* (Fig. 5b).

c) $\theta^- > \theta^+$ (Figs. 6).

We consider as natural from physical point of view for phase transforming materials the case **a)**, where the latent heat effect is more important than the dissipation effect along the traveling wave. That is way we discuss in the following some additional constitutive restrictions which are sufficient to ensure such behavior.

Example 1. Let us consider for simplicity the Kelvin-Voigt model (30). By making $E \to \infty$ in (97) we get that function $\theta = \Theta_{KV}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ satisfies relation

$$\frac{d\Theta_{KV}(\varepsilon)}{d\varepsilon} = \frac{1}{\rho C_{eq}(\varepsilon, \Theta_{KV}(\varepsilon))} \Big(\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon) + \Theta_{KV}(\varepsilon) \frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon, \Theta_{KV}(\varepsilon)) \Big). \tag{106}$$

Suppose that

$$\rho \Big| \frac{\partial C_{eq}(\varepsilon, \theta)}{\partial \varepsilon} \Big| \ll \Big| \frac{\partial \sigma_{eq}(\varepsilon, \theta)}{\partial \theta} \Big|, \tag{107}$$

that is, the variation of the specific heat $C_{eq}(\varepsilon, \theta)$ with respect to ε is negligible regarding the variation of $\sigma_{eq}(\varepsilon, \theta)$ with respect to the θ. The simplest case is when the specific heat does not depend on ε.

Taking into account that $\frac{\partial^2 \sigma_{eq}(\varepsilon,\theta)}{\partial \theta^2} = -\frac{\rho}{\theta} \frac{\partial C_{eq}(\varepsilon,\theta)}{\partial \varepsilon}$, condition (107) becomes $\theta \Big| \frac{\partial^2 \sigma_{eq}(\varepsilon,\theta)}{\partial \theta^2} \Big| \ll \Big| \frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} \Big|$. By using the mean value theorem we can show that for any $\varepsilon \in (\varepsilon^-, \varepsilon^+)$

$$\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon) + \Theta_{KV}(\varepsilon) \frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon, \Theta_{KV}(\varepsilon)) = \Theta_R(\varepsilon) \frac{\partial \sigma_{eq}(\varepsilon, \Theta_{KV}(\varepsilon))}{\partial \theta} + \frac{1}{2} (\Theta_R(\varepsilon) - \Theta_{KV}(\varepsilon))^2 \frac{\partial^2 \sigma_{eq}(\varepsilon, \bar{\theta})}{\partial \theta^2},$$

where $\bar{\theta}$ lies between $\Theta_R(\varepsilon)$ and $\Theta_{KV}(\varepsilon)$. Thus, it follows that $\frac{d\Theta_{KV}(\varepsilon)}{d\varepsilon} \approx \frac{\Theta_R(\varepsilon)}{\rho C_{eq}(\varepsilon, \Theta_{KV}(\varepsilon))} \frac{\partial \sigma_{eq}(\varepsilon, \Theta_{KV}(\varepsilon))}{\partial \theta} > 0$.

Consequently, $\theta = \Theta_{KV}(\varepsilon)$ is a *strictly increasing* function of $\varepsilon \in (\varepsilon^-, \varepsilon^+)$ and *the Hugoniot back state temperature has to be lower than the front state temperature*, i.e. $\theta^- < \theta^+$. Therefore, the corresponding *compressive wave discontinuity is of cooling type*. The result is in agreement with the fact that the reverse phase transformation $\mathscr{M}^+ \to \mathscr{A}$ is endothermic.

Remark. Let us note that the same condition (107) can be also used for the complementary *expansive case* ($\varepsilon^+ < \varepsilon^-$) with positive Grüneisen coefficients at the critical points, i.e. $\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^\pm, \theta^\pm) < 0$.

Indeed, in this case the chord condition requires that $\sigma_R(\varepsilon) - \sigma_{KV}(\varepsilon) > 0$ for any $\varepsilon \in (\varepsilon^+, \varepsilon^-)$. One gets from (106) and (107) that $\theta = \Theta_{KV}(\varepsilon)$ is a strictly decreasing function of ε and thus, the Hugoniot back state temperature has to be lower than the front state temperature, i.e. $\theta^+ > \theta^-$. Therefore, the corresponding *expansive wave discontinuity (rarefaction shock) is of cooling type*. Such behavior is natural and in agreement with the fact that the reverse phase transformation $\mathscr{M}^- \to \mathscr{A}$ is also endothermic. Physically that means the heating due to the intrinsic dissipation is dominated by the cooling due to latent heat.

For the Maxwellian rate-type model (29) if we suppose besides condition (107) that

$$\left|\varepsilon \frac{\partial^2 \sigma_{eq}(\varepsilon,\theta)}{\partial \varepsilon \partial \theta}\right| \ll \left|\frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta}\right| \quad \text{and} \quad \left|\varepsilon \frac{\partial^2 \sigma_{eq}(\varepsilon,\theta)}{\partial \varepsilon^2}\right| \ll E, \tag{108}$$

we can show, by using (96), that $\frac{d\Theta_{Mxw}(\varepsilon)}{d\varepsilon} \approx \frac{(E-\rho\dot{S}^2)\Theta_R(\varepsilon)}{\rho C_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\Theta_{Mxw}(\varepsilon))\left(E-\frac{\partial \sigma_{eq}}{\partial \varepsilon}(\varepsilon,\Theta_{Mxw}(\varepsilon))\right)} \frac{\partial \sigma_{eq}(\varepsilon,\Theta_{Mxw}(\varepsilon))}{\partial \theta} > 0.$

Thus $\theta = \Theta_{Mxw}(\varepsilon)$ is in case *C2* a strictly increasing function of ε and consequently $\theta^- < \theta^+$, which corresponds to a compressive shock discontinuity of cooling type.

Let us note that the explicit piecewise linear thermoelastic model considered in Part II [10] to describe phase transformation in a SMA alloy fulfills conditions (107) and (108).

Example 2. Another type of constitutive restriction has been considered by Pego [26] in order to prove the existence and uniqueness of shock layers in gas dynamics with non-convex equation of state and positive Grüneisen coefficient. In this case the shock layers are solution of the Navier-Stokes equation with viscosity and heat conduction for one-dimensional flows. Let us note that for fluids one considers only positive pressure. That corresponds in our notations to $\sigma = -p < 0$. Pego [26] proposed the following condition

$$\rho \frac{\partial e_{eq}(\varepsilon,\theta)}{\partial \varepsilon} = \sigma_{eq}(\varepsilon,\theta) - \theta \frac{\partial \sigma_{eq}(\varepsilon,\theta)}{\partial \theta} \geq 0. \tag{109}$$

Using (109) in relation (106) one obtains $\rho C_{eq}(\varepsilon,\theta)\frac{d\Theta_{KV}(\varepsilon)}{d\varepsilon} = \sigma_R(\varepsilon) - \frac{\partial e_{eq}(\varepsilon,\Theta_{KV}(\varepsilon))}{\partial \varepsilon} < 0.$

Thus, for the *expansive case* considered by Pego, i.e. for $\varepsilon^+ < \varepsilon^-$, one obtains that $\theta = \Theta_{KV}(\varepsilon)$ is strictly decreasing and the *rarefaction shock is of cooling type*, i.e $\theta^- < \theta^+$.

The inconvenience of this condition appears for solids, in the compressive case, when $\sigma = \sigma_R(\varepsilon)$ is in general positive and (109) does not more lead to the monotony of $\theta = \Theta_{KV}(\varepsilon)$. Thus, for our phase transforming material, in the compressive case *C2*, Pego's condition does not involve that $\theta = \Theta_{KV}(\varepsilon)$ is an increasing function of ε, i.e. the compressive discontinuity is of cooling type as it should be for a shock induced $\mathscr{M}^+ \to \mathscr{A}$ phase transformation.

We can show now that for the phase diagrams illustrated in Figs. 5 and Fig. 6a), which correspond to **cases a)**, **b)** and **c)**, respectively, the chord condition (95) is a necessary and sufficient requirement for the existence and uniqueness of a viscous, heat conducting profile layer for any given coefficients $\mu > 0$ and $\kappa > 0$. The proof, given below, is based on the properties of the vector field of the flow induced by (57) and topological considerations related to the corresponding figures.

On the other side, for the phase diagram illustrated by Fig. 6b, where the chord criterion is satisfied, but the curves $\theta = \Theta_{Mxw}(\varepsilon;\varepsilon^+\theta^+,\theta^-)$ and $\theta = \Theta_R(\varepsilon;\varepsilon^+\theta^+,\theta^-)$ meet again at a critical point (ε_0,θ_0), with the properties that $\varepsilon_0 > \varepsilon^+$ and $\theta_0 < \theta^-$, a trajectory connecting (ε^+,θ^+) and (ε^-,θ^-) no longer exists if the heat conduction dominates the viscosity. This phase diagram corresponds to the example given by Pego [26]) concerning the nonexistence of a shock layer in gas dynamics with a nonconvex equation of state.

Let us consider, for example, **case b)** represented by Fig. 5b). We denote by P the simply connected region bounded by $\theta = \Theta_{Mxw}(\varepsilon)$ and $\theta = \Theta_R(\varepsilon)$ for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$. Since $H_{Mxw} < 0$ and $R < 0$ in P any integral curves of (57) is monotone increasing inside P. On the boundary $R = 0$ all vector fields of the flow induced by (57) point vertically toward the region P while on the boundary $H_{Mxw} = 0$ all vector fields point horizontally right. Therefore, on the ascending branches of the curve $\theta = \Theta_{Mxw}(\varepsilon)$ the vector fields point toward the region P while on the descending branches of the curve $\theta = \Theta_{Mxw}(\varepsilon)$ they point horizontally outwards the region P.

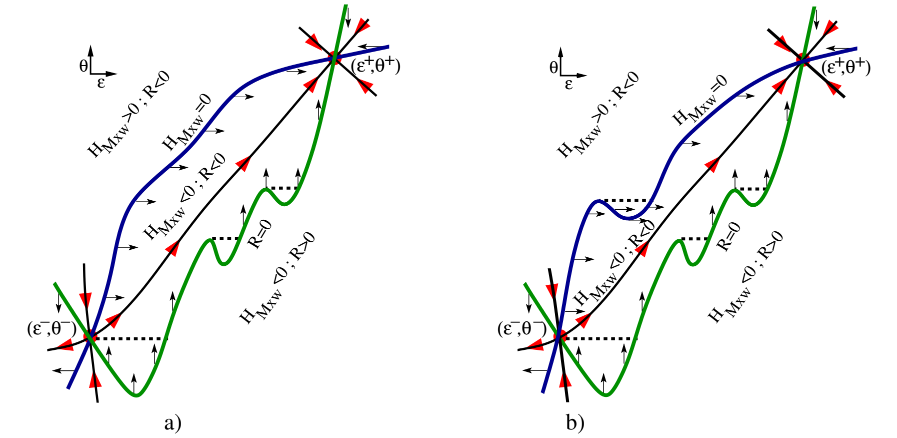

Fig. 5 Case C2 - phase portrait of (57) when: a) $\theta^- < \theta^+$ and $\theta = \Theta_{Mxw}(\varepsilon)$ monotone increasing; b) $\theta^- < \Theta_{Mxw}(\varepsilon) < \theta^+$, but $\theta = \Theta_{Mxw}(\varepsilon)$ is non-monotone.

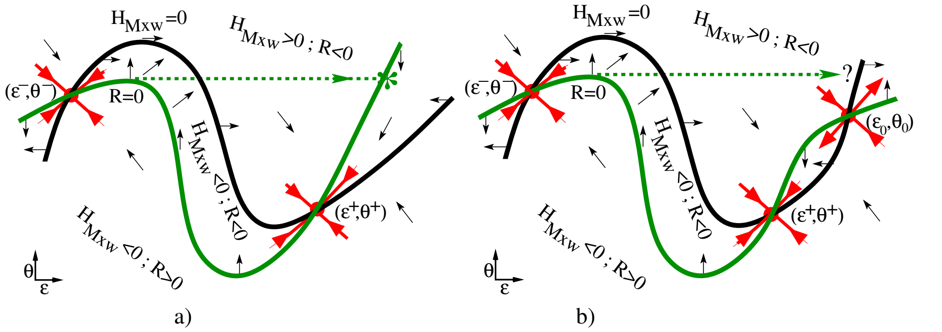

Fig. 6 Case C2 - possible phase portraits of (57) when $\theta^- > \theta^+$ and the chord criterion is satisfied: a) profile layers exists for any $\mu > 0$ and $\kappa > 0$; b) profile layers does not exists if heat conduction dominates the viscosity.

Let us note that if $\varepsilon \in (\varepsilon^-, \varepsilon^+)$ and $\theta > \Theta_{Mxw}(\theta)$ we have $H_{Mxw} > 0$ and $R < 0$. Consequently, any integral curve which leaves the domain P through the descending branches of the curve $\theta = \Theta_{Mxw}(\varepsilon)$ is monotone descending and when it meets again an ascending branch of this curve it is directed inside the region P.

Let us introduce the continuous and monotonic function

$$\theta = \bar{\Theta}_{Mxw}(\varepsilon) = \max_{\zeta \in [\varepsilon^-, \varepsilon]} \Theta_{Mxw}(\zeta), \quad \text{for } \varepsilon \in [\varepsilon^-, \varepsilon^+]. \tag{110}$$

This is the minimum among all monotone increasing curves bounded from below by the curve $\theta = \Theta_{Mxw}(\varepsilon)$. It is composed by ascending branches of $\theta = \Theta_{Mxw}(\theta)$ and by the horizontal lines marked with dotted lines in Fig. 5b. Let us denote by $\bar{P}$ the simply connected region bounded by $\theta = \bar{\Theta}_{Mxw}(\varepsilon)$ and $\theta = \Theta_R(\varepsilon)$. According to the above said, any integral curve of (57) cannot leave $\bar{P}$ and since there is no critical point in region $\bar{P}$ they must tend to the attractive point $(\theta^+, \varepsilon^+)$. Because $(\theta^-, \varepsilon^-)$ is a saddle point one obtains that a trajectory connecting $(\theta^+, \varepsilon^+)$ and $(\theta^-, \varepsilon^-)$ exists for any $\mu > 0$ and $\kappa > 0$ and lies inside $\bar{P}$.

The reverse implication can be proved in the same way as in case *C1* using relation (103). The uniqueness of the connecting orbit is based on the fact that a trajectory connecting $(\varepsilon^\pm, \theta^\pm)$ cannot lie outside $\bar{P}$.

Let us note that, for fixed viscosity $\mu = \bar{\mu}$ and sufficiently small κ, the connecting orbit $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \bar{\mu}, \kappa)$ is close to the non-monotone curve $\theta = \Theta_{Mxw}(\varepsilon)$. In this case the smooth profile layer has the property that $\varepsilon = \hat{\varepsilon}(\xi)$ is strictly monotone, but $\theta = \hat{\theta}(\xi)$ may become non-monotone as it happens in Fig. 5b. On the other side, when the heat conduction dominates the viscosity, the connecting orbit has to be close to the curve $\theta = \Theta_R(\varepsilon)$ if this is monotone increasing or to the curve $\theta = \bar{\Theta}_R(\varepsilon)$ defined by relation

$$\theta = \bar{\Theta}_R(\varepsilon) = \max_{\zeta \in [\varepsilon^-, \varepsilon]} \Theta_R(\zeta), \text{ for } \varepsilon \in [\varepsilon^-, \varepsilon^+], \tag{111}$$

if $\theta = \Theta_R(\varepsilon)$ is non-monotone. In this second case the limit trajectory $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); 0, \bar{\kappa})$ has the property that $\hat{\varepsilon}(\xi)$ is discontinuous (isothermal jumps inside the profile layer) and piecewise smooth, while $\hat{\theta}(\xi)$ is continuous and piecewise smooth. Its trajectory in the $\varepsilon - \theta$ plane is given by the ascending branches of $\theta = \Theta_R(\varepsilon)$ and by the isothermal strain jumps represented by dotted lines in Fig. 5b.

The case when the *non-monotone* function $\theta = \Theta_{Mxw}(\varepsilon)$ overcomes θ^+ can be treated as in **case c)** considered in the following.

Let us first consider the phase diagram in Fig. 6a. Since $\theta^- > \theta^+$ and $\frac{d\Theta_{Mxw}(\varepsilon^\pm)}{d\varepsilon} \geq 0$ it follows that $\theta = \Theta_{Mxw}(\varepsilon)$ must necessarily be *non-monotone*. According to relation (102) we have $\frac{d\Theta_R(\varepsilon^+)}{d\varepsilon} > \frac{d\Theta_{Mxw}(\varepsilon^+)}{d\varepsilon} > 0$ and $\frac{d\Theta_R(\varepsilon^-)}{d\varepsilon} < \frac{d\Theta_{Mxw}(\varepsilon^-)}{d\varepsilon}$. Consequently, $\theta = \Theta_R(\varepsilon)$ has to be also a non-monotone function for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$, but always ending with a positive slope in the neighborhood of ε^+. Since $H_{Mxw} < 0$ and $R < 0$ in the region P bounded by $\theta = \Theta_{Mxw}(\varepsilon)$ and $\theta = \Theta_R(\varepsilon)$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$ it follows that any integral curve of (57) is monotone increasing inside P. On the boundary $R = 0$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$, all vector fields of the flow point vertically toward the region P, while on the curve $\theta = \Theta_{Mxw}(\varepsilon)$ the vector fields point horizontally toward the region P on the ascending branches and point horizontally outwards P on the descending branches. In the region above $H_{Mxw} = 0$ for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$ we have $H_{Mxw} > 0$ and $R < 0$, and consequently the integral curves of (57) are monotone decreasing. Therefore, the integral curve starting from the saddle point $(\varepsilon^-, \theta^-)$ towards P has to be monotone increasing until it meets the descending branch of the curve $\theta = \Theta_{Mxw}(\varepsilon)$. After traversing it, this integral curve is monotone descending. If the viscosity dominates the heat conduction then this integral curve is close to the curve $\theta = \Theta_{Mxw}(\varepsilon)$. It may happen that $\hat{\theta}(\xi)$ descends below θ^+. Then, this integral curve will meet the ascending branch of the curve $\theta = \Theta_{Mxw}(\varepsilon)$ and will enter again inside P reaching the attractive point $(\varepsilon^+, \theta^+)$ by an ascending curve.

A different situation appears when the viscosity is dominated by the heat conduction. In this case the strain $\hat{\varepsilon}(\xi)$ may overcome ε^+. Thus, when $\mu \to 0$ the profile layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ will approach a pair of functions denoted by $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu = 0, \bar{\kappa})$ with the property that $\hat{\varepsilon}(\xi)$ is *discontinuous* while $\hat{\theta}(\xi)$ is continuous and piecewise smooth. Let us denote by $(\varepsilon^*, \theta^-)$ the unique point with the property that $R(\varepsilon^*, \theta^-) = 0$, where $\varepsilon^* > \varepsilon^+$ (Fig. 6a). Due to the above topological considerations this discontinuous limit solution, which corresponds to a non-viscous, heat conducting profile layer, is characterized by an isothermal jump from $(\varepsilon^-, \theta^-)$ to $(\varepsilon^*, \theta^-)$ and next is described by a pair of smooth functions $(\hat{\varepsilon}(\xi), \hat{\theta}(\xi))$ which are solution of the reduced system (104) and connect the point $(\varepsilon^*, \theta^-)$ with the attractive point $(\varepsilon^+, \theta^+)$.

The reverse implication that the chord criterion is also a necessary condition for the existence of a connecting orbit and its uniqueness can be also proved. We omit here the details.

Let us consider now the uncommon **case c)** illustrated in Fig. 6b. We denote again by $(\varepsilon^*, \theta^-)$ the unique point with the property that $R(\varepsilon^*, \theta^-) = 0$, where $\varepsilon^* > \varepsilon^+$. Unlike the diagram in Fig. 6a, we suppose that between ε^- and ε^* there exists another critical point $(\varepsilon_0, \theta_0)$ for the system (57), i.e. a point where the curves $\theta = \Theta_{Mxw}(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ and $\theta = \Theta_R(\varepsilon; \varepsilon^+, \theta^+, \varepsilon^-)$ intersect each other again. Let us note that this point has to be a saddle point like $(\varepsilon^-, \theta^-)$. The chord condition is satisfied for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$ and if the viscosity dominates heat conduction it is obvious that a shock layer connecting $(\varepsilon^\pm, \theta^\pm)$ exists. Instead, if the viscous effects are negligible with respect to heat conduction effects, it is possible that there is no trajectory connecting $(\varepsilon^-, \theta^-)$ with $(\varepsilon^+, \theta^+)$. Indeed, for fixed κ and $\mu \to 0$, the trajectory $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ leaving the saddle point $(\varepsilon^-, \theta^-)$ is almost horizontal and must enter in the region between $R(\varepsilon, \theta; \varepsilon^+, \theta^+, \varepsilon^-) = 0$ and $H_{Mxw}(\varepsilon, \theta; \varepsilon^+, \theta^+, \varepsilon^-) = 0$, for $\varepsilon > \varepsilon_0$. Since in this region $H_{Mxw} > 0$ and $R > 0$ the trajectory must be monotone increasing (ε increasing, θ increasing). Consequently, the trajectory can not more reach the critical point $(\varepsilon^+, \theta^+)$. Such an explicit example has been constructed by Pego [26] in order to exemplify that there may be a wave discontinuity which satisfies the chord criterion, but for which a profile layer doest not exist if the heat conduction dominates the viscosity.

Case C3. $\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^+, \theta^+) > 0$ and $\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^-, \theta^-) < 0$, i.e. different signs of the Grüneisen coefficient at the critical points.

That means, $(\varepsilon^-, \theta^-)$ belongs to the region $\mathscr{D}^-$ where $\frac{\partial \sigma_{eq}}{\partial \theta} < 0$, that is, where $\varepsilon < \varepsilon_t(\theta)$ and $(\varepsilon^+, \theta^+)$ belongs to the region $\mathscr{D}^+$ where $\frac{\partial \sigma_{eq}}{\partial \theta} > 0$, that is, where $\varepsilon > \varepsilon_t(\theta)$ (Fig. 2). A typical compressive jump discontinuity $\mathscr{M}^+ \to \mathscr{M}^-$ is illustrated in Fig. 3c and Fig. 7.

We remind that the chord condition (95) requires that $s'(\varepsilon^-) \leq 0$ and $s'(\varepsilon^+) \geq 0$. Let us suppose first that the inequalities are strict. According to (100), $(\varepsilon^-, \theta^-)$ is a *saddle point* and $(\varepsilon^+, \theta^+)$ is an *attractive point*.

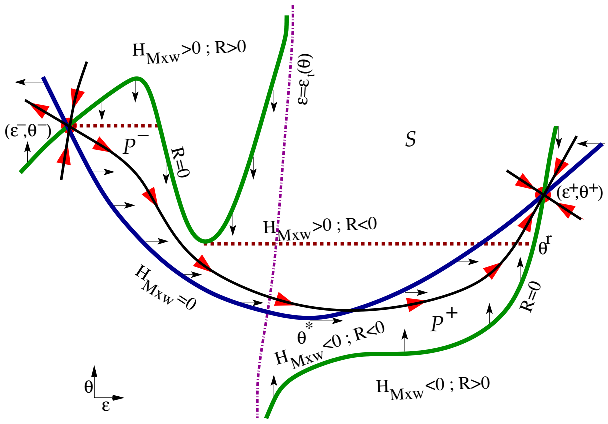

Fig. 7 Case C3 - phase portrait of (57) for the $\mathscr{M}^+ \to \mathscr{M}^-$ phase transformation illustrated in Fig. 3c.

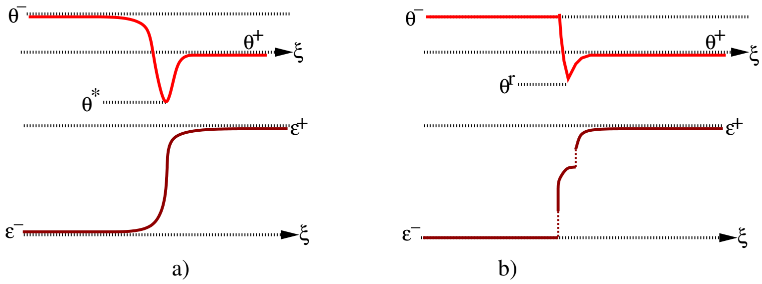

Fig. 8 Case C3 - temperature spike-layers and strain interphase transition layers corresponding to Fig. 7. a) viscous ($\mu > 0$), heat non-conducting layer ($\kappa = 0$); b) non-viscous ($\mu = 0$), heat conducting ($\kappa > 0$).

By using relation (99) one gets that $\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^-)}{\mathrm{d}\varepsilon} < 0$ and $\frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^+)}{\mathrm{d}\varepsilon} > 0$. Therefore, $\theta = \Theta_{Mxw}(\varepsilon)$ must necessarily be a *non-monotone* function. It is monotone decreasing in the neighborhood of ε^- and monotone increasing in the neighborhood of ε^+. Moreover, by using the chord condition (95) and relation (97) one gets that the branch of the curve $\theta = \Theta_{Mxw}(\varepsilon)$ which lies in $\mathscr{D}^-$ is monotone decreasing. This reflects the fact that, on this part of the trajectory of a viscous, heat non-conducting profile layer, both the intrinsic dissipation and the latent heat contribute to the heating process inside the layer. After the intersection with $\varepsilon = \varepsilon_t(\theta)$, the function $\theta = \Theta_{Mxw}(\varepsilon)$ reaches a minimum point at $(\varepsilon^*, \theta^*)$ and we suppose, that later on, it is monotone increasing as in Fig. 7. This situation always happens if the additional restrictions (107) and (108) are satisfied. On this part of the trajectory of the viscous, heat non-conducting profile layer, the intrinsic dissipation contributes to the heating, while the latent heat contribute to the cooling process inside the layer. Let us note that we can not say anything about the order relation between θ^+ and θ^-.

Since the Grüneisen coefficients have different signs at the critical points it follows that the implicit equation $R(\varepsilon, \theta) = 0$, given by (58), is representable, and we assume that globally, by two functions of ε. One branch $\theta = \Theta_R^-(\varepsilon)$, passing through $(\varepsilon^-, \theta^-)$, lies in the domain $\mathscr{D}^-$ and satisfies $R(\varepsilon, \Theta_R^-(\varepsilon)) = 0$ and $\sigma_R(\varepsilon) = \sigma_{eq}(\varepsilon, \Theta_R^-(\varepsilon))$ on the corresponding interval of definition. The second branch $\theta = \Theta_R^+(\varepsilon)$, passing through $(\varepsilon^+, \theta^+)$, lies in the domain $\mathscr{D}^+$ and satisfies $R(\varepsilon, \Theta_R^+(\varepsilon)) = 0$ and $\sigma_R(\varepsilon) = \sigma_{eq}(\varepsilon, \Theta_R^+(\varepsilon))$ on the corresponding interval of existence. Relations (101) and (102) are still valid for each branch and we get that $\frac{\mathrm{d}\Theta_R^-(\varepsilon^-)}{\mathrm{d}\varepsilon} > \frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^-)}{\mathrm{d}\varepsilon}$ and $\frac{\mathrm{d}\Theta_R^+(\varepsilon^+)}{\mathrm{d}\varepsilon} > \frac{\mathrm{d}\Theta_{Mxw}(\varepsilon^+)}{\mathrm{d}\varepsilon} > 0$. Therefore, $\theta = \Theta_R^+(\varepsilon)$ is an increasing function of ε in the neighborhood of ε^+ and moreover $\Theta_R^+(\varepsilon) < \Theta_{Mxw}(\varepsilon)$ in $\mathscr{D}^+$ for $\varepsilon < \varepsilon^+$. Indeed, this inequality can be proved by taking into account that it is satisfied in a neighborhood of ε^+ and from the fact that $\theta = \Theta_R^+(\varepsilon)$ (even non-monotone) cannot intersect $\theta = \Theta_{Mxw}(\varepsilon)$ a second time without violating the chord condition (95). In a similar way one shows that $\Theta_R^-(\varepsilon) > \Theta_{Mxw}(\varepsilon)$ in $\mathscr{D}^-$ for $\varepsilon > \varepsilon^-$.

Let us note that $\theta = \Theta_R^-(\varepsilon)$ is a non monotone function. Indeed, when $(\varepsilon, \Theta_R(\varepsilon)) \in \mathscr{I}^- \subset \mathscr{D}^-$ (see Fig. 2) we have $\frac{\partial \sigma_{eq}(\varepsilon, \Theta_R(\varepsilon))}{\partial \varepsilon} < 0$ and we get from (101) that $\theta = \Theta_R(\varepsilon)$ is monotone decreasing. When $\theta = \Theta_R(\varepsilon)$ enters in the domain $\mathscr{A} \subset \mathscr{D}^-$, it must end with a positive slope since otherwise would intersect the curve

$\theta = \Theta_{Mxw}(\varepsilon)$, which would contradict the chord criterion. Therefore, there exists a point $(\varepsilon^r, \theta^r) \in \mathscr{D}^-$ where the branch $\theta = \Theta_R^-(\varepsilon)$ reaches its minimum and for simplicity we assume that its form is as shown in Fig. 7.

We suppose first that $\theta^r < \theta^+$. That means, for $\bar{\theta} \in (\theta^r, \theta^+)$ the isotherm $\sigma = \sigma_{eq}(\varepsilon, \bar{\theta})$ intersects the Rayleigh line $\sigma = \sigma_R(\varepsilon)$ in three points in the interval $(\varepsilon^-, \varepsilon^+)$.

The existence of a profile layer, follows now from the following topological considerations. We denote by P^- the simply connected region bounded from above by $\theta = \Theta_R^-(\varepsilon)$, from the right by $\varepsilon = \varepsilon_t(\theta)$ and from below by $\theta = \Theta_{Mxw}(\varepsilon)$. We denote by P^+ the simply connected region bounded from below by $\theta = \Theta_R^+(\varepsilon)$, from the left by $\varepsilon = \varepsilon_t(\theta)$ and from above by $\theta = \Theta_{Mxw}(\varepsilon)$. We denote by S the simply connected region bounded from below by $\theta = \Theta_{Mxw}(\varepsilon)$, from the left by $\varepsilon = \varepsilon_t(\theta)$ and from the right by $\theta = \Theta_R^+(\varepsilon)$ for $\varepsilon > \varepsilon^+$. Let us note that $H_{Mxw} > 0$ on the curve $\theta = \Theta_R^-(\varepsilon)$ for $\varepsilon > \varepsilon^-$ and $H_{Mxw} < 0$ on the curve $\theta = \Theta_R^+(\varepsilon)$ for $\varepsilon < \varepsilon^+$. On the other side, $R < 0$ on the curve $\theta = \Theta_{Mxw}(\varepsilon)$ for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$. Therefore, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$, on the curve $H_{Mxw} = 0$ all vector fields of the flow induced by (57) point horizontally to the right. On the curve $\theta = \Theta_R^-(\varepsilon)$ the vector fields point vertically down, while on the curve $\theta = \Theta_R^+(\varepsilon)$ they point vertically up (Fig. 7). The integral curves of (57), which satisfy $\frac{d\theta}{d\varepsilon} = \frac{\mu(E-\rho\dot{S}^2)\dot{S}^2}{\kappa E}\frac{H_{Mxw}}{R}$, have the following properties. For given $\mu > 0$ and $\kappa > 0$ they are monotone decreasing in $P^- \cup S$ and are monotone increasing in P^+. Since $(\varepsilon^-, \theta^-)$ is a saddle point, the trajectory starting from this point toward region P^- is monotone decreasing and intersects the curve $\theta = \Theta_{Mxw}(\varepsilon)$ at a point in the region $\mathscr{D}^+$. At this point the temperature reaches a minimum value in the profile layer. After entering in the domain P^+ the trajectory is monotone increasing and since it cannot leave this domain it must end at the attractive point $(\varepsilon^+, \theta^+)$.

Therefore, when the Grüneisen coefficients have different signs at the critical points the temperature variation inside a viscous, heat conducting profile layer is *non-monotone*. Moreover, in the compressive case, the temperature reaches lower values than the initial and Hugoniot temperature (Fig. 7), while in the expansive case it will reach higher values. Thus, the profile layer of the temperature displays a narrow peak (or, spike) pointing down as it is illustrated for instance in Fig. 8. At the point ξ_0 where $\frac{d\hat{\theta}}{d\xi} = 0$ the heat flux q changes the sign, that means the heat flux q changes the direction inside the profile layer. This behavior is in agreement with the fact that, a continuous transformation from variant $\mathscr{M}^+$ to variant $\mathscr{M}^-$ passes through the phase $\mathscr{A}$ and the transformation $\mathscr{M}^+ \to \mathscr{A}$ is endothermic (cooling for $\xi > \xi_0$) while the transformation $\mathscr{A} \to \mathscr{M}^-$ is exothermic (heating for $\xi < \xi_0$) (Fig. 8).

The limit behavior of the profile layers $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \kappa)$ as $\mu \to 0$ and/or $\kappa \to 0$ can be studied in the same way as in the previous compressive cases.One observes that when the viscosity largely dominates the heat conduction, the trajectory of the connecting integral curve is closer to the curve $H_{Mxw} = 0$. Moreover, for fixed viscosity $\mu = \bar{\mu}$ and $\kappa = 0$ the connecting orbit $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \bar{\mu}, 0)$ is solution of the reduced system (66) and matches with the non-monotone curve $\theta = \Theta_{Mxw}(\varepsilon)$. The corresponding *viscous, heat non-conducting profile layer* is illustrated in Fig. 8a. It is worth to remark that when $\mu \to 0$ the limit of the strain profile $\hat{\varepsilon}(\xi; \mu, 0)$ is a discontinuous function whose value is ε^- for $\xi < \xi_0$ and ε^+ for $\xi > \xi_0$. On the other side, due to the spike-layer form of the temperature profiles, the limit of $\hat{\theta}(\xi; \mu, 0)$, for $\mu \to 0$, is a discontinuous function whose value is θ^- for $\xi < \xi_0$, θ^* at $\xi = \xi_0$ and θ^+ for $\xi > \xi_0$, where θ^* represents the minimum value of the function $\theta = \Theta_{Mxw}(\varepsilon)$, for $\varepsilon \in (\varepsilon^-, \varepsilon^+)$. For both strain and temperature profiles the convergence is uniform in ξ in any closed interval not containing the discontinuity point ξ_0. We have to remark that the adiabatic thermoelastic temperature structure with sharp interface does not inherit the temperature structure of the augmented theory. Indeed, it should be noted that the value θ^* does not play any role in solving a Riemann problem involving a compressive shock induced $\mathscr{M}^- \to \mathscr{M}^+$ phase transformation in the framework of the adiabatic thermoelastic wave theory. In this approach only the lateral limits across the discontinuities are relevant. On the other side, for an adiabatic rate-type approach of the same problem, with small viscosity μ, the minimum value θ^* and the corresponding spike-layer temperature structure may become extremely important, specially when one describes wave interaction phenomena. The prediction of the augmented theory of larger values of temperature than the initial and final one inside a profile layer could be also important in terms of experimental.

Let us consider the opposite case when the heat conduction largely dominates the viscosity. In a similar way as in the previous cases one shows that for fixed $\kappa = \bar{\kappa}$, as $\mu \to 0$, the trajectories $(\hat{\theta}(\xi), \hat{\varepsilon}(\xi); \mu, \bar{\kappa})$ of the integral curves of (57) in the region $\mathscr{D}^-$ tend to be closer to a monotone decreasing curve $\theta = \bar{\Theta}_R^-(\varepsilon)$ of the type (105), while the trajectories of the integral curves in the region $\mathscr{D}^+$ tend to be closer to a monotone increasing curve $\theta = \bar{\Theta}_R^+(\varepsilon)$ of the type (111). We also note that the intersection point between the trajectory of the profile layer $(\hat{\theta}(\xi),\hat{\varepsilon}(\xi);\mu,\bar{\kappa})$ with the curve $\theta=\Theta_{Mxw}(\varepsilon)$ moves up as $\mu\to 0$. The corresponding limit value of the temperature is θ^r, which is the minimum value of the function $\theta=\Theta_R^-(\varepsilon)$.

Since the sign of the slope of these trajectories is negative in $P^-\cup S$ and positive in P^+ one gets that the limit trajectory $(\hat{\theta}(\xi),\hat{\varepsilon}(\xi);0,\bar{\kappa})$ in the $\varepsilon-\theta$ plane, for the phase portrait in Fig. 7, is composed by: a horizontal line starting at (ε^-,θ^-) and ending at the intersection point with the curve $\theta=\Theta_R^-(\varepsilon)$ (dotted line in Fig. 7), then by the curve $\theta=\Theta_R^-(\varepsilon)$ until its minimum point having the temperature θ^r, next by a horizontal line which ends at the intersection point with the curve $\theta=\Theta_R^+(\varepsilon)$ (dotted line in Fig. 7) and finally by the curve $\theta=\Theta_R^+(\varepsilon)$ until the point (ε^+,θ^+). Thus, for this *non-viscous, heat conducting profile layer*, illustrated in Fig. 8b, the temperature is continuous, but the strain is discontinuous having isothermal jumps inside the profile layer.

The limit of the non-viscous, heat conducting profile layer $(\hat{\theta}(\xi),\hat{\varepsilon}(\xi);0,\bar{\kappa})$, as $\kappa\to 0$ has the following properties. The strain profile is a discontinuous function whose value is ε^- for $\xi<\xi_0$ and ε^+ for $\xi>\xi_0$, while the temperature profile is a discontinuous function whose value is θ^- for $\xi<\xi_0$, θ^r at $\xi=\xi_0$ and θ^+ for $\xi>\xi_0$. It should be noted that as $\mu,\ \kappa\to 0$ the iterated limits coincide and are equal with the double limit for any $\xi\neq\xi_0$. The difference appears at $\xi=\xi_0$ where the iterated limits do not coincide more.

The case $s'(\varepsilon^\pm)=0$, i.e. when $\dot{S}^2=\lambda^2(\varepsilon^\pm,\theta^\pm)$, can be treated as in the previous cases. The reverse implication that the chord criterion is also a necessary condition for the existence of a connecting orbit and its uniqueness can be also proved. We omit here the details.

Let us consider the case when $\theta^r>\theta^+$ in Fig. 7. This means that for any temperature $\bar{\theta}\in(\theta^*,\theta^+)$ the isotherm $\sigma=\sigma_{eq}(\varepsilon,\bar{\theta})$ intersects the Rayleigh line $\sigma=\sigma_R(\varepsilon)$ only at a point in the interval $(\varepsilon^-,\varepsilon^+)$. Two phase portraits similar with those illustrated in Figs. 6 are possible, but we disregard here their analysis.

5.2.2 The entropy production in a viscous, thermally conducting profile layer of Maxwell's type.

Let the pair $(\hat{\varepsilon}(\xi;\mu,\kappa),\hat{\theta}(\xi;\mu,\kappa))$ be a traveling wave solution of the system (57). According to the entropy identity (49) and the dissipation relations (34) established for the Maxwellian rate-type material coupled with the Fourier heat conduction law it follows that the total entropy production in a profile layer is given by

$$\begin{aligned}P_{Mxw}^{trav}&=-\dot{S}\int_{-\infty}^{\infty}\Big(\frac{(E-\rho\dot{S}^2)}{\hat{\theta}}\rho\frac{\partial\psi_{Mxw}(\hat{\varepsilon},\sigma_R(\hat{\varepsilon}),\hat{\theta})}{\partial\sigma}\hat{\varepsilon}'+\frac{1}{\hat{\theta}^2}H_{Mxw}(\hat{\varepsilon},\hat{\theta})\hat{\theta}'\Big)\mathrm{d}\xi=\\&=-\dot{S}\int_{\Gamma}\frac{(E-\rho\dot{S}^2)}{\theta}\rho\frac{\partial\psi_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\theta)}{\partial\sigma}\mathrm{d}\varepsilon+\frac{1}{\theta^2}H_{Mxw}(\varepsilon,\theta)\mathrm{d}\theta\geq 0,\end{aligned}\tag{112}$$

where $\Gamma=\{(\hat{\varepsilon}(\xi;\mu,\kappa),\hat{\theta}(\xi;\mu,\kappa))\mid\xi\in(-\infty,\infty)\ \}$ is the continuous piece-wise smooth curve connecting (ε^-,θ^-) and (ε^+,θ^+) in the $\varepsilon-\theta$ plane. Let us note that, by using relation $(33)_1$, we can prove that the integrand is a total differential. Moreover, one can show that

$$P_{Mxw}^{trav}=-\dot{S}\int_{\Gamma}\mathrm{d}\Big(-\frac{H_{Mxw}(\varepsilon,\theta)}{\theta}+\rho\eta_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\theta)\Big)=-\dot{S}\rho\big(\eta_{eq}(\varepsilon^+,\theta^+)-\eta_{eq}(\varepsilon^-,\theta^-)\big)\geq 0\tag{113}$$

Therefore, the total entropy production in a profile layer does not depend on viscosity or heat conductivity. It is just the entropy production (24) induced by a thermoelastic sharp discontinuity compatible with the second law. As a consequence, in a profile layer structured by Maxwellian viscosity and heat conductivity the entropy of the Hugoniot state (ε^-,θ^-) is never less than the entropy of the initial state (ε^+,θ^+). Therefore, a strong discontinuity which satisfies the selection criterion generated by the Maxwellian rate-type approach coupled with Fourier heat conduction law is compatible with the second law of thermodynamics.

5.2.3 The entropy variation inside a profile layer.

Let us denote by $\theta=\hat{\theta}(\varepsilon;\mu,\kappa)$ the trajectory in the $\varepsilon-\theta$ plane of the Maxwellian viscous, heat conducting traveling wave solution of the problem (57). By using the thermodynamic properties established in Sect. 4.1 and relation (89) one gets that, if the strain profile $\varepsilon=\hat{\varepsilon}(\xi;\mu,\kappa)$ is strictly monotone, the entropy along this trajectory, denoted by $\eta=\hat{\eta}(\varepsilon;\mu,\kappa)\equiv\eta_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\hat{\theta}(\varepsilon;\mu,\kappa))$, satisfies relation

$$\rho\frac{\mathrm{d}\hat{\eta}(\varepsilon;\mu,\kappa)}{\mathrm{d}\varepsilon}=\frac{\rho C_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\hat{\theta}(\varepsilon))}{\hat{\theta}(\varepsilon)}\frac{\mathrm{d}\hat{\theta}(\varepsilon)}{\mathrm{d}\varepsilon}-\frac{\big(E-\rho\dot{S}^2\big)}{\big(E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\tilde{\varepsilon},\hat{\theta}(\varepsilon))\big)}\frac{\partial\sigma_{eq}}{\partial\theta}(\tilde{\varepsilon},\hat{\theta}(\varepsilon)),\tag{114}$$

for ε between ε^- and ε^+, where $\tilde{\varepsilon}(\varepsilon)$ is the unique solution of equation (98) and $\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\varepsilon} = \frac{\mu(E-\rho\dot{S}^2)\dot{S}^2}{\kappa E}\frac{H_{Mxw}(\varepsilon,\hat{\theta})}{R(\varepsilon,\hat{\theta})}$.

Notice that if $\theta = \hat{\theta}(\varepsilon;\mu,\kappa)$ is the trajectory in the $\varepsilon-\theta$ plane of the Kelvin-Voigt viscous, heat conducting traveling wave solution of the problem (60), then the entropy variation along this trajectory, denoted $\eta = \hat{\eta}(\varepsilon;\mu,\kappa) \equiv \eta_{eq}(\varepsilon,\hat{\theta}(\varepsilon;\mu,\kappa))$, can be obtained by simply making $E \longrightarrow \infty$ in (114) and satisfies relation

$$\rho\frac{\mathrm{d}\hat{\eta}(\varepsilon;\mu,\kappa)}{\mathrm{d}\varepsilon} = \frac{\rho C_{eq}(\varepsilon,\hat{\theta}(\varepsilon))}{\hat{\theta}(\varepsilon)}\Big(\frac{\mathrm{d}\hat{\theta}(\varepsilon)}{\mathrm{d}\varepsilon} - \frac{\hat{\theta}(\varepsilon)\frac{\partial\sigma_{eq}}{\partial\theta}(\varepsilon,\hat{\theta}(\varepsilon))}{\rho C_{eq}(\varepsilon,\hat{\theta}(\varepsilon))}\Big), \tag{115}$$

for ε between ε^- and ε^+, where $\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\varepsilon} = \frac{\mu\dot{S}^2}{\kappa}\frac{H_{KV}(\varepsilon,\hat{\theta})}{R(\varepsilon,\hat{\theta})}$.

It is interesting to note that according to (115) the entropy for the Kelvin-Voigt model has a maximum or a minimum inside the profile layer whenever the trajectory $\theta = \hat{\theta}(\varepsilon;\mu,\kappa)$ is tangent to the isentrope (15) crossing it.

a) Monotonous variation of the entropy inside a viscous, heat non-conducting profile layer. We have seen that for fixed $\mu = \bar{\mu}$ and $\kappa \to 0$, $\hat{\varepsilon}(\xi;\bar{\mu},\kappa)$ and $\hat{\theta}(\xi;\bar{\mu},\kappa)$ approach the solution of the reduced system (66), which describes a viscous, heat non-conducting profile layer, and the curves $\theta = \hat{\theta}(\varepsilon;\bar{\mu},\kappa)$ are increasingly close to the curve $\theta = \Theta_{Mxw}(\varepsilon)$. Therefore, by making $\kappa \to 0$ in relation (114) and taking into account that $\lim_{\kappa\to 0}\hat{\theta}(\varepsilon;\bar{\mu},\kappa) = \Theta_{Mxw}(\varepsilon)$ and $\lim_{\kappa\to 0}\frac{\mathrm{d}\hat{\theta}(\varepsilon;\bar{\mu},\kappa)}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\Theta_{Mxw}(\varepsilon)}{\mathrm{d}\varepsilon}$, at the points where the derivative makes sense, we obtain by using relation (97) that

$$\rho\frac{\mathrm{d}\hat{\eta}(\varepsilon;\mu,0)}{\mathrm{d}\varepsilon} = \frac{(E-\rho\dot{S}^2)}{E}\frac{(\sigma_R(\varepsilon)-\sigma_{eq}(\tilde{\varepsilon},\Theta_{Mxw}(\varepsilon)))}{\Theta_{Mxw}(\varepsilon)}, \tag{116}$$

where $\tilde{\varepsilon}(\varepsilon)$ is given by relation (98). One observes that this relation is just relation (72) established when investigating the entropy production in a viscous, heat non-conducting profile layer of Maxwell's type. It reduces to (81) for profile layers of Kelvin-Voigt's type.

It is obvious now that for the compressive case, when $\varepsilon^- < \varepsilon^+$, the chord condition (95) requires that the entropy $\eta = \hat{\eta}(\varepsilon;\bar{\mu},0)$ in a viscous, heat non-conducting profile layer has to be a *strictly decreasing* function of ε, while for the expansive case, $\varepsilon^+ < \varepsilon^-$, it has to be a *strictly increasing* function of ε.

By using continuity arguments, we expect that this property of monotonicity of the entropy inside a profile layer remains valid when the viscosity effect largely dominates the heat conductivity effect.

b) Non-monotonous variation of the entropy inside a non-viscous, heat conducting profile layer. We show now that when the heat conductivity effect largely dominates the viscosity effect the variation of the entropy $\eta = \hat{\eta}(\xi)$, for $\xi \in (-\infty,\infty)$, is *non-monotone* and even more its value can become inside the profile layer lower than the front state value η^+ and/or larger than Hugoniot back state value $\eta^- > \eta^+$. This phenomenon of entropy overshoot or undershoot has been mentioned for instance by Landau and Lifschitz [19, Chap. IX, §87] in gas dynamics and by Dunn and Fosdick [6] in thermoelastic materials.

We remind that for fixed $\kappa = \bar{\kappa}$ and $\mu \to 0$ the pair $\hat{\varepsilon}(\xi;\mu,\bar{\kappa})$ and $\hat{\theta}(\xi;\mu,\bar{\kappa})$ approach the solution of the reduced system (104), which describes a non-viscous, heat conducting profile layer, and its trajectory in the $\varepsilon-\theta$ plane, $\theta = \hat{\theta}(\varepsilon;\bar{\mu},\kappa)$ is increasingly close to the curve $\theta = \bar{\Theta}_R(\varepsilon)$ defined by (105), or by (111), depending on the monotonicity of the function $\theta = \Theta_R(\varepsilon)$. Therefore, by making $\mu \to 0$ in relation (114) and taking into account that $\lim_{\mu\to 0}\hat{\theta}(\varepsilon;\mu,\bar{\kappa}) = \bar{\Theta}_R(\varepsilon)$, and $\lim_{\mu\to 0}\frac{\mathrm{d}\hat{\theta}(\varepsilon;\mu,\bar{\kappa})}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\bar{\Theta}_R(\varepsilon)}{\mathrm{d}\varepsilon}$, at the points where the derivative makes sense, we obtain

$$\rho\frac{\mathrm{d}\hat{\eta}(\varepsilon;0,\bar{\kappa})}{\mathrm{d}\varepsilon} = \frac{\rho C_{Mxw}(\varepsilon,\sigma_R(\varepsilon),\bar{\Theta}_R(\varepsilon))}{\bar{\Theta}_R(\varepsilon)}\frac{\mathrm{d}\bar{\Theta}_R(\varepsilon)}{\mathrm{d}\varepsilon} - \frac{(E-\rho\dot{S}^2)\frac{\partial\sigma_{eq}}{\partial\theta}(\tilde{\varepsilon},\bar{\Theta}_R(\varepsilon))}{\Big(E-\frac{\partial\sigma_{eq}}{\partial\varepsilon}(\tilde{\varepsilon},\bar{\Theta}_R(\varepsilon))\Big)} \tag{117}$$

where $\tilde{\varepsilon}(\varepsilon)$ is the unique solution of equation (37) for $\sigma = \sigma_R(\varepsilon)$ and $\theta = \bar{\Theta}_R(\varepsilon)$.

Let us note that, according to the definition (105), or (111) of function $\theta = \bar{\Theta}_R(\varepsilon)$, the expression of $\frac{\mathrm{d}\bar{\Theta}_R(\varepsilon)}{\mathrm{d}\varepsilon}$ is given by relation (101) on the open intervals on which $\bar{\Theta}_R(\varepsilon) \equiv \Theta_R(\varepsilon)$, or $\frac{\mathrm{d}\bar{\Theta}_R(\varepsilon)}{\mathrm{d}\varepsilon} = 0$ on the open intervals on which $\bar{\Theta}_R(\varepsilon) \neq \Theta_R(\varepsilon)$, i.e. on the intervals on which $\bar{\Theta}_R(\varepsilon)$ is constant.

We are interested to calculate the expression of (117) at $\varepsilon = \varepsilon^{\pm}$. By using relation (101) at the critical points $(\varepsilon^{\pm}, \theta^{\pm})$ one gets

$$\rho \frac{\mathrm{d}\hat{\eta}(\varepsilon^{\pm};0,\bar{\kappa})}{\mathrm{d}\varepsilon} = \begin{cases} \dfrac{\rho^2 C_{eq}(\varepsilon^{\pm},\theta^{\pm})\left(\dot{S}^2-\lambda^2(\varepsilon^{\pm},\theta^{\pm})\right)}{\theta^{\pm}\dfrac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^{\pm},\theta^{\pm})}, & \text{if } \dfrac{\mathrm{d}\bar{\Theta}_R(\varepsilon^{\pm})}{\mathrm{d}\varepsilon} \neq 0, \\ -\dfrac{(E-\rho\dot{S}^2)}{\left(E-\dfrac{\partial \sigma_{eq}}{\partial \varepsilon}(\varepsilon^{\pm},\theta^{\pm})\right)}\dfrac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^{\pm},\theta^{\pm}), & \text{if } \dfrac{\mathrm{d}\bar{\Theta}_R(\varepsilon^{\pm})}{\mathrm{d}\varepsilon} = 0. \end{cases} \tag{118}$$

where $\lambda(\varepsilon,\theta)$ is according to (19) the sound speed of the adiabatic thermoelastic system.

Let us consider for illustration the case *C3*, when $\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^{+},\theta^{+}) > 0$, $\frac{\partial \sigma_{eq}}{\partial \theta}(\varepsilon^{-},\theta^{-}) < 0$ and the phase portrait is illustrated in Fig. 7. We have already established that in this case $\frac{\mathrm{d}\Theta_R(\varepsilon^{+})}{\mathrm{d}\varepsilon} < 0$ and $\dot{S}^2 - \lambda^2(\varepsilon^{+},\theta^{+}) \geq 0$, since $(\varepsilon^{+},\theta^{+})$ is an attractive node. One gets from $(118)_1$ that $\hat{\eta}'(\varepsilon^{+};0,\bar{\kappa}) > 0$. At the critical point $(\varepsilon^{-},\theta^{-})$ where $S^2 - \lambda^2(\varepsilon^{-},\theta^{-}) \leq 0$ we have two possibilities concerning the value of $\frac{\mathrm{d}\Theta_R(\varepsilon^{-})}{\mathrm{d}\varepsilon}$. First, if $\frac{\mathrm{d}\Theta_R(\varepsilon^{-})}{\mathrm{d}\varepsilon} > 0$, like in Fig. 7, then $\frac{\mathrm{d}\bar{\Theta}_R(\varepsilon^{-})}{\mathrm{d}\varepsilon} = 0$ and according to $(118)_2$ it results that $\hat{\eta}'(\varepsilon^{-};0,\bar{\kappa}) > 0$. Second, if $\frac{\mathrm{d}\Theta_R(\varepsilon^{-})}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}\bar{\Theta}_R(\varepsilon^{-})}{\mathrm{d}\varepsilon} < 0$, then according to $(118)_1$ it follows that $\hat{\eta}'(\varepsilon^{-};0,\bar{\kappa}) > 0$.

Therefore, the entropy $\eta = \hat{\eta}(\varepsilon;0,\kappa)$, $\varepsilon \in (\varepsilon^{-},\varepsilon^{+})$, is an increasing function of ε in the neighborhoods of ε^{-} and ε^{+}, $\varepsilon^{-} < \varepsilon^{+}$. Since $\eta^{-} = \hat{\eta}(\varepsilon^{-};0,\kappa) > \eta^{+} = \hat{\eta}(\varepsilon^{+};0,\kappa)$ it results that the entropy in a neighborhood of ε^{-}, for $\varepsilon > \varepsilon^{-}$ is larger than the back state entropy η^{-} and its value in the neighborhood of ε^{+}, for $\varepsilon < \varepsilon^{+}$ is lower than the front state entropy η^{+}. As a result, the entropy has inside the profile layer an interior absolute maxima which overshoots the Hugoniot back state entropy η^{-} and an absolute minima which undershoots the front state entropy η^{+}.

In a similar way one shows that in case *C1* the entropy inside a non-viscous, heat conducting profile layer overshoots the back state entropy $\eta^{-} > \eta^{+}$ and in case *C2* the entropy undershoots the front state entropy η^{+}.

By using continuity arguments one gets that the non-monotonous variation of the entropy and the phenomena of entropy overshoot and entropy undershoot also occur when the heat conductivity effect dominates the viscosity effect.

6 Summary

We consider that knowledge of temperature variation is critical in studies of phase transition phenomena and that the transition from one stable phase to another does not occur instantaneously. For that reason we introduce a dissipative mechanism governed by a Maxwellian rate-type constitutive equation and by heat conduction. The equilibrium of this model is described by a thermoelastic relation with the typical feature that the Grüneisen coefficient changes its sign. The thermodynamic properties of the Maxwellian model are systematically used in investigating the existence, uniqueness and the structure of shock and interphase layers.

We show how steady profiles reflect, on one side, the exothermic or endothermic character of phase transitions, and on the other side, the effect of dissipative mechanisms. It is emphasized that the variation of the temperature inside a viscous, heat non-conducting profile layers results from a balance between the cooling/heating effect due to the latent heat, and the heating effect due to the intrinsic dissipation. Based on this observation additional constitutive assumptions are discussed for phase transforming materials.

For a $\mathscr{M}^{+} \to \mathscr{M}^{-}$ impact induced phase transformation, when the sign of the Grüneisen coefficient changes inside the layer, the temperature variation has a spike-layer form. Therefore, the experimental detection that a particle, during the passage of a wave, can experience lower or larger temperatures than that at its front state and back state could provide valuable information on the presence of an interphase layer and on the time of transition between phases.

We also discuss when the chord criterion with respect to the Hugoniot locus in the strain-stress space is a necessary and sufficient condition for the existence of a profile layer and its role as admissibility condition for discontinuous solutions of the adiabatic thermoelastic system.

The profound difference in the effect of viscosity and of heat conduction on the structure of the profile layers (possible existence of isothermal jumps inside a profile layer) and on the behavior of the entropy inside the profile layer (the phenomenon of entropy overshoot, or undershoot) have been discussed.

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References

1. Abeyaratne, R., Knowles, J. K.: A continuum model of a thermoelastic solid capable of undergoing phase transitions. J. Mech. Phys. Solids **41**, 541–571 (1993)
2. Abeyaratne, R., Knowles, J. K.: Impact-induced phase transitions in thermoelastic solids. Phil. Trans. R. Soc. Lond. A **355**, 843–867 (1997)
3. Abeyaratne, R., Kim, S-J., Knowles, J. K.: A one-dimensional continuum model for shape memory alloys. Int. J. Solids Structures **31**, 2229–2249 (1994)
4. Barker, L. M., in Behavior of Dense Media Under High Dynamic Pressures, 483–504, Gordon and Breach, New York (1968)
5. Coleman, B. D., Noll, W.: On the thermostatics of continuous media. Arch. Rat. Mech. Anal. **4**, 97–128 (1959)
6. Dunn, J. E., Fosdick, R. L.: Steady, structured shock waves. Part 1:Thermoelastic materials. Arch. Rat. Mech. Anal. **104**, 295–365 (1988)
7. Drumheller, D. S.: Introduction to Wave Propagation in Nonlinear Fluids and Solids. Cambridge University Press, (1998)
8. Ericksen, J. L.: Equilibrium of bars. J. Elasticity **5**, 191–201 (1975)
9. Făciu, C., Molinari, A.: On the longitudinal impact of two phase transforming bars. Elastic versus a rate-type approach. Part I: The elastic case. Part II: The rate-type case. Int. J. Solids Structures **43**, 497–522 and 523–550 (2006)
10. Făciu, C., Molinari, A.: The structure of shock and interphase layers for a heat conducting Maxwellian rate-type approach to solid-solid phase transitions. Part II: Numerical study for a SMA model (?)
11. Făciu, C., Mihăilescu-Suliciu, M.: On modelling phase propagation in SMAs by a Maxwellian thermo-viscoelastic approach. Int. J. Solids Structures **39**, 3811–3830 (2002)
12. Făciu, C., Mihăilescu-Suliciu, M.: On a rate-type themomechanical model for shape memory bars. Stress-controlled problems. Z. angew. Math. Phys. - ZAMP, **53**, 1014–1025, 2002
13. Făciu, C., Mihăilescu-Suliciu, M.: Shape Memory Effect: A Maxwellian rate-type constitutive approach. In: M. Mihăilescu-Suliciu, (ed) "New Trends in Continuum Mechanics", 139-145, Theta Publisher, Bucharest, (2005)
14. Gall, K., Sehitoglu, H., Chumlyakov, Y. I., Kireeva, I. V.: Tension-compression asymmetry of the stress-strain response in aged single crystal and polycrystalline NiTi. Acta Materialia, **47**, 1203–1217 (1999)
15. Gilbarg, D.: The existence and limit behavior of the one-dimensional shock layer. American Journal of Mathematics **73**(2), 256–274 (1951)
16. James, R.D.: The propagation of phase boundaries in elastic bars. Arch. Rational Mech. Anal. **73**, 125–158 (1980)
17. James, R.D.: A relation between the jump in temperature across a propagating phase boundary and the stability of solid phases. J. Elasticity **13**, 357–378 (1983)
18. Knowles, J. K.: On the structure of the Hugoniot relation for a shock-induced martensitic transformation. Shock Waves. **17**, 421–432 (2008)
19. Landau, L. D., Lifschitz, E. M.: Mécanique des fluides. Editions Mir, Moscou, (1971)
20. Lax, P. D.: Hyperbolic systems of conservation laws II. Commun. Pure Appl. Math. **10**, 537–566 (1957)
21. Liu, T.-P.: The entropy condition and the admissibility of shocks. J. Math. Anal. Appl. **53**, 78–88 (1976)
22. Liu, T.-P.: Shock waves in the nonisentopic gas flow. J. Diff. Eqs. **22**, 442–452(1976)
23. Menikoff, R., Plohr, J. B.: The Riemann problem for fluid flow of real materials. Rev. Modern Phys. **61**, 75–130 (1989)
24. Molinari, A., Ravichandran, G.: Fundamental structure of steady plastic shock waves in metals. J. Appl. Phys. **95**, 1718–1732 (2004)
25. Oleinik, O.: Uniqueness and stability of the generalized solution of the Cauchy problem for a quasi-linear equation. Usp. Mat. Nauk **14**, 165–170, (1959)
26. Pego, R. L.: Nonexistence of a shock layer in gas dynamics with a nonconvex equation of state. Arch. Rational Mech. Anal. **94**, 165–178 (1986)
27. Pego, R. L.: Phase transitions in one-dimensional nonlinear viscoelasticity: admissibility and stability. Arch. Rational Mech. Anal. **87**, 353–394 (1987)
28. Shaw, J.A.: Simulations of localized thermo-mechanical behavior in a NiTi shape memory alloy. Int. J. Plasticity **16**, 541–562 (2000)
29. Shaw, J.A., Kyriakides, S.: On the nucleation and propagation of phase transformation fronts in a NiTi alloy. Acta Materialia **45**, 683–700 (1997)
30. Slemrod, M.: Admissibility criteria for propagating phase boundaries in a van der Waals fluid. Arch. Rational Mech. Anal. **81**, 301–315 (1983)
31. Slemrod, M.: Dynamic phase transitions in a van der Waals fluid. J. Diff. Eqs. **52**, 1–23 (1984)
32. Tobushi, H., Iwanaga, H., Tanaka, K., Hori, T., Sawada, T.: Deformation behaviour of TiNi shape memory alloy subjected to variable stress and temperature. Continuum. Mech. Thermodyn. **3**, 79 – 93, (1991)
33. Truskinovski, L.: Dynamics of non-equilibrium phase boundaries in a heat-conducting non-linearly elastic medium. J. Appl. Math. Mech. (PMM USSR) **51**, 777–784 (1987)
34. Weyl, H.: Shock waves in arbitrary fluids. Comm. Pure Appl. Math. **2**, 103–122 (1949)
35. Uchil, J., Mohanchandra, K., Ganesh Kumara, K., Mahesh, K.K., Murali, T.P.: Thermal expansion in various phases of Nitinol using TMA. Physica B **270**, 289–297 (1999)